Equivariant homology decompositions for cyclic group actions on definite 4-manifolds

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ABSTRACT. In this paper, we study the equivariant homotopy type of a connected sum of linear actions on complex projective planes defined by Hambleton and Tanase. These actions are constructed for cyclic groups of odd order. We construct cellular filtrations on the connected sum using spheres inside unitary representations. A judicious choice of filtration implies a splitting on equivariant homology for general cyclic groups under a divisibility hypothesis, and in all cases for those of prime power order.

CONTENTS

1. Introduction

Simply connected 4-manifolds form an important category of spaces from the point of view of both topologists and geometers. Their homotopy type is determined by the intersection form. The ones with positive definite intersection form are homotopy equivalent to a connected sum of copies of $\mathbb{C}P^2$. This paper studies the equivariant homotopy type of certain cyclic group actions on these 4-manifolds defined in [9], and splitting results for the equivariant homology with constant coefficients.

Recall that a simply connected 4-manifold M possesses a CW-complex structure whose 2-skeleton is a wedge of spheres, and outside the 2-skeleton, there is a single 4-cell. It follows that the homology is torsion-free, and non-zero in only three degrees 0, 2 and 4, with $H_0(M) = \mathbb{Z}$ and $H_4(M) = \mathbb{Z}$. If k is the second Betti number of M, in the stable homotopy category, we obtain the decomposition $H\mathbb{Z} \wedge M_+ \simeq H\mathbb{Z} \vee (\bigvee_{i=1}^{k} \Sigma^2 H\mathbb{Z}) \vee \Sigma^4 H\mathbb{Z}$.

Homology decompositions akin to the above in equivariant stable homotopy category have been studied for complex projective spaces in [12] over the group C_p. The splitting therein is obtained with Burnside ring coefficients, which is denoted by $\underline{A}$. However, Lewis [13, Remark 2.2] shows that these decompositions are not likely if the group contains C_{p^2} or $C_p \times C_p$. On the other hand, for cyclic groups of square free order these splittings do exist ([1], [2]). There are generalizations of these results for the group C_p for even cell complexes ([6], [7]), and with constant $\mathbb{Z}/p$-coefficients [3]. Such results have also been extensively studied for the group C_2 with constant $\mathbb{Z}/2$-coefficients ([15], [11], [10]). In this paper, we prove decomposition results for cyclic group actions on a connected sum of copies of $\mathbb{C}P^2$ with constant $\mathbb{Z}$-coefficients.

The homology decompositions for X are usually proved by building up a cellular filtration of X, and then showing that after smashing with the spectrum $H\mathbb{Z}$, the connecting maps are all trivial. For this purpose, the cells are taken of the form $D(V)$, a disk in a unitary G-representation V, so that the filtration quotients are wedges of S^V, the one-point compactification of V. There is a different concept of G-CW-complexes with cells of the type $G/H \times \mathcal{D}^n$ for subgroups $H \leq G$, but they are not useful from the point of view of homology decompositions.

In this paper, the G-manifolds $X(\mathbb{T})$ are defined using *admissible weighted trees* $\mathbb{T}$ [9], which are directed rooted trees with G-action, with each vertex carrying a weight comprising 3 integers a, b, m of gcd 1 (see figure below). The underlying manifolds $\mathbb{C}P^2(a, b; m)$ are copies of $\mathbb{C}P^2$, which have an action of the group C_m, by identifying $\mathbb{C}P^2$ as the space of complex lines in a three-dimensional complex representation of C_m. The numbers a, b are used to describe the irreducible representations therein. We fix λ as a complex 1-dimensional representation where C_m acts via m^{th}-roots of unity, and in these terms $\mathbb{C}P^2(a, b; m) = P(\lambda^a + \lambda^b + 1_{\mathbb{C}})$. The admissible part of the definition of the tree allows us to construct the equivariant connected sum in the figure.
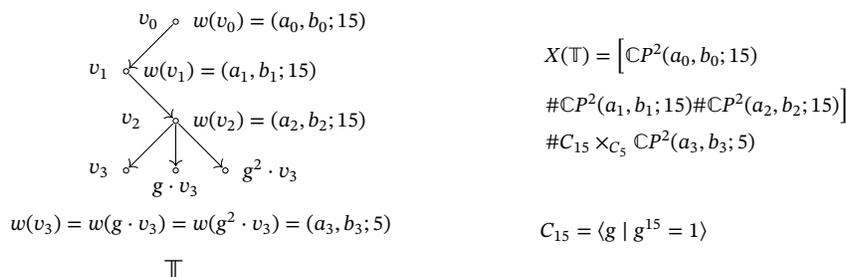

FIGURE 1.0. Example of an equivariant connected sum defined using a tree.

For a cyclic group C_m of odd order, we prove two decomposition results (see Theorem 4.5 and Theorem 4.9), where $\Sigma^V H\underline{\mathbb{Z}}$ denotes $H\underline{\mathbb{Z}} \wedge S^V$. In the theorem

below, the notation $\mathbb{T}_0$ stands for the C_m-fixed points of $\mathbb{T}$, and $\mathbb{T}_d$ refers to the vertices whose stabilizer is C_d for $d \mid m$.

Theorem A. *a) If $\mathbb{T}$ is an admissible weighted tree such that all fixed vertices v with weight $w(v) = (a_v, b_v; m_v)$ satisfy $\gcd(a_v - b_v, m_v) = 1$, then*

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda a_0 + \lambda b_0} H\underline{\mathbb{Z}} \vee \left(\bigvee_{\mathbb{T}_0} \Sigma^\lambda H\underline{\mathbb{Z}} \right)$$

$$\vee \left(\bigvee_{[\mu] \in \mathbb{T}_d / C_m, d \neq m} C_m / C_{d+} \wedge \Sigma^{\lambda a_\mu - b_\mu} H\underline{\mathbb{Z}} \right)$$

where $w(v_0) = (a_0, b_0; m)$ is the weight of the root vertex.
b) Let $\mathbb{T}$ be an admissible weighted tree such that for the root vertex v_0 with weight $w(v_0) = (a_0, b_0; m)$, one of a_0 or b_0 is zero. Then,

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \left(\bigvee_{[\mu] \in \mathbb{T}_d / C_m} C_m / C_{d+} \wedge \Sigma^{\lambda a_\mu - b_\mu} H\underline{\mathbb{Z}} \right).$$

For example in Figure 1.0, if $\gcd(a_i - b_i, 15) = 1$ for $0 \leq i \leq 2$, we are in the case a) of Theorem A which implies the decomposition

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda a_0 + \lambda b_0} H\underline{\mathbb{Z}} \vee \left((\Sigma^\lambda H\underline{\mathbb{Z}})^{\vee 3} \right) \vee \left(C_{15} / C_{5+} \wedge \Sigma^{\lambda a_3 - b_3} H\underline{\mathbb{Z}} \right).$$

On the other hand if $a_0 = 0$, we are in case b) of Theorem A, and the second summand here includes $d = m$. Also observe that the condition implies that $a_0 - b_0$ is relatively prime to 15. Therefore, we have

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^\lambda H\underline{\mathbb{Z}} \vee \Sigma^{\lambda a_1 - b_1} H\underline{\mathbb{Z}}$$

$$\vee \Sigma^{\lambda a_2 - b_2} H\underline{\mathbb{Z}} \vee \left(C_{15} / C_{5+} \wedge \Sigma^{\lambda a_3 - b_3} H\underline{\mathbb{Z}} \right).$$

The results in Theorem A depend on a hypothesis on the weights at vertices which are fixed under the C_m-action. We further prove that these hypotheses may be removed when the group is of prime power order. (See Theorem 5.7 and Theorem 5.10)

Theorem B. *a) Let $\mathbb{T}$ be an admissible weighted C_p-equivariant tree such that $p \nmid a_0, b_0$ but $p \mid a_v - b_v$ for some fixed vertex v. Then,*

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \left(\bigvee_{\phi(\mathbb{T})+1} \Sigma^\lambda H\underline{\mathbb{Z}} \right)$$

$$\vee \left(\bigvee_{\psi(\mathbb{T})-1} \Sigma^2 H\underline{\mathbb{Z}} \right) \vee \left(\bigvee_{\mathbb{T}_e / C_p} C_p / e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \right).$$

b) Let $\mathbb{T}$ be an admissible weighted C_{p^n}-equivariant tree. Suppose $\tau > 0$ is the maximum power of p that divides $a_v - b_v$ among the fixed vertices v and $p \nmid a_0, b_0$.

Then

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+\lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_{\mathbb{T}}(i)} \Big)$$

$$\vee \Big(\bigvee_{[\mu] \in \mathbb{T}_d/C_{p^n},\, d \neq p^n} C_{p^n}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big).$$

In the statements of Theorem B, we observe that the complementary cases are proved in Theorem A. The notations $\phi(\mathbb{T})$, $\psi(\mathbb{T})$ and $W_{\mathbb{T}}(i)$ are clarified later in the document. The techniques used in the proof are the cellular filtration of the manifolds $X(\mathbb{T})$, and the following result about the $RO(C_m)$-graded homotopy groups of $H\underline{\mathbb{Z}}$. (See Theorem 3.8)

Theorem C. *Let $\alpha \in RO(C_m)$ be such that $|\alpha|$ is odd, and $|\alpha^H| > -1$ implies $|\alpha^K| \geq -1$ for all subgroups $K \supseteq H$. Then $\underline{\pi}_\alpha^{C_m}(H\underline{\mathbb{Z}}) = 0$.*

1.1. Organization. In Section 2, we recall the equivariant connected sum construction leading to the definition of tree manifolds associated to admissible weighted trees in [9]. In Section 3, we introduce the facts from equivariant stable homotopy theory required in the subsequent sections and prove Theorem C. In Section 4, we prove the homology decompositions over the group C_m (Theorem A), and finally in Section 5, we prove the results over prime power groups (Theorem B).

1.2. Notation. Throughout this paper, $\mathcal{G}$ denotes the cyclic group of order m, where m is odd, and g denotes a fixed generator of $\mathcal{G}$. Whenever the notation p is used for a prime, it is supposed to be odd. For an orthogonal $\mathcal{G}$-representation V, $S(V)$ denotes the unit sphere, $D(V)$ the unit disk, and S^V the one-point compactification of V, which is $\cong D(V)/S(V)$. The notation $1_{\mathbb{C}}$ is used for the trivial complex representation and 1 for the real trivial representation.

2. Tree manifolds

In this section, we discuss the construction of connected sum of G-manifolds focussing on the special case of a connected sum of complex projective planes in the case $G = \mathcal{G}$. In the latter case, the construction is governed through a system of explicit combinatorial data expressed as admissible weighted trees (see [9] for details). We refer to these as *tree manifolds*.

2.1. Equivariant connected sums. Let X and Y be two smooth G-manifolds of the same dimension n. The equivariant connected sum $X \# Y$ depends on the

following data

1) Points $x \in X^G$, $y \in Y^G$.

2) An orientation reversing isomorphism of real G-representations $\varphi : T_x X \to T_y Y$.

Given the data above, one may conjugate φ with the exponential map to obtain a diffeomorphism of punctured disks near x and y. This identification is then performed on $X \setminus \{x\} \sqcup Y \setminus \{y\}$ to obtain the equivariant connected sum $X \# Y$. One readily observes the following homotopy cofibration sequences

$$X \setminus \{x\} \to X \# Y \to Y, \text{ and } Y \setminus \{y\} \to X \# Y \to X. \tag{2.2}$$

An additional feature in the G-equivariant situation is the orbit-wise connected sum. Let X be a G-manifold and Y an N-manifold for a subgroup N. The data underlying an orbit-wise connected sum is

1) A point $y \in Y^N$, and a point $x \in X$ such that the stabilizer of x is N.

2) An orientation reversing isomorphism of real N-representations $\varphi : T_x X \to T_y Y$.

