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2–dimensional Kähler-Einstein metrics induced by finite dimensional complex projective spaces

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ABSTRACT. In this paper we give a complete list of non-isometric bidimensional S^1-invariant Kähler-Einstein submanifolds of a finite dimensional complex projective space endowed with the Fubini-Study metric. This solves in the aforementioned case a classical and long-staying problem addressed among others in [7] and [31].

CONTENTS

1. Introduction

1.1. Description of the problem and state of the art. Holomorphic and isometric immersions (from now on *Kähler immersions*) into complex space forms (i.e. Kähler manifolds with constant holomorphic sectional curvature) are a classical topic in complex differential geometry. Even though it has been extensively studied starting from S. Bochner's work [5] and E. Calabi's seminal paper [6], a complete classification of Kähler manifolds admitting such type of immersions does not exist, even for Kähler manifolds of great interest, such as Kähler-Einstein manifolds and homogeneous Kähler ones.

In [32], M. Umehara classified Kähler-Einstein manifolds that are Kähler immersed into a finite dimensional complex space form with *non-positive* holomorphic sectional curvature: they are the totally geodesic submanifolds of either the complex Euclidean space or the complex hyperbolic one. In the case

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when the space form has *positive* holomorphic curvature, i.e., the complex projective space $\mathbb{C}P^n$ (endowed with the Fubini–Study metric g_{FS}), only some partial results exist (see for instance [27, 7, 31, 11, 13, 14]). Motivated by this, in the present paper we consider the problem to list those complex manifolds admitting a *projectively induced* Kähler-Einstein metric.

Definition 1.1. *We say that a Kähler metric on a connected complex manifold M is* projectively induced*, if M can be Kähler immersed into a finite dimensional*[1] *complex projective space $\mathbb{C}P^n$ endowed with the Fubini–Study metric g_{FS}, namely the metric associated to the Kähler form given in homogeneous coordinates by*

$$\frac{i}{2}\partial\bar{\partial}\log\left(|Z_0|^2+\ldots+|Z_n|^2\right).$$

The most relevant facts known so far about complex manifolds admitting projectively induced Kähler-Einstein metrics can be summarized by the following theorems:

Theorem A (S. S. Chern [7], K. Tsukada [31])**.** *Let (M,g) be a complete n-dimensional Kähler–Einstein manifold ($n \geq 2$). If (M,g) admits a Kähler immersion into $(\mathbb{C}P^{n+2}, g_{FS})$, in particular g is projectively induced, then M is either totally geodesic or the complex quadric in $(\mathbb{C}P^{n+1}, g_{FS})$.*

Theorem B (D. Hulin [14])**.** *If a compact Kähler-Einstein manifold is projectively induced then its Einstein constant is positive.*

Considering the previous results and taking also into account that all the explicit examples hitherto known are homogeneous manifolds (cfr. [28]), it has been proposed the following conjecture (see e.g. [19, Chap. 4]):

Conjecture 1.2. *If (M,g) is a Kähler-Einstein manifold endowed with a projectively induced metric, then it is an open subset of a complex flag manifold*[2]*.*

Remark 1.3. The conjecture cannot be extended to Kähler-Einstein manifolds embedded into the infinite dimensional complex projective space[3], indeed explicit examples of such non-homogeneous Kähler-Einstein manifolds can be found in [18, 12].

[1] Often in the literature, the definition of projectively induced metric does not exclude that ambient complex projective space may be infinite dimensional. Our choice is dictated by purely practical reasons, indeed we are going to study a conjecture that cannot be extended to the infinite dimensional setting (see Remark 1.3).

[2] A compact simply-connected Kähler manifold acted upon transitivity by its holomorphic isometry group.

[3] The classification of Kähler-Einstein manifolds admitting an immersion into an infinite dimensional complex space form is an open problem in all three cases (for some partial results see e.g. [8, 17, 16, 20]).

