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On the double of the (restricted) super Jordan plane

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ABSTRACT. In this paper we consider the super Jordan plane, a braided Hopf algebra introduced–to the best of our knowledge–in [AAH1], and its restricted version in odd characteristic introduced in [AAH2]. We show that their Drinfeld doubles give rise naturally to Hopf superalgebras justifying a posteriori the adjective *super* given in *loc. cit*. These Hopf superalgebras are extensions of super commutative ones by the enveloping, respectively restricted enveloping, algebra of $\mathfrak{osp}(1|2)$.

CONTENTS

1. Introduction

The context. Let $\Bbbk$ be an algebraically closed field. The super Jordan plane is the graded algebra $\mathcal{B}$ presented by generators x_1, x_2 with defining relations

$$x_1^2, \qquad x_2x_{21} - x_{21}x_2 - x_1x_{21}, \tag{1.1}$$

where $x_{21} = x_2x_1 + x_1x_2$. It is known that $\mathcal{B}$ has Gelfand-Kirillov dimension 2. The braided vector space (V, c) with basis $\{x_1, x_2\}$ and braiding

$$c(x_i \otimes x_1) = -x_1 \otimes x_i, \quad c(x_i \otimes x_2) = (-x_2 + x_1) \otimes x_i, \quad i = 1, 2, \tag{1.2}$$

determines a structure of braided Hopf algebra on $\mathcal{B}$ in the sense of [T].

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The Jordan plane and the super Jordan plane play a central role in the study of pointed Hopf algebras over abelian groups with finite Gelfand-Kirillov dimension in [AAH1], assuming $\operatorname{char}\Bbbk = 0$.

Let $p > 2$ be a prime. Assume that $\operatorname{char}\Bbbk = p$. Motivated by [CLW] that deals with the restricted Jordan plane, it is shown in [AAH2] that many analogues of the Nichols algebras from [AAH1] have finite dimension (see [ABDF] for examples in characteristic 2). Among them the *restricted* super Jordan plane i.e. the algebra presented by x_1, x_2 with relations (1.1) and

$$x_{21}^p, \qquad\qquad x_2^{2p}. \tag{1.3}$$

We began the study of the Drinfeld doubles of (suitable bosonizations of) the Jordan plane and its restricted analogue in [AP] showing among other results that they fit into exact sequences relating the enveloping algebra of $\mathfrak{sl}_2(\Bbbk)$, the algebras of functions on some algebraic groups and their restricted analogues. In the present paper we carry out a similar analysis of the super Jordan plane and its restricted version. We start in Section 2 with brief discussions on various topics needed later: Yetter-Drinfeld supermodules, restricted Lie superalgebras and Nichols algebras. In Section 3 we first record presentations of duals and doubles of finite-dimensional Hopf algebras arising as bosonizations; this is essentially straightforward but useful for further computations. Then we present the objects of our interest: the super Jordan plane, its restricted version and their duals. Finally we deal with different descriptions as bosonization of the same Hopf algebra. This allows to define alternatively the Hopf superalgebras $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ discussed below.

The double of the super Jordan plane. Here we just need that $\operatorname{char}\Bbbk \neq 2$. As in [AAH1] we realize the braided vector space (V, c) with c given by (3.6) as Yetter-Drinfeld module over the group algebra $\Bbbk\mathbb{Z}$, hence we have the Hopf algebra $\mathtt{H} := \mathcal{B}\#\Bbbk\mathbb{Z}$. Then we consider the dual super Jordan plane $\mathcal{B}^d$ described just before Lemma 3.6. The Sweedler dual of $\Bbbk\mathbb{Z}$ is spanned by the characters of $\mathbb{Z}$ and the Lie algebra of the one-dimensional torus. Hence its smallest Hopf subalgebra realizing $\mathcal{B}^d$ is isomorphic to $\Bbbk[\zeta] \otimes \Bbbk C_2$. Here C_N stands for the cyclic group of order N and ζ is an indeterminate. Thus we may consider $\mathtt{K} := \mathcal{B}^d\#(\Bbbk[\zeta] \otimes \Bbbk C_2)$ and define $\mathtt{D} = \mathtt{H} \bowtie \mathtt{K}^{\mathrm{op}}$ with respect to a suitable pairing between $\mathtt{H}$ and $\mathtt{K}^{\mathrm{op}}$. It turns out that there exists a Hopf superalgebra $\widetilde{\mathcal{D}}$ such that $\mathtt{D} \simeq \widetilde{\mathcal{D}}\#\Bbbk C_2$; this justifies a posteriori the adjective super given to $\mathcal{B}$. Thus the study of $\mathtt{D}$ reduces to that of $\widetilde{\mathcal{D}}$. We present basic properties of $\widetilde{\mathcal{D}}$ in Proposition 4.3 including the defining relations and a PBW-basis. Next we show in Theorem 4.5 that $\widetilde{\mathcal{D}}$ fits into an exact sequence of Hopf superalgebras $\mathcal{O}(\mathfrak{G}) \hookrightarrow \widetilde{\mathcal{D}} \twoheadrightarrow U(\mathfrak{osp}(1|2))$, where $\mathfrak{G}$ is an algebraic supergroup explicitly described. For the next result, Theorem 4.7, we need $\operatorname{char}\Bbbk = p > 2$; then $\widetilde{\mathcal{D}}$ is free module of finite rank over a central Hopf subalgebra $Z = \mathcal{O}(\mathbf{B})$ where $\mathbf{B}$ is a solvable connected algebraic group. We close Section 4 establishing some basic ring-theoretical properties of $\widetilde{\mathcal{D}}$.

The double of the restricted super Jordan plane. In Section 5 we assume that $\operatorname{char}\Bbbk = p > 2$. We realize (V, c) with the braiding (3.6) in ${}^{\Bbbk C_{2p}}_{\Bbbk C_{2p}}\mathcal{YD}$. Let $D(H)$ be the Drinfeld double of the bosonization $H = \mathscr{B}(V)\#\Bbbk C_p$. Again there exists a Hopf superalgebra $\mathcal{D}$ such that $D(H) \simeq \mathcal{D}\#\Bbbk C_2$, thus we may focus on $\mathcal{D}$. We present basic properties of $\mathcal{D}$ in Proposition 5.4 and show in Theorem 5.6 an exact sequence of Hopf superalgebras $\mathbf{R} \overset{\iota}{\hookrightarrow} \mathcal{D} \overset{\pi}{\twoheadrightarrow} \mathfrak{u}(\mathfrak{osp}(1|2))$ where $\mathbf{R}$ is a local commutative Hopf algebra and $\mathfrak{u}(\mathfrak{osp}(1|2))$ is the restricted enveloping algebra. We conclude that the simple $\mathcal{D}$-modules are the same as those of $\mathfrak{u}(\mathfrak{osp}(1|2))$ and we present them as quotients of Verma modules. See Theorem 5.7 and Proposition 5.12.

The extensions of Hopf superalgebras mentioned above fit into a 9-term commutative diagram where all columns and rows are exact sequences:

$$\begin{tikzcd}
\mathcal{O}(\mathbf{G}) \arrow[r, hook] \arrow[d, hook] & \mathcal{O}(\mathbf{B}) \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}(\mathbf{G}_a^3) \arrow[d, hook] \\
\mathcal{O}(\mathfrak{G}) \arrow[r, hook] \arrow[d, two heads] & \widetilde{\mathcal{D}} \arrow[r, two heads] \arrow[d, two heads] & U(\mathfrak{osp}(1|2)) \arrow[d, two heads] \\
\mathbf{R} \arrow[r, hook] & \mathcal{D} \arrow[r, two heads] & \mathfrak{u}(\mathfrak{osp}(1|2))
\end{tikzcd} \tag{1.4}$$

See Theorem 5.14. This is an analogue of [AP, (0.2)] for the Jordan plane.

2. Recollections

2.1. Conventions. We denote by $\mathbb{F}_p$ the field of p elements. If $\ell < n \in \mathbb{N}_0$, then we set $\mathbb{I}_{\ell,n} = \{\ell, \ell+1, \dots, n\}$, $\mathbb{I}_n = \mathbb{I}_{1,n}$. Let A be an algebra and $a_1, \dots, a_n \in A$, $n \in \mathbb{N}$. We denote by $\Bbbk\langle a_1, \dots, a_n\rangle$ the subalgebra generated by $a_1, \dots, a_n$. We identify $\mathcal{V}^* \otimes \mathcal{V}^* \simeq (\mathcal{V} \otimes \mathcal{V})^*$, where $\mathcal{V}$ is a finite-dimensional vector space, by

$$\langle f \otimes f', x \otimes y\rangle = \langle f, x\rangle\langle f', y\rangle, \qquad f, f' \in \mathcal{V}^*, x, y \in \mathcal{V}. \tag{2.1}$$

The cyclic group of order n is denoted by C_n and the infinite cyclic group by Γ. They are always written multiplicatively. The center of a group G is denoted $Z(G)$; similarly for the center of an algebra.

Let L be a Hopf algebra. The kernel of the counit ε is denoted L^+, the antipode (always assumed bijective) by $\mathcal{S}$ or by $\mathcal{S}_L$, the space of primitive elements of L is denoted by $\mathcal{P}(L)$ and the group of group-likes by $G(L)$. The space of g, h-primitives is $\mathcal{P}_{g,h}(L) = \{x \in L : \Delta(x) = x \otimes h + g \otimes x\}$ where $g, h \in G(L)$. We assume that the reader has some familiarity with the theory of Hopf algebras particularly with the notions of Yetter-Drinfeld module and bosonization (or biproduct), see e.g. [R]. The category of Yetter-Drinfeld modules over L is denoted by ${}^L_L\mathcal{YD}$.

The category of super vector spaces is denoted by sVec. If $\mathcal{X} \in \mathsf{sVec}$, and $x \in \mathcal{X}_i$, then we write $|x| = i$. We set $|\mathcal{X}| := \mathcal{X}_0 \cup \mathcal{X}_1$. The symmetric tensor category sVec is identified with a full tensor subcategory of ${}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$. An object $M \in {}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$ is in sVec if the two following conditions are satisfied:

(0) For every $a \in M$ such that $\epsilon \rightharpoonup a = a$, $\delta(a) = 1 \otimes a$ (then a is even).
(1) For every $a \in M$ such that $\epsilon \rightharpoonup a = -a$, $\delta(a) = \epsilon \otimes a$ (then a is odd).

Let $\mathcal{A}$ be a superalgebra, i.e. an algebra in sVec; $\mathcal{A}$ is super commutative if $ab = (-1)^{|a||b|}ba$ for all $a, b \in |\mathcal{A}|$.

The algebra of regular functions on an (affine) algebraic super group G is denoted $\mathcal{O}(G)$. This is by definition a finitely generated super commutative Hopf algebra. See [M] for the definitions. As usual, $\mathbf{G}_a$ is the additive algebraic group $(\Bbbk, +)$ and $\mathbf{G}_m$ is the multiplicative algebraic group $(\Bbbk^\times, \cdot)$.

Let R be the polynomial algebra $\Bbbk[\zeta]$ with the unique Hopf algebra structure such that $\zeta \in \mathcal{P}(R)$. Let R_p be the quotient Hopf algebra $R/(\zeta^p - \zeta)$.

Let $V = \Bbbk\{X_1, \dots, X_m\}$ be a vector space of dimension $m \in \mathbb{N}$. We denote by $\Lambda(V) = \Lambda(X_1, \dots, X_m)$ the exterior algebra of V.

We denote by $[k]^{[n]} := \prod_{i=1}^n (k + i - 1)$ $k \in \Bbbk$, $n \in \mathbb{N}$ the raising factorial, we also define $[k]^{[0]} := 1$ for every $k \in \Bbbk$. The unsigned Stirling numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined as the coefficients of the 'raising factorial' polynomial:

$$[X]^{[n]} = \prod_{i=1}^{n} (X + i - 1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} X^k \in \mathbb{Z}[X].$$

Recall that a sequence of morphisms of Hopf (super)algebras $A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ is exact, and C is an extension of A by B, if

(i) ι is injective.
(ii) π is surjective.
(iii) $\ker \pi = C\iota(A)^+$.
(iv) $\iota(A) = C^{\operatorname{co}\pi}$.

We write $B = C /\!/ A$ i.e. B is the Hopf algebra quotient of C by A.

Remark 2.1. If $A \overset{\iota}{\hookrightarrow} C$ is faithfully flat and $\iota(A)$ is stable by the left adjoint action of C then (i), (ii) and (iii) imply (iv), see [AD, 1.2.5, 1.2.14], [Sch].

2.2. A brief review of Yetter-Drinfeld supermodules. In this subsection, $\operatorname{char} \Bbbk \neq 2$. See [AAY] for more details. Let $\mathcal{A}$ be a Hopf superalgebra. The category of Yetter-Drinfeld supermodules over $\mathcal{A}$, denoted by ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}$, consists of super vector spaces $\mathcal{X}$ such that:

(i) $\mathcal{X}$ is a left supermodule over $\mathcal{A}$ with action $\rightharpoonup$.
(ii) $\mathcal{X}$ is a left supercomodule over $\mathcal{A}$ with coaction δ.
(iii) For every $a \in |\mathcal{A}|$ and $u \in |\mathcal{X}|$:

$$\delta(a \rightharpoonup x) = (-1)^{|v_{(-1)}|(|a_{(2)}|+|a_{(3)}|)+|a_{(2)}||a_{(3)}|} a_{(1)} x_{(-1)} \mathcal{S}(a_{(3)}) \otimes a_{(2)} \rightharpoonup x_{(0)}.$$

Then ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}$ is a braided tensor category. Namely, if $\mathcal{X}, \mathcal{Z} \in {}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}$, then the super vector space $\mathcal{X} \otimes \mathcal{Z}$ is an object in ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}$ via

$$a \rightharpoonup (x \otimes z) = (-1)^{|a_{(2)}||x|} a_{(1)} \rightharpoonup x \otimes a_{(2)} \rightharpoonup z,$$

$$\delta(x\otimes z)=(-1)^{|x_{(0)}||z_{(-1)}|}x_{(-1)}z_{(-1)}\otimes(x_{(0)}\otimes z_{(0)}),\quad \forall x\in|\mathcal{X}|, z\in|\mathcal{Z}|.$$

The braiding $c\colon \mathcal{X}\otimes\mathcal{Z}\longrightarrow\mathcal{Z}\otimes\mathcal{X}$ is given by:

$$c(x\otimes z)=(-1)^{|x_{(0)}||z|}x_{(-1)}\rightharpoonup z\otimes x_{(0)}. \tag{2.2}$$

There is an embedding functor of braided tensor categories

$$\mathfrak{i}\colon {}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}\longrightarrow {}^{\mathcal{A}\#\Bbbk C_2}_{\mathcal{A}\#\Bbbk C_2}\mathcal{YD}, \qquad \mathcal{X}\longmapsto \mathfrak{i}(\mathcal{X}) \tag{2.3}$$

where $\mathfrak{i}(\mathcal{X})=\mathcal{X}$ as vector space with action $\rightharpoonup$ and coaction δ as:

$$\begin{aligned} a\#\epsilon^k\rightharpoonup x&=(-1)^{|x|k}a\rightharpoonup x,\\ \delta(x)&=x_{(-1)}\#\epsilon^{|x_{(0)}|}\otimes x_{(0)}, \end{aligned}\qquad \forall a\in\mathcal{A}, x\in|\mathcal{X}|, k\in\mathbb{I}_{0,1}.$$

The super bosonization [AAY]**.** Let $\mathcal{R}$ be a Hopf algebra in ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}$. The comultiplication is written $\Delta_{\mathcal{R}}(h)=h^{(1)}\otimes h^{(2)}$ for $h\in\mathcal{R}$. The Hopf superalgebra $\mathcal{R}\natural\mathcal{A}$ is $\mathcal{R}\otimes\mathcal{A}$ as a super vector space with structure given by:

$$\begin{aligned} (h\natural a)(f\natural b)&\coloneqq(-1)^{|a_{(2)}||f|}h(a_{(1)}\rightharpoonup f)\natural a_{(2)}b,\\ \Delta(h\natural a)&\coloneqq(-1)^{|(h^{(2)})_{(0)}||a_{(1)}|}h^{(1)}\natural(h^{(2)})_{(-1)}a_{(1)}\otimes(h^{(2)})_{(0)}\natural a_{(2)},\\ \varepsilon(h\natural a)&=\varepsilon_{\mathcal{R}}(h)\varepsilon_{\mathcal{A}}(a),\qquad 1\coloneqq 1_{\mathcal{R}}\natural 1_{\mathcal{A}},\\ \mathcal{S}(h\natural a)&=(-1)^{|h_{(0)}||a|}(1\natural\mathcal{S}_{\mathcal{A}}(h_{(-1)}a))(\mathcal{S}_{\mathcal{R}}(h_{(0)})\natural 1). \end{aligned}$$

There is a canonical isomorphism $(\mathcal{R}\natural\mathcal{A})\#\Bbbk C_2\simeq\mathfrak{i}(\mathcal{R})\#(\mathcal{A}\#\Bbbk C_2)$.

