

Split metacyclic actions on surfaces

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ABSTRACT. Let $\text{Mod}(S_g)$ be the mapping class group of the closed orientable surface S_g of genus $g \geq 2$. In this paper, we derive necessary and sufficient conditions under which two torsion elements in $\text{Mod}(S_g)$ will have conjugates that generate a non-abelian finite split metacyclic subgroup of $\text{Mod}(S_g)$. As applications of the main result, we give a complete characterization of the finite dihedral and the generalized quaternionic subgroups of $\text{Mod}(S_g)$ up to a certain equivalence that we will call weak conjugacy. Furthermore, we show that any finite-order mapping class whose corresponding orbifold is a sphere has a conjugate that lifts under certain finite-sheeted regular cyclic covers of S_g . Moreover, for $g \geq 5$, we show the existence of an infinite dihedral subgroup of $\text{Mod}(S_g)$ that is generated by an involution and a root of a bounding pair map of degree 3. Finally, we provide a complete classification of the weak conjugacy classes of the non-abelian finite split metacyclic subgroups of $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$. We also describe nontrivial geometric realizations of some of these actions.

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1. Introduction

Let S_g be the closed orientable surface of genus $g \geq 0$, $\text{Homeo}^+(S_g)$ be the group of orientation-preserving homeomorphisms of S_g , and let $\text{Mod}(S_g)$ be the

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mapping class group of S_g . Given $F, G \in \text{Mod}(S_g)$ of finite order, a pair of conjugates F', G' (of F, G resp.) may (or may not) generate a subgroup isomorphic to $\langle F, G \rangle$. For example, consider the periodic mapping classes $F, G \in \text{Mod}(S_7)$ represented by homeomorphisms $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_7)$ (see [23] for details), as shown in the first subfigure of Figure 1 below. From Figure 1, it is apparent that

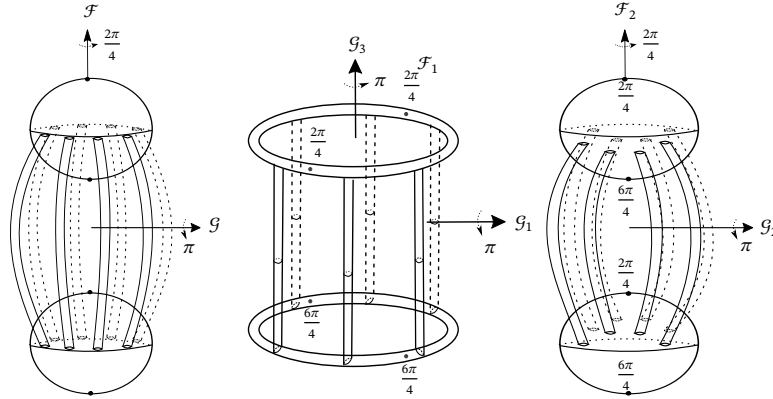


FIGURE 1. Split metacyclic subgroups of $\text{Mod}(S_7)$ with conjugate generators.

$\langle F, G \rangle \cong D_8$ (i.e. the dihedral group of order 8). For $1 \leq i \leq 3$, we consider the conjugates G_i of G , represented by the $\mathcal{G}_i \in \text{Homeo}^+(S_7)$ and for $1 \leq j \leq 2$, we consider the conjugates \mathcal{F}_j of \mathcal{F} indicated in the (second and third) subfigures. In the second subfigure, we have marked the fixed points of a conjugate \mathcal{F}_1 of \mathcal{F} (with the same local rotation angles as \mathcal{F}). Also, note that the third subfigure is different from the first (as an imbedding $S_7 \hookrightarrow \mathbb{R}^3$), since it has four pairs of tubes connecting the spheres, where in each pair, the tubes are aligned one behind the other. As it turns out, $\langle F_1, G_1 \rangle \cong \langle F_2, G_2 \rangle \cong D_8$, but since \mathcal{F}_1 and \mathcal{G}_3 commute, we have $\langle F_1, G_3 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Considering that the finite abelian subgroups of $\text{Mod}(S_g)$ have been extensively studied [9, 11, 14, 20], this example motivates the following natural question: Given $F', G' \in \text{Mod}(S_g)$ of orders n, m respectively, can one derive equivalent conditions under which there exist conjugates F, G (of F', G' resp.) such that $\langle F, G \rangle$ is a finite non-abelian *split metacyclic subgroup of order $m \cdot n$ and twist factor k* admitting the presentation

$$\langle F, G \mid F^n = G^m = 1, G^{-1}FG = F^k \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m?$$

The main result in this paper answers this question in the affirmative (see Theorem 3.3). This result is a generalization of an analogous result from [11] for two-generator finite abelian subgroups.

Given a finite split (non-abelian) metacyclic subgroup $H = \langle F, G \rangle$ of $\text{Mod}(S_g)$ as above, the Nielsen realization theorem [17, 22] asserts that we may also view H as a subgroup of $\text{Homeo}^+(S_g)$ with an *associated H -action* on S_g inducing the branched cover $S_g \rightarrow S_g/H$. Consider representatives $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$ of $F, G \in \text{Mod}(S_g)$ (resp.) with the same orders. Given a branched cover $S_g \rightarrow$

$S_g/\langle \mathcal{F} \rangle (= X)$ and a $\bar{G} \in \text{Mod}(X)$ that lifts under this cover to a $G \in \text{Mod}(S_g)$, it follows from Birman-Hilden theory [2, 3, 4, 5] that there is an exact sequence:

$$1 \rightarrow \langle F \rangle \rightarrow \langle F, G \rangle \rightarrow \langle \bar{G} \rangle \rightarrow 1. \quad (\dagger)$$

A key ingredient in the proof of the main result is the derivation of elementary number-theoretic conditions under which such a \bar{G} will have a conjugate that lifts so that the sequence (\dagger) splits (see Section 3). The proof integrates ideas from the theory of group actions on surfaces [17, 19] with elements of Thurston’s orbifold theory [26, Chapter 13]. Another crucial aspect of the proof (of the main result) is the analysis of the geometric properties of the automorphism $\bar{\mathcal{G}}$ induced by \mathcal{G} on $S_g/\langle \mathcal{F} \rangle$.

In Section 4, we provide several applications of our main theorem. The first application concerns the finite dihedral subgroups of $\text{Mod}(S_g)$. Let $D_{2n} = \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_2$ be the dihedral group of order $2n$. We derive the following characterization of dihedral subgroups of $\text{Mod}(S_g)$ in Subsection 4.1 (see Proposition 4.2).

Proposition 1. *Let $F \in \text{Mod}(S_g)$ be of order n . Then there exists an involution $G \in \text{Mod}(S_g)$ such that $\langle F, G \rangle \cong D_{2n}$ if and only if F and F^{-1} are conjugate in $\text{Mod}(S_g)$.*

It is worth mentioning here that dihedral actions on Riemann surfaces have been classified in [10].

For $n \geq 2$, the generalized quaternion group $Q_{2^{n+1}}$ is a metacyclic group of order 2^{n+1} that admits the presentation

$$\langle F, G \mid F^{2^n} = G^4 = 1, F^{2^{n-1}} = G^2, G^{-1}FG = F^{-1} \rangle.$$

In Subsection 4.2, we obtain the following characterization of generalized quaternionic actions on S_g (see Proposition 4.7).

Proposition 2. *For $g \geq 2$, let $F \in \text{Mod}(S_g)$ be of order 2^n . Then there exists a $G \in \text{Mod}(S_g)$ such that $\langle F, G \rangle \cong Q_{2^{n+1}}$ if and only if the $\langle \mathcal{F}, \mathcal{G} \rangle$ -action on S_g lifts to a $(\langle \tilde{\mathcal{F}}, \tilde{\mathcal{G}} \rangle \cong \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4)$ -action on S_{2g-1} under the 2-sheeted regular cyclic cover $S_{2g-1} \rightarrow S_g$ with deck transformation group $\langle \tilde{\mathcal{G}}^2 \tilde{\mathcal{F}}^{2^{n-1}} \rangle$.*

For a periodic mapping class $F \in \text{Mod}(S_g)$, the corresponding orbifold $\mathcal{O}_{\langle \mathcal{F} \rangle} := S_g/\langle \mathcal{F} \rangle \approx S_{g_0,r}$, where $S_{g_0,r}$ is the surface of genus $g_0 \geq 0$ with $r \geq 0$ marked points. It is known [13] that F is irreducible if and only if $\mathcal{O}_{\langle \mathcal{F} \rangle} \approx S_{0,3}$. In Subsection 4.3, we provide a characterization of the split metacyclic subgroups $\langle F, G \rangle$ of $\text{Mod}(S_g)$ when F is irreducible (see Corollary 4.11).

Let $\text{LMod}_p(S_g)$ be the liftable mapping class group of a finite n -sheeted regular cyclic cover $p : S_{n(g-1)+1} \rightarrow S_g$ with deck transformation group $\mathbb{Z}_n = \langle \mathcal{F} \rangle$, and let $\text{SMod}_p(S_{n(g-1)+1})$ be the symmetric mapping class group of p (see [21]), which in this case turns out to be the normalizer of $\langle F \rangle$ in $\text{Mod}(S_{n(g-1)+1})$. In this context, we have the following result.

Proposition 3. *For $g, n \geq 2$, let $p : S_{n(g-1)+1} \rightarrow S_g$ be a regular cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Then any involution $G' \in \text{Mod}(S_g)$ has a*

conjugate $G \in \text{LMod}_p(S_g)$ with a lift $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong D_{2n}$.

Moreover, we provide sufficient conditions for the liftability of a periodic mapping class (under p) whose corresponding orbifold is a sphere (see Propositions 4.15 - 4.16). As a consequence, we obtain the following corollary.

Corollary 1. *For $g \geq 2$ and prime n , let $p : S_{n(g-1)+1} \rightarrow S_g$ be a regular n -sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Let $G' \in \text{Mod}(S_g)$ be of order m such that the genus of $\mathcal{O}_{\langle G' \rangle}$ is zero. Then G' has a conjugate $G \in \text{LMod}_p(S_g)$ with a lift $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if there exists $k \in \mathbb{Z}_n^\times$ such that $|k| = m$.*

Consider an infinite metacyclic group [15] that admits a presentation of the form

$$\langle x, y \mid y^{2m} = 1, y^{-1}xy = x^{-1} \rangle.$$

When $m = 1$, we call such a group an *infinite dihedral group*. By a *root of a mapping class* $F \in \text{Mod}(S_g)$ of degree n , we mean a $G \in \text{Mod}(S_g)$ such that $G^n = F$. In Subsection 4.4, we use the theory developed in [24, 25] to construct roots of multitwists (i.e. products of powers of commuting Dehn twists) in $\text{Mod}(S_g)$ which together with certain mapping classes of order $2m$ generate infinite split metacyclic subgroups of $\text{Mod}(S_g)$ (of the form described above) for $g \geq 5$ (see Proposition 4.21). In particular, for $m = 1$, we have the following corollary.

Corollary 2. *For $g \geq 5$, there exists an infinite dihedral subgroup of $\text{Mod}(S_g)$ that is generated by an involution and a root of a bounding pair map of degree 3.*

In Section 5, we classify the non-abelian finite split metacyclic subgroups of $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$ up to a certain weaker notion of conjugacy that we call *weak conjugacy* (see Definition 2.10), which arises naturally in our setting. It may be noted that similar classifications for $2 \leq g \leq 4$ can also be obtained through the techniques developed in [6, 8, 18]. Finally, we apply the results in [23] to provide an algorithm for determining the hyperbolic structures that realize split metacyclic subgroups as groups of isometries. We conclude the paper by giving nontrivial geometric realizations of some non-abelian finite split metacyclic subgroups of $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$.

2. Preliminaries

2.1. Fuchsian groups. We let $\text{Homeo}^+(S_g)$ denote the group of orientation-preserving homeomorphisms of S_g , and let $H < \text{Homeo}^+(S_g)$ be a finite group. A faithful and properly discontinuous H -action on S_g induces a branched covering

$$S_g \rightarrow \mathcal{O}_H := S_g/H$$

with ℓ cone points x_1, \dots, x_ℓ on the quotient orbifold $\mathcal{O}_H \approx S_{g_0}$ (which we will call the *corresponding orbifold*) of orders n_1, \dots, n_ℓ , respectively. Then the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{O}_H)$ of \mathcal{O}_H has a presentation given by

$$\left\langle \alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \dots, \xi_\ell \mid \xi_1^{n_1}, \dots, \xi_\ell^{n_\ell}, \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \right\rangle. \quad (1)$$

In classical parlance, $\pi_1^{\text{orb}}(\mathcal{O}_H)$ is also known as a *Fuchsian group* [16, 19] with signature

$$\Gamma(\mathcal{O}_H) := (g_0; n_1, \dots, n_\ell),$$

and the relation $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i]$ appearing in its presentation is called the *long relation*. From Thurston’s orbifold theory [26, Chapter 13], we obtain an exact sequence

$$1 \rightarrow \pi_1(S_g) \rightarrow \pi_1^{\text{orb}}(\mathcal{O}_H) \xrightarrow{\phi_H} H \rightarrow 1. \quad (*)$$

In this context, we will require the following result due to Harvey [14].