1.2. Description of the main result. The present paper is a first step toward a more ambitious research plan aimed at approaching the problem described in Section 1.1 (in particular, Conjecture 1.2) from a different perspective compared to the past: we do not give any assumption about the codimension of the studied immersions (cfr. [27, 7, 31, 25]). Our only assumption involves the group of symmetries of the metric. Indeed, our goal will be to test the above mentioned conjecture in the case of S^1-*invariant*[4] Kähler metrics, namely those Kähler metrics admitting (around a suitable point and in suitable holomorphic coordinates) a local Kähler potential of the form $\Phi\left(|z_1|^2, \dots, |z_n|^2\right)$.

It is worth pointing out that our assumption fits well with the purely local nature of Conjecture 1.2 (no assumptions on the immersions or on the topology of the manifolds are required). Therefore, our problem can be viewed as a favorable situation in which we can investigate how local assumptions have global implications. In particular, we are going to prove that S^1-invariant and projectively induced Kähler-Einstein manifolds are open subsets of *Kähler toric manifolds* (Proposition 2.4). We recall that a complex manifold M of complex dimension n is said to be toric if it contains a complex torus $(\mathbb{C}^*)^n$ as a dense open subset, together with a holomorphic action $(\mathbb{C}^*)^n \times M \to M$ that extends the natural action of $(\mathbb{C}^*)^n$ on itself.

Since complex projective spaces are the only irreducible S^1-invariant flag manifolds and since only the integer multiples of the Fubini-Study metric are projectively induced (see [6, 19]), in the specific case of S^1-invariant Kähler metrics Conjecture 1.2 reads as:

Conjecture 1.4. *The only projectively induced and S^1-invariant Kähler-Einstein manifolds are open subsets of $\mathbb{C}P^{n_1} \times \dots \times \mathbb{C}P^{n_k}$ endowed with the Kähler metric*

$$q\left(c_1 g_{FS} \oplus \dots \oplus c_k g_{FS}\right),$$

where k and $q \in \mathbb{Z}^+$, $c_i = \frac{1}{G^{k-1}} \prod_{j\neq i}(n_j+1)$ for $i = 1, \dots, k$ and $G = \gcd(n_1 + 1, \dots, n_k + 1)$, namely the greatest common divisor between $n_1+1, \dots, n_k+1$.

Remark 1.5. The homogeneous spaces $\left(\mathbb{C}P^{n_1}\times\cdots\times\mathbb{C}P^{n_k}, q(c_1 g_{FS}\oplus\cdots\oplus c_k g_{FS})\right)$ are fully embedded into $\mathbb{C}P^{\binom{n_1+qc_1}{qc_1}\cdots\binom{n_k+qc_k}{qc_k}-1}$. A Kähler embedding can be explicitly described through a composition of suitable normalizations of the Veronese embeddings:

$$(\mathbb{C}P^n, c g_{FS}) \to (\mathbb{C}P^{\binom{n+c}{c}-1}, g_{FS})$$
$$[Z_i]_{0\le i\le n} \mapsto \sqrt{\frac{(c-1)!}{c^{c-2}}}\left[\frac{Z_0^{c_0}\dots Z_n^{c_n}}{\sqrt{c_0!\dots c_n!}}\right]_{c_0+\dots+c_n=c},$$

together with a Segre embedding (cfr. [6, 19]).

[4]Cfr. [17] for a list of projectively induced extremal metrics in the radial case, i.e. those Kähler metrics admitting a local potential depending only on the sum of the moduli of certain local coordinates.

Our main result is contained in the following theorem, that solves Conjecture 1.4 in the 2-dimensional case.

Theorem 1.6. *If (M,g) is a Kähler-Einstein surface whose metric is S^1-invariant and projectively induced, then (M,g) is an open subset of either $(\mathbb{C}\mathrm{P}^2, q\, g_{FS})$ or $\left(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, q(g_{FS} \oplus g_{FS})\right)$, where $q \in \mathbb{Z}^+$.*

Since the only compact Kähler toric surfaces with positive first Chern class that can be endowed with a Kähler-Einstein metric are the ones listed in the previous theorem and the Fermat cubic[5], i.e. the surface obtained by blowing up three noncollinear points of $\mathbb{C}\mathrm{P}^2$, the following corollary straightforwardly follows from our main result and Proposition 2.4

Corollary 1.7. *The Fermat cubic endowed with the Kähler-Einstein metric studied by Y. T. Siu in [26], cannot be holomorphically and isometrically immersed into a finite dimensional complex projective space.*

2. Proof of Theorem 1.6

The proof of Theorem 1.6 is organized in three subsections, described below.

