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Partial Poisson transforms on $SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$

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Abstract. In this article, we introduce a partial Poisson transform on the affine symmetric space $SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ and prove that this transform is a continuous $SU(n, n)$-homomorphism. We also give the form for the Fourier transform of the Poisson kernel.

Contents

1. Introduction

In a complex plane, the classical Poisson transform on the unit circle gives a harmonic function on the unit disk. More generally, each eigenfunction of the Laplace-Beltrami operator on the Poincaré disk can be represented by a generalized Poisson transform of a hyperfunction on the unit circle. The notion of the Poisson transform has been generalized to a Riemannian symmetric space G/K of the non-compact type, where G is a connected real reductive Lie group and K is its maximal compact subgroup (see [14, 15, 18, 23, 12] and the references given there).

Now we recall the *Helgason conjecture* for the classical Poisson transform on G/K.

Theorem 1.1. *Every joint eigenfunction of the invariant differential operators on G/K has a Poisson integral representation by a hyperfunction on the boundary of G/K.*

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In [14], Helgason proved Theorem 1.1 for Riemannian symmetric spaces G/K of rank one, except for an explicitly given set of spectral parameters determining the eigenvalues. In the same paper, Helgason conjectured that this theorem can be extended to higher-rank spaces by replacing the (geodesic) boundary with the Furstenberg boundary. In [15], Helgason proved such an extension for K-finite functions. In [18], Kashiwara et al. give a proof, which is based on the theory of boundary values for systems of differential equations with regular singularities developed by Kashiwara and Oshima in [17]. Recently, in [12], Hansen, Hilgert, and Parthasarathy have given a new proof of a generic version of Theorem 1.1. For a recent account of the theory, we refer the reader to [25, 20].

The affine symmetric case remains an open one. In this paper we only consider the affine symmetric space $\mathcal{X} = SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$. It is possible to define the Poisson transformation for some sets of parameters by an integral resembling those for $SU(n, n)/S(U(n) \times U(n))$ case.

Let $(\xi, V(\xi))$ be the irreducible representation of M. According to [13, p. 161], we choose the form of $w(M \cap H)w^{-1}$, which guarantees that the representation of $w(M \cap H \cap K)w^{-1}$ is trivial. Technically, by [6, Lemma 1] and Frobenius reciprocity, we prove that the restricted representation $\mathrm{Res}^{M}_{w(M\cap H)w^{-1}} V(\xi)$ of M is isomorphic to $\mathrm{Res}^{M\cap K}_{w(M\cap H\cap K)w^{(-1)}} V(\xi)$, where $w \in \mathcal{W}$ and $\mathcal{W}$ is as in subsection 3.2 below. Then, to prove that the Poisson transform is a continuous $SU(n,n)$-homomorphism, we just need to prove (4.3) below. We calculate the normalized coefficient of the Poisson transform by proving the limit convergence of (4.3). As an interesting result, we give the form for the Fourier transform of the Poisson kernel. We expect that some ideas of our article are applicable to Poisson transforms on more general semi-simple symmetric spaces, as in [9, 23, 21, 1, 5, 7, 8, 11]. We also plan to address Theorem 1.1 on $\mathcal{X}$ in forthcoming publications.

Precisely, this article is organized as follows.

In Section 2, we recall the affine symmetric space of Hermitian type.

In Section 3, we calculate the restricted root system of $SU(n,n)$ and obtain their Weyl groups and orbits, respectively. Then we define the Poisson kernel, Poisson transforms on these orbits, and obtain the normalized Eisenstein integrals.

In Section 4, a continuous $SU(n, n)$-homomorphism of the Poisson transform is obtained. Finally, we prove the relationship between the Poisson transform and Fourier transform.

2. Preliminaries

In what follows, we always define the conjugate transpose and the transpose of the matrix by $*$ and T, respectively. The group $SU(n,n)$ is a connected non-compact semi-simple Lie group defined by

$$SU(n,n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A^*A - C^*C = I_n, D^*D - B^*B = I_n, \right.$$
$$\left. B^*A - D^*C = 0, A, B, C, D \in M_n(\mathbb{C}) \right\}.$$

Let $S(U(n)\times U(n))$ be the maximal compact subgroup of $SU(n,n)$. The Lie algebra of $SU(n,\, n)$ is given by

$$\mathfrak{su}(n,\, n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \delta \end{pmatrix} :\ \alpha^* = -\alpha,\ \delta^* = -\delta,\ \mathrm{tr}(\alpha) + \mathrm{tr}(\delta) = 0 \right\}.$$

Here and hereafter, $\alpha,\ \beta,\ \delta \in M_n(\mathbb{C})$. Let $\mathfrak{su}(n,\, n) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, where

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{su}(n,\, n) :\ \theta(X) = -X^* = X\} \\ &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} :\ \alpha^* = -\alpha,\ \delta^* = -\delta,\ \mathrm{tr}(\alpha) + \mathrm{tr}(\delta) = 0 \right\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{p} &= \{X \in \mathfrak{su}(n,\, n) :\ \theta(X) = -X^* = -X\} \\ &= \left\{ \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix} : \beta \in M_n(\mathbb{C}) \right\}.\end{aligned} \tag{2.1}$$

Define the involution (non Cartan involution) on $SU(n,n)$ by

$$\tau\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} D & C \\ B & A \end{pmatrix},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,n)$. Let $\mathfrak{h} + \mathfrak{q}$ be the decomposition of $\mathfrak{su}(n,\, n)$ into the ± 1-eigenspaces of the involution τ, where

$$\mathfrak{h} = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} :\ \alpha^* = -\alpha,\ \beta^* = \beta, \mathrm{tr}(\alpha) = 0 \right\}$$

and

$$\mathfrak{q} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} :\ \alpha^* = -\alpha, \quad \beta^* = -\beta \right\}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} :\ \beta^* = -\beta \right\}.$$

The maximal Abel subspace of $\mathfrak{p} \cap \mathfrak{q}$ is given by

$$\mathfrak{a}_{\mathfrak{p}} = \left\{ i \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} : \ t = \operatorname{diag}(t_1, \cdots, t_n),\ t_i \in \mathbb{R} \right\}. \tag{2.3}$$

The Lie group H of $\mathfrak{h}$ is defined by

$$H = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} : \ A, B \in M_n(\mathbb{C}),\ A^*A - B^*B = I_n,\ B^*A = A^*B \right\}. \tag{2.4}$$

Let $\varphi : \mathfrak{h} \to \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}I_n$ be an isomorphism of Lie algebras, namely, $\varphi\left(\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\right) = \alpha - \beta$. And $\varphi : H \to SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ is an isomorphism of groups.