Lemma 2.1. *A finite group H acts faithfully on S_g with $\Gamma(\mathcal{O}_H) = (g_0; n_1, \dots, n_\ell)$ if and only if it satisfies the following two conditions:*

- (i) $\frac{2g-2}{|H|} = 2g_0 - 2 + \sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right)$, and
- (ii) *there exists a surjective homomorphism $\phi_H : \pi_1^{\text{orb}}(\mathcal{O}_H) \rightarrow H$ that preserves the orders of all torsion elements of $\pi_1^{\text{orb}}(\mathcal{O}_H)$.*

2.2. Cyclic actions on surfaces. For $g \geq 1$, let $F \in \text{Mod}(S_g)$ be of order n . The Nielsen-Kerckhoff theorem [17, 22] asserts that F is represented by a *standard representative* $\mathcal{F} \in \text{Homeo}^+(S_g)$ of the same order. We refer to both \mathcal{F} and the group it generates, interchangeably, as a \mathbb{Z}_n -*action on S_g* . Each cone point $x_i \in \mathcal{O}_{\langle \mathcal{F} \rangle}$ lifts to an orbit of size n/n_i on S_g , and the local rotation induced by \mathcal{F} around the points in each orbit is given by $2\pi c_i^{-1}/n_i$, where $\text{gcd}(c_i, n_i) = 1$ and $c_i c_i^{-1} \equiv 1 \pmod{n_i}$. Further, it is known (see [14] and the references therein) that the exact sequence in (*) takes the following form

$$1 \rightarrow \pi_1(S_g) \rightarrow \pi_1^{\text{orb}}(\mathcal{O}_{\langle \mathcal{F} \rangle}) \xrightarrow{\phi_{\langle \mathcal{F} \rangle}} \langle \mathcal{F} \rangle \rightarrow 1,$$

where $\phi_{\langle \mathcal{F} \rangle}(\xi_i) = \mathcal{F}^{(n/n_i)c_i}$, for $1 \leq i \leq \ell$. We will now introduce a tuple of integers that encodes the conjugacy class of a \mathbb{Z}_n -action on S_g .

Definition 2.2. *A data set of degree n is a tuple*

$$D = (n, g_0, r; (c_1, n_1), \dots, (c_\ell, n_\ell)),$$

where $n \geq 2$, $g_0 \geq 0$, and $0 \leq r \leq n - 1$ are integers, and each $c_i \in \mathbb{Z}_{n_i}^\times$ such that:

- (i) $r > 0$ if and only if $\ell = 0$ and $\text{gcd}(r, n) = 1$, whenever $r > 0$,
- (ii) each $n_i \mid n$,

(iii) $\text{lcm}(n_1, \dots, \hat{n}_i, \dots, n_\ell) = N$, for $1 \leq i \leq \ell$, where $N = n$ if $g_0 = 0$, and

$$(iv) \sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \pmod{n}.$$

The number g determined by the Riemann-Hurwitz equation

$$\frac{2-2g}{n} = 2-2g_0 + \sum_{j=1}^{\ell} \left(\frac{1}{n_j} - 1 \right) \quad (2)$$

is called the *genus* of the data set, denoted by $g(D)$.

Note that quantity r (in Definition 2.2) will be non-zero if and only if D represents a free rotation of S_g by $2\pi r/n$, in which case, D will take the form $(n, g_0, r;)$. We will not include r in the notation of a data set, whenever $r = 0$.

By the Nielsen-Kerckhoff theorem, the canonical projection $\text{Homeo}^+(S_g) \rightarrow \text{Mod}(S_g)$ induces a bijective correspondence between the conjugacy classes of finite-order maps in $\text{Homeo}^+(S_g)$ and the conjugacy classes of finite-order mapping classes in $\text{Mod}(S_g)$. This leads us to the following lemma (that follows from [25, Theorem 3.8] and [14]), which allows us to use data sets to describe the conjugacy classes of cyclic actions on S_g .

Lemma 2.3. *For $g \geq 1$ and $n \geq 2$, data sets of degree n and genus g correspond to conjugacy classes of \mathbb{Z}_n -actions on S_g .*

We will denote the data set corresponding to the conjugacy class of a periodic mapping class F by D_F . For compactness of notation, we also write a data set D (as in Definition 2.2) as

$$D = (n, g_0, r; ((d_1, m_1), \alpha_1), \dots, ((d_{\ell'}, m_{\ell'}), \alpha_{\ell'})),$$

where (d_i, m_i) are the distinct pairs in the multiset $S = \{(c_1, n_1), \dots, (c_\ell, n_\ell)\}$, and the α_i denote the multiplicity of the pair (d_i, m_i) in the multiset S . Further, we note that every cone point $[x] \in \mathcal{O}_{(\mathcal{F})}$ corresponds to a unique pair in the multiset S appearing in D_F , which we denote by $\mathcal{P}_x := (c_x, n_x)$.

Given $u \in \mathbb{Z}_m^\times$ and $\mathcal{G} \in H \leq \text{Homeo}^+(S_g)$ be of order m , let $\mathbb{F}_{\mathcal{G}}(u, m)$ denote the set of fixed points of \mathcal{G} with induced rotation angle $2\pi u/m$. Let $C_H(\mathcal{G})$ be the centralizer of $\mathcal{G} \in H$ and \sim denote the conjugation relation between any two elements in H . We conclude this subsection by stating the following result from the theory of Riemann surfaces [7], which we will use in the proof of our main theorem.

Lemma 2.4. *Let $H < \text{Homeo}^+(S_g)$ of finite order with $\Gamma(\mathcal{O}_H) = (g_0; n_1, \dots, n_\ell)$, and let $\mathcal{G} \in H$ be of order m . Then for $u \in \mathbb{Z}_m^\times$, we have*

$$|\mathbb{F}_{\mathcal{G}}(u, m)| = |C_H(\mathcal{G})| \cdot \sum_{\substack{1 \leq i \leq \ell \\ m | n_i \\ \mathcal{G} \sim \phi_H(\xi_i)^{n_i u/m}}} \frac{1}{n_i}.$$

2.3. Hyperbolic structures realizing cyclic actions. For a finite subgroup $H < \text{Mod}(S_g)$, let $\text{Fix}(H)$ denote the subspace of fixed points in the Teichmüller space $\text{Teich}(S_g)$ under the action of H . When H is cyclic, a method for constructing the hyperbolic metrics representing the points in $\text{Fix}(H)$ was described in [1] and [23], thereby yielding explicit solutions to the Nielsen realization problem [17, 22]. This method involved the construction of an arbitrary periodic element in $\text{Mod}(S_g)$ (that is not realizable as a rotation of S_g) by the “compatibilities” of irreducible periodic components, which are uniquely realized as rotations of certain special hyperbolic polygons with side-pairings.

A mapping class that is not reducible is called *irreducible*. Let $F \in \text{Mod}(S_g)$ be of order n . Gilman [13] showed that F is irreducible if and only if $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle})$ has the form $(0; n_1, n_2, n_3)$ (i.e. the quotient orbifold $\mathcal{O}_{\langle \mathcal{F} \rangle}$ is a sphere with three cone points.) Following the nomenclature in [1, 23], F is *rotational* if \mathcal{F} is either of order 2, or \mathcal{F} has at most 2 fixed points. A non-rotational F is said to be of *Type 1* if $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}) = (g_0; n_1, n_2, n)$, otherwise, it is called a *Type 2* action. The following result describes the unique hyperbolic structure that realizes an irreducible Type 1 action.

Theorem 2.5. *For $g \geq 2$, consider a irreducible Type 1 action $F \in \text{Mod}(S_g)$ with*

$$D_F = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).$$

Then F can be realized explicitly as the rotation $\theta_F = \frac{2\pi c_3^{-1}}{n}$ of a hyperbolic polygon \mathcal{P}_F with a suitable side-pairing $W(\mathcal{P}_F)$, where \mathcal{P}_F is a hyperbolic $k(F)$ -gon with

$$k(F) := \begin{cases} 2n, & \text{if } n_1, n_2 \neq 2, \text{ and} \\ n, & \text{otherwise,} \end{cases}$$

and for $0 \leq m \leq n - 1$,

$$W(\mathcal{P}_F) = \begin{cases} \prod_{i=1}^n a_{2i-1} a_{2i} \text{ with } a_{2m+1}^{-1} \sim a_{2z}, & \text{if } k(F) = 2n, \text{ and} \\ \prod_{i=1}^n a_i \text{ with } a_{m+1}^{-1} \sim a_z, & \text{otherwise,} \end{cases}$$

where $z \equiv m + qj \pmod{n}$ with $q = (n/n_2)c_3^{-1}$ and $j = n_2 - c_2$.

Further, it was shown [23] that the process of realizing an arbitrary non-rotational action F of order n using these unique hyperbolic structures realizing irreducible Type 1 components involved two broad types of processes.

- (a) *k-compatibility.* In this process, for $i = 1, 2$, we take a pair of irreducible Type 1 mapping classes $F_i \in \text{Mod}(S_{g_i})$ such that the $\langle \mathcal{F}_i \rangle$ -action on S_{g_i} induces a pair of *compatible orbits* of size k (where the induced local rotation angles add upto 0 modulo 2π). We remove (cyclically permuted) $\langle \mathcal{F}_i \rangle$ -invariant disks around points in the compatible orbits and then identify the resulting boundary components realizing a periodic mapping class

$F \in \text{Mod}(S_{g_1+g_2+k-1})$. An analogous construction can also be performed using a pair of orbits induced by a single $\langle \mathcal{F}' \rangle$ -action on S_g to realize a periodic mapping class $F \in \text{Mod}(S_{g+k})$.

- (b) *Permutation additions and deletions.* The *addition of a permutation component* involves the removal of (cyclically permuted) invariant disks around points in an orbit of size n induced by an $\langle \mathcal{F} \rangle$ -action on S_g and then pasting n copies of $S_{g'}^1$ (i.e. $S_{g'}$ with one boundary component) to the resultant boundary components. This realizes an action on $S_{g+ng'}$ with the same fixed point and orbit data as F . The reversal of this process is called a *permutation deletion*.

Thus, in summary, we have the following:

Theorem 2.6. [23, Theorem 2.24] *For $g \geq 2$, a non-rotational periodic mapping class in $\text{Mod}(S_g)$ can be realized through finitely many k -compatibilities, permutation additions, and permutation deletions on the unique structures of type \mathcal{P}_F realizing irreducible Type 1 mapping classes.*

A final but yet vital ingredient in the realization of split metacyclic actions is the following elementary lemma, which is a direct generalization of [11, Lemma 6.1].

Lemma 2.7. *Let $H = \langle F, G \rangle$ be a finite metacyclic subgroup of $\text{Mod}(S_g)$. Then*

$$\text{Fix}(H) = \text{Fix}(\langle F \rangle) \cap \text{Fix}(\langle G \rangle).$$

2.4. Split metacyclic actions on surfaces. Given integers $m, n \geq 2$, and $k \in \mathbb{Z}_n^\times$ such that $k^m \equiv 1 \pmod{n}$, a *finite split metacyclic action of order mn* (written as $m \cdot n$) on S_g is a tuple $(H, (\mathcal{G}, \mathcal{F}))$, where $H < \text{Homeo}^+(S_g)$, and

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^m = 1, \mathcal{G}^{-1} \mathcal{F} \mathcal{G} = \mathcal{F}^k \rangle.$$

The multiplicative class k will be called the *twist factor* of the split metacyclic action $(H, (\mathcal{G}, \mathcal{F}))$. As we are only interested in non-abelian split metacyclic subgroups, we will assume from here on that $k \neq 1$. Note that in classical notation $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$. As $\langle \mathcal{F} \rangle \triangleleft H$, it is known [7, 27] that \mathcal{G} would induce a $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_{\langle \mathcal{F} \rangle})$ that preserves the set of cone points in $\mathcal{O}_{\langle \mathcal{F} \rangle}$ along with their orders. We will call $\bar{\mathcal{G}}$, *the induced automorphism on $\mathcal{O}_{\langle \mathcal{F} \rangle}$* by \mathcal{G} , and we formalize this notion in the following definition.

Definition 2.8. Let $H < \text{Homeo}^+(S_g)$ be a finite cyclic group with $|H| = n$. We say a $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_H)$ is an *automorphism of \mathcal{O}_H* if for $[x], [y] \in \mathcal{O}_H$, $k \in \mathbb{Z}_n^\times$ and $\bar{\mathcal{G}}([x]) = [y]$, we have:

- (i) $n_x = n_y$, and
- (ii) $c_x = kc_y$.

We denote the group of automorphisms of \mathcal{O}_H by $\text{Aut}_k(\mathcal{O}_H)$.

We note that the concept of an induced orbifold automorphism in Definition 2.8 is more general than the one that was used in the abelian case ([11]), which

required a more rigid condition that $c_x = c_y$. The following lemma, which provides some basic properties of the induced map $\bar{\mathcal{G}}$, is a split metacyclic analog of [11, Lemma 3.1].

Lemma 2.9. *Let $\mathcal{G}, \mathcal{F} \in \text{Homeo}^+(S_g)$ be maps of orders m, n , respectively, such that $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$, and let $H = \langle \mathcal{F} \rangle$. Then:*

(i) \mathcal{G} induces a $\bar{\mathcal{G}} \in \text{Aut}_k(\mathcal{O}_H)$ such that

$$\mathcal{O}_H / \langle \bar{\mathcal{G}} \rangle = S_g / \langle \mathcal{F}, \mathcal{G} \rangle,$$

(ii) $|\bar{\mathcal{G}}|$ divides $|\mathcal{G}|$, and

(iii) $|\bar{\mathcal{G}}| < m$ if and only if $\mathcal{F}^l = \mathcal{G}^u$, for some $0 < l < n$ and $0 < u < m$.

We will now formalize the notion of weak conjugacy from Section 1.