The condition 1) implies that x induces the inclusion of an orbit $i_x : G/N \to X$. Now we may again use the exponential map to conjugate φ and identify punctured disks at points of $G/N \hookrightarrow X$ with those at points of $G/N \times \{y\} \hookrightarrow G \times_N Y$. The resulting connected sum is denoted by $X \# G \times_N Y$. The direct analogues of (2.2) are

$$X \setminus \{x\} \to X \# G \times_N Y \to \frac{G \times_N Y}{G/N \times \{y\}},$$

and,

$$G \times_N (Y \setminus \{y\}) \to X \# G \times_N Y \to X / i_x(G/N).$$

The second sequence has a refinement in the form of a homotopy pushout

$$\begin{array}{ccc} G \times_N (Y - \{y\}) & \longrightarrow & G \times_N C(Y - \{y\}) \\ \downarrow & & \downarrow {\scriptstyle i_x \circ \pi_1} \\ X \# G \times_N Y & \longrightarrow & X. \end{array} \tag{2.3}$$

2.4. Linear actions on projective spaces. The principal construction of interest in this paper is the equivariant connected sum of projective spaces. A method to construct a G-action on a complex projective space $\mathbb{C}P^n$ is to write it as $P(V)$, the projectivization of a unitary representation V. We call these *linear actions*. If ν is a 1-dimensional complex representation of G, there is an equivariant homeomorphism $P(V) \cong P(V \otimes \nu)$. In the case $G = \mathcal{G}$, we fix the following notation for its representations.

2.5. Notation. The irreducible complex representations of the group $\mathcal{G}$ are 1-dimensional, and up to isomorphism are listed as $1_{\mathbb{C}}, \lambda, \lambda^2, \dots, \lambda^{m-1}$ where λ sends g to $e^{2\pi i/m}$, the mth root of unity. The non-trivial real irreducible representations are realizations of these. The realization of λ^i is also denoted by the

same notation. Note that λ^i and λ^{m-i} are conjugate and hence their realizations are isomorphic by the natural $\mathbb{R}$-linear map $z \mapsto \bar{z}$ which reverses orientation.

In this paper, our principal objects of interest are linear $\mathcal{G}$-actions on $\mathbb{C}P^2$, that is, we write $\mathbb{C}P^2$ as $P(V)$ where V is a 3-dimensional complex representation of $\mathcal{G}$. In terms of the notation above, V is a sum $\lambda^a + \lambda^b + \lambda^c$ for some integers a, b and c viewed (mod m). As $P(V) \cong P(V \otimes \nu)$ for 1-dimensional ν, we may assume $c = 0$ in the expression for V. We denote this by $\mathbb{C}P^2(a, b; m)$. Often we use the notation $\mathbb{C}P^2(a, b; m')$ for a divisor m' of m. This denotes the $C_{m'}$-space $P(1_{\mathbb{C}} + \lambda^a + \lambda^b)$. In this expression, note that the restriction of $\mathbb{C}P^2(a, b; m)$ to the subgroup $C_{m'}$ is $\mathbb{C}P^2(a, b; m')$.

Proposition 2.6. The manifolds $\mathbb{C}P^2(a, b; m)$ satisfy the following properties.
1) If $\gcd(a, b, m) = d$, then $\mathbb{C}P^2(a, b; m) \cong \pi^*\mathbb{C}P^2(\frac{a}{d}, \frac{b}{d}; \frac{m}{d})$ the pullback via $\pi : C_m \to C_m/C_d \cong C_{m/d}$.
2) There are $\mathcal{G}$-homeomorphisms

$$\mathbb{C}P^2(a, b; m) \cong \mathbb{C}P^2(a - b, -b; m) \cong \mathbb{C}P^2(-a, b - a; m),$$

and $\mathbb{C}P^2(a, b; m) \cong \mathbb{C}P^2(b, a; m)$.
3) The points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, and $p_3 = [0, 0, 1]$ are fixed by $\mathcal{G}$. Their tangential representations are given by

$$T_{p_1}\mathbb{C}P^2(a, b; m) \cong \lambda^{b-a} + \lambda^{-a}, T_{p_2}\mathbb{C}P^2(a, b; m) \cong \lambda^{a-b} + \lambda^{-b},$$

$$T_{p_3}\mathbb{C}P^2(a, b; m) \cong \lambda^a + \lambda^b.$$

The proof of the above easily follows from the homeomorphism $P(V) \cong P(V \otimes \nu)$, and the identification of the tangent bundle of $\mathbb{C}P^2$ as $Hom(\gamma, \gamma^\perp)$, where γ is the canonical line bundle. In this paper, we call the numbers (a, b) associated to the representation $\lambda^a + \lambda^b$ *rotation numbers*. As in [9], we assume that for the manifold denoted by $\mathbb{C}P^2(a, b; m)$, $\gcd(a, b, m) = 1$.

We also denote $S^4(a, b; m)$ for the $\mathcal{G}$-action on S^4 by identifying it with $S^{\lambda^a + \lambda^b}$. This may also be described as $S(1 + \lambda^a + \lambda^b)$, where 1 is the trivial real representation of dimension 1. This action has fixed points 0 and ∞, and the tangential representations are $\lambda^a + \lambda^b$ and $\lambda^{-a} + \lambda^{-b}$ respectively.

We now list the conditions required to form equivariant connected sums of copies of $\mathbb{C}P^2(a, b; m)$ and $S^4(a, b; m)$.

Proposition 2.7. 1) The connected sum $\mathbb{C}P^2(a, b; m)\#\mathbb{C}P^2(a', b'; m)$ may be formed if and only if for one of the equivalent choices of (a', b') as in 2) of Proposition 2.6, $\pm(a', b') \in \{(a, -b), (a - b, b), (a, b - a)\}$. Once this condition is satisfied, there is a natural choice of data for the connected sum unless $a = b$ or one of a, b is 0.
2) The connected sum $\mathbb{C}P^2(a, b; m)\#S^4(a', b'; m)$ may be formed if and only if $\pm(a', b') \in \{(a, -b), (a - b, b), (a, b - a)\}$. Here, $\mathbb{C}P^2(a, b; m)\#S^4(a', b'; m)$ is $\mathcal{G}$-homeomorphic to $\mathbb{C}P^2(a, b; m)$.
3) For $m' \mid m$ but $m' \neq m$, the connected sum $\mathbb{C}P^2(a, b; m)\#\mathcal{G}\times_{C_{m'}}\mathbb{C}P^2(a', b'; m')$

may be defined if and only if the following are satisfied

a) One of a, b, or $a - b$ satisfies the equation $\gcd(x, m) = m'$.

b) One of a', b', or $a' - b'$ is 0, and the others are up to sign two of the numbers in $\{-b, -a, b - a\}$ not divisible by m'.

The statements 1) and 2) follow from the examination of tangential representations at fixed points of $\mathbb{C}P^2(a, b; m)$, and 3) follows from the results in [9, §1.C].

2.8. Admissible weighted trees. The observations above inform us that the $\mathcal{G}$-connected sums of different $\mathbb{C}P^2(a, b; m)$s and $S^4(a, b; m)$s may be formed only when certain relations are satisfied between the weights involved. We now lay down the sequence of combinatorial criteria which allow us to form such a connected sum. These are written in the form of weights attached to trees with a $\mathcal{G}$-action satisfying required conditions, called *admissible weighted trees*.

Recall that a group action on a tree is given by an action on the vertices which preserves the adjacency relation. We define two types of trees called *Type I* and *Type II*.

Definition 2.9. An *admissible weighted tree* is a tree with $\mathcal{G}$-action having the following properties

(1) There is a $\mathcal{G}$-fixed vertex v_0 called the root vertex of the tree. In case of a type II tree, v_0 is the unique $\mathcal{G}$-fixed vertex.

(2) The vertices of $\mathbb{T}$ are arranged in levels starting from zero according to the distance from the root vertex with edge length considered to be 1. Observe that $\mathcal{G}$ preserves the levels and every edge goes from level L to $L + 1$ for some L.

(3) Each vertex v is equipped with a weight $w(v) = (a_v, b_v; m_v)$ (defined up to equivalence $(a_v, b_v; m_v) \sim (b_v, a_v; m_v)$) such that $m_v \mid m$ and $m_{v_0} = m = |\mathcal{G}|$, and $\gcd(a_v, b_v, m_v) = 1$ for all v.

(4) For every vertex v, $Stab(v) = C_{m_v} \subset \mathcal{G}$. Also $w(g.v) = w(v)$, so that weights of vertices in the same orbit are equal.

(5) In the case of type I trees, there are at most three vertices v of level 1 such that $m_v = m$. Each of these vertices have distinct weights (up to equivalence) among $\{\pm(a_{v_0}, -b_{v_0}), \pm(a_{v_0}, b_{v_0} - a_{v_0}), \pm(b_{v_0}, a_{v_0} - b_{v_0})\}$.

(6) Vertices with the same weight (up to equivalence and sign) do not have a common neighbour unless they are related by the $\mathcal{G}$-action.

(7) Suppose there is an edge e from v in level L to u in level $L + 1$. Then $m_u \mid m_v$ and

(a) If $m_u = m_v$ and v is not the root vertex, then $\pm(a_u, b_u) \in \{(a_v, b_v - a_v), (a_v - b_v, b_v)\}$.

(b) If $m_u \neq m_v$ then one of $a_v, b_v, a_v - b_v$ satisfies the equation $\gcd(x, m) = m_u$, accordingly $b_u = 0$, and a_u is a value among $\pm\{-a_v, -b_v, b_v - a_v\}$ not divisible by m_u.

As far as Definition 2.9 is concerned, type II trees are only a special subset of type I trees, but the construction of the tree manifolds associated to them

$$v_0$$
$$w(v_0) = (a_0, b_0; 21)$$
$$v_1$$
$$g \cdot v_1$$
$$g^2 \cdot v_1$$
$$X(\mathbb{T}) = S^4(a_0, b_0; 21)$$
$$\#C_{21} \times_{C_7} \mathbb{C}P^2(a_1, b_1; 7)$$
$$w(v_1) = w(g \cdot v_1) = w(g^2 \cdot v_1) = (a_1, b_1; 7)$$
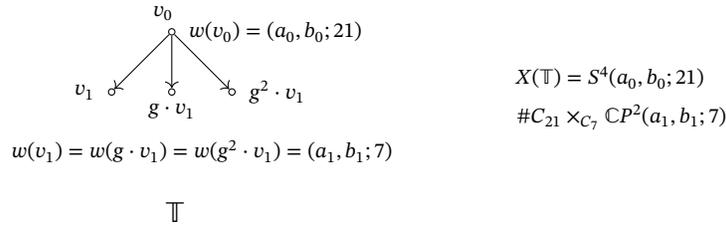$$\mathbb{T}$$

FIGURE 2.9. Example of a type II tree

will be different. In the case of type I trees (for example in Figure 1.0) the tree manifold associated to it is a connected sum of copies of $\mathbb{C}P^2$ with appropriate weights. For type II trees (see Figure 2.9), the root vertex gives a copy of S^4 in the connected sum and the rest of the vertices contribute $\mathbb{C}P^2$. One should observe here that if the root S^4 is connected to a $\mathbb{C}P^2$ associated to a fixed vertex, one may express the resultant tree manifold as one arising from a type I tree.

Remark 2.10. The definition of admissible, weighted tree above is the same definition as [9, §1.D]. To see this, one may observe the following

- The tree as defined inherits a direction, where an edge e moving from level L to $L + 1$ is directed so that $\partial_0 e$ lies in level L, and $\partial_1 e$ lies in level $L + 1$. One also observes that a vertex in level $L > 0$ is connected to a unique vertex in level $L - 1$.
- The partial order may be generated from the condition that $\partial_0 e < \partial_1 e$. This implies that two vertices are comparable if they are connected by a sequence of edges, and in this case the order relation is determined by the level.
- The weights $w(v) = (a_v, b_v; m_v)$ are so defined that we obtain an equivalent weight under the operations $(a_v, b_v) \mapsto (b_v, a_v)$. This is the equivalence of weights referred to in the definition above.
- The conditions (5) and (7) above reflect the condition "pair of matching fixed components" of [9]. As we shall see, this is a slightly stronger condition that also includes the data required for us to form the corresponding equivariant connected sum.