Lemma 2.1. *$SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ is an affine symmetric space of Hermitian type. Moveover,*

$$SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^* \simeq \mathcal{X} := \{(z, w) \in U(n) \times U(n) : \ \det(z - w) \neq 0\}.$$

Proof. Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2n, \mathbb{C})$ and the complexification of $\mathfrak{h}$ is

$$\mathfrak{h}_{\mathbb{C}} := \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \ \alpha \in M_n(\mathbb{C}),\ \beta \in M_n(\mathbb{C}), \operatorname{tr}(\alpha) = 0 \right\},$$

then Definition 1.1(i) of [22] is natural. In what follows, condition Definition 1.1(ii) of [22] can be replaced by saying that $\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}_{\mathbb{C}}$ has a non-trivial center. In fact,

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \ \alpha \in M_n(\mathbb{C}),\ \delta \in M_n(\mathbb{C}),\ \operatorname{tr}(\alpha + \delta) = 0 \right\}$$

and

$$\mathfrak{q}_{\mathbb{C}} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} : \ \alpha \in M_n(\mathbb{C}),\ \beta \in M_n(\mathbb{C}) \right\}.$$

Thus

$$\mathfrak{q}_{\mathfrak{k}}^{\mathbb{C}} := \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}_{\mathbb{C}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \ \alpha \in M_n(\mathbb{C}) \right\},$$

which implies that $\mathfrak{q}_{\mathfrak{k}}^{\mathbb{C}}$ has a non-trivial center:

$$\mathfrak{c}_{\mathbb{C}} = \left\{ \begin{pmatrix} \widetilde{\alpha} & 0 \\ 0 & \widetilde{\beta} \end{pmatrix} : \ \widetilde{\alpha} = aI_n,\ \widetilde{\beta} = -aI_n,\ a \in \mathbb{C} \setminus \{0\} \right\}.$$

And hence Definition 1.1(ii) of [22] holds and $SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ is an affine symmetric space of Hermitian type. Finally, similar to the proof of [16, Lemma 2.6.11, p. 68 and the table on p. 58], the bounded realization of the space $SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}_+^*$ is obtained. □

For the affine symmetric space of Hermitian type as in Lemma 2.1, we have an equal description, namely, $(\mathfrak{su}(n, n), \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}I_n)$ is an affine symmetric pair (also see [4]).

3. Frobenius reciprocity

3.1. Minimal parabolic subgroup.

Lemma 3.1. *The positive root system of* $\Delta := \Delta(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ *is given by*

$$\Delta^+ = \{2\gamma_j \,:\, 1 \le j \le n\} \cup \{\gamma_i \pm \gamma_j \,:\, 1 \le i < j \le n\}. \tag{3.1}$$

with multiplicity $m_\alpha = 2$ *for the short roots* $\alpha = \pm(\gamma_i \pm \gamma_j)$ *and* $m_\alpha = 1$ *for the long roots* $\alpha = \pm 2\gamma_i$*, where*

$$\gamma_i = \frac{\sqrt{-1}}{8n}(E_{i,n+i} - E_{n+i,i}) \quad \text{and} \quad (\gamma_i, \gamma_j) = \frac{1}{8n}\delta_{ij}. \tag{3.2}$$

Proof. Let $e_{ij} = E_{ij} + E_{ji}$ and $\widetilde{e}_{ij} = E_{ji} - E_{ij}$, where E_{ij} denote the $n\times n$ matrix with 1 on the (i,j)-th entry and zero on all other entries. Then, for $i < j$, the eigenvectors of $\pm(\gamma_i - \gamma_j)$, respectively, are given by

$$\begin{cases} a_1^{ij} = \begin{pmatrix} e_{ij}\sqrt{-1} & \widetilde{e}_{ij} \\ -\widetilde{e}_{ij} & e_{ij}\sqrt{-1} \end{pmatrix}, \\ a_2^{ij} = \begin{pmatrix} -\widetilde{e}_{ij} & e_{ij}\sqrt{-1} \\ -e_{ij}\sqrt{-1} & -\widetilde{e}_{ij} \end{pmatrix}, \end{cases} \tag{3.3}$$
$$\begin{cases} b_1^{ij} = \begin{pmatrix} e_{ij}\sqrt{-1} & -\widetilde{e}_{ij} \\ \widetilde{e}_{ij} & e_{ij}\sqrt{-1} \end{pmatrix}, \\ b_2^{ij} = \begin{pmatrix} -\widetilde{e}_{ij} & -e_{ij}\sqrt{-1} \\ e_{ij}\sqrt{-1} & -\widetilde{e}_{ij} \end{pmatrix}. \end{cases}$$

The eigenvectors of $\pm(\gamma_i + \gamma_j)$, respectively, are given by

$$\begin{cases} c_1^{ij} = \begin{pmatrix} e_{ij}\sqrt{-1} & e_{ij} \\ e_{ij} & -e_{ij}\sqrt{-1} \end{pmatrix}, \\ c_2^{ij} = \begin{pmatrix} -\widetilde{e}_{ij} & \widetilde{e}_{ij}\sqrt{-1} \\ \widetilde{e}_{ij}\sqrt{-1} & \widetilde{e}_{ij} \end{pmatrix}, \end{cases} \qquad \begin{cases} d_1^{ij} = \begin{pmatrix} -e_{ij}\sqrt{-1} & e_{ij} \\ e_{ij} & e_{ij}\sqrt{-1} \end{pmatrix}, \\ d_2^{ij} = \begin{pmatrix} \widetilde{e}_{ij} & \widetilde{e}_{ij}\sqrt{-1} \\ \widetilde{e}_{ij}\sqrt{-1} & -\widetilde{e}_{ij} \end{pmatrix}. \end{cases}$$

The eigenvectors of $\pm 2\gamma_i$, respectively, are given by

$$e^i = \begin{pmatrix} E_{ii}\sqrt{-1} & E_{ii} \\ E_{ii} & -E_{ii}\sqrt{-1} \end{pmatrix}, \quad f^i = \begin{pmatrix} -E_{ii}\sqrt{-1} & E_{ii} \\ E_{ii} & E_{ii}\sqrt{-1} \end{pmatrix}.$$