2.3. Restricted Lie superalgebras. A Lie superalgebra $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ is restricted if $\mathfrak{g}_0$ is a restricted Lie algebra with p-operation $x\mapsto x^{[p]}$ such that

$$[x^{[p]},z]=(\operatorname{ad}x)^p(z) \qquad \text{for all } x\in\mathfrak{g}_0, z\in\mathfrak{g}.$$

We refer to [BMPZ, Chapter 3] for a detailed exposition. The *restricted enveloping algebra* of the restricted Lie superalgebra $\mathfrak{g}$ is defined as

$$\mathfrak{u}(\mathfrak{g})\coloneqq U(\mathfrak{g})/\langle x^{[p]}-x^p : x\in\mathfrak{g}_0\rangle,$$

where $U(\mathfrak{g})$ is the enveloping algebra. Assume for simplicity that $\dim\mathfrak{g}<\infty$ and let $\{x_1,\dots,x_r\}$, $\{y_1,\dots,y_k\}$ be bases of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ respectively. Then the Hopf superalgebra $\mathfrak{u}(\mathfrak{g})$ has dimension p^r2^k; indeed it has a PBW-basis

$$\{x_1^{n_1}\cdots x_r^{n_r}y_1^{m_1}\cdots y_k^{m_k}\colon n_1,\dots,n_r\in\mathbb{I}_{0,p-1}, m_1,\dots,m_k\in\mathbb{I}_{0,1}\}.$$

2.3.1. The ortho-symplectic Lie superalgebra $\mathfrak{g}=\mathfrak{osp}(1|2)$. For our purposes we recall its structure: $\mathfrak{g}_0\simeq\mathfrak{sl}_2(\Bbbk)$ with Cartan generators $\{e,f,h\}$; $\mathfrak{g}_1=\Bbbk\{\psi_+,\psi_-\}$ is the natural $\mathfrak{sl}_2(\Bbbk)$-module, hence the bracket is

$$\begin{aligned} &[e,\psi_+]=0, && [h,\psi_+]=\psi_+, && [f,\psi_+]=\psi_-,\\ &[e,\psi_-]=\psi_+, && [h,\psi_-]=-\psi_-, && [f,\psi_-]=0,\\ &[\psi_+,\psi_+]=2e, && [\psi_-,\psi_-]=-2f, && [\psi_+,\psi_-]=-h. \end{aligned}$$

Assume now that $\operatorname{char}\Bbbk=p$. The algebra $\mathfrak{g}$ has a p-structure given by

$$e^{[p]}=0,\qquad h^{[p]}=h,\qquad f^{[p]}=0.$$

Below we consider the enveloping algebra $U(\mathfrak{osp}(1|2))$ and its restricted version $\mathfrak{u}(\mathfrak{osp}(1|2))$ which are Hopf superalgebras with suitable PBW-bases.

2.4. Nichols algebras. A braided vector space is a pair $(\mathcal{V}, c)$ where $\mathcal{V}$ is a vector space and $c \in GL(\mathcal{V}^{\otimes 2})$ satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Then the braid group $\mathbb{B}_n$ acts on $\mathcal{V}^{\otimes n}$; the Nichols algebra $\mathscr{B}(\mathcal{V})$ is defined taking the quotient of the quantum symmetrizer in each degree. See [A] for details. A realization of $(\mathcal{V}, c)$ over a Hopf algebra L is a structure of Yetter-Drinfeld L-module on $\mathcal{V}$ such that c coincides with the categorical braiding. Then $\mathscr{B}(\mathcal{V})$ is a Hopf algebra in ${}^L_L\mathcal{YD}$ and the bosonization $\mathscr{B}(\mathcal{V})\#L$ is a Hopf algebra. If $\mathcal{V}$ is finite-dimensional, then the dual vector space $\mathcal{V}^*$ is braided with the transpose braiding c^*–recall the identification (2.1):

$$\langle c^*(f \otimes f'), x \otimes y\rangle = \langle f \otimes f', c(x \otimes y)\rangle, \qquad f, f' \in \mathcal{V}^*, x, y \in \mathcal{V}. \tag{2.4}$$

Similarly a realization of $(\mathcal{V}, c)$ over a Hopf superalgebra $\mathcal{A}$ is a structure of Yetter-Drinfeld $\mathcal{A}$-supermodule on $\mathcal{V}$ such that c coincides with the categorical braiding. Then $\mathscr{B}(\mathcal{V})$ is a Hopf algebra in ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{YDS}$ and the bosonization $\mathscr{B}(\mathcal{V})\#\mathcal{A}$ is a Hopf superalgebra. Notice that there is an isomorphism $\mathfrak{i}(\mathscr{B}(\mathcal{V})) \simeq \mathscr{B}(\mathfrak{i}(\mathcal{V}))$ where $\mathfrak{i}$ is as in (2.3), see [AAY, §1.7].

If $(\mathcal{V}, c)$ admits a realization over a Hopf superalgebra, then necessarily $\mathcal{V}$ is C_2-graded and c preserves the grading of $\mathcal{V} \otimes \mathcal{V}$; such $(\mathcal{V}, c)$ might be called a super braided vector space, a concept already present in [KS, (53), p. 1610]. Indeed there is bijective correspondence between super braided vector spaces and solutions of the super Yang-Baxter equation, see *loc. cit.*

3. Preliminaries

3.1. The dual and the double of a bosonization. Let L be a Hopf algebra and $\mathcal{V} \in {}^L_L\mathcal{YD}$; we assume that $\dim L < \infty$ and $\dim \mathscr{B}(\mathcal{V}) < \infty$. In this subsection we describe explicitly the dual and the double of the bosonization $A := \mathscr{B}(\mathcal{V})\#L$. In particular we show that $A^* \simeq \mathscr{B}(\mathcal{V}^*)\#L^*$, and that $\mathcal{V}^*$ has the transpose braiding (2.4). We need some notation. First, we have morphisms of Hopf algebras $A \underset{\iota}{\overset{\pi}{\rightleftarrows}} L$ such that $\pi\iota = \mathrm{id}$. Next we fix

- a basis $\{v_1, \dots, v_n\}$ of $\mathcal{V}$; its dual basis is denoted by $\{w_1, \dots, w_n\}$;
- a basis $\{h_1, \dots, h_m\}$ of L; its dual basis is denoted by $\{f_1, \dots, f_m\}$.

The braided tensor categories ${}^L_L\mathcal{YD} \simeq {}^{L^*}_{L^*}\mathcal{YD}$ are equivalent via the functor

$$F\colon {}^L_L\mathcal{YD} \longrightarrow {}^{L^*}_{L^*}\mathcal{YD}$$

defined as follows. If $\mathcal{X} \in {}^L_L\mathcal{YD}$, then $F(\mathcal{X}) = \mathcal{X}$ with structure

$$f \rightharpoonup v = \langle f, \mathcal{S}(v_{(-1)})\rangle v_{(0)}, \quad \delta(v) = \sum_{i=0}^{m} \mathcal{S}^{-1}(f_i) \otimes h_i \rightharpoonup v, \quad v \in \mathcal{X}, f \in L^*,$$

see [AG, Prop. 2.2.1]. Let $\mathcal{W} := F(\mathcal{V}^*)$.

Lemma 3.1. [G, Lemma 2.6] *As a braided vector space, $\mathcal{W} \in {}^{L^*}_{L^*}\mathcal{YD}$ is isomorphic to $(\mathcal{V}^*, c^*)$. Hence* $\dim \mathscr{B}(\mathcal{W}) = \dim \mathscr{B}(\mathcal{V}) < \infty$. □

Proposition 3.2. $A^* \simeq \mathscr{B}(\mathcal{W})\#L^*$.

Proof. By transposing the maps π and ι above, we get maps $A^* \underset{\pi^*}{\overset{\iota^*}{\rightleftarrows}} L^*$ with $\iota^*\pi^* =$ id. Then $A^* \simeq \mathcal{R}\#L^*$, where $\mathcal{R} = (A^*)^{\operatorname{co}\iota^*}$ is a braided Hopf algebra in ${}^{L^*}_{L^*}\mathcal{YD}$. Since $A = \bigoplus_{n\in\mathbb{N}_0} A^n$ is graded, so are $A^* = \bigoplus_{n\in\mathbb{N}_0}(A^*)^n$ and $\mathcal{R} = \bigoplus_{n\in\mathbb{N}_0}\mathcal{R}^n$. We proceed in three steps:

(i) Find a basis of $\mathcal{R}^1 \in {}^{L^*}_{L^*}\mathcal{YD}$. Since $A^1 \simeq \mathcal{V}\otimes L$, $(A^*)^1 \simeq \mathcal{V}^*\otimes L^*$ as a vector space. Given $w \in \mathcal{V}^*$ and $f \in L^*$ we set $w\underline{\#}f := w\otimes f \in (A^*)^1$ by

$$\langle w\underline{\#}f, v\#h\rangle = \langle w, v\rangle\langle f, h\rangle, \qquad \forall v \in \mathcal{V}, h \in L.$$

Then $\{w_i\underline{\#}f_j : i \in \mathbb{I}_n, j \in \mathbb{I}_m\}$ is a basis of $(A^*)^1$. Let $\overline{w}_i := w_i\underline{\#}\varepsilon$, $i \in \mathbb{I}_n$. We claim that $\{\overline{w}_1, \dots, \overline{w}_n\}$ is a basis of $\mathcal{R}^1$. For this, we compute

$$\begin{aligned}\langle\Delta(\overline{w}_i), 1\#h_\ell\otimes v_j\#h_k\rangle &= \langle w_i, h_\ell \rightharpoonup v_j\rangle\varepsilon(h_k),\\ \langle\Delta(\overline{w}_i), v_j\#h_k\otimes 1\#h_\ell\rangle &= \varepsilon(h_kh_\ell)\delta_{i,j},\end{aligned}$$

for all $i,j \in \mathbb{I}_n$, $k,\ell \in \mathbb{I}_m$. Since Δ preserves the grading, we have

$$\begin{aligned}\Delta(\overline{w}_i) &= \sum_{\ell,k=1}^m \varepsilon(h_kh_\ell)w_i\underline{\#}f_k\otimes f_\ell + \sum_{\ell,k=1}^m\sum_{j=1}^n\langle w_i, h_\ell\rightharpoonup v_j\rangle\varepsilon(h_k)f_\ell\otimes w_j\underline{\#}f_k\\ &= \overline{w}_i\otimes 1 + \sum_{\ell=1}^m\sum_{j=1}^n\langle w_i, h_\ell\rightharpoonup v_j\rangle f_\ell\otimes\overline{w}_j.\end{aligned}$$

Then $\overline{w}_i \in \mathcal{R}^1$ because $\iota^*(\overline{w}_j) = 0$ for all $j \in \mathbb{I}_n$. Since the $\overline{w}_i$'s are linearly independent and $\dim\mathcal{R}^1 = \dim\mathscr{B}(\mathcal{V})^1 = \dim\mathcal{V}$, the claim follows.

(ii) $\mathcal{R}^1 \simeq \mathcal{W}$ in ${}^{L^*}_{L^*}\mathcal{YD}$. To compute the action and coaction of $\mathcal{R}^1$ we need the multiplication of elements of the form $(w\underline{\#}f)f'$ and $f'(w\underline{\#}f)$ for $f, f' \in L^*$, $w \in \mathcal{V}^*$. Since $A^0 \simeq L$, $(A^*)^0 \simeq L^*$, we see that

$$\begin{aligned}\langle(w\underline{\#}f)f', v\#h\rangle &= \langle w\underline{\#}ff', v\#h\rangle,\\ \langle f'(w\underline{\#}f), v\#h\rangle &= \langle f'_{(1)}\rightharpoonup w\underline{\#}f'_{(2)}f, v\#h\rangle\end{aligned}$$

for all $v \in \mathcal{V}$, $h \in L$. Then

$$f'(w\underline{\#}f) = f'_{(1)}\rightharpoonup w\underline{\#}f'_{(2)}f \qquad\text{and}\qquad (w\underline{\#}f)f' = w\underline{\#}ff'$$

for $w \in \mathcal{V}^*$, $f, f' \in L^*$. Hence the action of L^* in $\mathcal{R}^1$ is given by

$$\begin{aligned}f\rightharpoonup\overline{w}_i &= f_{(1)}(w_i\underline{\#}\varepsilon)\mathcal{S}(f_{(2)}) = f_{(1)}\rightharpoonup w_i\underline{\#}f_{(2)}\mathcal{S}(f_{(3)})\\ &= f_{(1)}\rightharpoonup w_i\underline{\#}\varepsilon(f_{(2)})\varepsilon = f\rightharpoonup w_i\underline{\#}\varepsilon\end{aligned}$$

$$= \sum_{j=1}^{n} \langle f, (v_j)_{(-1)} \rangle \langle w_i, (v_j)_{(0)} \rangle \overline{w}_j, \quad i \in \mathbb{I}_n, f \in L^*.$$

The formula for the coaction follows from the comultiplication above:

$$\delta(\overline{w}_i) = (\iota^* \otimes \mathrm{id}) \circ \Delta(\overline{w}_i) = \sum_{\ell=1}^{m} \sum_{j=1}^{n} \langle w_i, h_\ell \rightharpoonup v_j \rangle f_\ell \otimes \overline{w}_j, \qquad i \in \mathbb{I}_n.$$

Thus $\mathcal{W} \simeq \mathcal{R}^1$ in ${}^{L^*}_{L^*}\mathcal{YD}$.