In Section 2.1, we recall the definition of Calabi's diastasis function and Bochner's coordinates.

In Section 2.2, on account of the results recalled in Section 2.1, by proving several auxiliary lemmas, we rephrase in Proposition 2.6 the statement of Theorem 1.6 in terms of existence and uniqueness of polynomial solutions of a particular family of real Monge-Ampère equations, where the unknown function is the Calabi's diastasis function and the independent variables are the moduli of the Bochner's coordinates. The existence of polynomial solutions is a part of Proposition 2.6, whereas the proof of the uniqueness of such solutions is the core of Section 2.3.

In fact, in Section 2.3, we find a set of suitable initial conditions for the aforementioned family of Monge-Ampère equations: an arbitrary polynomial solution to a Monge-Ampère equation of this family needs to satisfy one and only one initial condition of such set. Taking this into account, in the end of the section, we prove that the solutions we listed in Proposition 2.6 are actually unique, thus getting the statement of Theorem 1.6.

2.1. Calabi's diastasis function. In order to prove Theorem 1.6, we need to recall the definition of Calabi's diastasis function and some of its properties.

Let (M,g) be a Kähler manifold with a local Kähler potential Φ, namely $\omega = \frac{i}{2}\partial\bar{\partial}\Phi$, where ω is the Kähler form associated to g. If g (and hence Φ) is assumed to be real analytic, by duplicating the variables z and $\bar{z}$, Φ can be

[5]There exist only five distinct compact complex toric Fano surfaces (up to isomorphisms). They are $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^2 \# k\overline{\mathbb{C}\mathrm{P}^2}$, with $0 \leq k \leq 3$ (see e.g. [23] and references therein). Since a complex Fano surface admits a Kähler-Einstein metric if and only if the Calabi-Futaki invariant vanishes (see e.g. [29]), $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# 3\overline{\mathbb{C}\mathrm{P}^2}$ are the only compact toric Fano surfaces admitting a Kähler-Einstein metric.

complex analytically extended to a function $\tilde{\Phi}$ defined in a neighbourhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). Thus one can consider the power expansion of Φ around the origin with respect to z and $\bar{z}$ and write it as

$$\Phi(z, \bar{z}) = \sum_{j,l=0}^{\infty} a_{jl} z^{m_j} \bar{z}^{m_l}, \tag{1}$$

where we arrange every n-tuple of nonnegative integers as a sequence $m_j = (m_{j,1}, \dots, m_{j,n})$ and order them as follows: $m_0 = (0, \dots, 0)$ and if $|m_j| = \sum_{\alpha=1}^{n} m_{j,\alpha}$, $|m_j| \leq |m_{j+1}|$ for all positive integer j. Moreover, z^{m_j} denotes the monomial in n variables $\prod_{\alpha=1}^{n} z_\alpha^{m_{j,\alpha}}$.

A Kähler potential is not unique, but it is defined up to an addition of the real part of a holomorphic function. The *diastasis function* D_0 for g is nothing but the Kähler potential around p such that each matrix (a_{jk}) defined according to equation (1) with respect to a coordinate system $z = (z_1, \dots, z_n)$ centered in p, satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer j.

Moreover, for any real analytic Kähler manifold there exists a coordinates system, in a neighbourhood of each point, such that

$$\mathrm{D}_0(z) = \sum_{\alpha=1}^{n} |z_\alpha|^2 + \psi_{2,2}, \tag{2}$$

where $\psi_{2,2}$ is a power series with degree ≥ 2 in both z and $\bar{z}$. These coordinates, uniquely determined up to unitary transformation (cfr. [5, 6]), are called *Bochner's coordinates* (cfr. [5, 6, 13, 14, 24, 30]).