Now we prove (3.2) holds true. Let ad be the adjoint representation of $\mathfrak{g}_+$. Then, for any $a \in \mathfrak{a}_\mathfrak{p}$, we have

$$\begin{cases} \mathrm{ad}_a^2 a_1^{ij} = (\gamma_i - \gamma_j)^2(a)(a_1^{ij}), \\ \mathrm{ad}_a^2 a_2^{ij} = (\gamma_i - \gamma_j)^2(a)(a_2^{ij}), \end{cases} \begin{cases} \mathrm{ad}_a^2 b_1^{ij} = (\gamma_i - \gamma_j)^2(a)(b_1^{ij}), \\ \mathrm{ad}_a^2 b_2^{ij} = (\gamma_i - \gamma_j)^2(a)(b_2^{ij}), \end{cases} \tag{3.4}$$

$$\begin{cases} \mathrm{ad}_a^2 c_1^{ij} = (\gamma_i + \gamma_j)^2(a)(c_1^{ij}), \\ \mathrm{ad}_a^2 c_2^{ij} = (\gamma_i + \gamma_j)^2(a)(c_2^{ij}), \end{cases} \begin{cases} \mathrm{ad}_a^2 d_1^{ij} = (\gamma_i + \gamma_j)^2(a)(d_1^{ij}), \\ \mathrm{ad}_a^2 d_2^{ij} = (\gamma_i + \gamma_j)^2(a)(d_2^{ij}), \end{cases}$$

and

$$\begin{cases} \mathrm{ad}_a^2 e^i = (2\gamma_i)^2(a)(e^i), \\ \mathrm{ad}_a^2 f^i = (2\gamma_i)^2(a)(f^i). \end{cases}$$

In particular, we let h_k, $e_k \in \mathfrak{a}_\mathfrak{p}$ be as follows:

$$h_k = \sqrt{-1}(E_{k,n+k} - E_{k+1,n+k+1} - E_{n+k,k} + E_{n+k+1,k+1}) \tag{3.5}$$

for $1 \le k \le n-1$, and

$$e_k = \sqrt{-1}(E_{k,n+k} - E_{n+k,k})$$

for $1 \le k \le n$. From this, the equations above and the Killing form $(x, y) = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$, it follows that

$$\begin{aligned} (h_k, h_k) &= \mathrm{Tr}(\mathrm{ad}_{h_k}\mathrm{ad}_{h_k}) \\ &= 4\sum_{1\le i<j\le n}\left[(\gamma_i-\gamma_j)^2(h_k) + (\gamma_i+\gamma_j)^2(h_k)\right] + 2\sum_{1\le i\le n}(2\gamma_i)^2(h_k) \\ &= 4(4+2(n-2)+2(n-2)) + 2(4+4) \\ &= 16n \end{aligned}$$

and

$$\begin{aligned} (e_k, e_k) &= \mathrm{Tr}(\mathrm{ad}_{e_k}\mathrm{ad}_{e_k}) \\ &= 4\sum_{1\le i<j\le n}\left[(\gamma_i-\gamma_j)^2(e_k) + (\gamma_i+\gamma_j)^2(e_k)\right] + 2\sum_{1\le i\le n}(2\gamma_i)^2(e_k) \\ &= 8n. \end{aligned}$$

Moreover, if $i \ne j$, then $(e_i, e_j) = \mathrm{Tr}(\mathrm{ad}_{e_i}\mathrm{ad}_{e_j}) = 0$.

Let $\gamma_k - \gamma_{k+1} = sh_k, \gamma_k = s\sqrt{-1}(E_{k,n+k} - E_{n+k,k})$. Together with $(\gamma_k - \gamma_{k+1})(h_k) = (sh_k, h_k)$ and $(\gamma_k - \gamma_{k+1})(h_k) = 2$, imply that $s = \frac{1}{8n}$. And hence

$$\gamma_k - \gamma_{k+1} = \frac{\sqrt{-1}}{8n}(E_{k,n+k} - E_{k+1,n+k+1} - E_{n+k,k} + E_{n+k+1,k+1})$$

for $1 \le k \le n-1$. Processing continues in this same way, we obtain

$$\gamma_k = \frac{\sqrt{-1}}{8n}(E_{k,n+k} - E_{n+k,k}) \text{ and } (\gamma_i, \gamma_j) = \frac{1}{64n^2}(e_i, e_j) = \begin{cases} \frac{1}{8n}, & i = j \\ 0, & i \ne j \end{cases}$$

This finishes the proof of Lemma 3.1. □

Let $\mathfrak{n} := \mathfrak{n}(\Delta^+)$ be the sum of the root spaces corresponding to the roots in this set, and put $N = N(\Sigma^+(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})) := \exp \mathfrak{n}$. Let M_1 denote the centralizer of $\mathfrak{a}_\mathfrak{p}$ in G, and put $P' = P'(\Sigma^+(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})) := M_1 N$. It is easily seen that M_1 normalizes N and hence P' is a subgroup of G. Let $\mathfrak{u}$ be complementary to $\mathfrak{n} \cap \mathfrak{h}$ in $\mathfrak{n}$, and write $U = \exp \mathfrak{u}$. Here we assume that $\mathfrak{u}$ is a unipotent radical. Let L be the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $SU(n, n)$, namely,

$$L = \left\{g \in SU(n,\, n) : gag^{-1} = a, a \in \mathfrak{a}_\mathfrak{p}\right\}. \tag{3.6}$$

Motivated by [19], we have the minimal $\tau\theta$-stable parabolic subgroup as follows:

$$P = LU. \tag{3.7}$$

3.2. Weyl groups and P-orbits. Since $\theta\tau = \tau\theta$, $\mathfrak{su}(n, n)$ has a $\theta\tau$ decomposition $\mathfrak{g}_+ \oplus \mathfrak{g}_-$, where

$$\mathfrak{g}_+ = \{X \in \mathfrak{su}(n, n) : \theta\tau(X) = X\} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p}) \tag{3.8}$$

and $\mathfrak{g}_- = \{X \in \mathfrak{su}(n, n) : \theta\tau(X) = -X\} = (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{q} \cap \mathfrak{k})$. Note that $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{su}(n, n)$. In fact, since $\mathfrak{g}_+$ is the subspace of $\mathfrak{su}(n, n)$, then, for any $X, Y \in \mathfrak{g}_+$, the bracket operation $[X, Y]$ is bilinear, and $[X, Y] \in \mathfrak{g}_+$, $[X, X] = 0$ for all $X \in \mathfrak{g}_+$, and Jacobi's identity holds. And hence $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{su}(n, n)$.