Definition 2.10. We say that two finite split metacyclic actions $(H_1, (\mathcal{G}_1, \mathcal{F}_1))$ and $(H_2, (\mathcal{G}_2, \mathcal{F}_2))$ of order $m \cdot n$ and twist factor k are *weakly conjugate* if there exists an isomorphism

$$\psi : \pi_1^{\text{orb}}(\mathcal{O}_{H_1}) \cong \pi_1^{\text{orb}}(\mathcal{O}_{H_2})$$

and an isomorphism $\chi : H_1 \rightarrow H_2$ such that the following conditions hold.

(i) $\chi((\mathcal{G}_1, \mathcal{F}_1)) = (\mathcal{G}_2, \mathcal{F}_2)$.

(ii) For $i = 1, 2$, let $\phi_{H_i} : \pi_1^{\text{orb}}(\mathcal{O}_{H_i}) \rightarrow H_i$ be the surface kernel (in the exact sequence (*) in Section 2). Then $(\chi \circ \phi_{H_1})(g) = (\phi_{H_2} \circ \psi)(g)$, whenever $g \in \pi_1^{\text{orb}}(\mathcal{O}_{H_1})$ is of finite order.

(iii) The pair $(\mathcal{G}_1, \mathcal{F}_1)$ is conjugate (component-wise) to the pair $(\mathcal{G}_2, \mathcal{F}_2)$ in $\text{Homeo}^+(S_g)$.

The notion of weak conjugacy defines an equivalence relation on split metacyclic actions on S_g and the equivalence classes thus obtained will be called *weak conjugacy classes*.

Remark 2.11. By virtue of the Nielsen-Kerckhoff theorem, the notion of weak conjugacy in Definition 2.10 naturally extends to an analogous notion in $\text{Mod}(S_g)$ via the natural association

$$(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F})) \leftrightarrow (\langle F, G \rangle, (G, F)).$$

For simplicity, we will now introduce the following notation.

Definition 2.12. Let $F, G \in \text{Mod}(S_g)$ be a finite order map of orders n, m , respectively. Then for some $k \in \mathbb{Z}_n^\times \setminus \{1\}$, we say (in symbols) that $\llbracket F, G \rrbracket_k = 1$ if there exists conjugates F', G' (of F, G resp.) such that $\langle F', G' \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ satisfying the relation $(G')^{-1}F'G' = (F')^k$.

We conclude this subsection with the following crucial remark.

Remark 2.13. Let $H < \text{Mod}(S_g)$ be a finite split metacyclic subgroup, and let $I(H)$ denote the isomorphism class of H (in $\text{Mod}(S_g)$). By Remark 2.11, we have

$$I(H) = \{H' : H' \cong H \text{ and } (H', (G', F')) \text{ represents a weak conjugacy class}\}$$

for some $F', G' \in H'$ such that $H' = \langle F', G' \rangle$.

Consequently, periodic mapping classes $F, G \in \text{Mod}(S_g)$ satisfy $\llbracket F, G \rrbracket_k = 1$ if and only if there exists conjugates F', G' (of F, G resp.) such that the triple $(\langle F', G' \rangle, (G', F'))$ represents a weak conjugacy class associated with a finite split metacyclic subgroup (of twist factor k) of $\text{Mod}(S_g)$.

3. Main theorem

In this section, we establish the main result of the paper by deriving equivalent conditions under which torsion elements $F, G \in \text{Mod}(S_g)$ would satisfy $\llbracket F, G \rrbracket_k = 1$. We will introduce an abstract tuple of integers that will capture each weak conjugacy class associated with a finite split metacyclic subgroup of $\text{Mod}(S_g)$.

Definition 3.1. A split metacyclic data set of degree $m \cdot n$, twist factor k , and genus $g \geq 2$ is a tuple

$$((m \cdot n, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

where $m, n \geq 2$, the n_{ij} are positive integers for $1 \leq i \leq \ell$, $1 \leq j \leq 2$, and $k \in \mathbb{Z}_n^\times$ such that $k^m \equiv 1 \pmod{n}$, satisfying the following conditions.

- (i) $\frac{2g-2}{mn} = 2g_0 - 2 + \sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right)$.
- (ii) (a) For each i, j , $n_{i1} \mid m$, $n_{i2} \mid n$, either $\gcd(c_{ij}, n_{ij}) = 1$ or $c_{ij} = 0$, and $c_{ij} = 0$ if and only if $n_{ij} = 1$.
- (b) For each i , $n_i = n_{i1} \cdot \beta_i$, where β_i is least positive integer such that

$$c_{i2} \frac{n}{n_{i2}} \left(\sum_{i'=0}^{n_{i1}\beta_i-1} k^{c_{i1} \frac{m}{n_{i1}} i'} \right) \equiv 0 \pmod{n}.$$

(iii) $\sum_{i=1}^{\ell} c_{i1} \frac{m}{n_{i1}} \equiv 0 \pmod{m}$.

(iv) Defining $A := \sum_{i=1}^{\ell} c_{i2} \frac{n}{n_{i2}} \prod_{s=i+1}^{\ell} k^{c_{s1} \frac{m}{n_{s1}}}$ and $d := \gcd(n, k-1)$, we have

$$A \equiv \begin{cases} 0 \pmod{n}, & \text{if } g_0 = 0, \text{ and} \\ d\theta \pmod{n}, \text{ for } \theta \in \mathbb{Z}_n, & \text{if } g_0 \geq 1. \end{cases}$$

- (v) If $g_0 = 0$, there exists $(p_1, \dots, p_{\ell v}), (q_1, \dots, q_{\ell v}) \in (\mathbb{N} \cup \{0\})^{\ell v}$ and $v \in \mathbb{N}$ such that the following conditions hold.

$$(a) \sum_{i'=1}^{\ell v} p_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 1 \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{p_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (p_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{p_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 0 \pmod{n}.$$

$$(b) \sum_{i'=1}^{\ell v} q_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 0 \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{q_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (q_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{q_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 1 \pmod{n}, \text{ where}$$

$$i \equiv \begin{cases} i' \pmod{\ell}, & \text{if } i' \not\equiv 0 \pmod{\ell}, \\ \ell & \text{otherwise,} \end{cases}$$

$$t \equiv \begin{cases} t' \pmod{\ell}, & \text{if } t' \not\equiv 0 \pmod{\ell}, \text{ and} \\ \ell, & \text{otherwise.} \end{cases}$$

(vi) If $g_0 = 1$, there exists $(p_1, \dots, p_{\ell v}), (q_1, \dots, q_{\ell v}) \in (\mathbb{N} \cup \{0\})^{\ell v}$ and $m', n' \in \mathbb{Z}$, $v \in \mathbb{N}$ such that $m' \mid m$ and $n' \mid n$, satisfying the following conditions.

$$(a) \sum_{i'=1}^{\ell v} p_{i'} c_{i1} \frac{m}{n_{i1}} \equiv m' \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{p_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (p_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{p_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 0 \pmod{n}.$$

$$(b) \sum_{i'=1}^{\ell v} q_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 0 \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left(\sum_{s=1}^{q_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (q_{i'} - s)} \right) \left(\prod_{t'=i'+1}^{\ell v} k^{q_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv n' \pmod{n}, \text{ where}$$

$$i \equiv \begin{cases} i' \pmod{\ell}, & \text{if } i' \not\equiv 0 \pmod{\ell}, \\ \ell & \text{otherwise,} \end{cases}$$

$$t \equiv \begin{cases} t' \pmod{\ell}, & \text{if } t' \not\equiv 0 \pmod{\ell}, \text{ and} \\ \ell, & \text{otherwise.} \end{cases}$$

(c) $A \equiv -\beta k^\alpha + \beta \pmod{n}$ for some non-negative integers α, β , where

$$\text{lcm} \left(\frac{m}{m'}, \frac{m}{\gcd(m, \alpha)} \right) = m \text{ and } \text{lcm} \left(\frac{n}{n'}, \frac{n}{\gcd(n, \beta)} \right) = n.$$

Furthermore, we set $\alpha = 1$, when $m' = 0$, and $\beta = 1$, when $n' = 0$.

We will now show that the split metacyclic data sets of genus g are in one-to-one correspondence with the weak conjugacy classes of split metacyclic subgroups of $\text{Mod}(S_g)$.

Proposition 3.2. *For integers $n, m, g \geq 2$, the split metacyclic data sets of degree $m \cdot n$ with twist factor k and genus g correspond to the weak conjugacy classes of $\mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ -actions on S_g .*

Proof. Let \mathcal{D} be a split metacyclic data set of degree $m \cdot n$ with twist factor k and genus g (as in Definition 3.1 above). We need to show that \mathcal{D} corresponds to the weak conjugacy class of a $\mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ -action on S_g represented by $(H, (\mathcal{G}, \mathcal{F}))$. To this effect, we first establish the existence of an epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$ which preserves the order of torsion elements. Let the presentations of H and $\pi_1^{orb}(\mathcal{O}_H)$ be given by

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^m = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m \text{ and}$$

$$\langle \alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \dots, \xi_\ell \mid \xi_1^{n_1} = \dots = \xi_\ell^{n_\ell} = \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1 \rangle,$$

respectively. We consider the map

$$\xi_i \xrightarrow{\phi_H} \mathcal{G}^{c_{i1} \frac{m}{n_{i1}}} \mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell.$$

As $|\mathcal{G}^{c_{i1} \frac{m}{n_{i1}}}| = n_{i1}$ and $|\mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}| = n_{i2}$, condition (ii) of Definition 3.1 would imply that ϕ_H is a map which preserves the order of torsion elements. For clarity, we break the argument for the surjectivity of ϕ_H into three cases.

First, we consider the case when $g_0 = 0$. Conditions (iii) and (iv) show that ϕ_H satisfies the long relation $\prod_{i=1}^{\ell} \xi_i = 1$ and the surjectivity of ϕ_H follows from condition (v).

When $g_0 \geq 2$, $\pi_1^{orb}(\mathcal{O}_H)$ has additional hyperbolic generators (viewing them as isometries of the hyperbolic plane), namely the α_i and the β_i . Extending ϕ_H by mapping $\alpha_1 \xrightarrow{\phi_H} \mathcal{G}, \beta_1 \xrightarrow{\phi_H} \mathcal{F}$ yields an epimorphism. Moreover, by carefully choosing the images of the α_i and the β_i under ϕ_H , for $i \geq 2$, conditions (iii) and (iv) would together ensure that the long relation $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1$ is satisfied.

When $g_0 = 1$, $\pi_1^{orb}(\mathcal{O}_H)$ has two additional hyperbolic generators, namely the α_1 and the β_1 . We extend ϕ_H by defining $\alpha_1 \xrightarrow{\phi_H} \mathcal{G}^\alpha$ and $\beta_1 \xrightarrow{\phi_H} \mathcal{F}^\beta$, and apply conditions (iii), (iv), and (vi) to obtain the desired epimorphism.

It remains to show that \mathcal{D} determines $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$ up to conjugacy (i.e. condition (iii) of Definition 2.10). Let $D_{\bar{G}} = (m, g_0; (c_{11}, n_{11}), \dots, (c_{\ell 1}, n_{\ell 1}))$ represent the conjugacy class of the action $\bar{\mathcal{G}}$ induced on the orbifold $\mathcal{O}_{\langle \mathcal{F} \rangle}$ by

the action $\mathcal{G} \in \text{Homeo}^+(S_g)$. We note that by Lemma 2.9, $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle})$ has the form

$$\left(g_1; \underbrace{\frac{n_1}{n_{11}}, \dots, \frac{n_1}{n_{11}}}_{\frac{m}{n_{11}} \text{ times}}, \dots, \underbrace{\frac{n_\ell}{n_{\ell 1}}, \dots, \frac{n_\ell}{n_{\ell 1}}}_{\frac{m}{n_{\ell 1}} \text{ times}} \right),$$

where if $n_i/n_{i1} = 1$, for some $1 \leq i \leq \ell$, then we exclude it from the signature, and $g_1 = g(D_{\bar{G}})$ is determined by Equation (2) of Definition 2.2. So, we get

$$D_F = (n, g_1; (d_{11}, \frac{n_1}{n_{11}}), \dots, (d_{1 \frac{m}{n_{11}}}, \frac{n_1}{n_{11}}), \dots, (d_{\ell 1}, \frac{n_\ell}{n_{\ell 1}}), \dots, (d_{\ell \frac{m}{n_{\ell 1}}}, \frac{n_\ell}{n_{\ell 1}})),$$

where

$$d_{i1}n_{i1} \equiv c_{i2} \frac{n_i}{n_{i2}} \sum_{j'=1}^{n_{i1}} k^{c_{i1} \frac{m}{n_{i1}}(j'-1)} \pmod{n_i} \text{ and}$$

$$d_{ij_i} \equiv d_{i1}k^{(j_i-1)} \pmod{\frac{n_i}{n_{i1}}} \quad 1 \leq i \leq l, \quad 1 \leq j_i \leq \frac{m}{n_{i1}}.$$

Moreover, by applying Lemma 2.4, we see that

$$D_G = (m, g_2; ((u_{ij}^{-1}, m_i), \frac{m_i |\mathbb{F}_{\frac{m}{g^{m_i}}}(u_{ij}, m_i)|}{m}) : u_{ij} \in \mathbb{Z}_{m_i}^\times \text{ and } m_i \mid m),$$

where

$$|\mathbb{F}_{\frac{m}{g^{m_i}}}(u_{ij}, m_i)| = |\mathbb{F}_{\frac{m}{g^{m_i}}}(u_{ij}, m_i)| - \sum_{\substack{m_{i'} \in \mathbb{N} \\ m_{i'} \neq m_i \\ m_i \mid m_{i'} \mid m}} \sum_{\substack{(u_{i'j'}, m_{i'})=1 \\ u_{ij} \equiv u_{i'j'} \pmod{m_i}}} |\mathbb{F}_{\frac{m}{g^{m_{i'}}}}(u_{i'j'}, m_{i'})|$$

and g_2 is determined by Equation (2) of Definition 2.2.