Notice that throughout this paper we will consider either projectively induced metrics or Kähler-Einstein metrics. In both cases these metrics are real analytic[6] and hence diastasis functions and Bochner's coordinates are defined. Moreover, in the particular case of S^1-invariant metrics, the diastasis function around the origin of the Bochner's coordinates system is a S^1-invariant Kähler potential.

2.2. Real Monge-Ampère equations. The lemmas contained in this section hold for manifolds of arbitrary dimension. By applying them to the bidimensional case, we show how the property of the projectively induced metrics to be S^1-invariant , allows us to address Conjecture 1.4 through real analysis' techniques. Indeed, we prove the equivalence of the statement of Theorem 1.6 to a uniqueness problem in a class of solutions of a family of real Monge-Ampère equations (Proposition 2.6).

[6]The condition of being a Kähler-Einstein metric reads locally, with respect to any local holomorphic coordinates (that clearly are real analytic), as a nonlinear overdetermined system of fourth-order elliptic PDEs. The real-analycity of ϕ follows from regularity results for elliptic PDEs (see e.g. [4]).

Lemma 2.1. *Let V be an open subset of $\mathbb{C}^n$ where it is defined a S^1-invariant potential for a Kähler metric g. Let $f : (V, g) \to (\mathbb{C}\mathrm{P}^N, g_{FS})$ be a full[7] Kähler immersion. Then $\mathrm{D}_0(z)$ can be written as*

$$\mathrm{D}_0(z) = \log\left(P(z)\right), \tag{3}$$

where

$$P(z) = 1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} a_j |z^{m_{h_j}}|^2 \tag{4}$$

with $a_j > 0$ and $h_j \neq h_l$ for $j \neq l$.

Proof. Recall that $Z_0, \dots, Z_N$ are the homogeneous coordinates on $\mathbb{C}\mathrm{P}^N$ (see Definition 1.1). Up to a unitary transformation of $\mathbb{C}\mathrm{P}^N$ and by shrinking V if necessary we can assume $f(p) = [1, 0 \dots, 0]$ and $f(V) \subset U_0 = \{Z_0 \neq 0\}$. Since the affine coordinates on U_0 are Bochner's coordinates for the Fubini–Study metric g_{FS}, by [6, Theorem 7], f can be written as:

$$f : V \to \mathbb{C}^N, \quad z = (z_1, \dots, z_n) \mapsto (z_1, \dots z_n, f_{n+1}(z), \dots, f_N(z)),$$

where

$$f_j(z) = \sum_{l=n+1}^{\infty} \alpha_{jl} z^{m_l},\ j = n+1, \dots, N.$$

Since the diastasis function is hereditary (see [6, Prop. 6]) and that of $\mathbb{C}\mathrm{P}^n$ around the point $[1, 0 \dots, 0]$ is given on U_0 by $\Phi(z) = \log(1 + \sum_{j=1}^{N} |z_j|^2)$, where $z_j = \frac{Z_j}{Z_0}$, one gets

$$\mathrm{D}_0(z) = \log\left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} |f_j(z)|^2\right).$$

The rotation invariance of $\mathrm{D}_0(z)$ and the fact that f is full imply that the f_j's are monomials of z of different degree and formula (3) follows. □

By setting

$$x = (x_1, \dots, x_n) = (|z_1|^2, \dots, |z_n|^2), \tag{5}$$

the diastasis function D_0 of a S^1-invariant Kähler metric g can be viewed as a function of the real variables x_i.

From now on we set, with a little abuse of notation,

$$P(x) = P(z(x)), \tag{6}$$

where $P(z)$ is given by (4) and x by (5).

A diastasis function of a S^1-invariant Kähler-Einstein metric satisfies the following lemma.

[7]A holomorphic immersion $f : U \to \mathbb{C}\mathrm{P}^n$ is said to be *full* provided $f(U)$ is not contained in any $\mathbb{C}\mathrm{P}^h$ for $h < n$.