Proposition 3.2. *Let $\mathfrak{g}_+$ be as in (3.8). Then*

$$\Delta_1 := \Delta(\mathfrak{g}_+, \mathfrak{a}_\mathfrak{p}) = \{\pm(\gamma_i - \gamma_j) : i < j\}$$

is the corresponding set of restricted roots of Δ with multiplicity $m_\alpha = 2$ for the roots $\alpha = \pm(\gamma_i - \gamma_j)$, where

$$\gamma_i = \frac{\sqrt{-1}}{4n}(E_{i,n+i} - E_{n+i,i}) \quad \text{and} \quad (\gamma_i, \gamma_j) = \begin{cases} \frac{n-1}{4n^2}, & i = j, \\ -\frac{1}{4n^2}, & i \neq j. \end{cases} \tag{3.9}$$

Proof. By (3.8), we have

$$\mathfrak{g}_+ = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} : X \in M_n(\mathbb{C}), Y \in M_n(\mathbb{C}), Y^* = -Y, X^* = -X \right\}.$$

Let $Z_{\mathfrak{g}_+}(\mathfrak{a}_\mathfrak{p})$ be the center of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{g}_+$. Then, by the form of $\mathfrak{g}_+$, we see that $\mathfrak{g}_+ = \mathfrak{a}_\mathfrak{p} + \mathfrak{m} + \mathfrak{n}^\pm$, where $\mathfrak{n}^\pm = \sum_{i<j} \mathfrak{g}_{\pm(\gamma_i - \gamma_j)}$ and

$$\mathfrak{m} = \left\{ \begin{pmatrix} \sqrt{-1}t & 0 \\ 0 & \sqrt{-1}t \end{pmatrix} : t = \mathrm{diag}(t_1, t_2, \dots, t_n),\ t_i \in \mathbb{R} \right\} \subset Z_{\mathfrak{g}^+}(\mathfrak{a}_\mathfrak{p}).$$

From this and Lemma 3.1, it follows that $\Delta_1 = \{\pm(\gamma_i - \gamma_j) : 1 \le i \le j \le n\}$.

Now we prove (3.9) holds true. For $i < j$, the eigenvectors of $\pm(\gamma_i - \gamma_j)$ are as in (3.3). Let ad be the adjoint representation of $\mathfrak{g}_+$. Then, by (3.3) again, for any $a \in \mathfrak{a}_\mathfrak{p}$, we have (3.4). In particular, we let h_k, $e_k \in \mathfrak{a}_\mathfrak{p}$ be as in (3.5). From this, (3.4) and the Killing form $(x, y) = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$, it follows that

$$(h_k, h_k) = \mathrm{Tr}(\mathrm{ad}_{h_k} \mathrm{ad}_{h_k}) = 4 \sum_{1 \le i < j \le n} (\gamma_i - \gamma_j)^2(h_k) = 4(4 + 2(n-2)) = 8n,$$

$$(e_k, e_k) = \mathrm{Tr}(\mathrm{ad}_{e_k} \mathrm{ad}_{e_k}) = 4 \sum_{1 \le i < j \le n} (\gamma_i - \gamma_j)^2(e_k) = 4(n-1)$$

and, if $i \neq j$, $(e_i, e_j) = \mathrm{Tr}(\mathrm{ad}_{e_i}\mathrm{ad}_{e_j}) = 4(\gamma_i - \gamma_j)(e_i)(\gamma_i - \gamma_j)(e_j) = -4$. Let $\gamma_k - \gamma_{k+1} = sh_k$. Together with $(\gamma_k - \gamma_{k+1})(h_k) = (sh_k, h_k)$ and $(\gamma_k - \gamma_{k+1})(h_k) = 2$, implies that $s = \frac{1}{4n}$. And hence

$$\gamma_k - \gamma_{k+1} = \frac{\sqrt{-1}}{4n}(E_{k,n+k} - E_{k+1,n+k+1} - E_{n+k,k} + E_{n+k+1,k+1})$$

for $1 \leq k \leq n-1$. Processing continues in this same way, we obtain (3.9). This finishes the proof of Proposition 3.2. □

Let H be as in (2.4) and

$$K = S(U(n) \times U(n)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \ A, D \in U(n), \det(AD) = 1 \right\}.$$

Then

$$L' = K \cap H = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : \ A \in U(n), \det(A) = \pm 1 \right\}.$$

Notice that L' is a maximal compact subgroup of $G_+ = \exp \mathfrak{g}_+$. Since $\mathfrak{g}_+ = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p})$ is the Cartan decomposition of $\mathfrak{g}_+$ as in (3.8), we can apply the theory of $L'\overline{A_{\mathfrak{p}}^+}L'$ decomposition to G_+ and obtain that $\mathfrak{a}_{\mathfrak{p}}$ is unique up to conjugacy by L', where $\overline{A_{\mathfrak{p}}^+}$ is the closure of $A_{\mathfrak{p}}^+$, $A_{\mathfrak{p}}^+ = \exp \mathfrak{a}_{\mathfrak{p}}^+$ and $\mathfrak{a}_{\mathfrak{p}}^+$ is the Weyl chamber defined by $\mathfrak{a}_{\mathfrak{p}}^+ = \{t \in \mathfrak{a}_{\mathfrak{p}} : \ t_k - t_\ell > 0, \ k < \ell\}$. The positive root system Δ_1^+ of Δ_1 is given by $\Delta_1^+ = \{\gamma_k - \gamma_\ell; \ k < \ell\}$. The Weyl group associated with the restricted positive roots Δ_1^+ is

$$W_{L'} := W_{L'}(\mathfrak{g}_+, \ \mathfrak{a}_{\mathfrak{p}}) = N_{L'}(\mathfrak{a}_{\mathfrak{p}})/Z_{L'}(\mathfrak{a}_{\mathfrak{p}}),$$

where $N_{L'}(\mathfrak{a}_{\mathfrak{p}})$ and $Z_{L'}(\mathfrak{a}_{\mathfrak{p}})$ are the normalizer and centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in L', respectively.

Define the Weyl group of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{su}(n, \ n)$ associated with Δ^+ by

$$W := W(\mathfrak{su}(n, \ n), \ \mathfrak{a}_{\mathfrak{p}}) = N_K(\mathfrak{a}_{\mathfrak{p}})/Z_K(\mathfrak{a}_{\mathfrak{p}}),$$

where $N_K(\mathfrak{a}_{\mathfrak{p}})$ and $Z_K(\mathfrak{a}_{\mathfrak{p}})$ are the normalizer and centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in K, respectively.

Lemma 3.3. *The Weyl groups W and $W_{L'}$ are isomorphic to the semidirect product groups $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ and S_n, respectively, where S_n is the n-th symmetric group.*

Proof. Let $\mathfrak{a}_{\mathfrak{p}}$ be as in (2.3). The root system of $(\mathfrak{su}(n, \ n), \ \mathfrak{a}_{\mathfrak{p}})$ is Δ as in Lemma 3.1. Then $\gamma_1, \gamma_2, \cdots, \gamma_n$ is a family basis of $\mathfrak{a}_{\mathfrak{p}}$. Let $h = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n \in \mathfrak{a}_{\mathfrak{p}}$, where $x_i \in \mathbb{R}$ for $1 \leq i \leq n$. Define the reflections by

$$r_{2\gamma_i}(h) = h - \frac{2(2\gamma_i, h)}{(2\gamma_i, 2\gamma_i)} 2\gamma_i.$$

By (3.2), we see that

$$r_{2\gamma_i}(h) = h - 2x_i\gamma_i = \sum_{j\neq i} x_j\gamma_j - x_i\gamma_i.$$