Conversely, consider the weak conjugacy class of $\mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ -actions on S_g represented by $(H, (\mathcal{G}, \mathcal{F}))$, where $H = \langle \mathcal{F}, \mathcal{G} \rangle$. So, Lemma 2.1 would imply that there exists a surjective homomorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$ defined by

$$\xi_i \xrightarrow{\phi_H} \mathcal{G}^{c_{i1} \frac{m}{n_{i1}}} \mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

which is order-preserving on the torsion elements. This yields a split metacyclic data set of degree $m \cdot n$ with twist factor k and genus g as in Definition 3.1. By Lemma 2.1, this tuple satisfies condition (i) of Definition 3.1, while condition (ii) follows from the fact that ϕ_H is order-preserving on torsion elements. Conditions (iii)-(iv) follow from the long relation satisfied by $\pi_1^{orb}(\mathcal{O}_H)$, and condition (v)-(vi) are implied by the surjectivity of ϕ_H . Thus, we obtain the split metacyclic data set of degree $m \cdot n$ with twist factor k and genus g , and the result follows. \square

We denote the data sets D_F and D_G (representing the cyclic factors of H) derived from the split metacyclic data set \mathcal{D} appearing in the proof of Proposition 3.2 by \mathcal{D}_1 and \mathcal{D}_2 , respectively. Thus, our main theorem will now follow from Remark 2.13 and Proposition 3.2.

Theorem 3.3 (Main theorem). *Let $F, G \in \text{Mod}(S_g)$ be of orders n, m , respectively. Then $\llbracket F, G \rrbracket_k = 1$ if and only if there exists a split metacyclic data set \mathcal{D} of degree $m \cdot n$, twist factor k , and genus g such that $\mathcal{D}_1 = D_F$ and $\mathcal{D}_2 = D_G$.*

We conclude this section with an example of a split metacyclic action of order 16 on S_5 .

Example 3.4. The split metacyclic data set $\mathcal{D} = ((4 \cdot 4, -1), 1; [(0, 1), (1, 2), 2])$ encodes the weak conjugacy class of a $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$ -action on S_5 represented by $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$, where

$$D_F = (4, 1; (1, 2), (1, 2), (1, 2), (1, 2)) \text{ and } D_G = (4, 2, 1;).$$

The geometric realization of this action is illustrated in Figure 2 below.

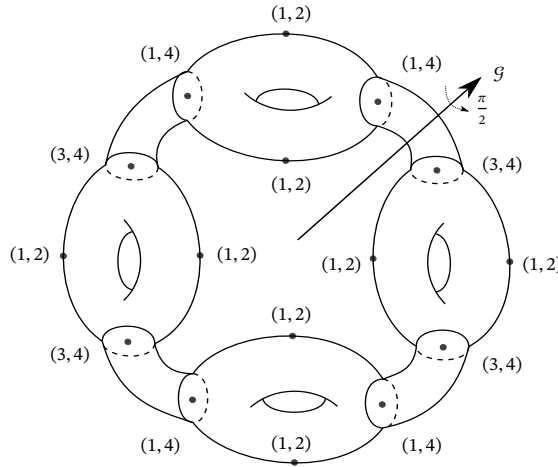


FIGURE 2. Realization of a $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$ -action on S_5 .

Note that the pairs of integers appearing in Figure 2 represent the compatible orbits involved in the realization of \mathcal{F} . Here, the action \mathcal{F} is realized via two 1-compatibilities between the action \mathcal{F}' on two copies of S_2 with

$$D_{F'} = (4, 0; ((1, 2), 2), (1, 4), (3, 4)).$$

Furthermore, the action \mathcal{F}' is realized by a 1-compatibility between the actions \mathcal{F}'' and $(\mathcal{F}'')^3$ on two copies of S_1 with

$$D_{F''} = (4, 0; (1, 2), (1, 4), (1, 4)).$$

4. Applications

4.1. Dihedral groups. Let $D_{2n} = \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_2$ be the dihedral group of order $2n$. We will call a split metacyclic data set of degree $2 \cdot n$ and twist factor -1 a *dihedral data set*. A simple computation reveals that a dihedral data set

$$((2 \cdot n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

would have the property that $(c_{j_1}, n_{j_1}) \in \{(0, 1), (1, 2)\}$, for $1 \leq j \leq \ell$. The following is an immediate consequence of Proposition 3.2.

Corollary 4.1. *For $g \geq 2$ and $n \geq 3$, dihedral data sets of degree $2 \cdot n$ and genus g correspond to the weak conjugacy classes of D_{2n} -actions on S_g .*

The following proposition provides an alternative characterization of a D_{2n} -action in terms of the generator of its factor subgroup of order n .

Proposition 4.2. *Let $F \in \text{Mod}(S_g)$ be of order n . Then there exists an involution $G \in \text{Mod}(S_g)$ such that $\langle F, G \rangle \cong D_{2n}$ if and only if D_F has the form*

$$(n, g_0, r; ((c_1, n_1), (-c_1, n_1), \dots, (c_s, n_s), (-c_s, n_s))). \tag{**}$$

Proof. Suppose that D_F has the form (**). Then $\mathcal{O}_{(\mathcal{F})}$ is an orbifold of genus g_0 with $2s$ cone points $[x_1], [y_1], \dots, [x_s], [y_s]$, where $\mathcal{P}_{x_i} = (c_i, n_i)$ and $\mathcal{P}_{y_i} = (-c_i, n_i)$, for $1 \leq i \leq s$. Up to conjugacy, let $\bar{\mathcal{G}} \in \text{Aut}_k(\mathcal{O}_{(\mathcal{F})})$ be the hyperelliptic involution so that $\bar{\mathcal{G}}([x_i]) = [y_i]$, for $1 \leq i \leq s$. To prove our assertion, it would suffice to show the existence of an involution $\mathcal{G} \in \text{Homeo}^+(S_g)$ that induces $\bar{\mathcal{G}}$. This amounts to showing that there exists a split metacyclic data set \mathcal{D} of degree $2 \cdot n$ with twist factor -1 encoding the weak conjugacy class $(H, (\mathcal{G}, \mathcal{F}))$ so that D_G has degree 2. Consider the tuple

$$\mathcal{D} = ((2 \cdot n, -1), 0; \underbrace{[(1, 2), (0, 1), 2], \dots, [(1, 2), (0, 1), 2]}_{t-2 \text{ times}}, [(1, 2), (c_{(t-1)2}, n_{(t-1)2}), 2], [(1, 2), (c_{t2}, n_{t2}), 2], [(0, 1), (c_1, n_1), n_1], \dots, [(0, 1), (c_s, n_s), n_s]),$$

where $t = 2g_0 + 2$,

$$(c_{(t-1)2}n/n_{(t-1)2}, c_{t2}n/n_{t2}) = \begin{cases} (0, -\sum_{i=1}^s c_i \frac{n}{n_i} \pmod{n}), & \text{if } g_0 = 0, \text{ and} \\ (1, 1 - \sum_{i=1}^s c_i \frac{n}{n_i} \pmod{n}), & \text{if } g_0 > 0. \end{cases}$$

It follows immediately that \mathcal{D} satisfies conditions (i)-(iv) of Definition 3.1. As $t \geq 2$, by taking $v = 1$, we may choose $(p_1, \dots, p_{t+s}) = (1, 0, \dots, 0)$ to conclude that \mathcal{D} also satisfies condition (v)(a). Since $t = 2 \iff g_0 = 0$, and when $g_0 = 0$, we have that $\text{lcm}(n_1, \dots, n_s) = n$, from which condition (v)(b) follows. Finally, for the case when $g_0 \neq 0$, (v)(b) follows by choosing $(q_1, \dots, q_{t-2}, q_{t-1}, \dots, q_{t+s}) = (0, \dots, 1, 1, \dots, 0)$. Thus, it follows that \mathcal{D} is a split metacyclic data set. Further, a direct application of Theorem 3.3 would show that \mathcal{D} indeed encodes the weak conjugacy represented by $(H, (\mathcal{G}, \mathcal{F}))$, as desired.

The converse follows immediately from Remark 2.11 and Proposition 3.2. □

We now provide a couple of examples of dihedral actions along with their realizations.

Example 4.3. Consider the $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$ -action $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_3 illustrated in Figure 3 below, where

$$D_F = (3, 1; (1, 3), (2, 3)) \text{ and } D_G = (2, 1; (1, 2), (1, 2), (1, 2), (1, 2)).$$

The weak conjugacy class of the action $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ is encoded by

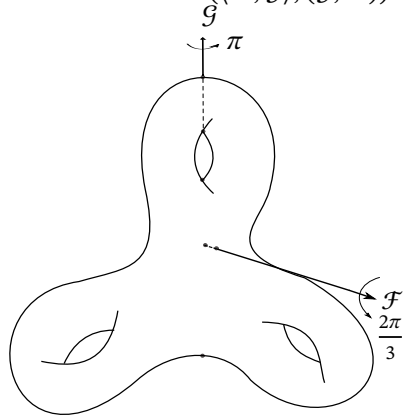


FIGURE 3. Realization of a D_6 -action on S_3 .

$$\mathcal{D} = ((2 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2], [(1, 2), (1, 3), 2], [(0, 1), (2, 3), 3]).$$

Example 4.4. Consider the $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2$ -actions $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ on S_3 illustrated in Figure 4 below, where $D_{\mathcal{F}} = (4, 0; (1, 4), (3, 4), (1, 4), (3, 4))$, $D_{\mathcal{G}} = (2, 1; (1, 2), (1, 2), (1, 2), (1, 2))$, and $D_{\mathcal{G}'} = (2, 2, 1;)$.

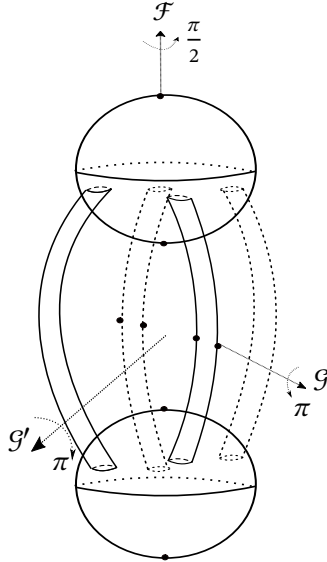


FIGURE 4. Realization of a D_8 -action on S_3 .

The weak conjugacy classes $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ and $(\langle \mathcal{F}', \mathcal{G}' \rangle, (\mathcal{G}', \mathcal{F}'))$ are encoded by

$$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2], [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4])$$

and

$((2 \cdot 4, -1), 0; [(1, 2), (1, 4), 2], [(1, 2), (1, 4), 2], [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4])$,
 respectively.

4.2. Generalized quaternions. For $n \geq 2$, the generalized quaternion group $Q_{2^{n+1}}$ is a metacyclic group of order 2^{n+1} that admits the presentation

$$\langle x, y \mid x^{2^n} = y^4 = 1, x^{2^{n-1}} = y^2, y^{-1}xy = x^{-1} \rangle.$$

Remark 4.5. Let \mathcal{D} be a split metacyclic data set of genus g , degree $4 \cdot 2^n$ and twist factor -1 (as in Definition 3.1) encoding a weak conjugacy class represented by $(H, (\mathcal{G}, \mathcal{F}))$. Suppose that \mathcal{D} has the property that

$$[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j] = [(1, 2), (1, 2), 2]$$

for some $1 \leq j \leq \ell$. Then it follows from the proof of Proposition 3.2 that under the epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$ which preserves the order of torsion elements, the tuple $[(1, 2), (1, 2), 2]$ would correspond to an involution $\mathcal{G}^2 \mathcal{F}^{2^{n-1}} \in H$ which defines a non-free action on S_g .

Remark 4.5 motivates the following definition.

Definition 4.6. A *quaternionic data set* is a split metacyclic data set of degree $4 \cdot 2^n$ that has the form

$$\mathcal{D} = ((4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

such that $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j] \neq [(1, 2), (1, 2), 2]$, for $1 \leq j \leq \ell$.

Proposition 4.7. For $g, n \geq 2$, quaternionic data sets of genus $2g - 1$ and degree $4 \cdot 2^n$ correspond to $Q_{2^{n+1}}$ -actions on S_g .

Proof. Suppose that there exists an action of $H = Q_{2^{n+1}}$ on S_g . By Lemma 2.1, there exists an epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$

$$\xi_i \xrightarrow{\phi_H} y^{c_{i1} \frac{m}{n_{i1}}} x^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

that is order-preserving on torsion elements. Let $H' = \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4$. Since the canonical projection $q : H' \rightarrow H (\cong H'/\mathbb{Z}_2)$ preserves the order of torsion elements on $H' \setminus \ker q$, the map ϕ_H naturally factors via q . Thus, as there are exactly two possible choices for $\phi_H|_{\{\xi_i : 1 \leq i \leq \ell\}}$ that preserves the order, at least one of which yields an action H' on $S_{g'}$ (for some $g' > g$). A weak conjugacy class associated with this action is encoded by a split metacyclic data set of genus g' and degree $2^{n+2} = 4 \cdot 2^n$, which has one of the following forms

$$((4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell])$$

or

$$(4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n_\ell]),$$

where $c'_{\ell_1 n'_{\ell_1}} \frac{4}{n'_{\ell_1}} \equiv c_{\ell_1} \frac{4}{n_{\ell_1}} + 2 \pmod{4}$ and $c'_{\ell_2 n'_{\ell_2}} \frac{2^n}{n'_{\ell_2}} \equiv c_{\ell_2} \frac{2^n}{n_{\ell_2}} + 2^{n-1} \pmod{2^n}$. Further, since $\ker q \cong \mathbb{Z}_2$ and q preserves the orders of all $x \in H' \setminus \ker q$, it follows that $\ker q$ acts freely on $S_{g'}$. Hence, it follows that $g' = 2g - 1$ and further by Remark 4.5, both (possible) tuples cannot contain a triple of the type $[(1, 2), (1, 2), 2]$.