(iii) $\mathcal{R} \simeq \mathscr{B}(\mathcal{W}) \simeq \mathscr{B}(\mathcal{V}^*)$. Let $\mathcal{R}'$ be the braided Hopf subalgebra of $\mathcal{R}$ generated by $\mathcal{R}^1$. Then $\mathcal{R}'$ is a pre-Nichols algebra of $\mathcal{R}^1 \simeq \mathcal{W}$. Thus

$$\dim \mathscr{B}(\mathcal{W}) \leq \dim \mathcal{R}' \leq \dim \mathcal{R} = \dim \mathscr{B}(\mathcal{V}) = \dim \mathscr{B}(\mathcal{W});$$

hence $\mathcal{R}' = \mathcal{R} = \mathscr{B}(\mathcal{W})$ and the result follows. $\square$

Next we describe the relations of the Drinfeld double $D(A)$ of A. Recall that this is the Hopf algebra whose underlying coalgebra is $A \otimes A^{*\,\mathrm{op}}$ and with multiplication defined as follows. Let $a \bowtie r := a \otimes r$ in $D(A)$. Then

$$(a \bowtie r)(a' \bowtie r') = \left\langle r_{(1)}, a'_{(1)} \right\rangle \left\langle r_{(3)}, \mathcal{S}(a'_{(3)}) \right\rangle (aa'_{(2)} \bowtie r' r_{(2)}),$$

$a, A' \in A$, $r, r' \in A^{*\,\mathrm{op}}$, where $rr' = m(r \otimes r')$ is in A^*, not in $A^{*\,\mathrm{op}}$.

Remark 3.3. [DT] If A is not finite-dimensional, we may define its double with respect to another Hopf algebra B provided with a skew-pairing, i. e. a linear map $\tau \colon A \otimes B \to \Bbbk$ satisfying

$$\begin{aligned} \tau(a\widetilde{a} \otimes b) &= \tau(a \otimes b_{(1)})\tau(\widetilde{a} \otimes b_{(2)}), \quad \tau(1 \otimes b) = \varepsilon(b), \quad a, \widetilde{a} \in A, \\ \tau(a \otimes \widetilde{b}b) &= \tau(a_{(1)} \otimes b)\tau(a_{(2)} \otimes \widetilde{b}), \quad \tau(a \otimes 1) = \varepsilon(a), \quad b, \widetilde{b} \in B. \end{aligned} \tag{3.1}$$

Let $\sigma \colon (A \otimes B) \otimes (A \otimes B) \to \Bbbk$ be the 2-cocycle associated to τ, where $A \otimes B$ has the structure of tensor product Hopf algebra, that is

$$\sigma(a \otimes b, \widetilde{a} \otimes \widetilde{b}) = \varepsilon(a)\varepsilon(\widetilde{b})\tau(\widetilde{a} \otimes b), \qquad a, \widetilde{a} \in A, b, \widetilde{b} \qquad \in B.$$

Then the double of A (with respect to B and τ) is the Hopf algebra $A \otimes B$ twisted by σ, i. e. $A \bowtie B =: (A \otimes B)_\sigma$.

We fix the following presentations for $\mathscr{B}(\mathcal{V})$, $\mathscr{B}(\mathcal{W})^{\mathrm{op}}$ and $D(L)$:

$$\begin{aligned} \mathscr{B}(\mathcal{V}) &= \Bbbk\langle v_1, \dots, v_n | r_1, \dots, r_{n_1} \rangle, \\ \mathscr{B}(\mathcal{W})^{\mathrm{op}} &= \Bbbk\langle w_1, \dots, w_n | r'_1, \dots, r'_{n_2} \rangle, \\ D(L) &= \Bbbk\langle s_1, \dots, s_{m_0}, s'_1, \dots, s'_{m_1} | \widetilde{r}_1, \dots, \widetilde{r}_{n_3} \rangle, \end{aligned} \tag{3.2}$$

with $s_1, \dots, s_{m_0}$ and $s'_1, \dots, s'_{m_1}$ generators of L and $(L^*)^{\mathrm{op}}$ respectively. Then we have the following presentation for $D(A)$.

Proposition 3.4. *The algebra $D(A)$ is presented by generators*

$$v_1, \dots, v_n, \quad w_1, \dots, w_n, \quad s_1, \dots, s_{m_0}, \quad s'_0, \dots, s'_{m_1},$$

with relations (3.2) *and*

$$\begin{aligned} s'_i v_j &= \langle (s'_i)_{(1)}, (v_j)_{(-1)} \rangle (v_j)_{(0)} (s'_i)_{(2)}, \quad (i,j) \in \mathbb{I}_{m_1} \times \mathbb{I}_n, \\ w_i s_j &= \sum_{\ell=1}^{n} \langle w_i, (s_j)_{(1)} \rightharpoonup v_\ell \rangle (s_j)_{(2)} w_\ell, \quad (i,j) \in \mathbb{I}_n \times \mathbb{I}_{m_0}, \\ w_i v_j &= \langle w_i, v_j \rangle 1 + \sum_{\ell=1}^{n} \langle w_i, (v_j)_{(-1)} \rightharpoonup v_\ell \rangle (v_j)_{(0)} w_\ell \end{aligned} \tag{3.3}$$

$$+ \sum_{k=1}^{m} \sum_{\ell,t=1}^{n} \langle w_i, (v_j)_{(-2)} \rightharpoonup v_\ell \rangle \langle w_\ell, h_k \rightharpoonup v_t \rangle \langle \overline{w}_t, \mathcal{S}((v_j)_{(0)}) \rangle (v_j)_{(-1)} f_k,$$

$$(i,j) \in \mathbb{I}_n \times \mathbb{I}_n.$$

Hence $D(A)$ admits a triangular decomposition

$$D(A) \simeq \mathscr{B}(\mathcal{V}) \otimes D(L) \otimes \mathscr{B}(\mathcal{W})^{\mathrm{op}}. \tag{3.4}$$

Proof. Let Υ be the algebra presented by generators and relations as above. By definition of $D(A)$ and the formulas above there exists an algebra map $\Upsilon \twoheadrightarrow D(A)$. Hence $\dim \Upsilon \geq \dim D(A)$. By definition of Υ there exist morphisms

$$\psi_1 \colon \mathscr{B}(\mathcal{V}) \longrightarrow \Upsilon, \qquad \psi_2 \colon \mathscr{B}(\mathcal{W})^{\mathrm{op}} \longrightarrow \Upsilon, \qquad \psi_3 \colon D(L) \longrightarrow \Upsilon.$$

Let $m \colon \Upsilon \otimes \Upsilon \longrightarrow \Upsilon$ be the multiplication. We have a linear map

$$\phi \colon \mathscr{B}(\mathcal{V}) \otimes D(L) \otimes \mathscr{B}(\mathcal{W})^{\mathrm{op}} \xrightarrow{\psi_1 \otimes \psi_2 \otimes \psi_3} \Upsilon \otimes \Upsilon \otimes \Upsilon \xrightarrow{m \circ (m \otimes \mathrm{id})} \Upsilon \tag{3.5}$$

which is surjective by (3.3), since every product of the generators can be rewritten to get a sum of elements of the form abc with $a \in \mathscr{B}(\mathcal{V})$, $b \in D(L)$ and $c \in \mathscr{B}(\mathcal{W})^{\mathrm{op}}$. Thus

$$\dim D(A) = \dim \mathscr{B}(\mathcal{V}) \cdot \dim D(L) \cdot \dim \mathscr{B}(\mathcal{W})^{\mathrm{op}} \geq \dim \Upsilon,$$

hence $D(A) \simeq \Upsilon$. As for (3.4), (3.5) provides the desired isomorphism. □

3.2. Objects of interest.

In this Subsection we introduce the braided vector spaces of our interest and their Nichols algebras, cf. [AAH1, AAH2].

Braided vector spaces. The braided vector space $(\mathcal{V}(-1,2), c) =: (V, c_V)$, called the -1-block in [AAH1], has a basis $\{x_1, x_2\}$ and braiding

$$c(x_i \otimes x_1) = -x_1 \otimes x_i, \quad c(x_i \otimes x_2) = (-x_2 + x_1) \otimes x_i, \quad i = 1, 2. \tag{3.6}$$

For simplicity we denote $(W, c_W) := (\mathcal{V}(-1,2)^*, c^*)$, cf. (2.4). Let $\{u_1, u_2\}$ be the basis of W given by $\langle u_i, x_j \rangle = 1 - \delta_{i,j}$, $i, j \in \mathbb{I}_2$. In this basis c_W is

$$c_W(u_1 \otimes u_i) = -u_i \otimes u_1, \quad c_W(u_2 \otimes u_i) = u_i \otimes (u_1 - u_2), \quad i = 1, 2. \tag{3.7}$$

Then $(W, c_W) \simeq (V, c_V^{-1})$ as braided vector spaces, via $u_1 \mapsto x_1$, $u_2 \mapsto -x_2$.

The super Jordan plane. This is the graded algebra

$$\mathcal{B} = \Bbbk\langle x_1, x_2 | x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21}\rangle$$

with the braided Hopf algebra structure extending the braiding (3.6), cf. [T].

Lemma 3.5. (a) [AAH1] *(*$\operatorname{char}\Bbbk \neq 2$*) The ordered monomials*

$$x_1^{n_1}x_{21}^{n_{21}}x_2^{n_2} \tag{3.8}$$

with $(n_1, n_{21}, n_2) \in \mathbb{I}_{0,1} \times \mathbb{N}_0 \times \mathbb{N}_0$ *form a basis of* $\mathcal{B}$ *and* $\operatorname{GKdim}\mathcal{B} = 2$*.*

(b) *(*$\operatorname{char}\Bbbk = 0$*)* [AAH1] $\mathcal{B}$ *is isomorphic to* $\mathscr{B}(V)$ *as braided Hopf algebras.*

(c) *(*$\operatorname{char}\Bbbk = p$*)* [AAH2] *The restricted super Jordan plane* $\mathcal{B}/\langle x_{21}^p, x_2^{2p}\rangle$ *is isomorphic to* $\mathscr{B}(V)$ *as braided Hopf algebras. The ordered monomials* (3.8) *with* $(n_1, n_{21}, n_2) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,p-1} \times \mathbb{I}_{0,2p-1}$ *form a basis of* $\mathscr{B}(V)$*.* □

Set $u_{21} := u_1u_2 + u_2u_1$. We define the *dual super Jordan plane* as the algebra $\mathcal{B}^d$ presented by generators u_1 and u_2 with defining relations

$$u_1^2 = 0, \qquad u_2u_{21} = u_{21}u_2 - u_1u_{21} \tag{3.9}$$

with the braided Hopf algebra structure extending the braiding (3.7), cf. [T]. Observe that $\mathcal{B}^d$ is not isomorphic to $\mathcal{B}$ as braided Hopf algebras. The *restricted dual super Jordan plane* is the quotient of $\mathcal{B}^d$ by the relations

$$u_2^{2p} = 0, \qquad u_{21}^p = 0. \tag{3.10}$$

Lemma 3.6. (a) *(*$\operatorname{char}\Bbbk = 0$*)* $\mathcal{B}^d \simeq \mathscr{B}(W)$ *as braided Hopf algebras.*

(b) *(*$\operatorname{char}\Bbbk = p$*) The restricted dual super Jordan plane is isomorphic to* $\mathscr{B}(W)$ *as braided Hopf algebras. It has dimension* $4p^2$*; indeed the monomials*

$$u_1^{n_1}u_{21}^{n_{21}}u_2^{n_2} \tag{3.11}$$

with $(n_1, n_{21}, n_2) \in \mathbb{I}_{0,1} \times \mathbb{I}_{0,p-1} \times \mathbb{I}_{0,2p-1}$ *form a basis of* $\mathscr{B}(W)$*.*

Proof. Since $\mathcal{B}^d \simeq \mathcal{B}$ as algebras, the monomials (3.11) with $(n_1, n_{21}, n_2) \in \mathbb{I}_{0,1}\times\mathbb{N}_0\times\mathbb{N}_0$ form a linear basis of $\mathcal{B}^d$. It is easy to see that u_1^2 and $u_{21}u_2 - u_2u_{21} + u_1u_{21}$ are primitive in $T(W)$, hence $\mathcal{B}^d$ is a pre-Nichols algebra of W. Now $\mathscr{B}(V)$ and $\mathscr{B}(W)$ have the same graded dimension by [G, Lemma 2.6]. Then (a) follows. For (b) we prove by induction that

$$\begin{aligned}
\Delta(u_{21}^n) &= \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k} n^\ell u_1^\ell u_{21}^{n-k-\ell} \otimes u_1^\ell u_{21}^k,\\
\Delta(u_2^{2n}) &= \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\sum_{t=0}^{k}\binom{n-\ell}{k}\binom{k}{t} n^\ell [k-n+\ell]^{[t]} u_2^{2(n-k)-\ell} \otimes u_1^\ell u_{21}^t u_2^{2(k-t)},
\end{aligned} \tag{3.12}$$

for all $n \in \mathbb{N}_0$. So $u_{21}^p, u_2^{2p} \in \mathcal{P}(\mathcal{B}^d)$ and $\mathcal{B}^d/(u_{21}^p, u_2^{2p})$ is a pre-Nichols algebra of W; hence $\mathcal{B}^d/(u_{21}^p, u_2^{2p}) \simeq \mathscr{B}(W)$ by dimension counting. □

3.3. Change of bosonization. Let L be a Hopf algebra and let C be a group provided with a group homomorphism $\mathrm{C} \to \mathrm{Aut}_{\mathrm{Hopf}}(L)$. Then L is a Hopf algebra in ${}^{\Bbbk\mathrm{C}}_{\Bbbk\mathrm{C}}\mathcal{YD}$ (with trivial coaction) and we may consider the smash product $L \rtimes \Bbbk\mathrm{C} := L\#\Bbbk\mathrm{C}$. Let $L \rtimes \Bbbk\mathrm{C} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \Bbbk\mathrm{C}$ be the natural projection and inclusion. We show that under certain conditions, $L \rtimes \Bbbk\mathrm{C}$ can be alternatively described as $\mathscr{L}^{\varphi}\#\Bbbk\mathrm{C}$ for a genuine braided Hopf algebra $\mathscr{L}^{\varphi}$ in ${}^{\Bbbk\mathrm{C}}_{\Bbbk\mathrm{C}}\mathcal{YD}$.

Namely, suppose that $L = R\#U$ where U is a Hopf algebra and R is a Hopf algebra in ${}^{U}_{U}\mathcal{YD}$. Let $\mathrm{Aut}_{\mathrm{Hopf}}{}^{U}_{U}(R)$ be the group of automorphisms of Hopf algebras in ${}^{U}_{U}\mathcal{YD}$, i.e. preserving multiplication, comultiplication, action and coaction. There is a morphism of groups

$$\mathrm{Aut}_{\mathrm{Hopf}}{}^{U}_{U}(R) \to \mathrm{Aut}_{\mathrm{Hopf}}(L), \quad \mathrm{Aut}_{\mathrm{Hopf}}{}^{U}_{U}(R) \ni \varsigma \mapsto \varsigma \otimes \mathrm{id} \in \mathrm{Aut}_{\mathrm{Hopf}}(L).$$

We fix a group C and a morphism of groups $\mathrm{C} \to \mathrm{Aut}_{\mathrm{Hopf}}(L)$ factorizing through $\mathrm{Aut}_{\mathrm{Hopf}}{}^{U}_{U}(R)$. Furthermore we assume that $U = U' \otimes \Bbbk\mathrm{G}$ where U' a Hopf algebra and G is a group.