2.11. Notation. The number $n(\mathbb{T})$ associated to an admissible, weighted tree $\mathbb{T}$ with vertex set $V(\mathbb{T})$ is defined as

$$n(\mathbb{T}) = \begin{cases} \#(V(\mathbb{T})) & \text{if } \mathbb{T} \text{ is of type I} \\ \#(V(\mathbb{T})) - 1 & \text{if } \mathbb{T} \text{ if of type II.} \end{cases}$$

The significance of the notation $n(\mathbb{T})$ is that we associate to an admissible weighted tree $\mathbb{T}$, a $\mathcal{G}$-manifold $X(\mathbb{T})$ whose underlying space is $\#^{n(\mathbb{T})}\mathbb{C}P^2$. We will use the notation $\mathbb{T}_0$ for $\mathbb{T}^{\mathcal{G}}$, and $\mathbb{T}_d = \{v \in \mathbb{T} \mid Stab(v) = C_d\}$. Observe that $\mathcal{G}/C_d$ acts freely on $\mathbb{T}_d$. Also note that $\mathbb{T}_0$ is always a sub-tree of $\mathbb{T}$, while $\mathbb{T}_d$ is not.

2.12. Construction of connected sums along trees.   The construction of $X(\mathbb{T})$, the $\mathcal{G}$-manifold obtained by the connected sum of linear actions according to the data described in the tree $\mathbb{T}$ is carried out in [9, Theorem 1.7]. We describe it's main features below. For a vertex $v \in V(\mathbb{T})$, we use the notation

$$\mathbb{C}P_v^2 := \mathbb{C}P^2(a_v, b_v; m_v).$$

Proposition 2.13. Given an admissible, weighted tree $\mathbb{T}$, there is a $\mathcal{G}$-manifold $X(\mathbb{T})$ such that

(1) The underlying space of $X(\mathbb{T})$ is $\#^{n(\mathbb{T})}\mathbb{C}P^2$.
(2) If $\mathbb{T}$ is of type I, then $X(\mathbb{T}) \cong$ a connected sum of copies of $\mathbb{C}P_v^2$ for every vertex v of $\mathbb{T}$. In case the vertex v is stabilized by a proper subgroup C_d of $\mathcal{G}$, the $\mathbb{C}P_{g \cdot v}^2$ assemble together equivariantly as $\mathcal{G} \times_{C_d} \mathbb{C}P_v^2$.
(3) If $\mathbb{T}$ is of type II, then $X(\mathbb{T}) \cong$ a connected sum of copies of $\mathbb{C}P_v^2$ for every non-root vertex v of $\mathbb{T}$, and a copy of $S^4(a_{v_0}, b_{v_0}; m)$. As in (2), if the vertex v is stabilized by a proper subgroup C_d of $\mathcal{G}$, the $\mathbb{C}P_{g \cdot v}^2$ assemble together equivariantly as $\mathcal{G} \times_{C_d} \mathbb{C}P_v^2$.
(4) For a non-root vertex v in level L, which is connected to w in level $L-1$ with $m_v = m_w$, the points where the connected sum is performed are $[0, 0, 1] \in \mathbb{C}P_v^2$, and the one in $\mathbb{C}P_w^2$ determined by the condition (7)(a) of Definition 2.9 if w is not the root vertex, or by (5) of Definition 2.9 if $w = v_0$.
(5) For a non-root vertex v in level L connected to w in level $L-1$ with $m_v < m_w$, the points where the connected sum is performed are $[0, 0, 1] \in \mathbb{C}P_v^2$, and some equivalent choice of point in $\mathbb{C}P_w^2$ determined by the condition (7) (b) of Definition 2.9. Equivalent choices of the latter give equivalent manifolds [9, Lemma 1.2].

We now elaborate further on (4) and (5) of Proposition 2.13 above. We start with an example.

Example 2.14. In order to see if $\mathbb{C}P^2(a, b; m) \# \mathbb{C}P^2(a', b'; m)$ is definable we may apply 1) of Proposition 2.7. Another method of saying this is that there is an expression of the second summand as $\mathbb{C}P^2(a', b'; m)$ such that $\pm(a', b') \in \{(a, -b), (a - b, b), (a, b - a)\}$. Once this choice is made, say $(a', b') = (a, -b)$, we get a natural data for the equivariant connected sum as

(1) The point $p \in \mathbb{C}P^2(a, b; m)$ used in the connected sum is $[0, 0, 1]$, and the corresponding tangential representation is $\lambda^a + \lambda^b$.
(2) The point $q \in \mathbb{C}P^2(a', b'; m)$ used in the connected sum is $[0, 0, 1]$, and the corresponding tangential representation is $\lambda^{a'} + \lambda^{b'} = \lambda^a + \lambda^{-b}$.
(3) The natural orientation reversing isomorphism

$$T_p\mathbb{C}P^2(a, b; m) \to T_q\mathbb{C}P^2(a', b'; m)$$

is given by identity on the factor λ^a and complex conjugation on the factor λ^b.

In (4) of Proposition 2.13, the choice of (a_v, b_v) implies that

$$T_{[0,0,1]}\mathbb{C}P_v^2 = \lambda^{a_v} + \lambda^{b_v}$$

equals one of $\lambda^{a_w} + \lambda^{b_w - a_w}, \lambda^{-a_w} + \lambda^{a_w - b_w}, \lambda^{b_w} + \lambda^{a_w - b_w}, \lambda^{-b_w} + \lambda^{b_w - a_w}$, in the case w is not the root vertex. We also note the tangential representations

$$T_{[1,0,0]}\mathbb{C}P_w^2 = \lambda^{-a_w} + \lambda^{b_w - a_w}, \ T_{[0,1,0]}\mathbb{C}P_w^2 = \lambda^{-b_w} + \lambda^{a_w - b_w}.$$

Among all the possibilities for $T_{[0,0,1]}\mathbb{C}P_v^2$, the first two are compatible with $T_{[1,0,0]}\mathbb{C}P_w^2$, and the second two are compatible with $T_{[0,1,0]}\mathbb{C}P_w^2$. This demonstrates how the weights imply the choice of connected sum point in $\mathbb{C}P_w^2$. The argument in Example 2.14 applies here to construct a canonical orientation reversing isomorphism among the tangential representations. Finally the condition (6) of Definition 2.9 implies that the choice of connected sum point is not the same as that of any other vertex.

We now look at (5) of Proposition 2.13. The condition $m_v < m_w$ implies that m_v is a proper divisor of m_w. Consider $\mathcal{O} \cong C_{m_w}/C_{m_v}$, the orbit of v under the C_{m_w}-action. From (4) of Definition 2.9, we observe that all the vertices in $\mathcal{O}$ have weight $w(v)$. The connected sum formed here is $\mathbb{C}P_w^2 \# C_{m_w} \times_{C_{m_v}} \mathbb{C}P_v^2$, which connects the manifolds at all the vertices in $\mathcal{O}$ to $\mathbb{C}P_w^2$ at one go by writing

$$C_{m_w} \times_{C_{m_v}} \mathbb{C}P^2 \cong \coprod_{h \in C_{m_w}/C_{m_v}} h \cdot \mathbb{C}P_v^2 = \coprod_{h \in C_{m_w}/C_{m_v}} \mathbb{C}P_{h \cdot v}^2.$$

In this case, we have $b_v = 0$ and m_v divides one of the numbers $a_w, b_w, b_w - a_w$ but not more than one (unless $m_v = 1$) as $\gcd(a_w, b_w, m_w) = 1$. We may assume $\gcd(b_w, m_w) = m_v$ without loss of generality, and it implies $a_v = \pm a_w$. The first part of the equivariant data for the connected sum is the point $[0,0,1]$ in $\mathbb{C}P_v^2$ with tangential C_{m_v}-representation $\lambda^{a_v} + 1_{\mathbb{C}}$ (as a complex representation). The next part is a choice of connected sum point which is required to have stabilizer C_{m_v}, and hence belongs to

$$P(\lambda^{b_w} + 1_{\mathbb{C}}) - \{[0,1,0], [0,0,1]\} \subset P(\lambda^{a_w} + \lambda^{b_w} + 1_{\mathbb{C}}) = \mathbb{C}P_w^2.$$

For any point

$$q \in P(\lambda^{b_w} + 1_{\mathbb{C}}) - \{[0,1,0], [0,0,1]\},$$

the tangential C_{m_v}-representation is $1_{\mathbb{C}} + \lambda^{a_w}$. We now have a canonical orientation reversing isomorphism between $T_{[0,0,1]}\mathbb{C}P_v^2$ and $T_q\mathbb{C}P_w^2$ which is conjugation on λ^{a_v} if $a_v = -a_w$, or conjugation on the other factor if $a_v = a_w$. As $P(\lambda^{b_w} + 1_{\mathbb{C}}) - \{[0,1,0], [0,0,1]\}$ is connected, this defines the equivariant connected sum up to diffeomorphism ([9, Lemma 1.2]). Note also that there is a completely analogous version of the above if w was the root vertex of a type II tree, and $\mathbb{C}P_w^2$ was replaced by $S^{\lambda^{a_w} + \lambda^{b_w}}$.

3. Equivariant homology for cyclic groups

In this section, we recall the definition of equivariant homology with coefficients in a Mackey functor. The main objective is to describe a theorem on $\underline{\mathbb{Z}}$-homology which allows us to construct the homology decompositions in the following sections. Equivariant homology and cohomology possess the richest structure when the coefficients are Mackey functors [5], which we summarize in explicit terms below.

Definition 3.1. A $\mathcal{G}$-Mackey functor[1] $\underline{M}$ is a collection of commutative $\mathcal{G}/H$-groups $\underline{M}(\mathcal{G}/H)$ for each subgroup $H \leq \mathcal{G}$, accompanied by *transfer* $\mathrm{tr}_K^H : \underline{M}(\mathcal{G}/K) \to \underline{M}(\mathcal{G}/H)$ and *restriction* $res_K^H : \underline{M}(\mathcal{G}/H) \to \underline{M}(\mathcal{G}/K)$ for $K \leq H \leq \mathcal{G}$ such that

(1) $\mathrm{tr}_J^H = \mathrm{tr}_K^H \mathrm{tr}_J^K$ and $res_J^H = res_J^K res_K^H$ for all $J \leq K \leq H$.
(2) $\mathrm{tr}_K^H(\gamma.x) = \mathrm{tr}_K^H(x)$ for all $x \in \underline{M}(\mathcal{G}/K)$ and $\gamma \in H/K$.
(3) $\gamma \cdot res_K^H(x) = res_K^H(x)$ for all $x \in \underline{M}(\mathcal{G}/H)$ and $\gamma \in H/K$.
(4) $res_K^H \mathrm{tr}_J^H(x) = \sum_{\gamma \in H/JK} \gamma \cdot \mathrm{tr}_{J \cap K}^K res_{J \cap K}^J(x)$ for all subgroups $J, K \leq H$.

Example 3.2. The Burnside ring Mackey functor denoted $\underline{A}$ is described by $\underline{A}(\mathcal{G}/H) = A(H)$, the Burnside ring of H. This is the group completion of the monoid of finite H-sets up to isomorphism. The restriction maps are given by restricting the action, and the transfer maps are given by inducing up the action $: S \mapsto H \times_K S$ for $K \leq H$.

In this paper, we work primarily with the constant Mackey functor $\underline{\mathbb{Z}}$ described by

$$\underline{\mathbb{Z}}(\mathcal{G}/H) = \mathbb{Z}, \ res_K^H = Id, \ \mathrm{tr}_K^H = [H : K],$$

for $K \leq H \leq \mathcal{G}$. One may make a dual construction to define the Mackey functor $\underline{\mathbb{Z}}^*$ by

$$\underline{\mathbb{Z}}^*(\mathcal{G}/H) = \mathbb{Z}, \ res_K^H = [H : K], \ \mathrm{tr}_K^H = Id,$$

for $K \leq H \leq \mathcal{G}$. For an Abelian group C, the Mackey functor $\langle C \rangle$ is described by

$$\langle C \rangle(\mathcal{G}/H) = \begin{cases} C & \text{if } H = \mathcal{G} \\ 0 & \text{otherwise.} \end{cases}$$

The importance of Mackey functors from the point of view of ordinary cohomology in the equivariant case is due to the following result.

Theorem 3.3. [8, Theorem 5.3] For a Mackey functor $\underline{M}$, there is an Eilenberg-MacLane $\mathcal{G}$-spectrum $H\underline{M}$ which is unique up to isomorphism in the equivariant stable homotopy category.