Conversely, if there exists a quaternionic data set \mathcal{D} of genus $g' = 2g - 1$ as in Definition 4.6. Then we obtain an epimorphism $\phi_{H'} : \pi_1^{orb}(\mathcal{O}_{H'}) \rightarrow H'$ which preserves the order of torsion elements, when composed with canonical projection $q : H' \rightarrow H$, yields an epimorphism $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$ which preserves the order of torsion elements, where $\pi_1^{orb}(\mathcal{O}_{H'}) = \pi_1^{orb}(\mathcal{O}_H)$. Further, as \mathcal{D} does not contain a triple of type $[(1, 2), (1, 2), 2]$, $\ker q$ acts freely on $S_{g'}$, thereby yielding an action of $Q_{2^{n+1}}$ on S_g , where $g' = 2g - 1$. \square

Remark 4.8. A crucial step in the proof (of Proposition 4.7) is the establishment of the fact that the canonical projection $q : \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4 \rightarrow Q_{2^{n+1}}$ is order-preserving on $(\mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4) \setminus \ker q$. However, it is interesting to note that this fact does not generalize to arbitrary metacyclic groups [15] arising as quotients of split metacyclic groups. This motivates the study of finite non-split metacyclic actions on surfaces, which we plan to undertake in future works.

Example 4.9. The split metacyclic data set in Example 3.4 is quaternionic. Hence, this represents the weak conjugacy class of an induced Q_8 -action on S_3 .

4.3. Lifting cyclic subgroups of mapping classes to split metacyclic groups. For $n, g \geq 2$, let $p : S_{\tilde{g}} \rightarrow S_g$ be a covering map (that is possibly branched) with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Let $\text{LMod}_p(S_g)$ (resp. $\text{SMod}_p(S_{\tilde{g}})$) denote the liftable (resp. symmetric) mapping class groups of S_g (resp. $S_{\tilde{g}}$) under p .

Remark 4.10. From Birman-Hilden theory [2], we have the exact sequence

$$1 \rightarrow \langle \mathcal{F} \rangle \rightarrow \text{SMod}_p(S_{\tilde{g}}) \rightarrow \text{LMod}_p(S_g) \rightarrow 1. \quad (\text{B})$$

Let $G \in \text{Mod}(S_g)$ be of finite order. Then $G \in \text{LMod}_p(S_g)$ if and only if G has a lift $\tilde{G} \in \text{SMod}_p(S_{\tilde{g}})$ of finite order so that the sequence (B) yields a sequence of the form

$$1 \rightarrow \langle \mathcal{F} \rangle \rightarrow \langle \mathcal{F}, \tilde{G} \rangle \rightarrow \langle G \rangle \rightarrow 1.$$

Thus, $G \in \text{LMod}_p(S_g)$ if and only if for any lift \tilde{G} of G , $\langle G \rangle$ lifts under p to a metacyclic group $\langle \mathcal{F}, \tilde{G} \rangle$.

In the following corollary, we characterize the finite cyclic subgroups in $\text{Mod}(S_{0,3})$ that lift to finite split metacyclic groups under branched covers induced by irreducible cyclic actions.

Corollary 4.11. For $g, n \geq 2$, let $p : S_g \rightarrow S_{0,3}$ be a cover with deck transformation group $\langle \mathcal{F} \rangle$ with $D_{\mathcal{F}} = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n_3))$. Then a $G' \in \text{Mod}(S_{0,3})$

of order m has a conjugate $G \in \text{LMod}_p(S_{0,3})$ with a lift $\tilde{G} \in \text{SMod}_p(S_g)$ such that $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if and only if one of the following conditions hold.

- (a) $D_F = (n, 0; (c_1, n_1), (c_2, n), (c_2k, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^2 \equiv 1 \pmod{n}$.
- (b) $D_F = (n, 0; (c_1, n), (c_1k, n), (c_1k^2, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^3 \equiv 1 \pmod{n}$.

Proof. Suppose that $G' \in \text{Mod}(S_{0,3})$ has a conjugate $G \in \text{LMod}_p(S_{0,3})$ with a lift $\tilde{G} \in \text{SMod}_p(S_g)$ such that $H = \langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$. First, we claim that the n_i , for $1 \leq i \leq 3$, are not distinct. Suppose that we assume on the contrary that the n_i , for $1 \leq i \leq 3$, are indeed distinct. Since $\mathcal{G}' \in \text{Aut}_k(\mathcal{O}_{\langle \mathcal{F} \rangle})$ and $|\mathcal{G}'| > 1$, it would have to fix all three cone points of $\mathcal{O}_{\langle \mathcal{F} \rangle}$, which contradicts the fact that any nontrivial automorphism of the sphere has exactly two fixed points. Thus, the following two cases arise.

Case 1: $n_2 = n_3 = n \neq n_1$. In this case, \mathcal{G}' fixes the cone point, say of order n_1 , and should permute the remaining 2 cone points of orders n_2 and n_3 . This implies that D_F takes the form in condition (a) in our hypothesis (by Definition 2.8), and hence $H = \langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_2$.

Case 2: $n_i = n, 1 \leq i \leq 3$. In this case, if \mathcal{G}' permutes all the three cone points cyclically, then D_F takes the form in condition (b) in our hypothesis, and hence $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_3$. Alternatively, \mathcal{G}' could also fix a cone point of order n and permute the remaining 2 cone points, in which case, D_F will take the form in condition (a).

Conversely, let $D_F = (n, 0; (c_1, n_1), (c_2, n), (c_2k, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^2 \equiv 1 \pmod{n}$. Up to conjugacy, let $\mathcal{G}' \in \text{Aut}_k(\mathcal{O}_{\langle \mathcal{F} \rangle})$ be an involution so that \mathcal{G}' maps the cone point represented by (c_2, n) to the cone point represented by (c_2k, n) . To prove our assertion, it would suffice to show the existence of an involution $\mathcal{G} \in \text{Homeo}^+(S_g)$ that induces \mathcal{G}' . This amounts to showing that there exists a split metacyclic data set \mathcal{D} of degree $2 \cdot n$ with twist factor k encoding the weak conjugacy class $(H, (\mathcal{G}, \mathcal{F}))$ so that D_G has degree 2. Consider the tuple $((2 \cdot n, k), 0; [(1, 2), (0, 1), 2], [(1, 2), (n - c_2, n), 2n_1], [(0, 1), (c_2, n), n])$. A simple computation would reveal that conditions (i) - (iv) of Definition 3.1 hold true. Condition (v) is true by taking $v = 1, (p_1, p_2, p_3) = (1, 0, 0)$ and $(q_1, q_2, q_3) = (0, 0, w)$ such that $wc_2 \equiv 1 \pmod{n}$, which proves our claim.

For the case when $D_F = (n, 0; (c_1, n), (c_1k, n), (c_1k^2, n))$ for some $k \in \mathbb{Z}_n^\times$ such that $k^3 \equiv 1 \pmod{n}$, let $\mathcal{G}' \in \text{Aut}_k(\mathcal{O}_{\langle \mathcal{F} \rangle})$ be of order 3 so that for $1 \leq i \leq 2, \mathcal{G}'^i$ maps the cone point represented by (c_1, n) to the cone point represented by (c_1k^{3-i}, n) . By similar argument as above, we can show that the tuple $((3 \cdot n, k), 0; [(1, 3), (0, 1), 3], [(2, 3), (n - c_1, n), 3], [(0, 1), (c_1, n), n])$ forms a split metacyclic data set of degree $3 \cdot n$ with twist factor k . \square

Example 4.12. For $i = 1, 2$, consider the branched cover $p : S_3 \rightarrow \mathcal{O}_{\langle \mathcal{F}_i \rangle} (\approx S_{0,3})$, where $D_{F_1} = (8, 0; (1, 4), (1, 8), (5, 8))$ and $D_{F_2} = (8, 0; (3, 4), (3, 8), (7, 8))$. Then (up to conjugacy) the order-2 mapping class $G \in \text{LMod}_p(S_{0,3})$ represented by an automorphism $\mathcal{G} \in \text{Aut}_5(S_{0,3})$, that permutes two cone points

of order 8 and fixes order 4 cone point, lifts to a $\tilde{G} \in \text{SMod}_p(S_3)$ with $D_{\tilde{G}} = (2, 1; ((1, 2), 4))$ such that $\langle F_i, \tilde{G} \rangle \cong \mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$. Moreover, the weak conjugacy class of $(\langle \mathcal{F}_i, \tilde{G} \rangle, (\tilde{G}, \mathcal{F}_i))$, for $i = 1, 2$, is encoded by

$$\begin{aligned} &((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (7, 8), 8], [(0, 1), (1, 8), 8]) \text{ and} \\ &((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 8), 8], [(0, 1), (7, 8), 8]), \end{aligned}$$

respectively. The geometric realization of these actions is illustrated in Figure 5 below, where for each i , the action \mathcal{F}_i is realized by the rotation of a polygon of type \mathcal{P}_{F_i} described in Theorem 2.5.

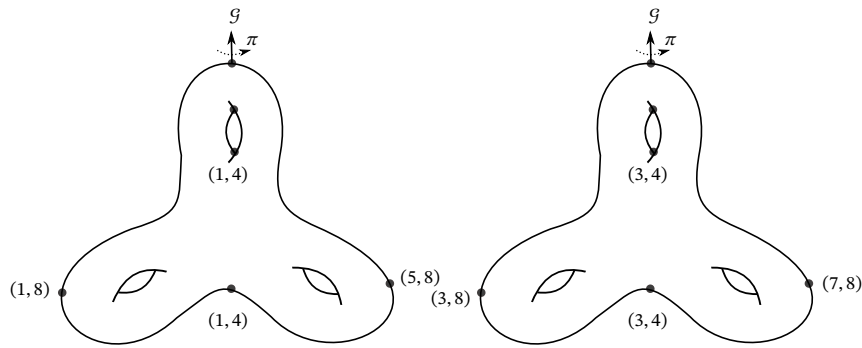


FIGURE 5. The realizations of two distinct $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$ -actions on S_3 .

Proposition 4.13. *For $g, n \geq 2$, let $p : S_{n(g-1)+1} \rightarrow S_g$ be a regular cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Then any involution $G' \in \text{Mod}(S_g)$ has a conjugate $G \in \text{LMod}_p(S_g)$ with a lift $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong D_{2n}$.*

Proof. Let $G' \in \text{Mod}(S_g)$ be an involution. When \mathcal{G}' generates a free action on S_g , it is easy to see that $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$ represents a weak conjugacy class in $\text{Mod}(S_{n(g-1)+1})$ with $\langle F, \tilde{G} \rangle \cong D_{2n}$. Now, we assume that \mathcal{G}' generates a non-free action with $D_{G'} = (2, g_0; ((1, 2), t))$. By Theorem 3.3 and Remark 4.10, it suffices to show that there exists a dihedral data set \mathcal{D} of degree $2 \cdot n$ and genus $n(g - 1) + 1$ representing the weak conjugacy class of $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$. When $g_0 \geq 1$, we take \mathcal{D} to be the tuple

$$((2 \cdot n, -1), g_0; \underbrace{[(1, 2), (0, 1), 2], \dots, [(1, 2), (0, 1), 2]}_{t \text{ times}}),$$

and when $g_0 = 0$, $t \geq 4$, and so we take \mathcal{D} to be the tuple

$$\begin{aligned} &((2 \cdot n, -1), 0; \underbrace{[(1, 2), (0, 1), 2], \dots, [(1, 2), (0, 1), 2]}_{t-2 \text{ times}}, \\ &[(1, 2), (1, n), 2], [(1, 2), (1, n), 2]). \end{aligned}$$

It is an easy computation to check that \mathcal{D} satisfies conditions (i)-(iv) of Definition 3.1 in both cases. When $g_0 = 0$, taking $\nu = 1$,

$$(p_1, \dots, p_t) = (1, 0, \dots, 0), \text{ and } (q_1, \dots, q_t) = (0, \dots, 0, 1, 1, 0)$$

we obtain condition (v). Moreover, when $g_0 = 1$, we take $\nu = 1$,

$$(p_1, \dots, p_t) = (1, 0, \dots, 0), \text{ and } (q_1, \dots, q_t) = (0, \dots, 0),$$

thereby verifying condition (vi). Thus, we have shown that \mathcal{D} is a dihedral data set as desired. Finally, it follows from Theorem 3.3 that \mathcal{D} encodes the weak conjugacy class of $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$. \square

Note that the same \mathbb{Z}_2 -action can lift to multiple non-isomorphic groups under a regular cyclic cover. We illustrate this phenomenon in the following example.

Example 4.14. Let $p : S_5 \rightarrow S_2$ be a regular 4-sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_4$ as illustrated in Figure 6 below.