Notice that $r_{2\gamma_i}r_{2\gamma_j} = r_{2\gamma_j}r_{2\gamma_i}$, we let W_1 be the subgroup of $GL(n,\mathbb{R})$ generated by the reflections $r_{2\gamma_i}$ for $1\leq i\leq n$. Then W_1 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Similarly, define the reflections by

$$r_{\gamma_i-\gamma_j}(h) = h - \frac{2(\gamma_i-\gamma_j,h)}{(\gamma_i-\gamma_j,\gamma_i-\gamma_j)}(\gamma_i-\gamma_j).$$

Then

$$r_{\gamma_i-\gamma_j}(h) = h-(x_i-x_j)(\gamma_i-\gamma_j) = \sum_{k\neq i,j} x_k\gamma_k + x_i\gamma_j + x_j\gamma_i.$$

Let W_2 be the subgroup of $GL(n,\mathbb{R})$ generated by the reflections $r_{\gamma_i-\gamma_j}$ for $i\neq j$. From this, it follows that W_2 is isomorphic to the symmetric group S_n. Since $r_{2\gamma_j}(\gamma_i-\gamma_j)=\gamma_i+\gamma_j$, so $r_{\gamma_i+\gamma_j}=r_{2\gamma_j}r_{\gamma_i-\gamma_j}r_{2\gamma_j}$,

$$r_{\gamma_i-\gamma_j}(2\gamma_k) = \begin{cases} 2\gamma_k, & \text{if } k\neq i,j,\\ 2\gamma_j, & \text{if } k=i,\\ 2\gamma_i, & \text{if } k=j,\end{cases} \qquad r_{\gamma_i-\gamma_j}r_{2\gamma_k}r_{\gamma_i-\gamma_j} = \begin{cases} r_{2\gamma_k}, & \text{if } k\neq i,j,\\ r_{2\gamma_j}, & \text{if } k=i,\\ r_{2\gamma_i}, & \text{if } k=j.\end{cases}$$

Let W be the Weyl group of Δ which is generated by W_1 and W_2. Then W is isomorphic to the semidirect product of $(\mathbb{Z}/2\mathbb{Z})^n\rtimes S_n$.

Moreover, let $\Delta_1 := \{\pm(\gamma_i-\gamma_j) : i\neq j\}$ be as in Proposition 3.2. Similar to the proof above, for $h=x_1\gamma_1+x_2\gamma_2+\cdots+x_n\gamma_n\in\mathfrak{a}_\mathfrak{p}$ and $x_i\in\mathbb{R}$ for $1\leq i\leq n$, we consider the reflections

$$r_{\gamma_i-\gamma_j}(h) = h - \frac{2(\gamma_i-\gamma_j,h)}{(\gamma_i-\gamma_j,\gamma_i-\gamma_j)}(\gamma_i-\gamma_j).$$

Then, by (3.9), we have

$$r_{\gamma_i-\gamma_j}(h) = h-(x_i-x_j)(\gamma_i-\gamma_j) = \sum_{k\neq i,j} x_k\gamma_k + x_i\gamma_j + x_j\gamma_i.$$

Hence $W_{L'}$ is isomorphic to S_n. □

Remark 3.4. It is worthwhile to discuss the quotient group $W/W_{L'}$. By using Lemma 3.3, we see that the quotient $W/W_{L'}$ has 2^n elements, which, together with [13, Theorem 3.3], implies that $SU(n,n)/H$ have 2^n open P-orbits.

The following lemma is just [21, Theorem 1].

Lemma 3.5. *Let P be a $\tau\theta$-minimal parabolic subgroup of $SU(n,n)$ as in (3.7). For $j=1,2,\dots,2^n$, we let $\mathcal{O}_j := Pm_jH$, where $m_j\in N_K(\mathfrak{a}_\mathfrak{p})$. Then the following holds:*

(i) *$\mathcal{O}_j$ is open in $SU(n,n)$.*
(ii) *$\mathcal{O}_i\cap\mathcal{O}_j=\emptyset, i\neq j$.*

(iii) $\bigoplus_{i=1}^{2^n} \mathcal{O}_i$ *is dense in* $SU(n, n)$*.*

In Remark 3.4, the open $P \times H$ double cosets on $SU(n, n)$ are given by Pm_jH for our fixed set $\mathcal{W}$ of representatives $m_j \in N_K(\mathfrak{a}_\mathfrak{p})$ for $W/W_{L'}$, where $j = 1, 2, \dots, 2^n$ (see [6, p. 521]). The map $\mathcal{W} \ni w \mapsto PwH$ sets up a bijective correspondence of $\mathcal{W}$ with $\bigoplus_{i=1}^{2^n} \mathcal{O}_i$.

3.3. Frobenius reciprocity. Since $L = MA_\mathfrak{p}$, then M is a unique τ-stable subgroup $M \subset L$, where L is as in (3.6) and $A_\mathfrak{p} = \exp \mathfrak{a}_\mathfrak{p}$. From the local structure theorem as in [19, Theorem 2.3], it follows that

$$P \cdot z_0 = MA_\mathfrak{p}N \cdot z_0, \tag{3.10}$$

where $N = \exp \mathfrak{n}$ and $z_0 = eH$ denotes the origin of the affine symmetric space of Hermitian type $SU(n, n)/H$. Given an irreducible unitary representation ξ of M, we denote it by $V(\xi)$. We have the formal orthogonal sum

$$V(\xi) = \bigoplus_{w \in \mathcal{W}} \mathcal{H}_\xi^{w(M \cap H)w^{-1}}$$

of the spaces of $w(M \cap H)w^{-1}$-fixed vectors for ξ (also see [6, (5)]). Note that conjugation by an element w from $N_K(\mathfrak{a}_\mathfrak{p})$ leaves M invariant, and that hence $M/(w(M \cap H)w^{-1}) = M/(M \cap wHw^{-1})$ is a symmetric space.

By using Lemma 3.5 and (3.10), we have the following definition.

Definition 3.6. Let $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$ be the dual of the complexification of $\mathfrak{a}_\mathfrak{p}$. Let $\rho_P \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$ be half the sum of positive roots. Let $C(\xi : \lambda)^H$ be the induced M-representation which is the set of all continuous functions $\eta \in V(\xi)$ satisfying, for $w \in \mathcal{W}$, $\lambda \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$ and $j = 1, 2, \dots, 2^n$,

$$\mathcal{K}_\lambda^j(\eta)(manwh) := \begin{cases} a^{\lambda+\rho_P}\xi(m)\eta_w, & \text{if } manwh \in \mathcal{O}_j, \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem is the Frobenius reciprocity.