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[1]This is a simplification in the case $\mathcal{G}$ is Abelian. Otherwise the double coset formula (4) has a slightly more complicated expression.

This Eilenberg-MacLane spectra are those whose integer-graded homotopy groups are concentrated in degree 0 in the category of equivariant orthogonal spectra [14]. The homotopy category of equivariant orthogonal spectra is called the equivariant stable homotopy category, where one has desuspension functors for one-point compactifications of orthogonal $\mathcal{G}$-representations.

Example 3.4. For a $\mathcal{G}$-spectrum X, the equivariant homotopy groups possess the structure of a Mackey functor $\underline{\pi}_n^{\mathcal{G}}(X)$, defined by the formula

$$\underline{\pi}_n^{\mathcal{G}}(X)(\mathcal{G}/H) := \pi_n(X^H).$$

In fact, the grading may be extended over $\alpha \in RO(\mathcal{G})$ via the formula

$$\underline{\pi}_\alpha^{\mathcal{G}}(X)(\mathcal{G}/K) \cong \text{Ho-}\mathcal{G}\text{-spectra}\,(S^\alpha \wedge \mathcal{G}/K_+, X),$$

which is in fact equal to $\pi_\alpha^K(X)$. Analogously the cohomology theory and homology theory associated to Mackey functors are $RO(\mathcal{G})$-graded and may also be equipped with the structure of a Mackey functor which on objects is described as

$$\underline{H}_{\mathcal{G}}^\alpha(X; \underline{M})(\mathcal{G}/K) \cong \text{Ho-}\mathcal{G}\text{-spectra}\,(X \wedge \mathcal{G}/K_+, \Sigma^\alpha H\underline{M}),$$

$$\underline{H}_\alpha^{\mathcal{G}}(X; \underline{M})(\mathcal{G}/K) \cong \text{Ho-}\mathcal{G}\text{-spectra}\,(S^\alpha \wedge \mathcal{G}/K_+, X \wedge H\underline{M}),$$

for $\alpha \in RO(\mathcal{G})$. For the Mackey functor $\underline{\mathbb{Z}}$, the integer graded groups at $\mathcal{G}/\mathcal{G}$ compute the cohomology of $X/\mathcal{G}$, the orbit space of X under the $\mathcal{G}$-action.

The Mackey functor $\underline{\mathbb{Z}}$ has a multiplicative structure which makes it a *commutative Green functor* [16, Chapter XIII.5]. The consequence of this multiplication is that the cohomology $H_{\mathcal{G}}^\star(X; \underline{\mathbb{Z}})$ has a graded commutative ring structure. The multiplicative structure also allows us to consider the Mackey functors which are $\underline{\mathbb{Z}}$-modules, and examples of these are the homology and the cohomology Mackey functors $\underline{H}_\alpha^{\mathcal{G}}(X; \underline{\mathbb{Z}})$ and $\underline{H}_{\mathcal{G}}^\alpha(X; \underline{\mathbb{Z}})$.

Remark 3.5. For any $\underline{M} \in \underline{\mathbb{Z}}\text{-Mod}_{\mathcal{G}}$, $\text{tr}_K^H\, \text{res}_K^H$ equals the multiplication by index $[H : K]$ for $K \le H \le \mathcal{G}$ [17, Theorem 4.3].

The spectrum $H\underline{\mathbb{Z}}$ also has the following well-known relation after smashing with representation spheres.

Proposition 3.6. If $\gcd(d, m) = 1$, then

$$H\underline{\mathbb{Z}} \wedge S^{\lambda^k} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{dk}}.$$

Proof. We check that $S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}}$ is an Eilenberg MacLane spectrum up to suspension, whose underlying Mackey functor depends on k but not on d. For this note that

$$\underline{\pi}_i^{\mathcal{G}}(S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}})(\mathcal{G}/L) \cong \underline{H}_{\mathcal{G}}^{-i}(S^{\lambda^{dk}}; \underline{\mathbb{Z}})(\mathcal{G}/L) \cong \tilde{H}_L^{-i}(S^{\lambda^{dk}}; \underline{\mathbb{Z}}) \cong \tilde{H}^{-i}(S^{\lambda^{dk}}/L; \mathbb{Z}).$$

Now observe that the orbit space $S^{\lambda^{dk}}/L \simeq S^2$, so that the groups above are $\mathbb{Z}$ for $i = -2$, and 0 otherwise. Therefore, we obtain

$$S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}} \simeq \Sigma^{-2} H\underline{M},$$

where $\underline{M}(\mathcal{G}/L) = \mathbb{Z}$ for $L \leq \mathcal{G}$. For a subgroup $K = C_{r_1}$ of another subgroup $L = C_{r_2}$ of $\mathcal{G}$, the restriction $\underline{M}(\mathcal{G}/L) \to \underline{M}(\mathcal{G}/K)$ is induced by the quotient $S^{\lambda^{dk}}/K \to S^{\lambda^{dk}}/L$, which is a map of degree $\frac{r_2 \cdot \gcd(r_1, dk)}{r_1 \cdot \gcd(r_2, dk)}$ which equals $\frac{r_2 \cdot \gcd(r_1, k)}{r_1 \cdot \gcd(r_2, k)}$ as $\gcd(d, m) = 1$. The transfer maps are then determined by Remark 3.5. It follows that $S^{-\lambda^{dk}} \wedge H\underline{\mathbb{Z}} \simeq S^{-\lambda^k} \wedge H\underline{\mathbb{Z}}$ if $\gcd(d, m) = 1$, so that $H\underline{\mathbb{Z}} \wedge S^{\lambda^k} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{dk}}$. $\qquad\square$

We may observe from Proposition 3.6 that $\Sigma^{\lambda^k - \lambda^{dk}} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}$. This means in the graded commutative ring $\pi_\star H\underline{\mathbb{Z}}$ (graded over $RO(\mathcal{G})$), there are invertible classes in degrees $\lambda^k - \lambda^{dk}$ whenever $(d, m) = 1$. As a consequence the ring $\pi_\star H\underline{\mathbb{Z}}$ is determined from it's values at the gradings which are linear combinations of λ^k for $k \mid m$. We recall the following computation of $\underline{\pi}_\star^{C_p}(H\underline{\mathbb{Z}})$ [6, Appendix B].

$$\underline{\pi}_\alpha^{C_p}(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & \text{if } |\alpha| = 0, \ |\alpha^{C_p}| \geq 0 \\ \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, \ |\alpha^{C_p}| < 0 \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0, \ |\alpha^{C_p}| \geq 0, \ \text{and } |\alpha| \text{ even} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0, \ |\alpha^{C_p}| < -1, \ \text{and } |\alpha| \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \qquad (3.7)$$

The homology decomposition theorems for $\mathcal{G}$-spaces with even cells require proving that certain odd degree homotopy groups of $H\underline{\mathbb{Z}}$ are 0. In this paper, we use the following result.

Theorem 3.8. Let $\alpha \in RO(\mathcal{G})$ be such that $|\alpha|$ is odd, and $|\alpha^H| > -1$ implies $|\alpha^K| \geq -1$ for all subgroups $K \supseteq H$. Then $\underline{\pi}_\alpha^{\mathcal{G}}(H\underline{\mathbb{Z}}) = 0$.

Proof. The proof relies on [4, Proposition 4.3], where the same result is proved for the groups C_{p^n} where p is an odd prime. Here, note that

$$\underline{\pi}_\alpha^{\mathcal{G}}(H\underline{\mathbb{Z}}) \cong \underline{H}_{\mathcal{G}}^{-\alpha}(S^0; \underline{\mathbb{Z}}),$$

so that the hypothesis in statement c) of [4, Proposition 4.3] corresponds to our hypothesis on α over here.

We now suppose that the result is true for all subgroups of $\mathcal{G}$ and then prove it for $\mathcal{G}$. In this way, the result will be true for all cyclic groups of odd order. The inductive hypothesis states that for all proper subgroups L of $\mathcal{G}$ and for all $\alpha \in RO(L)$ satisfying $|\alpha^H| > -1$ implies $|\alpha^K| \geq -1$ for all subgroups $L \supseteq K \supseteq H$, $\underline{\pi}_\alpha^L(H\underline{\mathbb{Z}}) = 0$.

Now suppose that α satisfies the hypothesis of the theorem, and let $x \in \pi_\alpha^{\mathcal{G}}(H\underline{\mathbb{Z}})$. This implies that the restriction of α to a subgroup L satisfies the hypothesis for L, so that by the inductive hypothesis we have $\underline{\pi}_\alpha^L(H\underline{\mathbb{Z}}) = 0$. As the result has already been proved at prime powers in [4, Proposition 4.3], we may assume that m is divisible by at least two distinct primes p and q. We now use the inductive hypothesis for $C_{m/p}$ and $C_{m/q}$, and compute

$$px = [\mathcal{G} : C_{m/p}]x = \mathrm{tr}_{C_{m/p}}^{\mathcal{G}} \, res_{C_{m/p}}^{\mathcal{G}}(x) = 0,$$

as $\underline{\pi}_\alpha^{\mathcal{G}}(H\underline{\mathbb{Z}})(\mathcal{G}/C_{m/p}) = \pi_\alpha^{C_{m/p}}(H\underline{\mathbb{Z}}) = 0$, and analogously,

$$qx = [\mathcal{G} : C_{m/q}]x = \mathrm{tr}_{C_{m/q}}^{\mathcal{G}} \, res_{C_{m/q}}^{\mathcal{G}}(x) = 0.$$

It follows that $x = 0$. $\qquad\square$

Theorem 3.8 is useful to prove that cohomology of $\mathcal{G}$-spaces which are constructed by attaching even cells of the type $D(V)$, is a free module over the cohomology of a point $\underline{H}_{\mathcal{G}}^\star(S^0; \underline{\mathbb{Z}})$. Such results have been proved in [12], [1], [2], in the context of equivariant projective spaces and Grassmannians. A more careful argument has also been used in [6] and [7], where the free module property has been proved for all finite complexes obtained by attaching cells of the type $D(V)$ in even dimensions.

4. Equivariant homology decompositions for tree manifolds

In this section, we obtain homology decompositions for the tree manifolds defined in Section 2. Recall that, $\mathbb{C}P^2(a, b; m)$ serves as a building block for these manifolds. We describe a cellular decomposition of complex projective spaces, which has been studied along with cohomology of such spaces in [12, §3] and [1, §8.1].

4.1. Cellular filtration of projective spaces. The equivariant complex projective space $P(V)$ is built up by attaching even dimensional cells of the type $D(W)$ for the realization of complex representations W. To see this, let V_n be a complex $\mathcal{G}$-representation that decomposes in terms of irreducible factors as $V_n = \sum_{i=0}^{n} \phi_i$, and let W_n denote the $\mathcal{G}$-representation $\phi_n^{-1} \otimes \sum_{i=0}^{n-1} \phi_i$. Consider the $\mathcal{G}$-equivariant map $D(W_n) \to P(V_n) \cong P(\phi_n^{-1} \otimes V_n)$ defined by

$$(z_0, z_1, \ldots, z_{n-1}) \mapsto [z_0, z_1, \ldots, z_{n-1}, 1 - \sum_{i=0}^{n-1} |z_i^2|],$$

where $z_i \in \phi_n^{-1} \otimes \phi_i$. Restricting this map to $S(W_n)$, we see that its image lies in $P(V_{n-1})$ (which may be regarded as a subspace of $P(V_n)$ in the obvious way), and it is a homeomorphism from $D(W_n) \setminus S(W_n)$ to $P(V_n) \setminus P(V_{n-1})$. Thus, $P(V_n)$ is obtained from $P(V_{n-1})$ by attaching the cell $D(W_n)$ along this boundary map. Observe that this filtration depends on the choice of the ordering of the ϕ_i's.

Returning to our example $\mathbb{C}P^2(a, b; m) = P(\lambda^a \oplus \lambda^b \oplus 1_{\mathbb{C}})$, we see that there are six possible ways to build it. This choices will play a crucial role in proving the homology decomposition theorems, as we will see below.