Proposition 3.7. (a) *Given $\varphi \in \mathrm{Hom}_{\mathrm{gps}}(\mathrm{G}, Z(\mathrm{C}))$, there exists a Hopf algebra $\mathscr{L}^{\varphi}$ in ${}^{\Bbbk\mathrm{C}}_{\Bbbk\mathrm{C}}\mathcal{YD}$ such that $L \rtimes \Bbbk\mathrm{C} \simeq \mathscr{L}^{\varphi}\#\Bbbk\mathrm{C}$.*

(b) *Let $\varpi^{\varphi} : \mathrm{G} \times \mathrm{C} \to \mathrm{C}$ be the morphism of groups given by $\varpi^{\varphi}(\gamma, c) = \varphi(\gamma)c$, $\gamma \in \mathrm{G}$, $c \in \mathrm{C}$, and let*

$$\mathrm{G}^{\varphi} := \ker \varpi^{\varphi} = \{(\gamma, \varphi(\gamma)^{-1}) : \gamma \in \mathrm{G}\}.$$

Then $\mathscr{L}^{\varphi}$ decomposes as $\mathscr{L}^{\varphi} \simeq R^{\varphi}\sharp(U' \otimes \Bbbk\mathrm{G}^{\varphi})$ (braided bosonization in ${}^{\Bbbk\mathrm{C}}_{\Bbbk\mathrm{C}}\mathcal{YD}$) where $R^{\varphi} = R$ as a subalgebra and U'-module but with $\mathrm{G}^{\varphi} \simeq \mathrm{G}$ acting by

$$\gamma \rightharpoonup r = \varphi(\gamma)^{-1} \cdot (\gamma \cdot r), \qquad r \in R, \gamma \in \mathrm{G}.$$

(c) *Assume that*

(i) *C and G are abelian and φ has a section $\vartheta \in \mathrm{Hom}_{\mathrm{gps}}(\mathrm{C}, \mathrm{G})$,*

(ii) *For every $r \in R$ and $c \in \mathrm{C}$, $c \cdot r = \vartheta(c) \cdot r$.*

Let $\mathrm{N}^{\varphi} := \ker \varphi \times \{e\}$. Then the subalgebra $R^{\varphi}\sharp U' \otimes \Bbbk\mathrm{N}^{\varphi}$ has a structure of Hopf algebra denoted $\mathcal{L}^{\varphi}$ such that $\Bbbk\mathrm{C} \hookrightarrow \mathscr{L}^{\varphi} \twoheadrightarrow \mathcal{L}^{\varphi}$ is exact. Furthermore R^{φ} is a Hopf algebra in ${}^{U'\otimes\Bbbk\mathrm{N}^{\varphi}}_{U'\otimes\Bbbk\mathrm{N}^{\varphi}}\mathcal{YD}$.

Proof. (a): Let $\pi^{\varphi} : L \rtimes \Bbbk\mathrm{C} \to \Bbbk\mathrm{C}$ be the Hopf algebra projection given by

$$\pi^{\varphi}(r\#u \otimes \gamma \otimes c) = \varepsilon(r)\varepsilon(u)\,\varphi(\gamma)c, \qquad r \in R,\ u \in U', \gamma \in \mathrm{G},\ c \in \mathrm{C}.$$

Then $\pi^{\varphi}\iota = \mathrm{id}_{\Bbbk\mathrm{C}}$ and the claim follows from [R] taking the subalgebra

$$\mathscr{L}^{\varphi} := (L \rtimes \Bbbk\mathrm{C})^{\mathrm{co}\,\pi^{\varphi}} = R \cdot U' \otimes \Bbbk\mathrm{G}^{\varphi}.$$

(b) The Hopf algebra maps $\mathscr{L}^{\varphi} \underset{\imath^{\varphi}}{\overset{p^{\varphi}}{\rightleftarrows}} U' \otimes \Bbbk\mathrm{G}^{\varphi}$ given by

$$\begin{aligned} p^{\varphi}(r\#u \otimes \gamma\varphi(\gamma)^{-1}) &= \varepsilon(r)\varepsilon(u)(1\#\gamma\varphi(\gamma)^{-1}), \\ \imath^{\varphi}(u \otimes \gamma\varphi(\gamma)^{-1}) &= 1\#u \otimes \gamma\varphi(\gamma)^{-1}, \end{aligned} \qquad r \in R,\ u \in U', \gamma \in \mathrm{G},$$

satisfy $p^\varphi \iota^\varphi = \mathrm{id}_{\Bbbk \mathrm{G}^\varphi}$. Hence $R^\varphi := (\mathscr{L}^\varphi)^{\operatorname{co} p^\varphi}$ and the claim follows.

(c) By (i), $\mathrm{G}^\varphi \simeq \mathrm{N}^\varphi \times \mathrm{C}^\vartheta$ where $\mathrm{C}^\vartheta = \{(\vartheta(c)^{-1}, c) : c \in \mathrm{C}\}$ and using also (i), C^ϑ is central in $\mathscr{L}^\varphi$. Now the multiplication provides a linear isomorphism

$$\mathscr{L}^\varphi \simeq R^\varphi \sharp (U' \otimes \Bbbk \mathrm{G}^\varphi) \simeq R^\varphi \sharp \big(U' \otimes (\Bbbk \mathrm{N}^\varphi \times \Bbbk \mathrm{C}^\vartheta)\big) \simeq \big(R^\varphi \sharp U' \otimes \Bbbk \mathrm{N}^\varphi\big) \otimes \Bbbk \mathrm{C}^\vartheta.$$

Then the quotient $\mathcal{L}^\varphi \simeq \mathscr{L}^\varphi / \Bbbk(\mathrm{C}^\vartheta)^+ \mathscr{L}^\varphi$ is isomorphic as an algebra to the subalgebra $R^\varphi \sharp U' \otimes \Bbbk \mathrm{N}^\varphi$. The rest is clear. □

The situation that we have in mind is when $\mathrm{C} = C_N$ and $R = \oplus_{i \in C_N} R_i$ has a C_N-grading of Hopf algebras in ${}^U_U\mathcal{YD}$. For instance, let $\mathcal{V} \in {}^{\Bbbk \mathrm{G}}_{\Bbbk \mathrm{G}}\mathcal{YD}$ be C_N-graded and take $R = \mathscr{B}(\mathcal{V})$ (or any pre-Nichols algebras whose defining relations are C_N-homogeneous). We fix $\omega \in \mathbb{G}_N$ (what amounts in this case to fix a quasi-triangular structure on $\Bbbk C_N$) and let act C_N on R_i by ω^i.

We discuss two specific examples of interest in this paper. In the rest of this Section $\operatorname{char} \Bbbk \neq 2$, in particular $\operatorname{char} \Bbbk = 0$ is allowed. Recall that $R = \Bbbk[\zeta]$ with ζ primitive and $\Gamma \simeq \mathbb{Z}$. We fix generators

$$C_2 = \langle \epsilon \rangle, \qquad \Gamma = \langle \mathbf{g} \rangle. \tag{3.13}$$

Example 3.8. We realize (V, c), with braiding (3.6), in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by

$$\mathbf{g} \rightharpoonup x_1 = -x_1, \quad \mathbf{g} \rightharpoonup x_2 = -x_2 + x_1, \quad \delta(x_i) = \mathbf{g} \otimes x_i, \quad i = 1, 2. \tag{3.14}$$

Since the ideal of $T(V)$ generated by the relations (1.1) belongs to ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, $\mathcal{B}$ is a Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ hence we have

$$\mathtt{H} := \mathcal{B} \# \Bbbk \Gamma. \tag{3.15}$$

Now thinking on V as purely odd and taking $\mathrm{C} = C_2$, $\mathrm{G} = \Gamma$, $L = \mathtt{H}$, $U = \Bbbk\Gamma$ and $\varphi : \Gamma \to C_2$ the standard projection, we have $\mathtt{H} \rtimes \Bbbk C_2 \simeq \widetilde{\mathcal{H}} \# \Bbbk C_2$ where

$$\widetilde{\mathcal{H}} \simeq \mathcal{B} \natural \Bbbk \Gamma, \tag{3.16}$$

that corresponds to the realization of V in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YDS}$ by

$$\mathbf{g} \rightharpoonup x_1 = x_1, \; \mathbf{g} \rightharpoonup x_2 = x_2 - x_1, \; \delta(x_i) = \mathbf{g} \otimes x_i, \; |x_i| = 1, \quad i \in \mathbb{I}_2. \tag{3.17}$$

Since φ admits no section, there is no further decomposition.

Example 3.9. Analogously we realize (W, c_W), cf. (3.7), in ${}^{R \otimes \Bbbk C_2}_{R \otimes \Bbbk C_2}\mathcal{YD}$ by

$$\begin{aligned} &\zeta \rightharpoonup u_i = u_i, && \epsilon \rightharpoonup u_i, = -u_i, \\ &\delta(u_1) = \epsilon \otimes u_1, && \delta(u_2) = \epsilon \otimes u_2 - \zeta\epsilon \otimes u_1, \end{aligned} \qquad i = 1, 2. \tag{3.18}$$

As before $\mathcal{B}^d$ is a Hopf algebra in ${}^{R \otimes \Bbbk C_2}_{R \otimes \Bbbk C_2}\mathcal{YD}$ hence we have

$$\mathtt{K} := \mathcal{B}^d \# (R \otimes \Bbbk C_2). \tag{3.19}$$

Now thinking on W as a purely odd super vector space and taking $\mathrm{C} = C_2 \simeq \mathrm{G}$, $L = \mathtt{K}$, $U = R \otimes \Bbbk C_2$ and $\varphi = \mathrm{id}_{C_2}$, we have $\mathtt{K} \rtimes \Bbbk C_2 \simeq \widetilde{\mathscr{K}} \# \Bbbk C_2$, where $\widetilde{\mathscr{K}} \simeq \mathcal{B}^d \natural (R \otimes \Bbbk C_2)$; furthermore $\Bbbk C_2$ is central and

$$\widetilde{\mathcal{K}} := \widetilde{\mathscr{K}} /\!/ \Bbbk C_2 \simeq \mathcal{B}^d \natural R. \tag{3.20}$$

Here the last corresponds to the realization of W in ${}^R_R\mathcal{YDS}$ by

$$\begin{aligned} \zeta \rightharpoonup u_i &= u_i, & |u_i| &= 1, & i &= 1,2, \\ \delta(u_1) &= 1 \otimes u_1, & \delta(u_2) &= 1 \otimes u_2 - \zeta \otimes u_1. \end{aligned} \tag{3.21}$$

The superalgebras $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{K}}$ admit a suitable Hopf pairing that allows an alternative characterization of the Hopf superalgebra $\widetilde{\mathcal{D}}$, see Remark 4.4.

4. The double of the super Jordan plane

4.1. The definition. We define the Drinfeld double $\mathtt{D}$ of $\mathtt{H} = \mathcal{B}\#\Bbbk\Gamma$, see (3.15), with respect to a suitable pairing with $\mathtt{K} = \mathcal{B}^d\#(R \otimes \Bbbk C_2)$, see (3.19). Then we show that there exists a Hopf superalgebra $\widetilde{\mathcal{D}}$ such that $\mathtt{D} \simeq \widetilde{\mathcal{D}}\#\Bbbk C_2$. In this way $\widetilde{\mathcal{D}}$ is fundamental to the study of the Drinfeld double $\mathtt{D}$; among other characteristics, it bears a triangular decomposition (4.14).

To start with the Hopf algebras $\mathtt{H}$ and $\mathtt{K}$ have PBW-basis, denoted by $\mathsf{B}_{\mathtt{H}}$, $\mathsf{B}_{\mathtt{K}}$, given by the ordered monomials (3.8) or (3.11) multiplied accordingly by elements of the groups Γ and C_2, or powers of ζ. The comultiplications of $\mathtt{H}$, respectively $\mathtt{K}$ satisfy $x_1, x_2 \in \mathcal{P}_{\mathbf{g},1}(\mathtt{H})$, $u_1 \in \mathcal{P}_{\epsilon,1}(\mathtt{K})$ and

$$\Delta(u_2) = u_2 \otimes 1 + \epsilon \otimes u_2 - \epsilon\zeta \otimes u_1. \tag{4.1}$$

Lemma 4.1. *The algebra* $\mathtt{H}$ *is presented by generators* $x_1, x_2, \mathbf{g}, \mathbf{g}^{-1}$ *and relations* (1.1),

$$\mathbf{g}x_1 = -x_1\mathbf{g}, \qquad \mathbf{g}x_2 = -x_2\mathbf{g} + x_1\mathbf{g}, \qquad \mathbf{g}^{\pm 1}\mathbf{g}^{\mp 1} = 1. \tag{4.2}$$

Also, $\mathtt{K}$ *is presented by generators* $u_1, u_2, \zeta, \epsilon$ *and relations* (1.1) *(in the* u_i*'s),*

$$\epsilon^2 = 1, \qquad \epsilon\zeta = \zeta\epsilon, \tag{4.3}$$

$$\epsilon u_1 = -u_1\epsilon, \qquad \epsilon u_2 = -u_2\epsilon, \tag{4.4}$$

$$\zeta u_1 = u_1\zeta + u_1, \qquad \zeta u_2 = u_2\zeta + u_2. \tag{4.5}$$

We define the Drinfeld double of $\mathtt{H}$ as $\mathtt{D} := \mathtt{H} \bowtie \mathtt{K}^{\mathrm{op}}$, see Remark 3.3, with respect to the unique skew-pairing $\tau : \mathtt{H} \otimes \mathtt{K}^{\mathrm{op}} \to \Bbbk$ such that

$$\begin{aligned} &\tau(x_1 \otimes u_1) = 0, && \tau(x_1 \otimes u_2) = 1, && \tau(x_1 \otimes \zeta) = 0, && \tau(x_1 \otimes \epsilon) = 0, \\ &\tau(x_2 \otimes u_1) = 1, && \tau(x_2 \otimes u_2) = 0, && \tau(x_2 \otimes \zeta) = 0, && \tau(x_2 \otimes \epsilon) = 0, \\ &\tau(\mathbf{g}^{\pm 1} \otimes u_1) = 0, && \tau(\mathbf{g}^{\pm 1} \otimes u_2) = 0, && \tau(\mathbf{g}^{\pm 1} \otimes \zeta) = \pm 1, && \tau(\mathbf{g}^{\pm 1} \otimes \epsilon) = -1. \end{aligned}$$

Proposition 4.2. *The algebra* $\mathtt{D}$ *is presented by generators* x_1, x_2, ζ, $\mathbf{g}$, ϵ, u_1, u_2 *with relations* (1.1) *(in the* x_i*'s and in the* u_i*'s),* (4.2), (4.3), (4.4),

$$u_1\zeta = \zeta u_1 + u_1, \qquad u_2\zeta = \zeta u_2 + u_2, \tag{4.6}$$

$$\epsilon\mathbf{g} = \mathbf{g}\epsilon, \qquad \zeta\mathbf{g} = \mathbf{g}\zeta, \tag{4.7}$$

$$\epsilon x_1 = -x_1\epsilon, \qquad \epsilon x_2 = -x_2\epsilon, \tag{4.8}$$

$$\zeta x_1 = x_1\zeta + x_1, \qquad \zeta x_2 = x_2\zeta + x_2, \tag{4.9}$$

$$u_2\mathbf{g} = -\mathbf{g}u_2 + \mathbf{g}u_1, \qquad u_1\mathbf{g} = -\mathbf{g}u_1, \tag{4.10}$$

$$u_1x_1 = -x_1u_1, \qquad u_1x_2 = -x_2u_1 + (1-\mathbf{g}\epsilon), \tag{4.11}$$

$$u_2x_2 = -x_2u_2 + \mathbf{g}\epsilon\zeta + x_2u_1, \qquad u_2x_1 = -x_1u_2 + (1-\mathbf{g}\epsilon) + x_1u_1, \tag{4.12}$$

The family $\mathsf{B}_{\mathtt{D}}$ *consisting of the monomials*

$$x_1^{n_1}x_{21}^{n_{21}}x_2^{n_2}\mathbf{g}^n\zeta^m u_1^{m_1}u_{21}^{m_{21}}u_2^{m_2}\epsilon^k \tag{4.13}$$

with $(k, n_1, m_1, n, n_2, n_{21}, m, m_2, m_{21}) \in \mathbb{I}_{0,1}^3 \times \mathbb{Z} \times \mathbb{N}_0^5$ *is a basis of* $\mathtt{D}$.