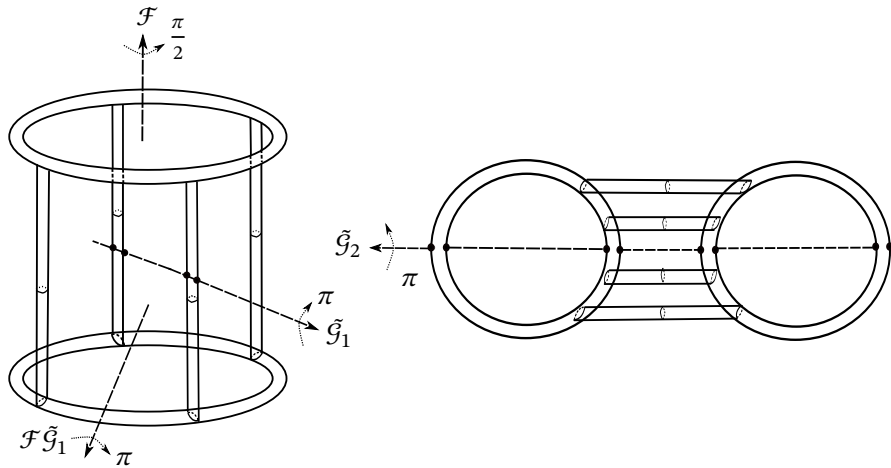


FIGURE 6. Two distinct lifts $\tilde{G}_1, \tilde{G}_2 \in \text{SMod}(S_5)$ of an involution $G \in \text{Mod}(S_2)$. Note that \tilde{G}_1 has four fixed points, while \tilde{G}_2 has eight fixed points.

The involution $G \in \text{Mod}(S_2)$ with $D_G = (2, 1; (1, 2), (1, 2))$ has two distinct lifts $\tilde{G}_1, \tilde{G}_2 \in \text{SMod}_p(S_5)$ (as indicated) such that $\langle F, \tilde{G}_1 \rangle \cong D_8$ and $\langle F, \tilde{G}_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Note that the weak conjugacy class of $(\langle \mathcal{F}, \tilde{G}_1 \rangle, (\tilde{G}_1, \mathcal{F}))$ is represented by $((2 \cdot 4, -1), 1; [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2])$.

The following proposition provides a sufficient condition for the liftability of \mathbb{Z}_m -actions whose corresponding orbifolds are spheres with a cone point of order m .

Proposition 4.15. For $g, n \geq 2$, let $p : S_{n(g-1)+1} \rightarrow S_g$ be a regular n -sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Let $G' \in \text{Mod}(S_g)$ be of order m such that $D_{G'} = (m, 0; (c_1, m_1), \dots, (c_\ell, m_\ell))$ with $m_\ell = m$ (say). Then G'

has a conjugate $G \in \text{LMod}_p(S_g)$ with a lift $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if the following conditions hold.

(a) There exists $a_1, \dots, a_{\ell-1} \in \mathbb{Z}$, and $k \in \mathbb{Z}_n^\times$, $k^m \equiv 1 \pmod{n}$ such that

$$\sum_{i=1}^{\ell-1} a_i (k^{c_i \frac{m}{m_i}} - 1) \prod_{s=i+1}^{\ell-1} k^{c_s \frac{m}{m_s}} \equiv 0 \pmod{n}.$$

(b) For $1 \leq i \leq \ell - 1$, there exists $d_i, n_i \in \mathbb{Z}$ such that $\gcd(d_i, n_i) = 1$, $n_i \mid n$,

$$d_i \frac{n}{n_i} \equiv a_i (k^{c_i \frac{m}{m_i}} - 1) \pmod{n}, \text{ and}$$

$$\text{lcm}(n_1, n_2, \dots, n_{\ell-1}) = n.$$

Proof. By Theorem 3.3 and Remark 4.10, it suffices to show that the tuple

$$\mathcal{D} = ((m \cdot n, k), 0; [(c_1, m_1), (d_1, n_1), m_1], \dots,$$

$$[(c_{\ell-1}, m_{\ell-1}), (d_{\ell-1}, n_{\ell-1}), m_{\ell-1}], [(c_\ell, m_\ell), (0, 1), m_\ell])$$

forms a split metacyclic data set of genus $n(g-1)+1$ that represents the weak conjugacy class of $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$ for some lift \tilde{G} of G under p . It can be verified easily that \mathcal{D} satisfies conditions (i)-(iii) of Definition 3.1, and further, condition (iv) follows from condition (a) in our hypothesis. By taking $v = 1$, $(p_1, \dots, p_\ell) = (0, \dots, 0, w)$ such that $wc_\ell \equiv 1 \pmod{m}$, we see that condition (v)(a) holds. Finally, condition (v)(b) follows from condition (b) in our hypothesis, and our assertion follows. \square

Using similar arguments, we can show the following.

Proposition 4.16. For $g, n \geq 2$, let $p : S_{n(g-1)+1} \rightarrow S_g$ be a regular n -sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Let $G' \in \text{Mod}(S_g)$ be of order m such that $D_{G'} = (m, 0; (c_1, m_1), \dots, (c_\ell, m_\ell))$ with $m_i \neq m$, for $1 \leq i \leq \ell$. Then G' has a conjugate $G \in \text{LMod}_p(S_g)$ with a lift $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if the following conditions hold.

(i) There exists $a_1, \dots, a_\ell \in \mathbb{Z}$, and $k \in \mathbb{Z}_n^\times$, $k^m \equiv 1 \pmod{n}$ such that

$$\sum_{i=1}^{\ell} a_i (k^{c_i \frac{m}{m_i}} - 1) \prod_{s=i+1}^{\ell} k^{c_s \frac{m}{m_s}} \equiv 0 \pmod{n}.$$

(ii) There exists $(p_1, \dots, p_{\ell v}), (q_1, \dots, q_{\ell v}) \in \mathbb{Z}^{\ell v}$ and $v \in \mathbb{N}$ such that condition (v)(b) of Definition 3.1 holds, where for $1 \leq i \leq \ell$, we have

$$c_{i1} \frac{m}{n_{i1}} \equiv c_i \frac{m}{m_i} \pmod{m} \text{ and } c_{i2} \frac{n}{n_{i2}} \equiv a_i (k^{c_i \frac{m}{m_i}} - 1) \pmod{n}.$$

A consequence of Propositions 4.15-4.16 is the following.

Corollary 4.17. For $g \geq 2$ and prime n , let $p : S_{n(g-1)+1} \rightarrow S_g$ be a regular n -sheeted cover with deck transformation group $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$. Let $G' \in \text{Mod}(S_g)$ be of order m such that the genus of $\mathcal{O}_{\langle G' \rangle}$ is zero. Then G' has a conjugate $G \in \text{LMod}_p(S_g)$ with a lift $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$ such that $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ if there exists $k \in \mathbb{Z}_n^\times$ such that $|k| = m$.

Proof. Let $D_{G'} = (m, 0; (c_1, m_1), \dots, (c_\ell, m_\ell))$. First, let us assume (without loss of generality) that $m_\ell = m$. By choosing

$$(a_1, \dots, a_{\ell-1}) = (0, \dots, 0, 1, -(k^{c_{\ell-2} \frac{m}{m_{\ell-2}}} - 1) \cdot k^{c_{\ell-1} \frac{m}{m_{\ell-1}}} \cdot (k^{c_{\ell-1} \frac{m}{m_{\ell-1}}} - 1)^{-1}),$$

we see that condition (i) of Proposition 4.15 holds true. Moreover, since $|k| = m$, we have $\gcd((k^{c_{\ell-2} \frac{m}{m_{\ell-2}}} - 1), n) = 1$, and so condition (ii) also holds, and our assertion follows.

Similarly, for the case when each $m_i < m$ for $1 \leq i \leq \ell$, the result follows by taking

$$(a_1, \dots, a_\ell) = (0, \dots, 0, 1, -(k^{c_{\ell-1} \frac{m}{m_{\ell-1}}} - 1) \cdot k^{c_\ell \frac{m}{m_\ell}} \cdot (k^{c_\ell \frac{m}{m_\ell}} - 1)^{-1}),$$

and applying Proposition 4.16. □

4.4. Infinite split metacyclic subgroups of $\text{Mod}(S_g)$. An infinite split metacyclic group that is isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ admits a presentation of the form

$$\langle x, y \mid y^{2m} = 1, y^{-1}xy = x^{-1} \rangle. \tag{3}$$

In this subsection, we give an explicit construction of an infinite metacyclic subgroup isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ of $\text{Mod}(S_g)$. Let $T_c \in \text{Mod}(S_g)$ denote the left-handed Dehn twist about a simple closed curve c in S_g . A root of T_c of degree s is an $F \in \text{Mod}(S_g)$ such that $F^s = T_c$. In the following lemma, by using some basic properties of Dehn twists [12, Chapter 3], we show that a root of Dehn twist cannot generate an infinite split metacyclic group that admits a presentation as in (3).

Lemma 4.18. *For $g \geq 2$, no root of T_c is a generator of any infinite split metacyclic subgroup of $\text{Mod}(S_g)$ that is isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$.*

Proof. Let F be a root of T_c of degree s . Suppose we assume on the contrary that for some $g \geq 2$, there exists an infinite split metacyclic subgroup $H \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ of $\text{Mod}(S_g)$ that admits the presentation

$$H = \langle F, G \mid G^{2m} = 1, G^{-1}FG = F^{-1} \rangle.$$

First, we consider the case when $s = 1$, that is, $F = T_c$. Then we have that

$$G^{-1}T_cG = T_c^{-1} \implies T_{G^{-1}(c)} = T_c^{-1},$$

which is impossible. Thus, we have that $H \neq \langle G, T_c \rangle$, which contradicts our assumption.

For $s > 1$, suppose that $H = \langle F, G \rangle$. Then the subgroup $\langle F^s, G \rangle$ of H would also be a split metacyclic group. Since $F^s = T_c$, this would contradict our conclusion in the previous case, and so our assertion follows. □

By a *multitwist* in $\text{Mod}(S_g)$, we mean a finite product of powers of commuting Dehn twists. In view of Lemma 4.18, a natural question that arises is whether a multitwist in $\text{Mod}(S_g)$ can generate an infinite split metacyclic group. In the following examples, we answer this question in the affirmative.

Example 4.19. Let $F' \in \text{Mod}(S_2)$ be of order 3 with

$$D_{F'} = (3, 0; ((1, 3), 2), ((2, 3), 2)).$$

First, we note that \mathcal{F}' has four fixed points on S_2 . Further, it induces a local rotation angle of $2\pi/3$ around two of these points (corresponding to the two $(1, 3)$ pairs in $D_{F'}$) and rotation angle of $4\pi/3$ around the remaining two points (corresponding to the two $(2, 3)$ pairs in $D_{F'}$), as indicated in Figure 7. Considering this action on two distinct copies of S_2 , we remove invariant disks around a distinguished $(1, 3)$ -type fixed point and a distinguished $(2, 3)$ -type fixed point in each of the two copies. We now attach two annuli connecting the resulting boundary components across the two surfaces so that:

- (a) each annulus connects a pair of boundary components where the induced rotation angle is the same, as shown in Figure 7 below, and further,
- (b) the annulus connecting the boundary components with rotation $4\pi/3$ (with the nonseparating curve c) has a $1/3^{rd}$ twist, while the other (with the nonseparating curve d) has a $-1/3^{rd}$ twist.

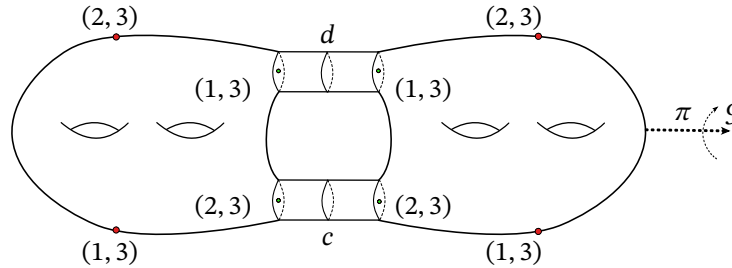


FIGURE 7. Realization of an infinite dihedral subgroup of $\text{Mod}(S_5)$.

Thus, by applying the theory developed in [25], we obtain an $F \in \text{Mod}(S_5)$, which is a root of the bounding pair map $T_c T_d^{-1}$ of degree 3. Now, we consider the hyperelliptic involution $G \in \text{Mod}(S_5)$ with $D_G = (2, 0; ((1, 2), 12))$ (also indicated in Figure 7). By our construction, it follows that $GFG^{-1} = F^{-1}$, and so we have $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_2$.

Example 4.20. Let $F' \in \text{Mod}(S_5)$ be of order 3 with

$$D_{F'} = (3, 1; ((1, 3), 2), ((2, 3), 2)).$$

First, we note that \mathcal{F}' has four fixed points on S_5 . Furthermore, it induces a local rotation angle of $2\pi/3$ around two of these points (corresponding to the two $(1, 3)$ pairs in $D_{F'}$) and rotation angle of $4\pi/3$ around the remaining two points (corresponding to the two $(2, 3)$ pairs in $D_{F'}$), as indicated in Figure 8. Considering this action on two distinct copies of S_5 , we remove invariant disks around all fixed point in each of the two copies. We now attach four annuli connecting the resulting boundary components across the two surfaces so that:

- (a) each annulus connects a pair of boundary components where the induced rotation angle is the same, as shown in Figure 8 below, and further,

- (b) the annulus connecting the boundary components with rotation $4\pi/3$ (with the nonseparating curve c_1 and c_3) has a $1/3^{rd}$ twist, while the other (with the nonseparating curve c_2 and c_4) has a $-1/3^{rd}$ twist.