Example 4.2. Writing $\mathbb{C}P^2(a, b; m) = P(\lambda^a \oplus \lambda^b \oplus 1_{\mathbb{C}})$ in this order, the cellular filtration above gives us the following cofibre sequence (using the fact that $P(\lambda^a \oplus \lambda^b) \cong S^{\lambda^{a-b}}$)

$$S^{\lambda^{a-b}} \to \mathbb{C}P^2(a, b; m) \to S^{\lambda^a + \lambda^b}.$$

Using the other orderings, we also obtain the following cofibre sequences

$$S^{\lambda^a} \to \mathbb{C}P^2(a, b; m) \to S^{\lambda^{a-b} + \lambda^{-b}}, \ S^{\lambda^b} \to \mathbb{C}P^2(a, b; m) \to S^{\lambda^{b-a} + \lambda^{-a}}.$$

The homology decomposition is obtained by smashing these cofibre sequences with $H\underline{\mathbb{Z}}$ and trying to prove a splitting. For example, in the cofibre sequence

$$\Sigma^{\lambda^{a-b}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \to \Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}}, \tag{4.3}$$

the connecting map $\Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}} \to \Sigma^{\lambda^{a-b}+1} H\underline{\mathbb{Z}}$ is a $H\underline{\mathbb{Z}}$-module map which is classified up to homotopy by $\pi_0^G(\Sigma^{\lambda^{a-b}+1-\lambda^a-\lambda^b} H\underline{\mathbb{Z}})$. This group is now analyzed using Theorem 3.8 at $\alpha = -\lambda^{a-b} - 1 + \lambda^a + \lambda^b$. Note that

$$|(-\lambda^{a-b}-1+\lambda^a+\lambda^b)^{C_d}| = \begin{cases} -1 & \text{if } d \text{ does not divide any of } a, \ b \text{ or } a-b \\ 1 & \text{if } d \text{ divides exactly one of } a, \ b \text{ but not } a-b \\ -3 & \text{if } d \text{ divides } a-b \text{ but not } a \text{ or } b. \end{cases}$$

Observe that $|\alpha| = 1 > 0$, so in order to show $\pi_\alpha^G H\underline{\mathbb{Z}} = 0$, we need $|\alpha^{C_d}| \geq -1$ for all $d \mid m$. Under the condition $\gcd(a, b, m) = 1$, this is true if and only if $a - b$ is relatively prime to m, and in this case,

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \simeq \Sigma^{\lambda^{a-b}} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}} \simeq \Sigma^\lambda H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}}.$$

The last equivalence follows from Proposition 3.6. Using the other two cofibre sequences for $\mathbb{C}P^2(a, b; m)$, we see that a homology decomposition is obtained if one of $a, b,$ or $a - b$ is relatively prime to m.

In the case of connected sums, we carry forward the homology decomposition argument of Example 4.2. We illustrate this in the following example.

Example 4.4. Let

$$X = \mathbb{C}P^2(a, b; m) \# \mathbb{C}P^2(a', b'; m)$$

where $\gcd(a'-b', m) = 1$ and $\gcd(a-b, m) = 1$. We assume that the connected sum point p in $\mathbb{C}P^2(a', b'; m)$ has tangential representation $\lambda^{a'} \oplus \lambda^{b'}$ as in 1) of Proposition 2.7. Example 4.2 shows that

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \simeq \Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}} \vee \Sigma^\lambda H\underline{\mathbb{Z}}.$$

To compute $H\underline{\mathbb{Z}} \wedge X$, we use the cofibre sequence (2.2). We note that

$$\mathbb{C}P^2(a', b'; m) \setminus \{p\} \simeq P(\lambda^{a'} \oplus \lambda^{b'}) \simeq S^{\lambda^{a'-b'}}.$$

Therefore, we obtain a cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda^{a'-b'}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X \to \Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}} \vee \Sigma^\lambda H\underline{\mathbb{Z}}.$$

From Proposition 3.6, we have $\Sigma^{\lambda^{a'-b'}} H\underline{\mathbb{Z}} \simeq \Sigma^\lambda H\underline{\mathbb{Z}}$, and now Theorem 3.8 implies that the above sequence splits. Consequently, we obtain

$$H\underline{\mathbb{Z}} \wedge X_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^a + \lambda^b} H\underline{\mathbb{Z}} \vee \Sigma^\lambda H\underline{\mathbb{Z}} \vee \Sigma^\lambda H\underline{\mathbb{Z}}.$$

We now prove the main theorems of this section. Example 4.2 points out the necessity of the hypothesis in the theorem.

Theorem 4.5. If $\mathbb{T}$ is an admissible weighted tree of type I with $\mathcal{G}$-action such that for all vertices $v \in \mathbb{T}_0$ with $w(v) = (a_v, b_v; m_v)$, $\gcd(a_v - b_v, m_v) = 1$, then, the $H\underline{\mathbb{Z}}$-module $H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+$ admits the decomposition

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \vee \left(\bigvee_{\mathbb{T}_0} \Sigma^\lambda H\underline{\mathbb{Z}} \right)$$

$$\vee \left(\bigvee_{[\mu] \in \mathbb{T}_d / \mathcal{G}, d \neq m} \mathcal{G}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \right)$$

where $w(v_0) = (a_0, b_0; m)$. If $\mathbb{T}$ is of type II,

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \vee \left(\bigvee_{[\mu] \in \mathbb{T}_d / \mathcal{G}, \, d \neq m} \mathcal{G}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \right).$$

Proof. We proceed by induction on $L(\mathbb{T})$, the maximum level reached by vertices of the tree. The induction starts from a tree with only the root vertex. In the type I case, this is computed in Example 4.2. In case of type II, the manifold is $S^4(a_0, b_0; m)$, for which we have the following decomposition

$$H\underline{\mathbb{Z}} \wedge S^4(a_0, b_0; m)_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}}.$$

Assume that the statement holds whenever $L(\mathbb{T}) \leq L$. We prove it for trees with $L(\mathbb{T}) = L + 1$. Given a tree $\mathbb{T}$ we denote by $\mathbb{T}(L)$ the part of it up to level L, so that the result holds for $X(\mathbb{T}(L))$. We attach orbits of the level $L + 1$ vertices one at a time. We write down the argument for a type I tree, as the other case is entirely analogous. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ denote the orbits of the level $L + 1$ vertices. It suffices to prove the case when an orbit $\mathcal{O}_i$ is added to $\mathbb{T}(L)$ together with the attaching edges, which we denote by $\mathbb{T}(L) + \mathcal{O}_i$. The stabilizer for the vertices in $\mathcal{O}_i$ can be either the whole group $\mathcal{G}$ or a smaller subgroup C_d. We deal these cases separately.

Case 1: The stabilizer for the vertices in $\mathcal{O}_i$ is $\mathcal{G}$, that is, $\mathcal{O}_i = \{v_i\}$. Suppose $w(v_i) = (a_i, b_i; m)$. By Proposition 2.13, this implies that the tangential representation at the connected sum point of $\mathbb{C}P^2_{v_i}$ is $\lambda^{a_i} \oplus \lambda^{b_i}$. We have the following

cofibre sequence from (2.2)

$$\mathbb{C}P^2(a_i, b_i; m) \setminus D(\lambda^{a_i} \oplus \lambda^{b_i}) \to X(\mathbb{T}(L) + \mathcal{O}_i) \to X(\mathbb{T}(L)).$$

The left hand term can be simplified further as

$$\mathbb{C}P^2(a_i, b_i; m) \setminus D(\lambda^{a_i} \oplus \lambda^{b_i}) \simeq P(\lambda^{a_i} \oplus \lambda^{b_i}) \simeq S^{\lambda^{a_i-b_i}}.$$

We now apply Proposition 3.6 to note that $\Sigma^{\lambda^{a_i-b_i}} H\underline{\mathbb{Z}} \simeq \Sigma^\lambda H\underline{\mathbb{Z}}$. Applying the induction hypothesis on $X(\mathbb{T}(L))$, we get a cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^\lambda H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L) + \mathcal{O}_i) \to \Sigma^{\lambda^{a_0}+\lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}(L)_d/\mathcal{G}} \mathcal{G}/C_{d_+} \wedge \Sigma^\lambda H\underline{\mathbb{Z}} \Big). \quad (4.6)$$

In the second summand of the right side of (4.6), d is also allowed to equal m. Next we observe that the cofibre sequence splits by showing that up to homotopy, the connecting map from each summand of the right hand side of equation (4.6) to $\Sigma^{\lambda+1} H\underline{\mathbb{Z}}$ is zero. For the first summand, this follows from Example 4.2. In the second summand, if $d = m$, the connecting map is classified up to homotopy by an element of $\underline{\pi}_{-1}(H\underline{\mathbb{Z}}) = 0$. If $d \neq m$, this is classified by an element of $\underline{\pi}_{-1}(H\underline{\mathbb{Z}})(\mathcal{G}/C_d)$, which is also 0. Using Theorem 3.8 in this manner, we obtain the required homology decomposition for $X(\mathbb{T}(L) + \mathcal{O}_i)$.

Case 2: The vertices in $\mathcal{O}_i$ have stabilizer $C_{m_i} < \mathcal{G}$ and $w(v_i) = (a_i, b_i; m_i)$. Here, we are considering the connected sum of the form

$$X(\mathbb{T}(L)) \# \mathcal{G} \times_{C_{m_i}} \mathbb{C}P^2(a_i, b_i; m_i).$$

Consider the following homotopy pushout square of $\mathcal{G}$-spaces (2.3)

$$\begin{array}{ccc}
\mathcal{G} \times_{C_{m_i}} S^{\lambda^{a_i-b_i}} & \longrightarrow & \mathcal{G} \times_{C_{m_i}} C(S^{\lambda^{a_i-b_i}}) \\
\downarrow & & \downarrow \\
X(\mathbb{T}(L) + \mathcal{O}_i) & \longrightarrow & X(\mathbb{T}(L)),
\end{array}$$

where $C(S^{\lambda^{a_i-b_i}})$ denotes the cone of $S^{\lambda^{a_i-b_i}}$. Note that the C_{m_i}-representation λ is also the restriction of a C_m-representation that we have denoted also using λ. Further using the shearing homeomorphism, we may write $\mathcal{G} \times_{C_{m_i}} S^{\lambda^{a_i-b_i}} \cong G/C_{m_i} \times S^{\lambda^{a_i-b_i}}$. In $\mathcal{G}$-spectra, this gives rise to the cofibre sequence

$$\mathcal{G}/C_{m_{i+}} \wedge S^{\lambda^{a_i-b_i}}_+ \to X(\mathbb{T}(L) + \mathcal{O}_i)_+ \vee \mathcal{G}/C_{m_{i+}} \to X(\mathbb{T}(L))_+.$$

We use

$$\mathcal{G}/C_{m_{i+}} \wedge S^{\lambda^{a_i-b_i}}_+ \simeq (\mathcal{G}/C_{m_{i+}} \wedge S^{\lambda^{a_i-b_i}}) \vee \mathcal{G}/C_{m_{i+}},$$

to deduce the cofibre sequence

$$\mathcal{G}/C_{m_{i+}} \wedge S^{\lambda^{a_i-b_i}} \to X(\mathbb{T}(L) + \mathcal{O}_i)_+ \to X(\mathbb{T}(L))_+.$$

Now we take the smash product with $H\underline{\mathbb{Z}}$ to get the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\mathcal{G}/C_{m_{i}}{}_{+} \wedge \Sigma^{\lambda^{a_i - b_i}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X\big(\mathbb{T}(L) + \mathcal{O}_i\big)_{+} \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L))_{+}. \quad (4.7)$$

Note that the summands of the right hand side of the form $\Sigma^\lambda H\underline{\mathbb{Z}}$ and $\mathcal{G}/C_{d}{}_{+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}}$ support a trivial connecting map using Theorem 3.8, and the facts
a) $|(\lambda^r - \lambda^s - 1)^K| > -1$ if $|K| \mid r$ but not s.
b) $|(\lambda^r - \lambda^s - 1)^K| < -1$ if $|K| \mid s$ but not r.
    We now note that if the stabilizer C_{m_v} of a vertex v satisfies $m_v < m$, then m_v must divide a_0 or b_0 under the given hypothesis. In the case of type II trees, this is clear from (7)(b) of Definition 2.9 as the maximum value of m_v is reached among the vertices at level 1. For a type I tree, the analogous role is played by vertices v with $m_v < m$ that are joined to a vertex w of $\mathbb{T}_0$. The same condition now implies that one of $a_w, b_w, a_w - b_w$ is divisible by m_v. The hypothesis rules out the third case. Now we repeatedly apply condition (7)(a) of Definition 2.9 along the path from w to the root vertex v_0 with the hypothesis ruling out the fact that m_v divides $a_u - b_u$ for any vertex u along the path. It follows that m_v divides either a_u or b_u for every vertex along this path. Therefore, m_v divides either a_0 or b_0. Now by Theorem 3.8 using the fact that m_i divides either a_0 or b_0, the connecting map on the summand $\Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}}$ is 0. This completes the proof.   □

Remark 4.8. Observe that Theorem 4.5 has no hypothesis if the tree $\mathbb{T}$ is of type II. Henceforth, we prove further results for trees of type I. The hypothesis in Theorem 4.5 is required crucially in the proof. For example, observe that if $a_0 - b_0 \equiv 0 \pmod{m}$, then the the cofibre sequence (4.3) gives rise to the connecting map

$$\Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \to \Sigma^3 H\underline{\mathbb{Z}},$$

which is determined by

$$\pi^{\mathcal{G}}_{\lambda^{a_0} + \lambda^{b_0} - 3}(H\underline{\mathbb{Z}}).$$

This group may be non-zero.