Proof. Let A be the algebra presented as in the statement; then $A \twoheadrightarrow \mathtt{D}$. As $\mathsf{B}_{\mathtt{D}}$ is a basis of $\mathtt{D}$ by construction, the corresponding monomials are linearly independent in A. Let $S = \{x_1, x_{21}, x_2, \mathbf{g}, \zeta, u_1, u_{21}, u_2, \epsilon\}$ with x_{21} and u_{21} as before, ordered by $x_1 < x_{21} < x_2 < \mathbf{g} < \zeta < u_1 < u_{21} < u_2 < \epsilon$. From the defining relations we have

$$\begin{aligned}
&u_{21}x_1 = x_1u_{21}, \quad \epsilon u_{21} = u_{21}\epsilon, \quad u_1x_{21} = x_{21}u_1, \quad \epsilon x_{21} = x_{21}\epsilon,\\
&u_{21}x_2 = x_2u_{21} + (\mathbf{g}\epsilon + 1)u_1, \quad \zeta x_{21} = x_{21}\zeta + 2x_{21},\\
&u_2x_{21} = x_{21}u_2 - 2x_{21}u_1 + x_1(\mathbf{g}\epsilon + 1).
\end{aligned}$$

If $a > b \in S$, then ab is a linear combination of monomials $c_1 \cdots c_s$ with $c_1 \leq c_2 \leq \cdots \leq c_s \in S$. Thus the monomials (4.13) generate A and $A \simeq \mathtt{D}$. $\square$

Let $\widetilde{\mathbf{g}} := \mathbf{g}\epsilon$ and $\widetilde{\mathcal{D}} := \Bbbk\langle x_1, x_2, u_1, u_2, \widetilde{\mathbf{g}}, \zeta\rangle \hookrightarrow \mathtt{D}$.

Proposition 4.3. (i) *A basis of* $\widetilde{\mathcal{D}}$ *is given by the family* B *consisting of monomials*

$$x_1^{n_1}x_{21}^{n_{21}}x_2^{n_2}\widetilde{\mathbf{g}}^n\zeta^m u_1^{m_1}u_{21}^{m_{21}}u_2^{m_2},$$

where

$$(n_1, m_1, n, n_2, n_{21}, m, m_2, m_{21}) \in \mathbb{I}_{0,1}^2 \times \mathbb{Z} \times \mathbb{N}_0^5.$$

(ii) $\widetilde{\mathcal{D}}$ *has a triangular decomposition i.e. a linear isomorphism induced by multiplication*

$$\widetilde{\mathcal{D}} \simeq \mathcal{B} \otimes (\Bbbk\Gamma \otimes R) \otimes (\mathcal{B}^d)^{\mathrm{op}}. \tag{4.14}$$

(iii) *The algebra* $\widetilde{\mathcal{D}}$ *is presented by generators* x_1, x_2, u_1, u_2, $\widetilde{\mathbf{g}}^{\pm1}$, ζ *with defining relations* (1.1) *(in the* x_i*'s and the* u_i*'s),* (4.6), (4.9), (4.11) *(with* $\widetilde{\mathbf{g}}$ *instead of* $\mathbf{g}\epsilon$*) and*

$$\begin{aligned}
&\widetilde{\mathbf{g}}^{\pm1}\widetilde{\mathbf{g}}^{\mp1} = 1, && \zeta\widetilde{\mathbf{g}} = \widetilde{\mathbf{g}}\zeta, && \widetilde{\mathbf{g}}x_1 = x_1\widetilde{\mathbf{g}},\\
&\widetilde{\mathbf{g}}x_2 = x_2\widetilde{\mathbf{g}} - x_1\widetilde{\mathbf{g}}, && u_2\widetilde{\mathbf{g}} = \widetilde{\mathbf{g}}u_2 - \widetilde{\mathbf{g}}u_1, && u_1\widetilde{\mathbf{g}} = \widetilde{\mathbf{g}}u_1.
\end{aligned}$$

(iv) $\widetilde{\mathcal{D}}\#\Bbbk C_2 \simeq \mathtt{D}$ *as Hopf algebras; here* $\widetilde{\mathcal{D}}$ *is a Hopf superalgebra with comultiplication given by* $\widetilde{\mathbf{g}} \in G(\widetilde{\mathcal{D}})$, $\zeta, u_1 \in \mathcal{P}(\widetilde{\mathcal{D}})$, $x_1, x_2 \in \mathcal{P}_{\widetilde{\mathbf{g}},1}(\widetilde{\mathcal{D}})$ *and*

$$\Delta_{\widetilde{\mathcal{D}}}(u_2) = u_2 \otimes 1 + 1 \otimes u_2 - \zeta \otimes u_1, \tag{4.15}$$

Proof. (i) Argue as in the proof of Proposition 4.2; in turn (i) implies (ii).

(iii) Let A be the algebra as in the statement. Using the commutation relations we have a surjective map of algebras $\phi\colon A \twoheadrightarrow \widetilde{\mathcal{D}}$. Clearly the family B' consisting of the monomials analogous to those in B generates A and is linearly independent because $\phi(\mathsf{B}') = \mathsf{B}$. Hence $A \simeq \widetilde{\mathcal{D}}$.

(iv) Let us identify $\Bbbk C_2$ with the subalgebra of $\mathtt{D}$ generated by $\epsilon \in G(\mathtt{D})$ by $\iota\colon \Bbbk C_2 \hookrightarrow \mathtt{D}$. There is a Hopf algebra map $\pi\colon \mathtt{D} \twoheadrightarrow \Bbbk C_2$ given by

$$x_1 \mapsto 0, \quad x_2 \mapsto 0, \quad u_1 \mapsto 0, \quad u_2 \mapsto 0, \quad \mathbf{g} \mapsto \epsilon, \quad \zeta \mapsto 0, \quad \epsilon \mapsto \epsilon.$$

Then $\pi\iota = \mathrm{id}$ and $\mathtt{D} \simeq \mathtt{D}^{\operatorname{co}\pi} \# \Bbbk C_2$. Clearly $\widetilde{\mathcal{D}} \subseteq \mathtt{D}^{\operatorname{co}\pi}$ because every generator of $\widetilde{\mathcal{D}}$ is coinvariant, and $\widetilde{\mathcal{D}}$ is a braided Hopf algebra in ${}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$ with comultiplication $\Delta_{\widetilde{\mathcal{D}}}$ as in (4.15). But $\widetilde{\mathcal{D}} \# \Bbbk C_2$ contains $\mathsf{B}_{\mathtt{D}}$, thus $\widetilde{\mathcal{D}} = \mathtt{D}^{\operatorname{co}\pi}$. Since every generator of $\widetilde{\mathcal{D}}$ is either even or odd, $\widetilde{\mathcal{D}}$ is a Hopf superalgebra. $\square$

Remark 4.4. Recall the Hopf superalgebras $\widetilde{\mathcal{H}}$ (3.16) and $\widetilde{\mathcal{K}}$ (3.20). There are Hopf superalgebra maps $\widetilde{\mathcal{H}} \hookrightarrow \widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{K}}^{\mathrm{op}} \hookrightarrow \widetilde{\mathcal{D}}$ (by the presentation of $\widetilde{\mathcal{D}}$) and $\widetilde{\mathcal{D}}$ is isomorphic to the double of $\widetilde{\mathcal{H}}$ with respect to a suitable skew-pairing, cf. [GZB].

4.2. The double as a super abelian extension. We show that $\widetilde{\mathcal{D}}$ fits into an exact sequence of Hopf superalgebras $R \hookrightarrow \widetilde{\mathcal{D}} \twoheadrightarrow U$ with R super commutative and U super cocommutative.

Let $\mathfrak{G}$ be the super algebraic group such that its algebra of functions is the commutative Hopf superalgebra $\mathcal{O}(\mathfrak{G}) := \Bbbk[X_1, X_2, T^{\pm}] \otimes \Lambda(Y_1, Y_2)$ with

$$|X_1| = |X_2| = |T| = 0, \qquad |Y_1| = |Y_2| = 1,$$

and comultiplication

$$\begin{aligned} &\Delta(X_1) = X_1 \otimes 1 + T^2 \otimes X_1 + Y_1 T \otimes Y_1, \quad \Delta(T) = T \otimes T, \\ &\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + Y_2 \otimes Y_2, \\ &\Delta(Y_2) = Y_2 \otimes 1 + 1 \otimes Y_2, \quad \Delta(Y_1) = Y_1 \otimes 1 + T \otimes Y_1. \end{aligned} \tag{4.16}$$

Theorem 4.5. *There is an exact sequence of Hopf superalgebras*

$$\mathcal{O}(\mathfrak{G}) \hookrightarrow \widetilde{\mathcal{D}} \twoheadrightarrow U(\mathfrak{osp}(1|2)). \tag{4.17}$$

Proof. From the defining relations of $\widetilde{\mathcal{D}}$ we deduce that

$$\begin{aligned} x_{21}x_1 &= x_1x_{21}, & x_{21}u_1 &= u_1x_{21}, & u_{21}u_1 &= u_1u_{21}, \\ \widetilde{\mathbf{g}}x_{21} &= x_{21}\widetilde{\mathbf{g}}, & x_{21}u_{21} &= u_{21}x_{21}, & \widetilde{\mathbf{g}}u_{21} &= u_{21}\widetilde{\mathbf{g}}. \end{aligned}$$

Hence the map $\iota : \mathcal{O}(\mathfrak{G}) \hookrightarrow \widetilde{\mathcal{D}}$ given by

$$Y_1 \mapsto x_1, \quad Y_2 \mapsto u_1, \quad X_1 \mapsto x_{21}, \quad X_2 \mapsto u_{21}, \quad T \mapsto \widetilde{\mathbf{g}}$$

is a well defined injective morphism of Hopf superalgebras. Next the map $\pi\colon \widetilde{\mathcal{D}} \longrightarrow U(\mathfrak{osp}(1|2))$ given by

$$x_1 \mapsto 0, \quad x_2 \mapsto \psi^-, \quad u_1 \mapsto 0, \quad u_2 \mapsto \psi^+, \quad \widetilde{\mathbf{g}} \mapsto 1, \quad \zeta \mapsto -h.$$

is a well defined surjective morphism of Hopf superalgebras, since $\pi(x_{21}) = \pi(u_{21}) = 0$, $\pi(-x_2^2) = f$ and $\pi(u_2^2) = e$. We check that $\ker\pi = \widetilde{\mathcal{D}}\iota(\mathcal{O}(\mathfrak{G}))^+$ using the PBW-basis of $\widetilde{\mathcal{D}}$. Thus (4.17) is exact. $\square$

4.3. A central Hopf subalgebra. In this Subsection $\operatorname{char}\Bbbk = p > 2$. We show that $\widetilde{\mathcal{D}}$ has a central Hopf subalgebra $Z = \mathcal{O}(\mathbf{B})$, with $\mathbf{B}$ a solvable algebraic group. We shall need the following commutation relations in $\widetilde{\mathcal{D}}$.

Lemma 4.6. *The following equalities are valid for all $n, m \in \mathbb{N}_0$:*

$$\begin{aligned}
x_2^{2m}x_{21}^n &= \sum_{k=0}^{m}\binom{m}{k}[n]^{[k]}x_{21}^{n+k}x_2^{2(m-k)},\\
x_2^{2m+1}x_{21}^n &= \sum_{\ell=0}^{1}\sum_{k=0}^{m}\binom{m}{k}[n]^{[k+\ell]}x_1^{\ell}x_{21}^{n+k}x_2^{2(m-k)-\ell+1},\\
x_2^{2m}x_1 &= \sum_{k=0}^{m}(-1)^k[-m]^{[k]}x_1x_{21}^{k}x_2^{2(m-k)},\\
x_2^{2m+1}x_1 &= \sum_{\ell=0}^{1}\sum_{k=0}^{m}(-1)^{k+\ell}[-m]^{[k]}x_1^{\ell}x_{21}^{k-\ell+1}x_2^{2(m-k)+\ell}\\
\widetilde{\mathbf{g}}^n x_2^{2m} &= \sum_{k=0}^{m}\binom{m}{k}[-n]^{[k]}x_{21}^{k}x_2^{2(m-k)}\widetilde{\mathbf{g}}^n,\\
\widetilde{\mathbf{g}}^n x_2^{2m+1} &= \sum_{\ell=0}^{1}\sum_{k=0}^{m}\binom{m}{k}(-1)^{n+1}[-n]^{[k+\ell]}x_1^{\ell}x_{21}^{k}x_2^{2(m-k)-\ell+1}\widetilde{\mathbf{g}}^n,\\
x_2^{2m}x_{21}^n &= \sum_{k=0}^{m}\binom{m}{k}[n]^{[k]}x_{21}^{n+k}x_2^{2(m-k)},\\
x_2^{2m+1}x_{21}^n &= \sum_{\ell=0}^{1}\sum_{k=0}^{m}\binom{m}{k}[n]^{[k+\ell]}x_1^{\ell}x_{21}^{n+k}x_2^{2(m-k)-\ell+1},\\
\zeta^n x_2^m &= \sum_{\ell=0}^{n}\binom{n}{\ell}m^{n-\ell}x_2^m\zeta^{\ell},\\
\zeta^n x_{21}^m &= \sum_{\ell=0}^{n}\binom{n}{\ell}(2m)^{n-\ell}x_{21}^m\zeta^{\ell},\\
u_2^{2m}u_1 &= \sum_{k=0}^{m}(-1)^k[-m]^{[k]}u_1u_{21}^{k}u_2^{2(m-k)},\\
u_2^{2m+1}u_1 &= \sum_{\ell=0}^{1}\sum_{k=0}^{m}(-1)^{k+\ell}[-m]^{[k]}x_1^{\ell}u_{21}^{k-\ell+1}u_2^{2(m-k)+\ell}
\end{aligned}$$