Lemma 2.2. *If g is a S^1-invariant Kähler-Einstein metric, its diastasis $\mathrm{D}_0(x)$, where x is given by (5), is a solution of the real Monge-Ampère equation*

$$\det\left(\frac{\partial^2 \mathrm{D}_0}{\partial x_\alpha \partial x_\beta} x_\alpha + \frac{\partial \mathrm{D}_0}{\partial x_\alpha}\delta_{\alpha\beta}\right) = e^{-\frac{\lambda}{2}\mathrm{D}_0} \tag{7}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and λ is the Einstein constant.

Proof. A Kähler metric g with diastasis function $\mathrm{D}_0(z)$ is Einstein (see e.g. [21]) if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\lambda \frac{i}{2}\partial\bar\partial \mathrm{D}_0 = -i\partial\bar\partial \log\det(g_{\alpha\bar\beta}).$$

Hence, by the $\partial\bar\partial$-lemma, there exists a holomorphic function φ such that

$$\det(g_{\alpha\bar\beta}) = e^{-\frac{\lambda}{2}(\mathrm{D}_0+\varphi+\bar\varphi)}. \tag{8}$$

Once Bochner's coordinates are set, by comparing the series expansions of both sides of the previous equation, we get that $\varphi + \bar\varphi$ is forced to be zero (cfr. [3, 14, 25]). The PDE (8), in coordinates (5), coincides with (7). □

Lemma 2.3. *The Einstein constant λ of a projectively induced and S^1-invariant Kähler-Einstein manifold of dimension n is a positive rational number less than or equal to $2(n+1)$.*

Proof. By Lemma 2.1, the diastasis of a S^1-invariant and projectively induced Kähler metric can be written as $\mathrm{D}_0(x) = \log(P(x))$, where P is a polynomial of type (6). By Lemma 2.2, we have

$$\mathcal{D}_n(P) = P^{-\frac{\lambda}{2}+n+1}, \tag{9}$$

where we denote by $\mathcal{D}_n$ the following differential operator

$$\mathcal{D}_n(P) = \frac{\det\left[\left(P\frac{\partial^2 P}{\partial x_\alpha \partial x_\beta} - \frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}\right)x_\alpha + P\frac{\partial P}{\partial x_\alpha}\delta_{\alpha\beta}\right]_{1\le\alpha,\beta\le n}}{P^{n-1}}.$$

By multilinearity of determinants and by considering that $(\frac{\partial P}{\partial x_\alpha}\frac{\partial P}{\partial x_\beta}x_\alpha)_{1\le\alpha,\beta\le n}$ is a rank-1 matrix, we get that left side of (9) is a polynomial. Therefore λ needs to be a rational number satisfying the inequality $-\frac{\lambda}{2}+n+1 \ge 0$. Then we obtain the upper bound for the Einstein constant λ. Furthermore, by comparing the degrees of both sides of (9), we get $\lambda \ge 2\frac{n}{\deg P} > 0$. □

Proposition 2.4. *An S^1-invariant Kähler-Einstein manifold endowed with a projectively induced metric is an open subset of a simply-connected compact Kähler toric Fano manifold.*

Proof. According to Hulin's results [13], every Kähler-Einstein manifold embedded into a (possibly infinite dimensional) complex projective space can be extended to a complete Kähler-Einstein manifold M. Since by Lemma 2.3 the

Einstein constant of M needs to be positive, then it follows from the Myers' theorem that M is compact. Moreover, M is also simply connected by a well-known theorem of Kobayashi [15].

The existence of a S^1-invariant Kähler potential yields the existence of $n = \dim M$ local commuting Killing vector fields X_i corresponding to the S^1-action on the special holomorphic coordinates. Being M a real-analytic and simply-connected manifold, each Killing vector field X_i can be extended to a unique Killing vector field $\tilde{X}_i$ defined on the whole manifold by Nomizu's extension theorems [22]. Furthermore, we have that $[\tilde{X}_i, \tilde{X}_j] \equiv 0$.