    In the following theorem, we observe that if one of the rotation numbers at the root vertex is 0, then we obtain a decomposition result with no further hypothesis on the weights.

Theorem 4.9. Let $\mathbb{T}$ be an admissible weighted tree with $\mathcal{G}$-action of type I such that for the root vertex v_0 with $w(v_0) = (a_0, b_0; m)$, one of a_0 or b_0 is zero. Then, $H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_{+}$ admits the following decomposition

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_{+} \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda + 2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{[\mu] \in \mathbb{T}_d/\mathcal{G}} \mathcal{G}/C_{d}{}_{+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big).$$

Note that in the rightmost summand d may be equal to m.

Proof. We proceed by induction on $L(\mathbb{T})$, the maximum level reached by the tree as in Theorem 4.5. We may assume b_0 is zero, so we have, $\gcd(a_0, m) = 1$. At the initial case $L(\mathbb{T}) = 0$, $X(\mathbb{T}) = \mathbb{C}P^2(a_0, 0; m)$, and for this manifold, the cellular decomposition gives us the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda^{a_0}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_0, 0; m) \to \Sigma^{\lambda^{a_0}+2} H\underline{\mathbb{Z}}. \tag{4.10}$$

Note that $\Sigma^{\lambda^{a_0}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda} H\underline{\mathbb{Z}}$. The connecting map in (4.10),

$$\Sigma^{\lambda+2} H\underline{\mathbb{Z}} \to \Sigma^{\lambda+1} H\underline{\mathbb{Z}}$$

is trivial up to homotopy, hence the cofibre sequence splits and we obtain

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_0, 0; m)_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

For the inductive step, assume the statement is true for the tree up to level L, $\mathbb{T}(L)$ and we attach one orbit $\mathcal{O}_i$ of the level $L + 1$ vertices to $\mathbb{T}(L)$. We prove the case when the stabilizer for the vertices in $\mathcal{O}_i$ is $\mathcal{G}$. An analogous reasoning applies to the other cases.

Suppose $\mathcal{O}_i = \{v_i\}$ and $w(v_i) = (a_i, b_i; m)$. As in the proof of Theorem 4.5, we get the following cofibre sequence

$$S^{\lambda^{a_i - b_i}} \to X(\mathbb{T}(L) + \mathcal{O}_i) \to X(\mathbb{T}(L)).$$

Applying the induction hypothesis on $X(\mathbb{T}(L))$, we get a cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\Sigma^{\lambda^{a_i - b_i}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L) + \mathcal{O}_i) \to \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{[\mu] \in \mathbb{T}(L)_d / \mathcal{G}} \mathcal{G}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big).$$
$$\tag{4.11}$$

We claim that the connecting map is zero from each summand of the right hand side of the equation to $\Sigma^{\lambda^{a_i - b_i}+1} H\underline{\mathbb{Z}}$. For the first summand note that the group

$$\pi^{\mathcal{G}}_{\lambda+1-\lambda^{a_i - b_i}}(H\underline{\mathbb{Z}}) = 0$$

as it satisfies the condition given in Theorem 3.8. To show the map

$$\mathcal{G}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \to \Sigma^{\lambda^{a_i - b_i}+1} H\underline{\mathbb{Z}}$$

is trivial consider the group

$$\pi^{C_d}_{\alpha}(H\underline{\mathbb{Z}}), \quad \alpha = \lambda^{a_\mu - b_\mu} - \lambda^{a_i - b_i} - 1$$

Then $|\alpha| = -1$, and for all subgroups C_j, $|\alpha^{C_j}| \leq 1$. Equality holds if and only if $j \mid (a_\mu - b_\mu)$ and $j \nmid (a_i - b_i)$. Then for any subgroup $C_k \supset C_j$, k does not divide $(a_i - b_i)$. So $|\alpha^{C_k}| \geq -1$. Thus, α satisfies the condition given in Theorem 3.8. Hence, the cofibre sequence in (4.11) splits and we obtain the required decomposition. This completes the proof. $\qquad\square$

5. Homology decompositions for $\mathcal{G} = C_{p^n}$

In this section, we derive homology decompositions for tree manifolds in the case $\mathcal{G} = C_{p^n}$ without any restriction on weight. We start with an example pointing out the need for a judicious choice of cellular filtration and later, we discuss how a reorientation may help to solve this. As a result, we observe some dimension shifting phenomena among the summands in the homology decomposition.

Example 5.1. Let $p \mid a - b$. We know from the cellular filtration of projective spaces (Example 4.2) that $\mathbb{C}P^2(a, b; p) = P(\lambda^a \oplus 1_{\mathbb{C}} \oplus \lambda^b)$ gives us the cofibre sequence

$$\Sigma^{\lambda^a} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \to \Sigma^{\lambda^{a-b} + \lambda^{-b}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda + 2} H\underline{\mathbb{Z}}.$$

Since the connecting map is zero, we obtain

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \simeq \Sigma^{\lambda + 2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}. \tag{5.2}$$

One may also write $\mathbb{C}P^2(a, b; p) = P(\lambda^a \oplus \lambda^b \oplus 1_{\mathbb{C}})$ which yields the cofibre sequence

$$\Sigma^{\lambda^{a-b}} H\underline{\mathbb{Z}} \simeq \Sigma^2 H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \to \Sigma^{2\lambda} H\underline{\mathbb{Z}}. \tag{5.3}$$

We claim that the connecting map of (5.3) is non-zero. Suppose on the contrary that the connecting map is trivial. Then we have the splitting

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m) \simeq \Sigma^{2\lambda} H\underline{\mathbb{Z}} \vee \Sigma^2 H\underline{\mathbb{Z}}. \tag{5.4}$$

The Mackey functor

$$\underline{\pi}_{\lambda}^{C_p}(H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a, b; m))$$

is isomorphic (3.7) to $\underline{\mathbb{Z}}$ from (5.2), while isomorphic to $\underline{\mathbb{Z}}^* \oplus \langle \mathbb{Z}/p \rangle$ if (5.4) were true, a contradiction. Hence, the connecting map of (5.3) should be non-trivial.

Recall from Proposition 2.13 (4) that for a non-root vertex $v \in V(\mathbb{T})$, the connected sum is performed at the point $[0, 0, 1] \in \mathbb{C}P_v^2$. At the root vertex v_0 with weight $(a_0, b_0; m)$ we may change the weights to $(-a_0, b_0 - a_0; m)$ or $(a_0 - b_0, -b_0; m)$ to obtain a $\mathcal{G}$-homeomorphic manifold. This fact will be used in the proof of the theorems below.

The following example summarizes how a reorientation is performed and the resulting modifications of weights. We also demonstrate how this allows us to make a judicious choice of cellular filtration of $X(\mathbb{T})$.

Example 5.5. Suppose we have an admissible weighted $\mathcal{G}$-equivariant tree $\mathbb{T}$ depicted as in the left of the figure below with root vertex v_0, $w(v_0) = (a_0, b_0; m)$, and all other vertices have weight as mentioned therein. We reorient $\mathbb{T}$ to obtain a new tree $\mathbb{T}'$ whose root vertex is $u_0 = v_3$ with $w(u_0) = (a_3 - b_3, -b_3; m)$, and let for $i = 1, 2, 3$, the vertices $u_i \in \mathbb{T}'$ represent the vertices v_{3-i}. Proposition 2.13 (4) tells us that the connected sum is performed at $[0, 0, 1] \in \mathbb{C}P_{v_3}^2$ for which

$$T_{[0,0,1]}\mathbb{C}P_{v_3}^2 = \lambda^{a_3} + \lambda^{b_3}.$$

This means there exists a $\mathcal{G}$-fixed point p in $\mathbb{C}P^2_{v_2}$ so that

$$T_p\mathbb{C}P^2_{v_2} = \lambda^{a_3} + \lambda^{-b_3} \text{ or } \lambda^{-a_3} + \lambda^{b_3}.$$

Then, if necessary, we apply a suitable $\mathcal{G}$-homeomorphism to map the point p to $[0, 0, 1]$, which allow us to perform the connected sum at the point $[0, 0, 1]$ of $\mathbb{C}P^2_{u_1}$ with $\mathbb{C}P^2_{u_0}$. This explains the weights in the new tree $\mathbb{T}'$. Note that $X(\mathbb{T})$ is $\mathcal{G}$ homeomorphic to $X(\mathbb{T}')$.

$$v_0 : w(v_0) = (a_0, b_0; m) \qquad\qquad u_3 : w(u_3) = \pm(a_1, -b_1; m)$$

$$v_1 : w(v_1) = (a_1, b_1; m) \qquad\qquad u_2 : w(u_2) = \pm(a_2, -b_2; m)$$

$$v_2 : w(v_2) = (a_2, b_2; m) \qquad\qquad u_1 : w(u_1) = \pm(a_3, -b_3; m)$$

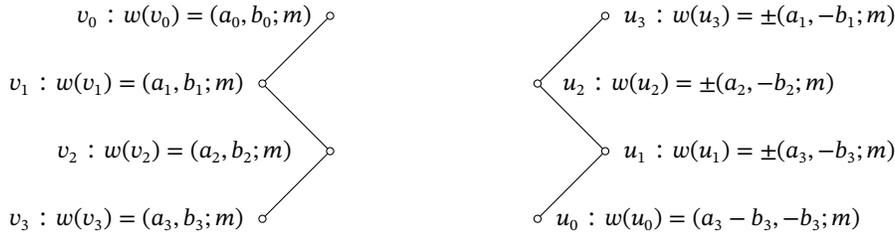$$v_3 : w(v_3) = (a_3, b_3; m) \qquad\qquad u_0 : w(u_0) = (a_3 - b_3, -b_3; m)$$

FIGURE 5.5. An example of reorientation: The left tree has root vertex v_0 and the right one has root vertex $u_0 = v_3$.