$$
\begin{aligned}
u_2^{2m}\widetilde{\mathbf{g}}^n &= \sum_{k=0}^{m}\binom{m}{k}[-n]^{[k]}\widetilde{\mathbf{g}}^n u_{21}^k u_2^{2(m-k)},\\
u_2^{2m+1}\widetilde{\mathbf{g}}^n &= \sum_{\ell=0}^{1}\sum_{k=0}^{m}\binom{m}{k}(-1)^{n+1}[-n]^{[k+\ell]}x_1^\ell\widetilde{\mathbf{g}}^n u_{21}^k u_2^{2(m-k)-\ell+1},\\
u_2^m\zeta^n &= \sum_{\ell=0}^{n}\binom{n}{\ell}m^{n-\ell}\zeta^\ell u_2^m,\\
u_{21}^m\zeta^n &= \sum_{\ell=0}^{n}\binom{n}{\ell}(2m)^{n-\ell}\zeta^\ell u_{21}^m,\\
u_2x_{21}^n &= x_{21}u_2 - 2nx_{21}^n u_1 + nx_1x_{21}^{n-1}(1+\widetilde{\mathbf{g}}),\\
u_{21}^n x_2 &= x_2u_{21}^n + n(\widetilde{\mathbf{g}}+1)u_1u_{21}^{n-1},\\
u_2^{2n}x_1 &= x_1u_2^{2n} - nx_1u_{21}u_2^{2(n-1)} + nu_1u_2^{2n-1},\\
u_2^{2n}x_2 &= x_2u_2^{2n} - nx_2u_{21}u_2^{2(n-1)} + n\widetilde{\mathbf{g}}u_2^{2n-1}\\
&\quad - n\widetilde{\mathbf{g}}u_1u_2^{2(n-1)} - n(n-1)\widetilde{\mathbf{g}}u_{21}u_2^{2(n-1)-1},\\
u_1x_2^{2n} &= x_2^{2n}u_1 + nx_2^{2(n-1)}x_1\widetilde{\mathbf{g}} - n(n-1)x_{21}x_2^{2(n-2)}x_1\widetilde{\mathbf{g}},\\
u_2x_2^{2n} &= x_2^{2n}u_2 - 2nx_2^{2n}u_1 + nx_2^{2n-1} - nx_2^{2(n-1)}x_1\widetilde{\mathbf{g}}\zeta\\
&\quad + n(n-1)x_{21}x_2^{2(n-2)}x_1\widetilde{\mathbf{g}}\zeta - n(2n-1)x_2^{2(n-1)}x_1\widetilde{\mathbf{g}}\\
&\quad + n(n-1)(2n-1)x_{21}x_2^{2(n-2)}x_1\widetilde{\mathbf{g}},
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{\mathbf{g}}^n x_1 = x_1\widetilde{\mathbf{g}}^n, && \widetilde{\mathbf{g}}^n x_{21}^m = x_{21}^m\widetilde{\mathbf{g}}^n, && x_1x_{21}^n = x_{21}^n x_1,\\
&u_{21}^n x_1 = x_1u_{21}^n, && x_{21}^n u_1 = u_1x_{21}^n.
\end{aligned}
$$

Proof. Straightforward by induction. □

For our next statement we need to set up the notation. Let

$$\mathbf{B} := ((\mathbf{G}_a \times \mathbf{G}_a) \rtimes \mathbf{G}_m) \times \mathbf{H}_3 \tag{4.18}$$

be the algebraic group that in the first factor has the semidirect product where $\mathbf{G}_m$ acts on $\mathbf{G}_a \times \mathbf{G}_a$ by $\lambda.(r_1, r_2) = (\lambda^2 r_1, \lambda^2 r_2)$, $\lambda \in \Bbbk^\times$, $r_1, r_2 \in \Bbbk$ while in the second factor appears the Heisenberg group $\mathbf{H}_3$ i.e. the group of upper triangular matrices with ones in the diagonal. Let $\zeta^{(p)} := \zeta^p - \zeta$ and $Z := \Bbbk\langle x_{21}^p, x_2^{2p}, u_{21}^p, u_2^{2p}, \widetilde{\mathbf{g}}^p, \zeta^{(p)}\rangle \hookrightarrow \widetilde{\mathcal{D}}$. Note that Z is an even subalgebra.

Theorem 4.7. (i) *Z is a central Hopf subalgebra of $\widetilde{\mathcal{D}}$.*

(ii) *$\widetilde{\mathcal{D}}$ is a finitely generated free Z-module.*

(iii) *$Z \simeq \Bbbk[T^\pm, X_1, \dots, X_5]$ as an algebra. In particular Z is a domain.*

(iv) *$Z \simeq \mathcal{O}(\mathbf{B})$ as Hopf algebras.*

Proof. (i) By Lemma 4.6 Z is a even central subalgebra of $\widetilde{\mathcal{D}}$. We need to verify that is a subcoalgebra and invariant by the antipode. We have the following comultiplication formulas for every $n \in \mathbb{N}$

$$\begin{aligned}
\Delta(x_{21}^n) &= \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k} n^\ell x_1^\ell x_{21}^k \widetilde{\mathbf{g}}^{2(n-k)-\ell} \otimes x_1^\ell x_{21}^{n-k-\ell},\\
\Delta(x_2^{2n}) &= \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\sum_{t=0}^{k}\binom{n-\ell}{k}\binom{k}{t} n^\ell [k-n+\ell]^{[t]}\\
&\qquad x_1^\ell x_{21}^t x_2^{2(k-t)} \widetilde{\mathbf{g}}^{2(n-k)-\ell} \otimes x_2^{2(n-k)-\ell},\\
\Delta(u_{21}^n) &= \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k} n^\ell u_1^\ell u_{21}^k \otimes u_1^\ell u_{21}^{n-k-\ell}.
\end{aligned}$$

Hence $u_{21}^p, \zeta^{(p)} \in \mathcal{P}(\widetilde{\mathcal{D}})$, x_2^{2p}, x_{21}^p are $(1, \widetilde{\mathbf{g}}^{2p})$-primitive and $\widetilde{\mathbf{g}}^p \in G(\widetilde{\mathcal{D}})$. It only remains to calculate $\Delta(u_2^{2p})$. Recall the formula (3.12) and

$$\delta(u_2^{2n}) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\begin{bmatrix}k\\ j\end{bmatrix}(-1)^k \zeta^j \otimes u_{21}^k u_2^{2(n-k)}, \qquad n \in \mathbb{N},$$

where $\begin{bmatrix}k\\ j\end{bmatrix}$ are the Stirling numbers. Since $[\zeta]^{[p]} = \prod_{i=1}^{p}(\zeta+i-1) = \sum_{k=0}^{p}\begin{bmatrix}p\\ k\end{bmatrix}\zeta^k = \zeta^p - \zeta$, we have

$$\begin{bmatrix}p\\ k\end{bmatrix} = 0,\ k = 2, \dots, p-1, \qquad \begin{bmatrix}p\\ p\end{bmatrix} = 1, \qquad \text{and } \begin{bmatrix}p\\ 1\end{bmatrix} = -1. \tag{4.19}$$

Then we get $\Delta(u_2^{2p}) = u_2^{2p} \otimes 1 + 1 \otimes u_2^{2p} - \zeta^{(p)} \otimes u_{21}^p$.

(ii) To prove this we consider another basis of $\widetilde{\mathcal{D}}$ using a different basis of $\Bbbk[\zeta]$. The family of polynomials

$$(\zeta^{(p)})^k \zeta^j, \qquad k \in \mathbb{N}_0, j \in \mathbb{I}_{0,p-1},$$

is a basis of $\Bbbk[\zeta]$, see the proof of [AP, Prop. 2.6]. Thus the elements

$$x_1^{n_1} x_{21}^{n_{21}} x_2^{n_2} \widetilde{\mathbf{g}}^n (\zeta^{(p)})^k \zeta^j u_1^{m_1} u_{21}^{m_{21}} u_2^{m_2},$$

with $(n_1, m_1, n, n_2, n_{21}, k, m_2, m_{21}, j) \in \mathbb{I}_{0,1}^2 \times \mathbb{N}_0^6 \times \mathbb{I}_{0,p-1}$ form a basis of $\widetilde{\mathcal{D}}$. Hence a basis of $\widetilde{\mathcal{D}}$ as a Z-module is given by

$$x_1^{n_1} x_{21}^{n_{21}} x_2^{n_2} \widetilde{\mathbf{g}}^n \zeta^j u_1^{m_1} u_{21}^{m_{21}} u_2^{m_2},$$

with $(n_1, m_1, n_2, m_2, n_{21}, m_{21}, j, n) \in \mathbb{I}_{0,1}^2 \times \mathbb{I}_{0,2p-1}^2 \times \mathbb{I}_{0,p-1}^4$.

(iii) The map $\phi\colon \Bbbk[T^{\pm}, X_1, \dots, X_5] \longrightarrow Z$ given by

$$T \mapsto \widetilde{\mathbf{g}}^p, \quad X_1 \mapsto x_{21}^p, \quad X_2 \mapsto x_2^{2p}, \quad X_3 \mapsto -\zeta^{(p)}, \quad X_4 \mapsto u_{21}^p, \quad X_5 \mapsto u_2^{2p},$$

is the desired isomorphism of algebras.

(iv) It is easy to see that $\mathcal{O}(\mathbf{B}) \simeq \Bbbk[T^{\pm}, X_1, \dots, X_5]$ with comultiplication determined by $T \in G(\mathcal{O}(\mathbf{B}))$, $X_1, X_2 \in \mathcal{P}_{T^2,1}(\mathcal{O}(\mathbf{B}))$, $X_3, X_5 \in \mathcal{P}(\mathcal{O}(\mathbf{B}))$ and $\Delta(X_5) = X_5 \otimes 1 + 1 \otimes X_5 + X_3 \otimes X_4$. The claim follows. □

4.4. Ring theoretical properties of the double.

Proposition 4.8. (i) *The algebra $\widetilde{\mathcal{D}}$ admits an exhaustive ascending filtration $(\widetilde{\mathcal{D}}_n)_{n\in\mathbb{N}_0}$ such that* $\operatorname{gr}\widetilde{\mathcal{D}} \simeq \Bbbk[X_1, \dots, X_5, T^{\pm 1}] \otimes \Lambda(Y_1, \dots, Y_4)$.

(ii) *The algebras $\mathtt{D}$ and $\widetilde{\mathcal{D}}$ are noetherian.*

(iii) *If* $\operatorname{char}\Bbbk = p > 2$ *then $\widetilde{\mathcal{D}}$ is a PI-algebra.*

Proof. (i) Let A be the algebra presented by generators x_1, x_2, u_1, u_2, ζ, $\widetilde{\mathbf{g}}^{\pm 1}$ and Z_i, $i \in \mathbb{I}_4$, with relations $\widetilde{\mathbf{g}}^{\pm 1}\widetilde{\mathbf{g}}^{\mp 1} = 1$. The algebra A is graded with

$$\deg x_1 = \deg u_1 = 3, \qquad \deg x_2 = \deg u_2 = 4, \qquad \deg \widetilde{\mathbf{g}}^{\pm 1} = \pm 1,$$
$$\deg Z_1 = \deg Z_2 = \deg \zeta = 1, \qquad \deg Z_3 = \deg Z_4 = 2.$$

The filtration associated to this grading induces a filtration on $\widetilde{\mathcal{D}}$ via the epimorphism $A \twoheadrightarrow \widetilde{\mathcal{D}}$ given by

$$Z_1 \mapsto x_{21}, \qquad Z_2 \mapsto u_{21}, \qquad Z_3 \mapsto x_2^2, \qquad Z_4 \mapsto u_2^2,$$

the remaining generators being mapped to their homonyms. The relations of $\widetilde{\mathcal{D}}$ imply that $\operatorname{gr}\widetilde{\mathcal{D}}$ is super conmutative with the same parity as $\widetilde{\mathcal{D}}$. Then

$$\phi\colon \Bbbk[X_1, \dots, X_5, T^{\pm 1}] \otimes \Lambda(Y_1, \dots, Y_4) \to \operatorname{gr}\widetilde{\mathcal{D}}$$

given by $X_i \mapsto Z_i$, $i \in \mathbb{I}_4$, and

$$T \mapsto \widetilde{\mathbf{g}}, \quad Y_1 \mapsto x_1, \quad Y_2 \mapsto x_2, \quad Y_3 \mapsto u_1, \quad Y_4 \mapsto u_2, \quad X_5 \mapsto \zeta$$

is an isomorphism of algebras by comparison of the Hilbert series.

(ii) It is well-known that $\Bbbk[X_1, \dots, X_5, T^{\pm 1}] \otimes \Lambda(Y_1, \dots, Y_4)$ is noetherian, hence so is $\widetilde{\mathcal{D}}$ by (i) and a fortiori $\mathtt{D}$ which is a finitely generated $\widetilde{\mathcal{D}}$-module. (iii) follows from Theorem 4.7, Proposition 5.4 and [MR, Corollary 1.13]. □

5. The double of the restricted super Jordan plane

In this Section, $\operatorname{char}\Bbbk = p > 2$.

5.1. The bosonizations. Recall that $R_p = \Bbbk[\zeta]/(\zeta^p - \zeta)$ is a Hopf algebra with ζ primitive. Besides (3.13) we also fix the generators

$$C_p = \langle g \rangle, \qquad C_{2p} = \langle \gamma \rangle. \tag{5.1}$$

It is well-known that $\Bbbk^{C_p} \simeq R_p$, see e.g. [AP, 1.3]. Hence $\Bbbk^{C_{2p}} \simeq R_p \otimes \Bbbk C_2$ and the algebra $\Bbbk^{C_{2p}}$ is presented by generators ϵ and ζ with relations

$$\epsilon^2 = 1, \tag{5.2}$$
$$\zeta^p = \zeta, \tag{5.3}$$
$$\epsilon\zeta = \zeta\epsilon. \tag{5.4}$$

We consider the realizations of V in ${}^{\Bbbk C_{2p}}_{\Bbbk C_{2p}}\mathcal{YD}$ and W in ${}^{\Bbbk^{C_{2p}}}_{\Bbbk^{C_{2p}}}\mathcal{YD}$ given by

$$\gamma \rightharpoonup x_1 = -x_1, \quad \gamma \rightharpoonup x_2 = -x_2 + x_1, \quad \delta(x_i) = \gamma \otimes x_i, \quad i = 1, 2; \tag{5.5}$$

$$\begin{aligned} &\zeta \rightharpoonup u_i = u_i, \qquad \epsilon \rightharpoonup u_i = -u_i, \qquad i = 1, 2, \\ &\delta(u_1) = \epsilon \otimes u_1, \qquad \delta(u_2) = \epsilon \otimes u_2 - \zeta\epsilon \otimes u_1, \end{aligned} \tag{5.6}$$

Thus we have the Hopf algebras

$$H := \mathscr{B}(V)\#\Bbbk C_{2p}, \qquad K := \mathscr{B}(W)\#\Bbbk^{C_{2p}}.$$

They have PBW-basis, denoted by B_H and B_K, given by the ordered monomials (3.8) or (3.11) with $(n_1, n_{21}, n_2) \in \mathbb{I}_{0,1} \times \mathbb{N}_0 \times \mathbb{N}_0$, multiplied accordingly by elements of the groups C_p, C_{2p}, or powers of ζ. Then

$$\dim H = \dim K = 8p^3.$$

The comultiplications are given by $x_1, x_2 \in \mathcal{P}_{\gamma,1}(H)$, $u_1 \in \mathcal{P}_{\epsilon,1}(K)$ and $\Delta_K(u_2)$ by (4.1) Therefore there are surjective Hopf algebra maps

$$\mathtt{H} \twoheadrightarrow H, \qquad \mathtt{K} \twoheadrightarrow K.$$

The basic properties of H and $K \simeq H^*$ follow without difficulties.