Theorem 3.7. *Let* $V(\xi)$ *and* $C(\xi : \lambda)^H$ *be as above, representations denote by* ξ_1, ξ*, and* $\mathcal{H}_\xi$ *the representation of* $w(M \cap H)w^{-1}$*. Then there is a canonical isomorphism*

$$\mathrm{Hom}_M\left(V(\xi), C(\xi : \lambda)^H\right) \simeq \mathrm{Hom}_{w(M\cap H)w^{-1}}\left(\mathrm{Res}^M_{w(M\cap H)w^{-1}} V(\xi), \mathcal{H}_\xi\right),$$

where $\mathrm{Res}^M_{w(M\cap H)w^{-1}} V(\xi)$ *denotes the restriction of* $V(\xi)$ *to* $\mathcal{H}_\xi$*.*

Proof. By Definition 3.6 with $ma \in MA_\mathfrak{p} \cap wHw^{-1}$, [13, p. 140] and $\lambda \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$, the evaluation map $e_w : C(\xi : \lambda)^H \to \mathcal{H}_\xi$ given by $e_w(F) = F(w)$ for $F \in C(\xi : \lambda)^H$, satisfies, for all $m_0 \in w(M \cap H)w^{-1}$ and $F \in C(\xi : \lambda)^H$,

$$e_w(\xi(m_0)F) = (\xi(m_0)F)(w) = F(m_0w) = F(w) = e_w(F),$$

namely, e_w is a morphism of representation of $w(M\cap H)w^{-1}$. Let $V(\xi)$ be a representation of M and $\mathcal{K}_\lambda^j : V(\xi)\to C(\xi:\lambda)^H$ be a morphism of representation of M as in Definition 3.6. Then the composition $e_w\circ\mathcal{K}_\lambda^j$ is a morphism of $\mathrm{Res}^M_{w(M\cap H)w^{-1}}V(\xi)$ into $\mathcal{H}_\xi$. Denote the linear map $A : \mathcal{K}_\lambda^j\to e_w\circ\mathcal{K}_\lambda^j$ by

$$\mathrm{Hom}_M\left(V(\xi), C(\xi:\lambda)^H\right)\hookrightarrow \mathrm{Hom}_{w(M\cap H)w^{-1}}\left(\mathrm{Res}^M_{w(M\cap H)w^{-1}}V(\xi), \mathcal{H}_\xi\right).$$

Let $v\in V(\xi)$ and $\varphi : V(\xi)\to\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$ be a morphism of representation of $w(M\cap H)w^{-1}$. Then, we now consider the function $F_v : M\to\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$ given by $F_v(m)=\varphi(\xi(m)v)$ for any $m\in M$. First, for $m_0\in w(M\cap H)w^{-1}$ and any $m\in M$, we have

$$F_v(m_0m)=\varphi(\xi(m_0m)v)=\varphi(\xi(m_0)\xi(m)v)=\varphi(\xi(m)v)=F_v(m),$$

which implies that F_v is a function in $C(\xi:\lambda)^H$. Write the map $\mathcal{K}_\lambda^j : V(\xi)\to C(\xi:\lambda)^H$ given by $\mathcal{K}_\lambda^j(v)=F_v$. Clearly, for each $v, v_0\in V(\xi)$ and any $m\in M$,

$$F_{v+v_0}(m)=\varphi(\xi(m)(v+v_0))=\varphi(\xi(m)v)+\varphi(\xi(m)v_0)=F_v(m)+F_{v_0}(m).$$

Then $\mathcal{K}_\lambda^j(v+v_0)=\mathcal{K}_\lambda^j(v)+\mathcal{K}_\lambda^j(v_0)$. In addition, for any $m\in M$, $\alpha\in\mathbb{C}$ and $v\in V(\xi)$, $\mathcal{K}_\lambda^j(\alpha v)(m)=\alpha\varphi(\xi(m)v)=\alpha\mathcal{K}_\lambda^j(v)(m)$, which implies that $\mathcal{K}_\lambda^j(\alpha v)=\alpha\mathcal{K}_\lambda^j(v)$. From this, it follows that $\mathcal{K}_\lambda^j$ is a linear map $V(\xi)\hookrightarrow C(\xi:\lambda)^H$. Moreover, for all $m'\in M$,

$$\varphi(\xi(m')\xi(m)v)=\varphi(\xi(m'm)v)=\mathcal{K}_\lambda^j(v)(m'm).$$

And hence, for any $m\in M$ and $v\in V(\xi)$, $\mathcal{K}_\lambda^j(\xi(m)v)=\xi_1(m)\mathcal{K}_\lambda^j(v)$. Therefore, $\mathcal{K}_\lambda^j$ is a morphism of representations $V(\xi)$ and $C(\xi:\lambda)^H$ of M.

On the other hand, denote the map $B : \varphi\to\mathcal{K}_\lambda^j$ by

$$\mathrm{Hom}_{w(M\cap H)w^{-1}}(\mathrm{Res}^M_{w(M\cap H)w^{-1}}V(\xi), \mathcal{H}_\xi)\hookrightarrow\mathrm{Hom}_M(V(\xi), C(\xi:\lambda)^H).$$

Clearly, for any $\varphi\in\mathrm{Hom}_{w(M\cap H)w^{-1}}(\mathrm{Res}^M_{w(M\cap H)w^{-1}}V(\xi), \mathcal{H}_\xi)$, we obtain, for any $v\in V(\xi)$,

$$((A\circ B)(\varphi))(v)=(A(\mathcal{K}_\lambda^j))(v)=\mathcal{K}_\lambda^j(v)(w)=F_v(w)=\varphi(v).$$

Thus $A\circ B$ is the identity map. By this, for any $m\in M$, $\mathcal{K}_\lambda^j(\xi(m)v)(w)=\xi(m)(\mathcal{K}_\lambda^j(v))(w)$ and $\mathcal{K}_\lambda^j\in\mathrm{Hom}_M\left(V(\xi), C(\xi:\lambda)^H\right)$, we have

$$\begin{aligned}(((B\circ A)(\mathcal{K}_\lambda^j))(v))(mw)&=(B(A(\mathcal{K}_\lambda^j))(v))(mw)\\&=A(\mathcal{K}_\lambda^j)(\xi(m)v)\\&=\mathcal{K}_\lambda^j(v)(mw).\end{aligned}$$

Hence, we have $(B\circ A)(\mathcal{K}_\lambda^j) = \mathcal{K}_\lambda^j$ for all $\mathcal{K}_\lambda^j$ and $B\circ A$ is also the identity map. $\square$

Corollary 3.8. *Let $\mathcal{H}_\xi = \mathbb{C}^{\mathcal{W}}$ be the trivial one-dimensional representation of $w(M\cap H)w^{-1}$. Let $C^0(M/w(M\cap H)w^{-1}, \mathbb{C}^{\mathcal{W}})$ be the space of continuous functions $M/w(M\cap H)w^{-1} \to \mathbb{C}^{\mathcal{W}}$ with M-action given by left translation. Then*

$$C(\xi : \lambda)^H \simeq C^0(M/w(M\cap H)w^{-1}, \mathbb{C}^{\mathcal{W}}) \tag{3.11}$$

and the multiplicity of $V(\xi)$ in $C^0(M/w(M\cap H)w^{-1}, \mathbb{C}^{\mathcal{W}})$ is equal to 1.