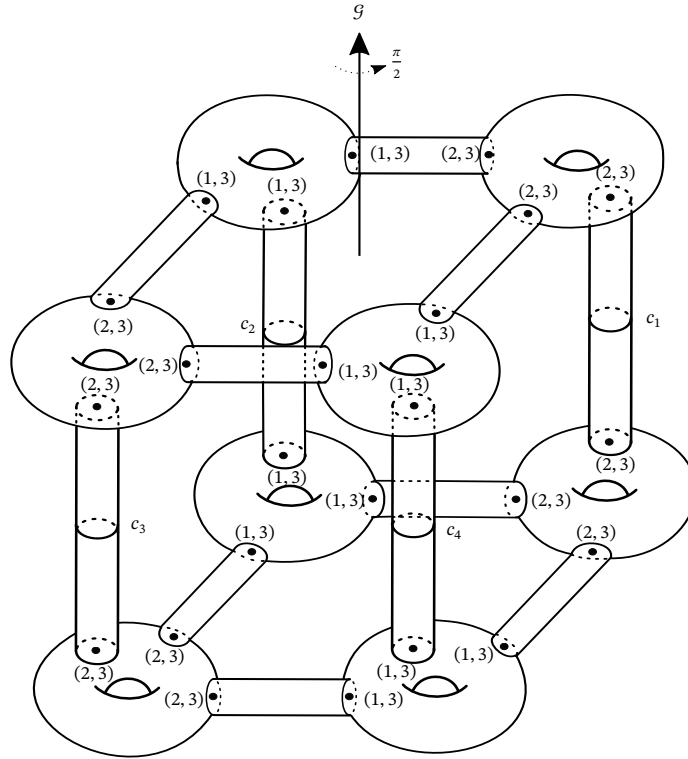


FIGURE 8. Realization of an infinite metacyclic subgroup of $\text{Mod}(S_{13})$.

Thus, by applying the theory developed in [25], we obtain an $F \in \text{Mod}(S_{13})$, which is a root of the multitwist $T_{c_1} T_{c_2}^{-1} T_{c_3} T_{c_4}^{-1}$ of degree 3. Now, we consider a $G \in \text{Mod}(S_{13})$ with $D_G = (4, 4, 1;)$ (also indicated in Figure 8). By our construction, as $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4 \cong \langle F', G' \rangle \leq \text{Mod}(S_5)$, where $D_{G'} = (4, 2, 1;)$, it follows that $GFG^{-1} = F^{-1}$, and so we have $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_4$.

Generalizing the above all constructions in Example 4.19 and Example 4.20, we have the following.

Proposition 4.21. *For $i = 1, 2$, let $H_i = \langle F_i, G_i \rangle \leq \text{Mod}(S_{g_i})$ with $H_i \cong \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_{2m}$, such that the weak conjugacy class $(H_i, (G_i, F_i))$ is represented by a split metacyclic data set \mathcal{D}_{H_i} containing a tuple $[(0, 1), (a_i, n), n]$. Then there exists an infinite metacyclic subgroup of $\text{Mod}(S_{g_1+g_2+2m-1})$ isomorphic to $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ that is generated by a periodic mapping class of order $2m$ and a root of a multitwist of degree n .*

Proof. As \mathcal{D}_{H_i} contains a tuple $[(0, 1), (a_i, n), n]$, by Proposition 3.2, we have

$$D_{F_1} = (n, g_0; (c_1, n_1), \dots, (c_s, n_s), \underbrace{(a_1, n), (n - a_1, n), \dots, (a_1, n), (n - a_1, n)}_{m \text{ times}})$$

and

$$D_{F_2} = (n, g'_0; (c'_1, n'_1), \dots, (c'_t, n'_t), \underbrace{(a_2, n), (n - a_2, n), \dots, (a_2, n), (n - a_2, n)}_{m \text{ times}}).$$

Taking inspiration from the theory developed in [24, 25] and Examples 4.19-4.20, we glue $2m$ annuli connecting the boundary components resulting from removing invariant disks around the orbit points corresponding to the pairs (a_1, n) and

$$\begin{cases} (a_2, n), & \text{if } a_2 \neq n - a_1, \text{ or} \\ (n - a_2, n), & \text{if } a_2 = n - a_1. \end{cases}$$

This yields a degree- n root F of a multitwist of the form

$$\begin{cases} \prod_{i=1}^{2m} T_{c_i}^{(-1)^{i+1}(a_1^{-1}+a_2^{-1})}, & \text{if } a_2 \neq n - a_1, \text{ or} \\ \prod_{i=1}^{2m} T_{c_i}^{(-1)^{i+1}(a_1^{-1}+(n-a_2)^{-1})}, & \text{if } a_2 = n - a_1, \end{cases}$$

where $a_i a_i^{-1} \equiv 1 \pmod{n}$ and $a_1^{-1} + a_2^{-1} \in \mathbb{Z}_n$. By considering the action \mathcal{G} obtained by performing a $2m$ -compatibility on \mathcal{G}_1 and \mathcal{G}_2 (see Section 2), we see that $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$, as desired. \square

The group for $m = 1$ in the presentation of the infinite split metacyclic group of the type in the Equation (3) is known as the *infinite dihedral group*. Here is the corollary, which directly follows from Proposition 4.21.

Corollary 4.22. *For $g \geq 5$, there exists an infinite dihedral subgroup of $\text{Mod}(S_g)$ that is generated by an involution and a root of a bounding pair map of degree 3.*

5. Hyperbolic structures realizing split metacyclic actions

We begin this section by providing an algorithm for obtaining the hyperbolic structures that realize finite split metacyclic subgroups of $\text{Mod}(S_g)$ (up to weak conjugacy) as groups of isometries.

Step 1. Consider a weak conjugacy class represented by $(H, (\mathcal{G}, \mathcal{F}))$.

Step 2. Use Theorem 3.3 to determine the conjugacy classes D_F (resp. D_G) of the generators F (resp. G).

Step 3. We apply Lemma 2.7, and Theorems 2.5-2.6, to obtain the hyperbolic structures that realize H as a group of isometries.

We now describe the geometric realizations of some split metacyclic actions on S_3 and S_5 represented by the split metacyclic data sets listed in Tables 1 and 2 in Section 6.

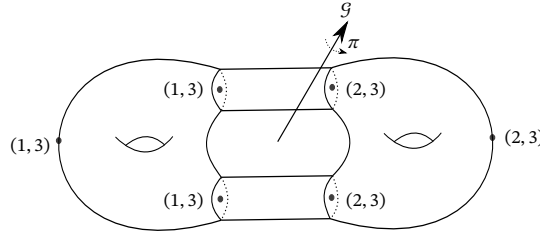


FIGURE 9. A realization of a D_6 -action $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_3 , where $D_G = (2, 2, 1;)$ and $D_F = (3, 1; (1, 3), (2, 3))$. The action \mathcal{F} is realized through two 1-compatibilities between two actions \mathcal{F}' and \mathcal{F}'' on S_1 with $D_{F'} = (3, 0; ((1, 3), 3))$ and $D_{F''} = (3, 0; ((2, 3), 3))$. The weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ is encoded by the first split metacyclic data set in Table 1.

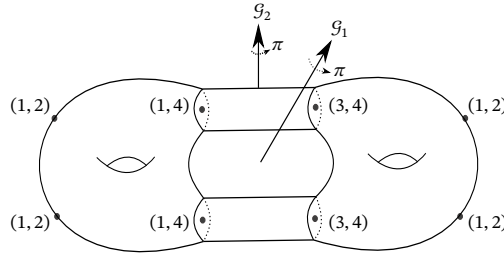


FIGURE 10. The realizations of two distinct D_8 -actions $\langle \mathcal{F}, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}, \mathcal{G}_2 \rangle$ on S_3 , where $D_F = (4, 1; ((1, 2), 2))$, $D_{G_1} = (2, 2, 1;)$, and $D_{G_2} = (2, 1; ((1, 2), 4))$. The action \mathcal{F} is realized via two 1-compatibilities between two actions \mathcal{F}' and \mathcal{F}'' on S_1 , where $D_{F'} = (4, 0; ((1, 4), 2), (1, 2))$ and $D_{F''} = (4, 0; ((3, 4), 2), (1, 2))$. The weak conjugacy classes of $(\langle \mathcal{F}, \mathcal{G}_1 \rangle, (\mathcal{G}_1, \mathcal{F}))$ and $(\langle \mathcal{F}, \mathcal{G}_2 \rangle, (\mathcal{G}_2, \mathcal{F}))$ are encoded by split metacyclic data sets nos. 3 and 6, respectively, in Table 1.

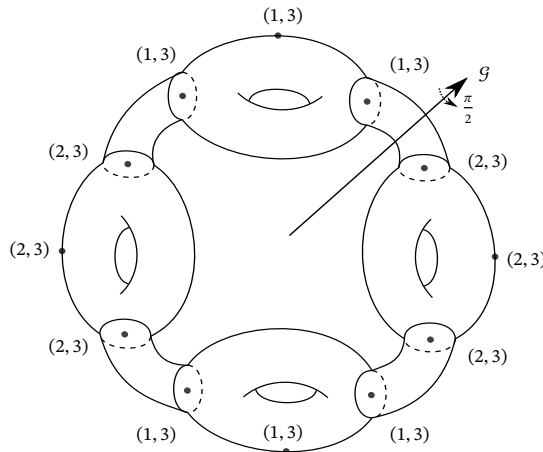


FIGURE 11. A realization of a $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4$ -action $\langle \mathcal{F}, \mathcal{G} \rangle$ on S_5 , where $D_G = (4, 2, 1;)$ and $D_F = (3, 1; ((1, 3), 2), ((2, 3), 2))$. The action \mathcal{F} is realized via two 1-compatibilities between the action \mathcal{F}' on two copies of S_2 with $D_{F'} = (3, 0; ((1, 3), 2), ((2, 3), 2))$. Furthermore, the action \mathcal{F}' is realized by a 1-compatibility between the actions \mathcal{F}'' and \mathcal{F}''' on S_1 , where $D_{F''} = (3, 0; ((1, 3), 3))$ and $D_{F'''} = (3, 0; ((2, 3), 3))$. The weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ is encoded by the split metacyclic data set no. 14 in Table 2.

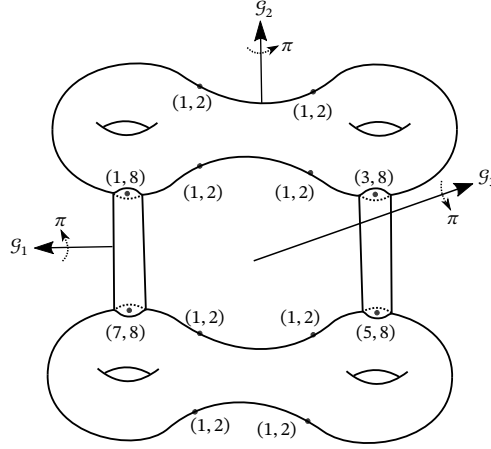


FIGURE 12. Realization of $\mathbb{Z}_8 \rtimes_{-1} \mathbb{Z}_2$ -action $\langle \mathcal{F}, \mathcal{G}_1 \rangle$, $\mathbb{Z}_8 \rtimes_3 \mathbb{Z}_2$ -action $\langle \mathcal{F}, \mathcal{G}_2 \rangle$ and $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$ -action $\langle \mathcal{F}, \mathcal{G}_3 \rangle$ on S_5 , where $D_{G_1} = D_{G_2} = (2, 2; ((1, 2), 4))$, $D_{G_3} = (2, 3, 1;)$ and $D_{\mathcal{F}} = (8, 1; ((1, 2), 2))$. The action \mathcal{F} is realized via two 1-compatibilities between two actions \mathcal{F}' and \mathcal{F}'' on S_2 where $D_{\mathcal{F}' } = (8, 0; (1, 2), (1, 8), (3, 8))$ and $D_{\mathcal{F}''} = (8, 0; (1, 2), (5, 8), (7, 8))$. The weak conjugacy class of $(\langle \mathcal{F}, \mathcal{G}_i \rangle, (\mathcal{G}_i, \mathcal{F}))$ $1 \leq i \leq 3$ is encoded by the split metacyclic data set nos. 26, 25, and 22, respectively, in Table 2.

6. Classification of the weak conjugacy classes in $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$

In this section, we will use Theorem 3.3 to classify the weak conjugacy classes in $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$. For brevity, we will further assume the following equivalence of the split metacyclic data sets (i.e. the weak conjugacy classes).

Definition 6.1. Two split metacyclic data sets

$$\mathcal{D} = ((m \cdot n, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_{\ell}])$$

$$\mathcal{D}' = ((m \cdot n, k), g_0; [(c'_{11}, n'_{11}), (c'_{12}, n'_{12}), n'_1], \dots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n'_{\ell}])$$

are said to be *equivalent* if for each tuple $[(c'_{i1}, n'_{i1}), (c'_{i2}, n'_{i2}), n'_i]$, there exists a unique tuple $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j]$ satisfying the following conditions:

- (i) $(c'_{i1}, n'_{i1}) = (c_{j1}, n_{j1})$,
- (ii) $n'_i = n_j$, and
- (iii) $c'_{i2} \frac{n}{n'_{i2}} \equiv c_{j2} \frac{n}{n_{j2}} k^{a_i} + b_i (k^{c_{j1} \frac{m}{n_{j1}}} - 1) \pmod{n}$ for some $a_i, b_i \in \mathbb{Z}$.

Note that equivalent data sets \mathcal{D} and \mathcal{D}' as in Definition 6.1 satisfy $\mathcal{D}'_i = \mathcal{D}_i$, for $i = 1, 2$. We will now provide a classification of the weak conjugacy classes of finite split metacyclic subgroups of $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$ (up to this equivalence) in Tables 1 and 2, respectively.