Suppose in the tree $\mathbb{T}$, $m \nmid a_0, b_0, a_1 - b_1, a_2 - b_2$ but $m \mid a_3 - b_3$. For the $\mathcal{G}$-manifold $X(\mathbb{T}) = \#^3_{i=0}\mathbb{C}P^2_{v_i}$ if we proceed as in the proof of Theorem 4.5, we obtain the cofibre sequence

$$
\begin{array}{ccc}
\Sigma^{\lambda^{a_3-b_3}}H\underline{\mathbb{Z}} \longrightarrow H\underline{\mathbb{Z}} \wedge X(\mathbb{T}) \longrightarrow H\underline{\mathbb{Z}} \wedge \#^2_{i=0}\mathbb{C}P^2_{v_i} \\
\downarrow \simeq \qquad\qquad\qquad\qquad \downarrow \simeq \\
\Sigma^2 H\underline{\mathbb{Z}} \qquad\qquad\qquad \Sigma^{2\lambda}H\underline{\mathbb{Z}} \vee \left(\bigvee^2_{i=0} \Sigma^\lambda H\underline{\mathbb{Z}} \right)
\end{array}
$$

Observe that the connecting map may be non-zero here. On the other hand, for $\mathbb{T}'$, we obtain

$$
\begin{array}{ccc}
\Sigma^{\lambda^{\pm(a_1+b_1)}}H\underline{\mathbb{Z}} \longrightarrow H\underline{\mathbb{Z}} \wedge X(\mathbb{T}') \longrightarrow H\underline{\mathbb{Z}} \wedge \#^2_{i=0}\mathbb{C}P^2_{u_i} \\
\downarrow \simeq \qquad\qquad\qquad\qquad \downarrow \simeq \\
\Sigma^\lambda H\underline{\mathbb{Z}} \qquad\qquad\qquad \Sigma^{\lambda+2}H\underline{\mathbb{Z}} \vee \left(\bigvee^2_{i=0} \Sigma^\lambda H\underline{\mathbb{Z}} \right)
\end{array} \qquad (5.6)
$$

The right vertical equivalence comes from Theorem 4.9 and the equivalence $\Sigma^{\lambda^{a_i+b_i}}H\underline{\mathbb{Z}} \simeq \Sigma^\lambda H\underline{\mathbb{Z}}$ for $i = 1, 2, 3$. To see this, note from $7(a)$ of 2.9 that

$$\pm(a_1, b_1) \in \{(a_0, -b_0), (a_0 - b_0, b_0), (a_0, b_0 - a_0)\}.$$

Our condition $m \nmid a_0, b_0, a_0 - b_0$ implies $m \nmid a_1, b_1, a_1 + b_1$. Iterating this process the desired equivalence follows. Observe that, the connecting map in cofibre sequence (5.6) is trivial.

We now proceed towards the decomposition result in the case $\mathcal{G} = C_p$. For the tree $\mathbb{T}$, if $p \nmid a_v - b_v$ for all vertices $v \in \mathbb{T}_0$, the result is obtained from Theorem 4.5 as

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}_0} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_e/\mathcal{G}} \mathcal{G}/e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \Big)$$

where $\mathbb{T}_e = \{v \in \mathbb{T} \mid Stab(v) = e\}$. In the complementary situation p must divide $a_v - b_v$ for some $v \in \mathbb{T}_0$. If further $p \mid a_0$ or b_0, we are in the situation dealt in Theorem 4.9, so that we have

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\phi(\mathbb{T})} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\psi(\mathbb{T})} \Sigma^2 H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_e/\mathcal{G}} \mathcal{G}/e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \Big)$$

where $\phi(\mathbb{T}) = \#\{v \in \mathbb{T}_0 \mid p \nmid a_v - b_v\}$, and $\psi(\mathbb{T}) = \#\{v \in \mathbb{T}_0 \mid p \text{ divides } a_v - b_v\}$. For the remaining case, we prove

Theorem 5.7. Let $\mathcal{G} = C_p$ and $\mathbb{T}$ be an admissible weighted $\mathcal{G}$-equivariant tree of type I such that $p \nmid a_0, b_0$ but $p \mid a_v - b_v$ for some $v \in \mathbb{T}_0$. Then,

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\phi(\mathbb{T})+1} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big)$$

$$\vee \Big(\bigvee_{\psi(\mathbb{T})-1} \Sigma^2 H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_e/\mathcal{G}} \mathcal{G}/e_+ \wedge \Sigma^2 H\underline{\mathbb{Z}} \Big).$$

Proof. By given hypothesis, there exist a vertex $v_\ell \in \mathbb{T}_0$ with $w(v_\ell) = (a_\ell, b_\ell; p)$ such that $a_\ell - b_\ell \equiv 0 \pmod{p}$. We further assume that v_ℓ is closest to the root vertex v_0 in terms of number of edges from v_0 to v_ℓ. We prove the statement for $\mathbb{T}_0 \subseteq \mathbb{T}$. The result follows from this because when we attach a free orbit to any level, the resulting connecting map becomes a map of underlying non-equivariant spectra, which is trivial.

Let Γ denote the path from v_0 to v_ℓ passing through vertices $v_0, v_1, \ldots, v_\ell$, and let for $v_i \in \Gamma$, $w(v_i) = (a_i, b_i; p)$. We reorient Γ, as in Example 5.5, so that now v_ℓ becomes the root vertex u_0; $v_i \in \Gamma$ becomes the vertex $u_{\ell-i}$. Observe that the weight at $u_i = v_{\ell-i}$ becomes $\pm(a_{\ell-i+1}, -b_{\ell-i+1}; p)$ and the weight at u_0 is $(a_\ell - b_\ell, -b_\ell; p)$. Since $a_\ell - b_\ell \equiv 0 \pmod{p}$, the choice of cellular filtration of $\mathbb{C}P^2(a_\ell, b_\ell; p)$, as in Example 5.1, leads to

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_\ell, b_\ell; p) \simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

Theorem 4.9 implies

$$H\underline{\mathbb{Z}} \wedge X(\Gamma) \simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=1}^{\ell} \Sigma^{\lambda^{a_{\ell-i+1} + b_{\ell-i+1}}} H\underline{\mathbb{Z}} \Big)$$

$$\simeq \Sigma^{\lambda+2} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\ell} \Sigma^{\lambda} H\underline{\mathbb{Z}} \Big)$$

To deduce the second equivalence we show that for $i = 1$ to ℓ, $p \nmid a_i + b_i$. The condition $p \nmid a_0, b_0, a_0 - b_0$ in turn implies $p \nmid a_1, b_1, a_1 + b_1$ as by 7(a) of

Definition 2.9 $\pm(a_1, b_1) \in \{(a_0, -b_0), (a_0 - b_0, b_0), (a_0, b_0 - a_0)\}$. The fact that for $1 \le i \le \ell$, $p \nmid a_{i-1} - b_{i-1}$, let us continue this process up to $\ell - 1$.

Next we attach vertices of $\mathbb{T} \setminus \Gamma$ to Γ proceeding by induction as in Theorem 4.9, and observe that in each step the connecting map for the cofibre sequence is null. This completes the proof. $\qquad\square$

The rest of the section is devoted to proving homology decompositions in the case $\mathcal{G} = C_{p^n}$. We start with the following observation

Lemma 5.8. Let $\mathcal{G} = C_{p^n}$ and $\mathbb{T}$ be an admissible weighted $\mathcal{G}$-equivariant tree of type I. Let τ be the maximum power of p that divides $a_v - b_v$ among the vertices of $\mathbb{T}_0$, and p^τ does not divide a_0 and b_0. Then for $v \in \mathbb{T} \setminus \mathbb{T}_0$, $Stab(v) \le C_{p^\tau}$.

Proof. Choose a vertex $u \in \mathbb{T} \setminus \mathbb{T}_0$ such that $Stab(u)$ is maximum among vertices of $\mathbb{T} \setminus \mathbb{T}_0$, and u is closest to the root vertex. This implies if u is in level L, then the vertex v in level $L - 1$ connected to u belongs to $\mathbb{T}_0$. So let $w(v) = (a_v, b_v, p^n)$. We claim that $C_{m_u} = Stab(u) \le C_{p^\tau}$. On the contrary suppose $C_{m_u} > C_{p^\tau}$. From 7(b) of Definition 2.9, we get m_u divides one of $a_v, b_v, a_v - b_v$. Since m_u can not divide $a_v - b_v$, m_u divides a_v or b_v. Further, if v is connected to a level $L - 2$ vertex v' with $w(v') = (a_{v'}, b_{v'}; p^n)$, then

$$\pm(a_v, b_v) \in \{(a_{v'}, -b_{v'}), (a_{v'}, b_{v'} - a_{v'}), (a_{v'} - b_{v'}, b_{v'})\}.$$

This means m_u divides $a_{v'}$ or $b_{v'}$. Continuing this process we end up with p^τ dividing a_0 or b_0, hence, a contradiction.

$\qquad\square$

Proceeding as in the C_p-case, we have that if $p \nmid a_v - b_v$ for all $v \in \mathbb{T}_0$, by Theorem 4.5

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{\mathbb{T}_0} \Sigma^\lambda H\underline{\mathbb{Z}} \Big) \vee \Big(\bigvee_{\mathbb{T}_d/\mathcal{G}, \, d \ne p^n} \mathcal{G}/C_{d+} \wedge \Sigma^\lambda H\underline{\mathbb{Z}} \Big).$$

In the complementary case p divides $a_v - b_v$ for some $v \in \mathbb{T}$. Let $\tau > 0$ be the maximum power of p that divides $a_v - b_v$ among the vertices of $\mathbb{T}_0$. If further $p^\tau \mid a_0$ or b_0, then proceeding exactly as in Theorem 4.9 we obtain

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda^{a_0} + \lambda^{b_0}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee Z_{\mathbb{T}}(i)} \Big)$$

$$\vee \Big(\bigvee_{[\mu] \in \mathbb{T}_d/\mathcal{G}, \, d \ne p^n} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big) \tag{5.9}$$

where for fixed $0 \le i \le n$, $Z_{\mathbb{T}}(i) := \#\{v \in \mathbb{T}_0, w(v) = (a_v, b_v; p^n) \mid \gcd(a_v - b_v, p^n) = p^i\}$.

We also define

$$W_{\mathbb{T}}(i) = \begin{cases} Z_{\mathbb{T}}(i) + 1 & \text{if } i = 0 \\ Z_{\mathbb{T}}(i) - 1 & \text{if } i = \tau \\ Z_{\mathbb{T}}(i) & \text{otherwise.} \end{cases}$$

Observe that the conditions on weights in Definition 2.9 does not change if we replace the weight $(a_0, b_0; p^n)$ at the root vertex by one of $(a_0 - b_0, -b_0; p^n)$ or $(b_0 - a_0, -a_0; p^n)$. This allows us to further assume $p \nmid a_0, b_0$ in the theorem below.

Theorem 5.10. Let $\mathcal{G} = C_{p^n}$ and $\mathbb{T}$ be an admissible weighted $\mathcal{G}$-equivariant tree of type I. Suppose $\tau > 0$ is the maximum power of p that divides $a_v - b_v$ among the vertices of $\mathbb{T}_0$ and $p \nmid a_0, b_0$. Then

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T})_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}})^{\vee W_{\mathbb{T}}(i)} H\underline{\mathbb{Z}} \Big)$$

$$\vee \Big(\bigvee_{[\mu] \in \mathbb{T}_d / \mathcal{G}, \, d \neq p^n} \mathcal{G}/C_{d+} \wedge \Sigma^{\lambda^{a_\mu - b_\mu}} H\underline{\mathbb{Z}} \Big).$$

Proof. Let $v_\ell \in \mathbb{T}_0$ with $w(v_\ell) = (a_\ell, b_\ell; p^n)$ be a vertex for which $p^\tau \mid a_\ell - b_\ell$, and v_ℓ is closest to the root in terms of number of edges. Let Γ denote the path from v_0 to v_ℓ passing through the vertices $v_0, v_1, \ldots, v_\ell$ with $w(v_i) = (a_i, b_i; p^n)$. We first compute $H\underline{\mathbb{Z}} \wedge X(\Gamma)$, then $H\underline{\mathbb{Z}} \wedge X(\mathbb{T}_0)$ and finally $H\underline{\mathbb{Z}} \wedge X(\mathbb{T})$.

We reorient Γ so that now v_ℓ becomes the root vertex u_0, and $v_{\ell-i}$ becomes the vertex u_i. Note that the weight at $u_i = v_{\ell-i}$ becomes $\pm(a_{\ell-i+1}, -b_{\ell-i+1}; p^n)$ and the weight at u_0 is $(a_\ell - b_\ell, -b_\ell; p^n)$. Since $\gcd(a_\ell - b_\ell, p^n) = p^\tau$,

$$\Sigma^{\lambda^{a_\ell - b_\ell}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^{p^\tau}} H\underline{\mathbb{Z}}. \tag{5.11}$$

We use the following cofibre sequence for $\mathbb{C}P^2(a_\ell, b_\ell; p^n)$

$$S^{\lambda^{a_\ell}} \to \mathbb{C}P^2(a_\ell, b_\ell; p^n) \to S^{\lambda^{a_\ell - b_\ell} + \lambda^{-b_\ell}}.$$

This together with the identification (5.11) leads to the following decomposition

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^2(a_\ell, b_\ell; p^n)_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}}.$$

Proceeding as in the proof of Theorem 5.7 we see that for $i = 0$ to $\ell - 1$, $p^\tau \nmid a_{i+1} + b_{i+1}$. Now (5.9) implies

$$H\underline{\mathbb{Z}} \wedge X(\Gamma) \simeq \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Sigma^{\lambda} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=1}^{\ell} \Sigma^{\lambda^{a_{\ell-i+1} + b_{\ell-i+1}}} H\underline{\mathbb{Z}} \Big)$$

$$\simeq \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_{\Gamma}(i)} \Big).$$

The last equivalence is obtained from Proposition 3.6 and the following claim

Claim: Given τ and Γ as above, we have for $s < \tau$

$$\#\{1 \le i \le \ell \mid \nu_p(a_i + b_i) = s\} = \#\{0 \le j \le \ell - 1 \mid \nu_p(a_j - b_j) = s\},$$

where $\nu_p(r) = \max\{k \mid p^k \text{ divides } r\}$ is the p-adic valuation.