Proof. By using Theorem 3.7, we see that (3.11) holds true. Moreover, we also have

$$\begin{aligned}&\dim \mathrm{Hom}_M \left(V(\xi), C(\xi : \lambda)^H\right)\\ &\quad = \dim \mathrm{Hom}_{w(M\cap H)w^{-1}} \left(\mathrm{Res}^M_{w(M\cap H)w^{-1}} V(\xi), \mathbb{C}^{\mathcal{W}}\right).\end{aligned}$$

Notice that $\mathrm{Res}^M_{w(M\cap H)w^{-1}} V(\xi) = \mathcal{H}_\xi^{w(M\cap H)w^{-1}}$. From this and Lemma 5.3 as in [13, p. 140], it follows that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{w(M\cap H)w^{-1}} \left(\mathcal{H}_\xi^{w(M\cap H)w^{-1}}, \mathbb{C}^{\mathcal{W}}\right) = 1.$$

This finishes the proof of Corollary 3.8. $\square$

Corollary 3.9. *For any $w \in \mathcal{W}$, if $V(\xi)$ is non-zero then the restriction of ξ to $M\cap K$ is irreducible and*

$$\mathrm{Res}^M_{w(M\cap H)w^{-1}} V(\xi) \simeq \mathrm{Res}^{M\cap K}_{w(M\cap H\cap K)w^{-1}} V(\xi).$$

Proof. By [6, Lemma 1] and [13, Lemma 3.2, p. 115], we see that $M/w(M\cap H)w^{-1}$ for $w \in \mathcal{W}$ is a compact symmetric space and

$$M/w(M\cap H)w^{-1} \simeq M\cap K/w(M\cap H\cap K)w^{-1},$$

which, together with [6, p, 522], implies that Corollary 3.9 holds. $\square$

4. Partial Poisson transform

By Theorem 3.7, Corollary 3.9 and [13, p. 161], we consider the principal series with the trivial $w(M\cap H\cap K)w^{-1}$-type 1 for H-invariant function on $\mathcal{X}$.

Definition 4.1. For all $m \in K\cap M, k \in K$, the spaces $C^\infty(K : \xi) = C^\infty(K : 1)$ and $C^{-\infty}(K : 1)$ are the set of all smooth and generalized $\mathbb{C}^{\mathcal{W}}$-valued functions on K, respectively, transforming according to $f(mk) = \xi(m)f(k) = f(k)$. The Hilbert space $L^2(K)$ is defined similarly and the inner product is given by

$$(f, g) = \int_K \langle f(k), g(k)\rangle dk$$

with respect to invariant measure on K.

Definition 4.2. Let ρ_P be as in Definition 3.6. The Poisson kernel is given by, for $w\in\mathcal{W}$, $\lambda\in\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$ and $j=1,2,\dots,2^n$,

$$p^j_\lambda(manwh):=\begin{cases} a^{\lambda+\rho_P}, & \text{if } manwh\in\mathcal{O}_j,\\ 0, & \text{otherwise.}\end{cases}$$

The following lemma is just [13, p. 145, Proposition 6.1].

Lemma 4.3. *For $j=1,2,\dots,2^n$, let p^j_λ be as in Definition 4.2. Suppose that $\langle Re\lambda+\rho_P,\alpha\rangle<0$, where $\alpha\in\Delta^+$ and Δ^+ is as in (3.1), then, for $\eta_w\in\mathbb{C}^{\mathcal{W}}$, $p^j_\lambda\eta_w$ as a $C^\infty(K:1)$-valued function of λ is a holomorphic function.*

Let $\mathbb{D}(\mathcal{X})$ be the commutative algebra of invariant differential operators on $\mathcal{X}$. Let $S(\mathfrak{a}_\mathfrak{p})^W$ be the W-invariant symmetric algebra. By [13, Lemma 4.6, p. 130], we have an algebra homomorphism γ_p of $\mathbb{D}(\mathcal{X})$ into $S(\mathfrak{a}_\mathfrak{p})^W$. This is independence of the choice of Δ^+ as in (3.1).

By using Definition 4.2 and Lemma 4.3, for any $x\in\mathcal{X}$, $\langle Re\lambda+\rho_P,\alpha\rangle<0$ and any $f\in C^\infty(K:1)$, we now define the Poisson transform $\mathcal{P}^j_\lambda$ as follows:

$$\mathcal{P}^j_\lambda f(x)=\int_K f(k)p^j_{-\lambda}(x^{-1}k)\,dk, \tag{4.1}$$

where p^j_λ for $j=1,2,\dots,2^n$ is as in Definition 4.2.

The following lemma is just [13, Proposition 7.4, p. 162].

Lemma 4.4. *For any $D\in\mathbb{D}(\mathcal{X})$, $\lambda\in\mathfrak{a}^*_{\mathfrak{p},c}$ and $f\in C^\infty(K:1)$, $D\mathcal{P}^j_\lambda f=\gamma^j_\mathfrak{p}(D,\lambda)\mathcal{P}^j_\lambda f$, where $\mathcal{P}^j_\lambda f$ is as in (4.1) for $j=1,2,\dots,2^n$, and $\gamma^j_\mathfrak{p}(D,\lambda)$ denotes the eigenvalue of D.*

Definition 4.5. For $\lambda\in\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$, we define $\mathcal{A}(\mathcal{X},\mathcal{M}_\lambda)$ to be the space of functions $\mathcal{P}^j_\lambda f\in C^\infty(\mathcal{X})$ satisfying the system of differential equation: $\mathcal{M}_{\lambda,j}:\ D\mathcal{P}^j_\lambda f=\gamma^j_\mathfrak{p}(D,\lambda)\mathcal{P}^j_\lambda f$, where D is as in Lemma 4.4.