Group	Weak conjugacy classes in $\text{Mod}(S_3)$	Cyclic factors $[D_G; D_F]$
$Z_3 \rtimes_{-1} Z_2$	$((2 \cdot 3, -1), 0; [(1, 2), (0, 1), 2]_3, [(0, 1), (1, 3), 2], [(0, 1), (2, 3), 3])^*$	$[(2, 2, 1); (3, 1; (1, 3), (2, 3))]$
$Z_4 \rtimes_{-1} Z_2$	$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 2), 2], [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4])$	$[(2, 2, 1); (4, 1; (1, 2), 2)]$
$Z_3 \rtimes_{-1} Z_4$	$((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4], [(0, 1), (1, 6), 6])$	$[(2, 1; ((1, 2), 4); (4, 0; ((1, 4), 2), (3, 4), 2))]$
$Z_6 \rtimes_{-1} Z_2$	$((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 4), 2], [(1, 2), (3, 4), 2], [(0, 1), (1, 2), 2])$	$[(2, 1; ((1, 2), 4); (4, 1; (1, 2), 2))]$
$Z_4 \rtimes_{-1} Z_4$	$((4 \cdot 4, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 3), 4], [(1, 2), (2, 3), 6])$	$[(4, 0; ((1, 4), 2), ((1, 2), 3)); (3, 1; (1, 3), (2, 3))]$
$Z_8 \rtimes_{-1} Z_4$	$((4 \cdot 4, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 4), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 6), 6])$	$[(4, 0; ((1, 4), 2), ((1, 2), 3)); (4, 0; ((1, 4), 2), (3, 4), 2)]$
$Z_4 \rtimes_{-1} Z_4$	$((4 \cdot 4, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 4), 4], [(1, 2), (3, 4), 4])^\dagger$	$[(4, 0; ((1, 4), 2), ((1, 2), 3)); (4, 1; ((1, 2), 2))]$
$Z_4 \rtimes_{-1} Z_4$	$((4 \cdot 4, -1), 0; [(3, 4), (0, 1), 4], [(0, 1), (1, 4), 4], [(1, 4), (1, 4), 4])^\dagger$	$[(4, 0; ((3, 4), 2), ((1, 2), 3)); (4, 0; ((1, 4), 2), (3, 4), 2)]$
$Z_8 \rtimes_{-1} Z_2$	$((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (7, 8), 8], [(0, 1), (1, 8), 8])$	$[(4, 0; ((3, 4), 2), ((1, 2), 3)); (4, 1; (1, 2), 2)]$
$Z_7 \rtimes_{-2} Z_3$	$((3 \cdot 7, 2), 0; [(1, 3), (0, 1), 3], [(2, 3), (6, 7), 3], [(0, 1), (1, 7), 7])$	$[(2, 1; ((1, 2), 4)); (8, 0; (1, 4), (1, 8), (5, 8))]$
$Z_{12} \rtimes_{-5} Z_2$	$((2 \cdot 12, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (11, 12), 4], [(0, 1), (1, 12), 12])$	$[(2, 1; ((1, 2), 4)); (7, 0; (3, 7), (6, 7), (5, 7))]$
	$((2 \cdot 12, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (5, 12), 4], [(0, 1), (7, 12), 12])$	$[(2, 1; ((1, 2), 4)); (12, 0; (1, 12), (5, 12), (11, 12), (1, 2))]$
	$((2 \cdot 12, 5), 0; [(1, 2), (1, 6), 2], [(1, 2), (1, 12), 4], [(0, 1), (1, 12), 12])$	$[(2, 2, 1); (12, 0; (1, 12), (5, 12), (1, 2))]$
	$((2 \cdot 12, 5), 0; [(1, 2), (1, 6), 2], [(1, 2), (7, 12), 4], [(0, 1), (7, 12), 12])$	$[(2, 2, 1); (12, 0; (7, 12), (11, 12), (1, 2))]$

TABLE 1. The weak conjugacy classes of finite non-abelian split metacyclic subgroups of $\text{Mod}(S_3)$. Note that each data set of type \dagger is quaternionic, and therefore corresponds to the weak conjugacy action of a Q_8 -action on S_2 . (*The suffix refers to the multiplicity of the tuple in the split metacyclic data set.)

Group	Weak conjugacy classes in $\text{Mod}(S_5)$	Cyclic factors $[D_G; D_F]$
$Z_3 \rtimes_{-1} Z_2$	$((2 \cdot 3, -1), 1; [(0, 1), (1, 3), 3]_2)^*$ $((2 \cdot 3, -1), 0; [(1, 2), (0, 1), 2]_4, [(0, 1), (1, 3), 3], [(0, 1), (2, 3), 3])$ $((2 \cdot 4, -1), 1; [(1, 2), (0, 1), 2]_2)$	$[(2, 3, 1); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(2, 2; ((1, 2), 4)); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(2, 2; ((1, 2), 4)); (4, 2, 1); [(2, 3, 1); (4, 2, 1);]]$
$Z_4 \rtimes_{-1} Z_2$	$((2 \cdot 4, -1), 1; [(1, 2), (1, 4), 2]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 2), 2], [(0, 1), (1, 4), 4]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (1, 4), 2]_2, [(0, 1), (1, 2), 2], [(0, 1), (1, 4), 4]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_4, [(1, 2), (1, 4), 2]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 2), (1, 4), 2]_4)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 4), 2]_2, [(0, 1), (1, 2), 2]_2)$ $((2 \cdot 5, -1), 1; [(0, 1), (1, 5), 5])$ $((2 \cdot 5, -1), 1; [(0, 1), (2, 5), 5])$	$[(2, 3, 1); (4, 2, 1);]$ $[(2, 2; ((1, 2), 4)); (4, 0; ((1, 2), 2), ((1, 4), 2), ((3, 4), 2))]$ $[(2, 3, 1); (4, 0; ((1, 2), 2), ((1, 4), 2), ((3, 4), 2))]$ $[(2, 1; ((1, 2), 8)); (4, 2, 1);]$ $[(2, 2; ((1, 2), 4)); (4, 2, 1);]$ $[(2, 2; ((1, 2), 4)); (4, 1; ((1, 2), 4))]$ $[(2, 3, 1); (5, 1; (1, 5), (4, 5))]$ $[(2, 3, 1); (5, 1; (2, 5), (3, 5))]$
$Z_5 \rtimes_{-1} Z_2$	$((2 \cdot 5, -1), 0; [(1, 2), (0, 1), 2]_3, [(1, 2), (4, 5), 2], [(0, 1), (1, 5), 5])$ $((2 \cdot 5, -1), 0; [(1, 2), (0, 1), 2]_3, [(1, 2), (3, 5), 2], [(0, 1), (2, 5), 5])$ $((4 \cdot 3, -1), 1; [(0, 1), (1, 3), 3])$ $((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(1, 4), (0, 1), 4], [(1, 4), (2, 3), 4])$ $((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(3, 4), (0, 1), 4], [(3, 4), (2, 3), 4])$ $((2 \cdot 6, -1), 1; [(0, 1), (1, 3), 3])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (1, 6), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (2, 3), 2], [(1, 6), 2]_2, [(0, 1), (1, 3), 3])$ $((4 \cdot 4, -1), 1; [(0, 1), (1, 2), 2])$ $((2 \cdot 8, 5), 1; [(0, 1), (1, 2), 2])$	$[(2, 2; ((1, 2), 4)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 3, 1); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 2; ((1, 2), 4)); (6, 1; (1, 3), (2, 3))]$ $[(4, 2, 1); (6, 1; (1, 3), (2, 3))]$ $[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 0; (1, 2), (3, 4), (1, 8), (5, 8))]$ $[(2, 3, 1); (8, 0; (1, 2), (1, 4), (3, 8), (7, 8))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(2, 2; (1, 2), 4); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 2; (1, 2), 4); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$
$Z_3 \rtimes_{-1} Z_4$	$((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(1, 4), (0, 1), 4], [(1, 4), (2, 3), 4])$ $((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(3, 4), (0, 1), 4], [(3, 4), (2, 3), 4])$ $((2 \cdot 6, -1), 1; [(0, 1), (1, 3), 3])$	$[(4, 2, 1); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(4, 0; ((1, 4), 2), ((1, 2), 5)); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(4, 0; ((3, 4), 2), ((1, 2), 5)); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(2, 3, 1); (6, 1; (1, 3), (2, 3))]$
$Z_6 \rtimes_{-1} Z_2$	$((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (1, 6), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (2, 3), 2], [(1, 6), 2]_2, [(0, 1), (1, 3), 3])$ $((4 \cdot 4, -1), 1; [(0, 1), (1, 2), 2])$ $((2 \cdot 8, 5), 1; [(0, 1), (1, 2), 2])$	$[(2, 2; ((1, 2), 4)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 3, 1); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 2; ((1, 2), 4)); (6, 1; (1, 3), (2, 3))]$ $[(4, 2, 1); (4, 1; ((1, 2), 4))]$ $[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 0; (1, 2), (3, 4), (1, 8), (5, 8))]$ $[(2, 3, 1); (8, 0; (1, 2), (1, 4), (3, 8), (7, 8))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(2, 2; (1, 2), 4); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 2; (1, 2), 4); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$
$Z_8 \rtimes_5 Z_2$	$((2 \cdot 8, 5), 0; [(1, 2), (1, 4), 4], [(0, 1), (1, 8), 8], [(1, 2), (1, 8), 8])$ $((2 \cdot 8, 5), 0; [(1, 2), (1, 4), 4], [(1, 2), (3, 8), 8], [(0, 1), (3, 8), 8])$ $((2 \cdot 8, 3), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 4), 2], [(1, 2), (1, 8), 4], [(1, 2), (3, 8), 4])$ $((2 \cdot 8, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 2), (1, 8), 2], [(1, 2), (5, 8), 2], [(0, 1), (1, 2), 2])$ $((4 \cdot 5, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 5), 4], [(1, 2), (4, 5), 10])$ $((4 \cdot 5, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (2, 5), 4], [(1, 2), (3, 5), 10])$ $((4 \cdot 5, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (1, 5), 4], [(1, 2), (4, 5), 10])$ $((4 \cdot 5, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (2, 5), 4], [(1, 2), (3, 5), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 5), 2], [(0, 1), (1, 2), 2], [(0, 1), (3, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (2, 5), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (1, 10), 2], [(1, 2), (3, 10), 2], [(0, 1), (1, 2), 2], [(0, 1), (3, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (1, 10), 2], [(1, 2), (1, 2), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (1, 10), 2], [(1, 2), (1, 2), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 10), 10])$	$[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 0; (1, 2), (3, 4), (1, 8), (5, 8))]$ $[(2, 3, 1); (8, 0; (1, 2), (1, 4), (3, 8), (7, 8))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(2, 2; (1, 2), 4); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 2; (1, 2), 4); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$

TABLE 2. The weak conjugacy classes of finite non-abelian split metacyclic subgroups of $\text{Mod}(S_5)$. (*The suffix refers to the multiplicity of the tuple in the split metacyclic data set.)

Continuation of Table 2.

Group	Weak conjugacy classes in $\text{Mod}(S_5)$	Cyclic factors $[D_G; D_F]$
$\mathbb{Z}_6 \rtimes_{-1} \mathbb{Z}_4$	$((4 \cdot 6, -1), 0; [(1, 4), (0, 1), 4], [(3, 4), (1, 6), 4], [(0, 1), (5, 6), 6])$	$[(4, 0; ((1, 4), 2), ((1, 2), 5)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$
	$((4 \cdot 6, -1), 0; [(3, 4), (0, 1), 4], [(1, 4), (1, 6), 4], [(0, 1), (5, 6), 6])$	$[(4, 0; ((3, 4), 2), ((1, 2), 5)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$
	$((4 \cdot 6, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 6), 4], [(1, 2), (5, 6), 6])$	$[(4, 0; ((1, 4), 2), ((1, 2), 5)); (6, 1; (1, 3), (2, 3))]$
	$((4 \cdot 6, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (1, 6), 4], [(1, 2), (5, 6), 6])$	$[(4, 0; ((3, 4), 2), ((1, 2), 5)); (6, 1; (1, 3), (2, 3))]$
	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (14, 15), 6], [(0, 1), (1, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (1, 15), (4, 15), (2, 3))]$
$\mathbb{Z}_{15} \rtimes_4 \mathbb{Z}_2$	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (13, 15), 6], [(0, 1), (2, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (2, 15), (8, 15), (1, 3))]$
	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (8, 15), 6], [(0, 1), (7, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (7, 15), (13, 15), (2, 3))]$
	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (4, 15), 6], [(0, 1), (11, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (11, 15), (14, 15), (1, 3))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (19, 20), 4], [(0, 1), (1, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (1, 20), (9, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (1, 20), 4], [(0, 1), (1, 20), 20])$	$[(2, 3, 1; (20, 0; (1, 20), (9, 20), (1, 2))]$
$\mathbb{Z}_{20} \rtimes_9 \mathbb{Z}_2$	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (17, 20), 4], [(0, 1), (3, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (3, 20), (7, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (19, 20), 4], [(0, 1), (3, 20), 20])$	$[(2, 3, 1; (20, 0; (3, 20), (7, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (9, 20), 4], [(0, 1), (11, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (11, 20), (19, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (11, 20), 4], [(0, 1), (11, 20), 20])$	$[(2, 3, 1; (20, 0; (11, 20), (19, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (7, 20), 4], [(0, 1), (13, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (13, 20), (17, 20), (1, 2))]$
$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (9, 20), 4], [(0, 1), (13, 20), 20])$	$[(2, 3, 1; (20, 0; (13, 20), (17, 20), (1, 2))]$	

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