Proof of the claim. Suppose p^s divides $a_i + b_i$ for some $1 \le i \le \ell$ and v_{i-1} is not the root vertex. Applying 7(a) of Definition 2.9, we see p^s divides a_{i-1} or b_{i-1} (this implies $p \nmid a_{i-1} - b_{i-1}$). Going one step further we see p^s divides one

of $a_{i-2} - b_{i-2}$, a_{i-2} or b_{i-2}, and the other two are relatively prime to p. Since $p \nmid a_0, b_0$, continuing this process we end up with p^s dividing $a_j - b_j$ for some $0 \le j < i - 1$ and $p \nmid a_q - b_q$ or $a_q + b_q$ for $j < q \le i - 1$. If $p^s \mid a_1 + b_1$, we must have $p^s \mid a_0 - b_0$ by (5) of 2.9 and the fact that $p \nmid a_0, b_0$. Therefore, the left-hand side is less than or equal to right-hand side.

In the reverse direction, suppose $p^s \mid a_j - b_j$ for some $0 \le j \le \ell - 1$ and v_j is not the root vertex. Then by $7(a)$ of Definition 2.9, we see p^s divides one of a_{j+1}, b_{j+1}, and hence, p divides neither of $a_{j+1}+b_{j+1}$ or $a_{j+1}-b_{j+1}$. Continuing further we see that p^s divides one of $a_{j+2} + b_{j+2}$, a_{j+2} or b_{j+2}. Since $p \mid a_\ell - b_\ell$, $p \nmid a_\ell, b_\ell$. Thus iterating this process, we see p^s divides $a_i + b_i$ for some $j + 1 < i < \ell$ and $p \nmid a_q + b_q$ or $a_q - b_q$ for $j + 1 \le q < i$. If $p^s \mid a_0 - b_0$, then by (5) of Definition 2.9, p^s divides one $a_1 + b_1$, a_1 or b_1. If $p^s \nmid a_1 + b_1$, then continuing one step further we see p^s divides one of $a_2 + b_2$, a_2 or b_2. Iterating this way we obtain the required. This completes the proof of the claim.

Next we attach vertices of $\mathbb{T}_0 \setminus \Gamma$ to Γ proceeding by induction as in the proof of Theorem 4.9, except the fact that now levels are defined according to the distance from Γ instead of the root vertex. For the inductive step suppose the statement holds for the tree up to level L, $\mathbb{T}_0(L)$ and we adjoin an orbit $\mathcal{O}_x$ containing a level $L + 1$-vertex v_x to $\mathbb{T}_0(L)$. Suppose $w(v_x) = (a_x, b_x; p^n)$. By Proposition 3.6, we may write $\Sigma^{\lambda^{a_x - b_x}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^{p^t}} H\underline{\mathbb{Z}}$ for some $0 \le t < \tau$. Then we obtain the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules after applying the induction hypothesis

$$\Sigma^{\lambda^{p^t}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}_0(L) + \mathcal{O}_x) \to \Sigma^{\lambda + \lambda^{p^\tau}} H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n} (\Sigma^{\lambda^{p^i}} H\underline{\mathbb{Z}})^{\vee W_{\mathbb{T}_0(L)}(i)} \Big).$$

The fact that p^τ is the highest power ensures the connecting maps are zero by Theorem 3.8. Hence, the above cofibre sequence splits and the required homology decomposition is obtained.

Finally, to complete the proof, we adjoin vertices of $\mathbb{T} \setminus \mathbb{T}_0$ to $\mathbb{T}_0$, i.e., vertices on which $\mathcal{G}$ acts non-trivially. Again we proceed by induction on levels where levels are defined according to the distance from $\mathbb{T}_0$. Assume the statement holds for the tree up to level L', $\mathbb{T}(L')$ and we attach an orbit $\mathcal{O}_y$ of the level $L'+1$-vertices to $\mathbb{T}(L')$. Suppose for $v_y \in \mathcal{O}_y$, $w(v_y) = (a_y, b_y; C_{m_y})$, and $\Sigma^{\lambda^{a_y - b_y}} H\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^{p^{t'}}} H\underline{\mathbb{Z}}$ for some $0 \le t' \le n$. Proceeding along the lines of Theorem 4.5, we obtain the following cofibre sequence of $H\underline{\mathbb{Z}}$-modules

$$\mathcal{G}/C_{m_y}{}_+ \wedge \Sigma^{\lambda^{p^{t'}}} H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \wedge X\big(\mathbb{T}(L') + \mathcal{O}_y\big)_+ \to H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L'))_+ \quad (5.12)$$

where by the induction hypothesis

$$H\underline{\mathbb{Z}} \wedge X(\mathbb{T}(L'))_+ \simeq H\underline{\mathbb{Z}} \vee \Sigma^{\lambda+\lambda^{p^\tau}}H\underline{\mathbb{Z}} \vee \Big(\bigvee_{i=0}^{n}(\Sigma^{\lambda^{p^i}}H\underline{\mathbb{Z}})^{\vee W_{\mathbb{T}(L')}(i)}\Big)$$
$$\vee \Big(\bigvee_{[\mu]\in\mathbb{T}(L')_d/\mathcal{G}}\mathcal{G}/C_{d_+} \wedge \Sigma^{\lambda^{a_\mu-b_\mu}}H\underline{\mathbb{Z}}\Big).$$

The connecting map

$$\Sigma^{\lambda+\lambda^{p^\tau}}H\underline{\mathbb{Z}} \to \mathcal{G}/C_{m_{y}{}_+} \wedge \Sigma^{\lambda^{p^{t'}}+1}H\underline{\mathbb{Z}}$$

is classified up to homotopy by $\pi_0^{C_{m_y}}(\Sigma^{\lambda^{p^{t'}}+1-\lambda-\lambda^{p^\tau}}H\underline{\mathbb{Z}})$. Since Lemma 5.8 asserts that $C_{m_y} \leq C_{p^\tau}$, the above group reduces to $\pi_0^{C_{m_y}}(\Sigma^{\lambda^{p^{t'}}-\lambda-1}H\underline{\mathbb{Z}})$, which is trivial by Theorem 3.8. Analogously all other connecting maps can be seen to be zero. Hence, the cofibre sequence (5.12) splits and we obtain the required decomposition. □

References

[1] BASU, SAMIK; GHOSH, SUROJIT. Computations in C_{pq}-Bredon cohomology. *Math. Z.* **293** (2019), no. 3-4, 1443–1487. MR4024594, Zbl 1435.55003, doi: 10.1007/s00209-019-02248-2. 1555, 1567

[2] BASU, SAMIK; GHOSH, SUROJIT. Equivariant cohomology for cyclic groups of square-free order. Preprint, 2020. arXiv:2006.09669. 1555, 1567

[3] BASU, SAMIK; GHOSH, SUROJIT. Bredon cohomology of finite dimensional C_p-spaces. *Homology Homotopy Appl.* **23** (2021), no. 2, 33–57. MR4236689, Zbl 07420242, arXiv:1911.08137, doi: 10.4310/hha.2021.v23.n2.a3. 1555

[4] BASU, SAMIK; GHOSH, SUROJIT. Non-trivial extensions in equivariant cohomology with constant coefficients. Preprint, 2021. arXiv:2108.12763. 1566, 1567

[5] DRESS, ANDREAS W. M. Contributions to the theory of induced representations. *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic* (Battelle Memorial Inst., Seattle, Wash., 1972), pp.183-240. Lecture Notes in Math., 342. *Springer, Berlin*, 1973. MR0384917, Zbl 0331.18016, doi: 10.1007/BFb0073727. 1564

[6] FERLAND, KEVIN K. On the RO(G)-graded equivariant ordinary cohomology of generalized G-cell complexes for G = Z/p. *Thesis (Ph.D.)–Syracuse University*, 1999. 176 pp. ISBN: 978-0599-42024-3. MR2699528. 1555, 1566, 1567

[7] FERLAND, KEVIN K.; LEWIS, L. GAUNCE, JR. The $RO(G)$-graded equivariant ordinary homology of G-cell complexes with even-dimensional cells for $G = \mathbb{Z}/p$. *Mem. Amer. Math. Soc.* **167** (2004), no. 794, viii+129. MR2025457, Zbl 1052.55007, doi: 10.1090/memo/0794. 1555, 1567

[8] GREENLEES, JOHN P. C.; MAY, J. PETER. Equivariant stable homotopy theory. *Handbook of algebraic topology*, 277–323. *North-Holland, Amsterdam*, 1995. MR1361893, Zbl 0866.55013, doi: 10.1016/B978-044481779-2/50009-2. 1564

[9] HAMBLETON, IAN; TANASE, MIHAIL. Permutations, isotropy and smooth cyclic group actions on definite 4-manifolds. *Geom. Topol.* **8** (2004), 475–509. MR2033485, Zbl 1072.58003, arXiv:math/0307297, doi: 10.2140/gt.2004.8.475. 1554, 1555, 1557, 1559, 1560, 1561, 1562, 1563

[10] HAZEL, CHRISTY. The $RO(C_2)$-graded cohomology of C_2-surfaces in $\mathbb{Z}/2$-coefficients. *Math. Z.* **297** (2021), no. 1-2, 961–996. MR4204721, Zbl 1459.57027, arXiv:1907.07280, doi: 10.1007/s00209-020-02542-4. 1555

[11] KRONHOLM, WILLIAM C. A freeness theorem for $RO(\mathbb{Z}/2)$-graded cohomology. *Topology Appl.* **157** (2010), no. 5, 902–915. MR2593703, Zbl 1194.55011, arXiv:0908.3825, doi:10.1016/j.topol.2009.12.006. 1555

[12] LEWIS, L. GAUNCE, JR. The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbf{Z}/p$ actions. *Algebraic topology and transformation groups* (Göttingen, 1987), 53—122. Lecture Notes in Math., 1361, Math Gottingensis. *Springer, Berlin*, 1988. MR0979507, Zbl 0669.57024, doi:10.1007/BFb0083034. 1555, 1567

[13] LEWIS, L. GAUNCE, JR. The equivariant Hurewicz map. *Trans. Amer. Math. Soc.* **329** (1992), no. 2, 433–472. MR1049614, Zbl 0769.54042, doi10.2307/2153946. 1555

[14] MANDELL, MICHAEL A.; MAY, J. PETER. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.* **159** (2002), no. 755, x+108 pp. MR1922205, Zbl 1025.55002, doi:10.1090/memo/0755. 1565

[15] MAY, CLOVER. A structure theorem for $RO(C_2)$-graded Bredon cohomology. *Algebr. Geom. Topol.* **20** (2020), no. 4, 1691–1728. MR4127082, Zbl 1447.55009, arXiv:1804.03691, doi:10.2140/agt.2020.20.1691. 1555

[16] MAY, J. PETER. Equivariant homotopy and cohomology theory. CBMS Regional Conference Series in Mathematics, 91. *American Mathematical Society, Providence, RI*, 1996. xiv+366 pp. ISBN: 0-8218-0319-0. MR1413302, Zbl 0890.55001, doi:10.1090/cbms/091. 1565

[17] YOSHIDA, TOMOYUKI. On G-functors. II. Hecke operators and G-functors. *J. Math. Soc. Japan.* **35** (1983), no. 1, 179–190. MR679083, Zbl 0507.20010, doi:10.2969/jmsj/03510179. 1565

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