5.2. The double. In this subsection we show that the Drinfeld double of H fits into an exact sequence of Hopf algebras $\Bbbk C_2 \hookrightarrow D(H) \twoheadrightarrow D$.

We first give a presentation of $D(H)$; for this we need that of $D(\Bbbk C_{2p})$ which follows easily since $D(\Bbbk C_{2p}) \simeq \Bbbk C_{2p} \otimes \Bbbk^{C_{2p}} \simeq \Bbbk C_{2p} \otimes R_p \otimes \Bbbk C_2$.

Proposition 5.1. *$D(H)$ is generated by $x_1, x_2, \zeta, \epsilon, \gamma, u_1, u_2$ with relations (1.1), (1.3), (3.10), (4.8), (4.9), (5.2), (5.3), (5.4) and*

$$\gamma^{2p} = 1, \qquad \epsilon\gamma = \gamma\epsilon, \quad \zeta\gamma = \gamma\zeta, \tag{5.7}$$

$$\gamma x_1 = -x_1\gamma, \qquad \gamma x_2 = (-x_2 + x_1)\gamma, \tag{5.8}$$

$$u_1^2 = 0, \qquad u_2 u_{21} = u_{21}u_2 + u_1 u_{21}, \tag{5.9}$$

$$u_i\zeta = \zeta u_i + u_i, \qquad i = 1, 2. \tag{5.10}$$

$$u_1 x_1 = -x_1 u_1, \qquad u_1 x_2 = -x_2 u_1 + (1 - \gamma\epsilon), \tag{5.11}$$

$$u_2 x_1 = -x_1 u_2 + (1 - \gamma\epsilon) + x_1 u_1, \qquad u_2 x_2 = -x_2 u_2 + \gamma\zeta\epsilon + x_2 u_1, \tag{5.12}$$

$$u_1\gamma = -\gamma u_1, \qquad u_2\gamma = -\gamma u_2 + \gamma u_1. \tag{5.13}$$

The comultiplication is determined by $\gamma, \epsilon \in G(D(H))$, $\zeta \in \mathcal{P}(D(H))$, $x_1, x_2 \in \mathcal{P}_{\gamma,1}(D(H))$, $u_1 \in \mathcal{P}_{\epsilon,1}(D(H))$ and (4.1). The monomials

$$x_1^{n_1} x_{21}^{n_{21}} x_2^{n_2} \gamma^i \epsilon^j \zeta^k u_1^{m_1} u_{21}^{m_{21}} u_2^{m_2}$$

with $(n_1, m_1, j, n_{21}, k, m_{21}, i, n_2, m_2) \in \mathbb{I}_{0,1}^3 \times \mathbb{I}_{0,p-1}^3 \times \mathbb{I}_{0,2p-1}^3$ is a PBW-basis of $D(H)$.

Proof. This is a direct application of Proposition 3.4. $\square$

Let $t := \gamma^p\epsilon$ and $g := \gamma^{p+1}$. The Hopf subalgebra generated by t is isomorphic to $\Bbbk C_2$ and the one generated by g to $\Bbbk C_p$. By the defining relations t is a central element. Let $Z_0 = \Bbbk\langle t\rangle$ and $D := D(H)/D(H)Z_0^+$. We use the same symbol for an element in $D(H)$ and its class in D.

Proposition 5.2. (a) *The algebra D is generated by $x_1, x_2, g, \zeta, u_1, u_2, \epsilon$ with relations* (1.1), (1.3), (3.10), (4.8), (4.9), (5.9), (5.10), (5.2), (5.3), (5.4)

$$\zeta g = g\zeta, \qquad g^p = 1, \tag{5.14}$$
$$gx_1 = x_1 g, \qquad gx_2 = (x_2 - x_1)g, \tag{5.15}$$
$$u_2 g = gu_2 - gu_1, \qquad u_1 g = gu_1, \tag{5.16}$$
$$u_1 x_2 = -x_2 u_1 + (1-g), \qquad u_2 x_2 = -x_2 u_2 + g\zeta + x_2 u_1, \tag{5.17}$$
$$u_2 x_1 = -x_1 u_2 + (1-g) + x_1 u_1, \qquad u_1 x_1 = -x_1 u_1, \tag{5.18}$$
$$\epsilon g = g\epsilon. \tag{5.19}$$

(b) *The sequence of Hopf algebras $Z_0 := \Bbbk C_2 \hookrightarrow D(H) \twoheadrightarrow D$ is exact.*

(c) *The algebra D has dimension $32p^6$ and basis consisting in the monomials*

$$x_1^{n_1} x_{21}^{n_{21}} x_2^{n_2} g^n \zeta^m u_1^{m_1} u_{21}^{m_{21}} u_2^{m_2} \epsilon^k$$

with $(n_1, m_1, k, n_{21}, m, m_{21}, n, n_2, m_2) \in \mathbb{I}_{0,1}^3 \times \mathbb{I}_{0,p-1}^4 \times \mathbb{I}_{0,2p-1}^2$

Proof. (a): Very similar to the proof of Proposition 4.2. (b) Conditions (i) and (ii) above are evident while (iii) is proved considering the basis of $D(H)$

$$x_1^{n_1} x_{21}^{n_{21}} x_2^{n_2} g^n \zeta^m u_1^{m_1} u_{21}^{m_{21}} u_2^{m_2} \epsilon^k (1-t)^\ell$$

with $(n_1, m_1, \ell, k, n, m, n_{21}, m_{21}, n_2, m_2) \in \mathbb{I}_{0,1}^4 \times \mathbb{I}_{0,p-1}^4 \times \mathbb{I}_{0,2p-1}^2$. Now Remark 2.1 gives (iv) since Z_0 is central. The proof of (c) is direct. □

Remark 5.3. There is an exact sequence of Hopf algebras $\mathcal{O}(\mathbf{B}) \hookrightarrow \mathtt{D} \twoheadrightarrow D$.

5.3. The Hopf superalgebra $\mathcal{D}$. Here we show that D is the bosonization of the Hopf superalgebra $\mathcal{D} := \Bbbk\langle x_1, x_2, u_1, u_2, g, \zeta\rangle \hookrightarrow D$ and that $\mathcal{D}$ is a restricted analogue of the Hopf superalgebra $\widetilde{\mathcal{D}}$.

Proposition 5.4. (i) *A PBW-basis of $\mathcal{D}$ is given by the monomials*

$$x_1^{n_1} x_{21}^{n_{21}} x_2^{n_2} g^n \zeta^m u_1^{m_1} u_{21}^{m_{21}} u_2^{m_2},$$

$(n_1, m_1, n, m, n_{21}, m_{21}, n_2, m_2) \in \mathbb{I}_{0,1}^2 \times \mathbb{I}_{0,p-1}^4 \times \mathbb{I}_{0,2p-1}^2$. *Thus* $\dim \mathcal{D} = 16p^6$.

(ii) *$\mathcal{D}$ has a triangular decomposition i.e. a linear isomorphism induced by multiplication*

$$\mathcal{D} \simeq \mathcal{B}(V) \otimes (\Bbbk C_p \otimes R_p) \otimes \mathcal{B}(W)^{\mathrm{op}}. \tag{5.20}$$

(iii) *$\mathcal{D}$ is generated by x_1, x_2, u_1, u_2, g, ζ with relations* (1.1), (1.3), (3.10), (4.9), (5.3), (5.15), (5.9), (5.10), (5.14), (5.16), (5.17), (5.18).

(iv) *$\mathcal{D}$ is a Hopf superalgebra with comultiplication given by $g \in G(\mathcal{D})$, $\zeta, u_1 \in \mathcal{P}(\mathcal{D})$, $x_1, x_2 \in \mathcal{P}_{g,1}(\mathcal{D})$ and* (4.15). *Indeed $\mathcal{D}\#\Bbbk C_2 \simeq D$ as Hopf algebras.*

(v) *There is an exact sequence of Hopf superalgebras*

$$\mathcal{O}(\mathbf{B}) \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{D}} \overset{\pi}{\twoheadrightarrow} \mathcal{D}.$$

Proof. (i), (ii), (iii) are analogous to the ones of Proposition 4.3. (iv) As in Proposition 4.3, show that $\mathcal{D} \simeq D^{\mathrm{co}\,\varpi}$ where $\varpi\colon D \twoheadrightarrow \Bbbk C_2$ is given by

$$x_1 \mapsto 0, \quad x_2 \mapsto 0, \quad u_1 \mapsto 0, \quad u_2 \mapsto 0, \quad g \mapsto 1, \quad \zeta \mapsto 0, \quad \epsilon \mapsto \epsilon.$$

(v) By the presentations of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ there exist $\pi\colon \widetilde{\mathcal{D}} \twoheadrightarrow \mathcal{D}$ such that

$$\mathbf{g} \mapsto g, \qquad \zeta \mapsto \zeta, \qquad x_i \mapsto x_i, \qquad u_i \mapsto u_i, \qquad i = 1, 2.$$

The map $\iota\colon \mathcal{O}(\mathbf{B}) \hookrightarrow \widetilde{\mathcal{D}}$ is defined as the ϕ in the proof of Theorem 4.7. Clearly $\ker \pi = \widetilde{\mathcal{D}}\iota(\mathcal{O}(\mathbf{B}))^+$ by the PBW bases. Since $\iota(\mathcal{O}(\mathbf{B}))$ is central and $\widetilde{\mathcal{D}}$ is a free module over $\iota(\mathcal{O}(\mathbf{B}))$, the claim follows from Remark 2.1. $\square$

Remark 5.5. We may realize V in ${}^{\Bbbk C_p}_{\Bbbk C_p}\mathcal{YDS}$ and W in ${}^{R_p}_{R_p}\mathcal{YDS}$ by

$$g \rightharpoonup x_1 = x_1,\ g \rightharpoonup x_2 = x_2 - x_1,\ \delta(x_i) = g \otimes x_i,\ |x_i| = 1,\ i = 1, 2;$$
$$\zeta \rightharpoonup u_i = u_i, \qquad |u_i| = 1, \qquad i = 1, 2,$$
$$\delta(u_1) = 1 \otimes u_1, \quad \delta(u_2) = 1 \otimes u_2 - \zeta \otimes u_1.$$

Thus we have the Hopf superalgebras $\mathcal{H} := \mathscr{B}(V) \natural \Bbbk C_p$, $\mathcal{K} := \mathscr{B}(W) \natural R_p$; clearly $\dim \mathcal{H} = \dim \mathcal{K} = 4p^3$. Their comultiplications are determined by $x_1, x_2 \in \mathcal{P}_{g,1}(\mathcal{H})$, $u_1 \in \mathcal{P}(\mathcal{K})$, while $\Delta_{\mathcal{K}}(u_2)$ is given by (4.15). Therefore there are surjective Hopf algebra maps $\widetilde{\mathcal{H}} \twoheadrightarrow \mathcal{H}$, $\widetilde{\mathcal{K}} \twoheadrightarrow \mathcal{K}$. The basic properties of $\mathcal{H}$ and $\mathcal{K}$ follow at once. Also, there are isomorphisms of Hopf algebras $H \simeq \mathcal{H}\#\Bbbk C_2$, $K \simeq \mathcal{K}\#\Bbbk C_2$. Finally, there are morphisms of Hopf superalgebras $\mathcal{H} \hookrightarrow \mathcal{D}$, $\mathcal{K}^{\mathrm{op}} \hookrightarrow \mathcal{D}$, hence $\mathcal{D} \simeq D(\mathcal{H})$, see [GZB].

5.4. $\mathcal{D}$ as an extension. We next show that $\mathcal{D}$ fits into an exact sequence of Hopf superalgebras $\mathbf{R} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathfrak{u}$ with $\mathbf{R}$ super commutative and $\mathfrak{u}$ super cocommutative. First let $\mathbf{R}$ be the super commutative Hopf superalgebra

$$\mathbf{R} := \Bbbk[X_1, X_2, T]/(X_1^p, X_2^p, T^p - 1) \otimes \Lambda(Y_1, Y_2)$$

with $|X_1| = |X_2| = |T| = 0$, $|Y_1| = |Y_2| = 1$ and comultiplication (4.16). Arguing as in Proposition 4.17, we have:

Theorem 5.6. *There exist Hopf superalgebra maps ι and π such that*

$$\mathbf{R} \overset{\iota}{\hookrightarrow} \mathcal{D} \overset{\pi}{\twoheadrightarrow} \mathfrak{u}(\mathfrak{osp}(1|2)) \tag{5.21}$$

is an exact sequence of Hopf superalgebras. $\square$

The simple $\mathcal{D}$-supermodules can be determined from the previous result.

Theorem 5.7. *There are exactly p isomorphism classes of simple $\mathcal{D}$-modules which have dimensions $1, 3, 5, \dots, 2p-1$.*

Proof. Being nilpotent, the ideal $\mathcal{D}\mathbf{R}^+ = \langle x_1, x_{21}, u_1, u_{21}, g - 1\rangle$ is contained in the Jacobson radical of $\mathcal{D}$; thus $\operatorname{Irrep} \mathcal{D} \simeq \operatorname{Irrep} \mathfrak{u}(\mathfrak{osp}(1|2))$ and [WZ, Prop. 6.3] applies. $\square$

5.5. Simple modules. We describe the simple $\mathcal{D}$-modules as quotients of Verma modules reproving Theorem 5.7. Let $\mathcal{D} = \oplus_{n\in\mathbb{Z}}\mathcal{D}^n$ be $\mathbb{Z}$-graded by

$$\deg x_1 = \deg x_2 = -1, \qquad \deg u_1 = \deg u_2 = 1, \qquad \deg g = \deg\zeta = 0.$$

Recall that $D(\Bbbk C_p) \simeq R_p \otimes \Bbbk C_p$. Consider the triangular decomposition (5.20) and the graded subalgebras $\mathcal{D}^{>0} := \mathscr{B}(W)^{\mathrm{op}}$, $\mathcal{D}^{<0} := \mathscr{B}(V)$. Then

(1) $\mathcal{D}^{>0} \subseteq \oplus_{n\in\mathbb{N}_0}\mathcal{D}^n$, $\mathcal{D}^{<0} \subseteq \oplus_{n\in-\mathbb{N}_0}\mathcal{D}^n$ and $D(\Bbbk C_p) \subseteq \mathcal{D}^0$.
(2) $(\mathcal{D}^{>0})^0 = \Bbbk = (\mathcal{D}^{<0})^0$.
(3) $\mathcal{D}^{\geq 0} := D(\Bbbk C_p)\mathcal{D}^{>0}$ and $\mathcal{D}^{\leq 0} := \mathcal{D}^{<0}D(\Bbbk C_p)$ are subalgebras of $\mathcal{D}$.