Since Killing vector fields vanish at most on $(n-1)$ - complex dimensional submanifolds, there exists an open dense subset $\mathcal{U}$ of M where

$$\mathrm{D} = \mathrm{span}\{\tilde{X}_1, \dots, \tilde{X}_n\}$$

is a distribution of rank n.

Being M compact, every Killing vector field is also Hamiltonian and real holomorphic. Then

$$d(\omega(\tilde{X}_i, \tilde{X}_j)) = -i_{[\tilde{X}_i, \tilde{X}_j]}\omega + i_{\tilde{X}_i}\mathcal{L}_{\tilde{X}_j}\omega - i_{\tilde{X}_j}\mathcal{L}_{\tilde{X}_i}\omega \equiv 0,$$

where ω is the Kähler form on M. Since $\omega(X_i, X_j)|_0 = 0$, $\mathrm{D}|_{\mathcal{U}}$ is an integrable Lagrangian distribution. Hence, M is toric. □

Remark 2.5. It's worth to notice that every local immersion of a simply-connected manifold into a complex space form can be extended to a global one (cfr. [6]).

Now, let λ be the Einstein constant of a projectively induced and S^1-invariant Kähler-Einstein manifold of dimension n. In view of Lemma 2.3, $\lambda = 2\frac{s}{q}$, where $\gcd(s, q) = 1$. Since $\gcd(2nq, s) = 1$, a polynomial solution of type (6) to (9), is forced to be the q-th power of a polynomial $R(x)$. After the change of variables $x = \frac{\tilde{x}}{q}$, we easily check that $R(\tilde{x})$ is a solution for (9) with $q = 1$. Vice versa, every solution $R(\tilde{x})$ of (9) for $q = 1$ gives rise to a solution of (9) for $q \neq 1$ by taking the q-th power of $R(\tilde{x})$ and by considering the same changing of variables $\tilde{x} = qx$. Hence, we are going to study from now on the real Monge-Ampère equations (9) just when $q = 1$.

By restricting (9) to the case $n = 2$, by recalling that, for our purposes, we consider only solutions belonging to the polynomial class (6) and that the upper bound for the above parameter s can be obtained by Lemma 2.3, we have that the statement of Theorem 1.6 can be get by proving the following proposition.

Proposition 2.6. *The only solutions of type*

$$P(x) = P(x_1, x_2) = 1 + x_1 + x_2 + \xi(x_1, x_2), \tag{10}$$

where ξ is a polynomial with positive coefficients and no terms of degree less than 2, to the real Monge-Ampère equation

$$\mathcal{D}_2(P) = P^{3-s} \tag{11}$$

for some integer $s \in \{1,2,3\}$, *are*

$$\begin{cases} 1 + x_1 + x_2, & \textit{when } s = 3; \\ (1+x_1)(1+x_2) & \textit{when } s = 2; \\ (1+\frac{x_1+x_2}{3})^3 \textit{ and } (1+\frac{x_1}{2})^2(1+\frac{x_2}{2})^2 & \textit{when } s = 1. \end{cases} \tag{12}$$

Remark 2.7. The holomorphic coordinate system we choose to study Kähler-Einstein metrics on compact Kähler toric surfaces might appear unnatural, if we consider the existence of coordinate systems more studied and more suitable for this purpose, such as symplectic coordinates (see e.g. [1]). On the one hand, symplectic coordinates would lead to re-interpret our problem as a classification of particular solutions to a special case of the Abreu equation, a PDE better suited than (9) to be studied from an analytical point of view and extensively studied in the last few decades (see e.g. [2, 10, 9]). On the other hand, we would lose the polynomial nature of our problem, making harder the algebraic considerations on which our approach is based on.

2.3. Proof of Proposition 2.6. As a first step towards the proof of Proposition 2.6, we characterize the initial conditions that an arbitrary polynomial solution of type (10) to the Monge-Ampère equation (11) needs to satisfy on the coordinate axes. These conditions will be given by the Corollary 2.9 of the following lemma, that holds true for any dimension.