Theorem 4.6. *The Poisson transform $\mathcal{P}^j_\lambda$ for $j=1,2,\dots,2^n$ is a continuous $SU(n,n)$-homomorphism from $C^\infty(K:1)$ to $\mathcal{A}(\mathcal{X},\mathcal{M}_{\lambda,j})$.*

Proof. For any $f\in C^\infty(K:1)$, let $\mathcal{P}^j_\lambda f$ be as above. Then, by [21, Theorem 1.2, Lemma 1.3] and [24, Lemma 7.6.6], we have

$$\mathcal{P}^j_\lambda f(x)=\int_K f(xk)p^j_{-\lambda}(k)\,dk. \tag{4.2}$$

In fact, let $\psi(k)=f(xk)p^j_{-\lambda}(k)$ and $\psi(mk)=\psi(k)$. Then,

$$\int_K\psi(k)\,dk=\int_K\psi(\kappa(x^{-1}k))\exp(-2\rho_P(\mathbf{a}(x^{-1}k)))\,dk,$$

where $\kappa(x^{-1}k) \in K$ and $\mathbf{a}$ is the Iwasawa projection $\mathbf{a} : xK \to \mathfrak{a}_{\mathfrak{p}}$. From this,

$$\psi(\kappa(x^{-1}k)) = f(x\kappa(x^{-1}k))p^{j}_{-\lambda}(\kappa(x^{-1}k)),$$

the change of variables $x\kappa(x^{-1}k) \to \kappa(x^{-1}k)$ and the M-invariant (left action) of ψ on K, it follows that

$$\begin{aligned}\int_K & f(x\kappa(x^{-1}k))p^{j}_{-\lambda}(\kappa(x^{-1}k)) \exp(-2\rho_P(\mathbf{a}(x^{-1}k)))\,dk \\ &= \int_K f(\kappa(x^{-1}k))p^{j}_{-\lambda}(x^{-1}\kappa(x^{-1}k))\,dk,\end{aligned}$$

which implies that (4.2) holds true. And it means that $\mathcal{P}^{j}_{\lambda}$ is an $SU(n,n)$-homomorphism.

We prove the continuous of $SU(n,n)$-map. Let $P, \bar{P}$ be two minimal $\tau\theta$-stable parabolic subgroups. If, for any $\alpha \in \Delta^+$, there is a constant $C \geq 0$ such that $\langle \mathrm{Re}\lambda, \alpha\rangle > C$, then, for any $f \in C^\infty(K : 1)$, $A(\bar{P} : P : 1 : \lambda)f(g) = \int_{\bar{N}} f(g\bar{n})\,d\bar{n}$ converges absolutely (see [13, Proposition 5.5, p. 144]). From this, [13, Theorem 5.6, p. 144] and $A(\bar{P} : P : 1 : \lambda)f$ extends to a meromorphic $C^\infty(K : 1)$-valued function of λ in $\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$, it follows that $A(\bar{P} : P : 1 : \lambda)$ is a continuous intertwining operator from $C^{-\infty}(P : 1 : \lambda)$ to $C^{-\infty}(\bar{P} : 1 : \lambda)$. Then, the normalization of $\mathcal{P}^{j}_{\lambda}f$ is given by

$$_0\mathcal{P}^{j}_{\lambda}f(gH) = \int_K f(k)\,{}_0p^{j}_{-\lambda}(g^{-1}k)dk,$$

where ${}_0p^{j}_{-\lambda} = c^{-1}(\lambda)p^{j}_{-\lambda}$ for some meromorphic function $c^{-1}(\lambda)$. Now we calculate the constant $c^{-1}(\lambda)$ via the asymptotic of the properties $\mathcal{P}^{j}_{\lambda}f$, where $\mathcal{P}^{j}_{\lambda}f$ is as in (4.1). Let $\lambda \in \mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$ with $\mathrm{Re}\lambda$ strictly dominant and $w \in \mathcal{W}$. From [10, Section 6], [2, p. 316] and [3, pp. 276-278], it follows that

$$\lim_{a\to\infty} a^{\lambda-\rho_P}\mathcal{P}^{j}_{\lambda}f(aw) = c(\lambda)f(w), \lambda = (\lambda_1, \cdots, \lambda_n) \mapsto \sum_j \lambda_j\gamma_j, \tag{4.3}$$

where

$$c(\lambda) = \prod_j \frac{\Gamma(\lambda_j + \frac{1}{2})}{\Gamma(\lambda_j + 1)} \prod_{i<j}(\lambda_j^2 - \lambda_i^2)^{-1}.$$

Moreover, by using [13, Proposition 7.7, p. 167], we have

$$\lim_{a\to\infty} a^{\lambda-\rho_P}\,{}_0\mathcal{P}^{j}_{\lambda}f(aw) = f(w),$$

which, together with (4.3), implies that $\mathcal{P}^{j}_{\lambda}f = c(\lambda)_0\mathcal{P}^{j}_{\lambda}f$. And hence ${}_0p^{j}_{-\lambda} = c^{-1}(\lambda)p^{j}_{-\lambda}$. This finishes the proof of Theorem 4.6. □

The Fourier transform of $f\in C_c^\infty(\mathcal{X})$ is given by, for $j=1,2,\dots,2^n$,

$$\widehat{f}^j(1:\lambda)(\eta)(k)=\int_{\mathcal{X}} f(x)\,{}_0\mathcal{K}^j_{-\lambda}(\eta)(x^{-1}k)dx, \tag{4.4}$$

where ${}_0\mathcal{K}^j_{-\lambda}(\eta)=c(\lambda)\mathcal{K}^j_{-\lambda}(\eta)$ and $\mathcal{K}^j_{-\lambda}(\eta)$ is as in Definition 3.6.

Theorem 4.7. *Let* $\mathcal{E}:=\left\{\lambda\in\mathfrak{a}^*_{\mathfrak{p},c}:\ \forall\alpha\in\Delta^+,\ \mathrm{Re}(\lambda+\rho_P,\alpha)<0\right\}$. *Let* $\lambda\in\mathcal{E}$. *Then, for* $w\in\mathcal{W}$, $j=1,2,\dots,2^n$,

$$\int_K \widehat{1}^j(1:\lambda)(\eta)(k)dk=c(\lambda)\int_{\mathcal{X}}\mathcal{P}^j_\lambda\eta_w(y)dy.$$

Proof. From (4.4), it follows that

$$\int_K \widehat{1}^j(1:\lambda)(\eta)(k)\,dk=\int_K\left[\int_{\mathcal{X}} {}_0\mathcal{K}^j_{-\lambda}(\eta)(y^{-1}k)dy\right]dk.$$

Notice that, as the function of variables $y\mapsto {}_0\mathcal{K}^j_{-\lambda}(\eta)(y^{-1}k)$ is continuous with compact support it is integrable. By Lemma 4.3 and Fubini's theorem, we have

$$\begin{aligned}\int_K\int_{\mathcal{X}} {}_0\mathcal{K}^j_{-\lambda}(\eta)(y^{-1}k)\,dydk&=\int_{\mathcal{X}}\int_K {}_0\mathcal{K}^j_{-\lambda}(\eta)(y^{-1}k)\,dkdy\\&=c(\lambda)\int_{\mathcal{X}}\mathcal{P}^j_\lambda\eta_w(y)dy,\end{aligned}$$

this completes that proof of Theorem 4.7. □

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