In this context the simple modules of $\mathcal{D}$ arise inducing from $\mathcal{D}^{\geq 0}$. The elements of $\Lambda := \operatorname{Irrep} D(\Bbbk C_p)$ are called *weights*. Since $\mathcal{D}^{>0}$ is local, the (homogeneous) projection $\mathcal{D}^{\geq 0} \twoheadrightarrow D(\Bbbk C_p)$ allows to identify $\Lambda \simeq \operatorname{Irrep}\mathcal{D}^{\geq 0}$. The Verma module associated to $\lambda \in \Lambda$ is

$$M(\lambda) = \operatorname{Ind}_{\mathcal{D}^{\geq 0}}^{\mathcal{D}} \lambda = \mathcal{D} \otimes_{\mathcal{D}^{\geq 0}} \lambda.$$

By a standard argument, $M(\lambda)$ is indecomposable. Let $L(\lambda)$ be the head of $M(\lambda)$. The following result is well-known, see for instance [V, Theorem 2.1].

Lemma 5.8. *The map $\lambda \mapsto L(\lambda)$ gives a bijection $\Lambda \simeq \operatorname{Irrep}\mathcal{D}$.* □

The set Λ is easy to compute since $D(\Bbbk C_p) \simeq \Bbbk C_p \otimes R_p$ and $\Bbbk C_p$ is local. Given $k \in \mathbb{F}_p$, let $\lambda_k = \Bbbk w_k$ be the one-dimensional vector space with action

$$g \cdot w_k = w_k, \qquad\qquad \zeta \cdot w_k = k w_k.$$

Lemma 5.9. *The map $k \mapsto \lambda_k$ provides a bijection $\mathbb{F}_p \simeq \Lambda$.* □.

We fix $k \in \mathbb{F}_p$ and compute $L(\lambda_k)$. Since $M(\lambda_k)$ is free as a $\mathcal{D}^{<0}$-module with basis w_k, the elements

$$w_k^{(n_1,n_{21},n_2)} := x_1^{n_1}x_{21}^{n_{21}}x_2^{n_2} \cdot w_k, \qquad n_1 \in \mathbb{I}_{0,1}, n_{21} \in \mathbb{I}_{0,p-1}, n_2 \in \mathbb{I}_{0,2p-1},$$

form a linear basis of $M(\lambda_k)$. This makes $M(\lambda_k)$ a graded module by

$$\deg w_k^{(n_1,n_{21},n_2)} = \deg(x_1^{n_1}x_{21}^{n_{21}}x_2^{n_2}),$$

so $M(\lambda_k) = \oplus_{n\leq 0}M(\lambda_k)_n$ and $M(\lambda_k)_0 = \Bbbk w_k$. Any proper submodule of $M(\lambda_k)$ is then contained in $\oplus_{n\leq -1}M(\lambda_k)_n$. Since $L(\lambda_k)$ is the unique simple quotient of $M(\lambda_k)$, we divide the later by proper submodules until we get a simple one. We start by the submodule N_k of $M(\lambda_k)$ generated by $S_k := \{w_k^{(1,0,0)}, w_k^{(0,1,0)}\}$.

Lemma 5.10. *The submodule N_k is proper.*

Proof. The action of u_1 and u_2 gives

$$\begin{aligned}
u_1x_1 \cdot w_k &= -x_1u_1 \cdot w_k = 0, \qquad u_1x_{21} \cdot w_k = x_{21}u_1 \cdot w_k = 0,\\
u_2x_1 \cdot w_k &= -x_1u_2 \cdot w_k + (1-g)\cdot w_k + x_1u_1 \cdot w_k = 0,\\
u_2x_{21} \cdot w_k &= x_{21}u_2 \cdot w_k - 2x_{21}u_1 \cdot w_k + x_1(1+g)\cdot w_k = 2x_1 \cdot w_k.
\end{aligned}$$

So $\mathcal{D}^{>0} \cdot S_k \subseteq S_k$. Then $N_k = \mathcal{D}^{\leq 0} \cdot S_k \subseteq \oplus_{n\leq -1}M(\lambda_k)_n$ is proper. □

Let $V_k = M(\lambda_k)/N_k$ and let y_j be the class of $w_k^{(0,0,j)}$ in V_k.

Lemma 5.11. *The family $(y_j)_{j\in\mathbb{I}_{0,2p-1}}$ generates linearly V_k and g, u_1, u_{21}, x_1 and x_{21} act trivially on V_k.*

Proof. We claim that the class of $w_k^{(n_1,n_{21},n_2)} = 0$ in V_k if $(n_1, n_{21}) \neq (0,0)$. It suffices to show that $w_k^{(0,1,n_2)} = 0$ and $w_k^{(1,0,n_2)} = 0$ in V_k for every n_2. This follows by induction on n_2 using the analogues for $\mathcal{D}$ of the relations in Lemma 4.6. Since x_{21} and x_1 commute, they both act trivially on V_k:

$$x_{21}\cdot w_k^{(n_1,n_{21},n_2)} = x_1^{n_1}x_{21}^{n_{21}}\cdot w_k^{(0,1,n_2)} = 0,$$
$$x_1\cdot w_k^{(n_1,n_{21},n_2)} = x_1^{n_1}x_{21}^{n_{21}}\cdot w_k^{(1,0,n_2)} = 0.$$

Then $(y_j)_{j\in\mathbb{I}_{0,2p-1}}$ generates linearly V_k. Also g, u_1 and u_{21} act trivially on these generators by Lemma 4.6. $\square$

Set $y_{-1} = y_{2p} = 0$. The action of $\mathcal{D}$ on V_k can be computed inductively:

$$\begin{aligned}
&\zeta\cdot y_j = (k-j)y_j, \quad g\cdot y_j = y_j, \quad x_2\cdot y_j = y_{j+1}, \quad x_1\cdot y_j = 0,\\
&u_1\cdot y_j = 0, \quad u_2\cdot y_j = \begin{cases}\frac{j}{2}y_{j-1} & \text{if } j \text{ is even},\\ (\frac{j-1}{2}-k) & \text{if } j \text{ is odd},\end{cases} \quad j\in\mathbb{I}_{0,2p-1}.
\end{aligned}\tag{5.22}$$

Now $\widetilde{V}_k := \mathcal{D}y_{2k+1}$ is a proper submodule of V_k because $\mathcal{D}^{>0}\cdot y_{2k+1} = 0$.

Proposition 5.12. *The module $L_k = V_k/\widetilde{V}_k$ is simple of dimension $2k+1$.*

It follows that $L_k = L(\lambda_k)$, the head of the Verma module $M(\lambda_k)$.

Proof. Let z_j be the class of y_j in L_k; the action of $\mathcal{D}$ on the z_j's is still given by (5.22). Then $(z_j)_{j\in\mathbb{I}_{0,2k}}$ is a basis of L_k. To see that L_k is simple, we show that every $0\neq z\in L_k$ generates L_k. Let $z = \sum_{j=0}^{m} c_j\, z_j$ with $m\leq 2k$ and $c_m\neq 0$. Then $u_2^m\cdot z\in\Bbbk^\times z_0$, and $\mathcal{D}\cdot z = L_k$. $\square$

5.6. $\mathcal{O}(\mathfrak{G})$ as an extension. Let $\mathbf{G} = (\mathbf{G}_a\times\mathbf{G}_a)\rtimes\mathbf{G}_m$ be the semidirect product where $\mathbf{G}_m$ acts on $\mathbf{G}_a\times\mathbf{G}_a$ by $\lambda.(t_1,t_2) = (\lambda^2 t_1, t_2)$, $\lambda\in\Bbbk^\times$, $t_1,t_2\in\Bbbk$. Then $\mathcal{O}(\mathbf{G})$ is isomorphic to $A := \Bbbk[\mathrm{X}_1,\mathrm{X}_2,\mathrm{T}^{\pm1}]$ with comultiplication given by $\mathrm{T}\in G(A)$, $\mathrm{X}_1\in\mathcal{P}_{\mathrm{T}^2,1}(A)$, $\mathrm{X}_2\in\mathcal{P}(A)$.

Proposition 5.13. *There is a short exact sequence of Hopf superalgebras*

$$\mathcal{O}(\mathbf{G}) \overset{\iota}{\hookrightarrow} \mathcal{O}(\mathfrak{G}) \overset{\pi}{\twoheadrightarrow} \mathbf{R},\tag{5.23}$$

with morphisms $\iota\colon \mathcal{O}(\mathbf{G})\hookrightarrow\mathcal{O}(\mathfrak{G})$, $\pi\colon\mathcal{O}(\mathfrak{G})\longrightarrow\mathbf{R}$ given by

$$\begin{aligned}
&\iota(\mathrm{X}_1) = X_1^p, \quad \iota(\mathrm{X}_2) = X_2^p, \quad \iota(\mathrm{T}) = T^p,\\
&\pi(X_1) = X_1, \quad \pi(X_2) = X_2, \quad \pi(T) = T, \quad \pi(Y_1) = Y_1, \quad \pi(Y_2) = Y_2.
\end{aligned}$$

Proof. Clearly π is of Hopf superalgebras by the definition of comultiplication in $\mathcal{O}(\mathfrak{G})$ and $\mathbf{R}$. In $\mathcal{O}(\mathfrak{G})$ we have the following comultiplication formulas

$$\Delta(X_1^n) = \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k} n^\ell Y_1^\ell X_1^k T^{2(n-k)-\ell} \otimes Y_1^\ell X_1^{n-k-\ell},$$
$$\Delta(X_2^n) = \sum_{\ell=0}^{1}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k} n^\ell Y_2^\ell X_2^k \otimes Y_2^\ell X_2^{n-k-\ell},$$

$n \in \mathbb{N}$. Thus $X_1^p \in \mathcal{P}_{\mathrm{T}^{2p},1}(\mathcal{O}(\mathfrak{G}))$, $X_2^p \in \mathcal{P}(\mathcal{O}(\mathfrak{G}))$ and ι is of Hopf superalgebras. Then $\ker\pi = \mathcal{O}(\mathfrak{G})\iota(\mathcal{O}(\mathbf{G}))^+$, ι is injective and π is surjective, using the PBW bases. Since $\mathcal{O}(\mathfrak{G})$ is a free $\mathcal{O}(\mathbf{G})$-module, Remark 2.1 applies. $\square$

5.7. A commutative square. We summarize the relationship between the Hopf superalgebras studied so far in the commutative diagram

$$\begin{tikzcd}
\mathcal{O}(\mathbf{G}) \arrow[r, hook] \arrow[d, hook] & \mathcal{O}(\mathbf{B}) \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}(\mathbf{G}_a^3) \arrow[d, hook] \\
\mathcal{O}(\mathfrak{G}) \arrow[r, hook] \arrow[d, two heads] & \widetilde{\mathcal{D}} \arrow[r, two heads] \arrow[d, two heads] & U(\mathfrak{osp}(1|2)) \arrow[d, two heads] \\
\mathbf{R} \arrow[r, hook] & \mathcal{D} \arrow[r, two heads] & \mathfrak{u}(\mathfrak{osp}(1|2))
\end{tikzcd} \tag{5.24}$$

Proposition 5.14. *All columns and rows in (5.24) are exact sequences.*

Proof. Theorems 4.5 and 5.6, and and Propositions 5.4 and 5.13 covers everything except the topmost row and the rightmost column.

For the rightmost column we need to prove that the even Hopf subalgebra $Z' = \langle e^p, f^p, h^p - h\rangle$ of $U(\mathfrak{osp}(1|2))$ is $\mathcal{O}(\mathbf{G}_a^3) \simeq \Bbbk[X_1, X_2, X_3]$. Taking the basis of $U(\mathfrak{osp}(1|2))$ consisting of monomials

$$f^n(h^p - h)^k h^\ell e^m \psi_+^i \psi_-^j$$

with $(n, m, k, \ell, i, j) \in \mathbb{N}_0^3 \times \mathbb{I}_{0,p-1} \times \mathbb{I}_{0,1}^2$, we see that the assignment

$$X_1 \mapsto f^p, \qquad X_2 \mapsto h^p - h, \qquad X_3 \mapsto e^p,$$

gives an algebra isomorphism $Z' \simeq \Bbbk[X_1, X_2, X_3] \simeq \mathcal{O}(\mathbf{G}_a^3)$. Comparing comultiplications, the previous isomorphism is of Hopf algebras. $\mathcal{O}(\mathbf{G}_a^3)$ is stable by the adjoint action of $U(\mathfrak{osp}(1|2))$ and a free module over $\mathcal{O}(\mathbf{G}_a^3)$ using the previous basis, then Remark 2.1 applies and the column is exact.

We next describe explicitly the top row. $\phi\colon \mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{B})$ is given by

$$\mathrm{X}_1 \mapsto X_1, \qquad \mathrm{X}_2 \mapsto X_4, \qquad \mathrm{T} \mapsto T.$$

Recall that $\mathcal{O}(\mathbf{B}) \simeq \Bbbk[T^\pm, X_1, \dots, X_5]$ and $\mathcal{O}(\mathbf{G}) \simeq \Bbbk[\mathrm{X}_1, \mathrm{X}_2, \mathrm{T}^{\pm 1}]$, cf. Theorem 4.7 and Proposition 5.13. Take $\psi\colon \mathcal{O}(\mathbf{B}) \longrightarrow \mathcal{O}(\mathbf{G}_a^3)$ given by

$$T \mapsto 1, \quad X_1 \mapsto 0, \quad X_2 \mapsto -f^p, \quad X_3 \mapsto h^p - h, \quad X_4 \mapsto 0, \quad X_5 \mapsto e^p.$$

Now ϕ is a injective because it maps a basis into a linearly independent set while ψ is surjective since the PBW-basis of $\mathcal{O}(\mathbf{G}_a^3)$ is in its image. Clearly $\ker \psi \supseteq \mathcal{O}(\mathbf{B})\phi(\mathcal{O}(\mathbf{G}))^+$, the other inclusion follows using the basis of $\mathcal{O}(\mathbf{B})$

$$\tau_m X_1^n X_2^r X_3^k X_4^i X_5^j, \qquad i, j, k, n, r \in \mathbb{N}_0, m \in \mathbb{Z},$$

where τ_m is given by the formula

$$\tau_m = \begin{cases} (T-1)^m & \text{if } m \geq 0, \\ (T^{-1}-1)^{-m} & \text{if } m < 0 \end{cases}.$$

The adjoint action of $\mathcal{O}(\mathbf{B})$ is trivial, so $\phi(\mathcal{O}(\mathbf{G}))$ is invariant. Considering the PBW-bases of $\mathcal{O}(\mathbf{B})$ and $\mathcal{O}(\mathbf{G})$, ϕ is faithfully flat. Thus the row is exact. □

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