Lemma 2.8. *The restriction* p *on a coordinate axis of a polynomial solution of type* (6) *to the Monge-Ampère equation* (9) *reads as:*

$$\begin{cases} p(t) = 1 + t, & \textit{when } s = n+1; \\ p(t) = \left(1+\frac{t}{k}\right)^k, \textit{ with } k \in \{1,2\} & \textit{when } s = n; \\ p(t) = \left(1+\frac{t}{k}\right)^k, \textit{ with } k \in \mathbb{Z}^+ & \textit{when } 1 \le s \le n-1. \end{cases} \tag{13}$$

Proof. Let p be the restriction on the i-th coordinate axis (i.e. the line $x_j = 0$, for $j \neq i$) of a polynomial solution P of type (6) to the Monge-Ampère equation (9). Hence, we have that

$$\mathcal{D}_1\left(p(t)\right) q(t) = p(t)^{n-s+1}, \tag{14}$$

where the polynomial $q(t)$ is the restriction on the i-th coordinate axis of $\prod_{j\neq i} \frac{\partial P}{\partial x_j}$. Let $\{-r_1, \dots, -r_R\}$ be the (possibly complex) distinct roots of p, namely[8]

$$p(t) = \frac{1}{\prod_{i=1}^R r_i^{k_i}} \prod_{i=1}^R (t + r_i)^{k_i}.$$

[8]Notice that the constant term of $p(x)$ and $q(x)$ are fixed to be equal to 1 by the definition of (6).

Considering that

$$\mathcal{D}_1\left(\prod_{i=1}^{R}(t+r_i)^{k_i}\right)=\prod_{i=1}^{R}(t+r_i)^{2k_i-2}\sum_{i=1}^{R}k_ir_i\prod_{\substack{j=1\\ j\neq i}}^{R}(t+r_j)^2,$$

the equation (14) can be written as

$$\left(\sum_{i=1}^{R}k_ir_i\prod_{\substack{j=1\\ j\neq i}}^{R}(t+r_j)^2\right)q(x)=\frac{1}{\prod_{i=1}^{R}r_i^{k_i(n-s-1)}}\prod_{i=1}^{R}(t+r_i)^{k_i(n-s-1)+2}.$$

Therefore we get

$$q(t)=\frac{1}{\prod_{i=1}^{R}r_i^{k_i(n-s-1)+2}}\prod_{i=1}^{R}(t+r_i)^{k_i(n-s-1)+2} \tag{15}$$

and

$$\sum_{i=1}^{R}k_ir_i\prod_{\substack{j=1\\ j\neq i}}^{R}(t+r_j)^2-\prod_{i=1}^{R}r_i^2=0. \tag{16}$$

Let us now consider (16) as a linear system in the variables $k_1,\dots,k_R$. If $R=1$, such a system consists of just one equation, which has a unique solution: $k_1=r_1$. If $R\geq 2$, it cannot be compatible for any t. Indeed, being the left hand side of (16) a polynomial in t of degree $2R-2$, in particular its first R higher order coefficients have to vanish. Therefore, $k_1,\dots,k_R$ need to satisfy a homogeneous system, whose determinant of the coefficients matrix can be easily computed:

$$R!\prod_{i=1}^{R}r_i\prod_{1\leq i<j\leq R}(r_i-r_j).$$

In view of our hypotheses, such determinant is always different from zero. Therefore our system admits only the trivial solution, leading to a contradiction, since k_i represent the multiplicity of a root of a polynomial, so they should be positive. □

Corollary 2.9. *An arbitrary polynomial solution of type* (10) *to the Monge-Ampère equation* (11) *satisfies one and only one of the following initial conditions on the coordinate axis $x_2=0$:*

$$\begin{cases} P(x_1,0)=1+x_1,\ \frac{\partial P}{\partial x_2}(x_1,0)=1 & \text{when } s=3;\\ P(x_1,0)=1+x_1,\ \frac{\partial P}{\partial x_2}(x_1,0)=1+x_1 & \text{when } s=2;\\ P(x_1,0)=\left(1+\frac{x_1}{2}\right)^2,\ \frac{\partial P}{\partial x_2}(x_1,0)=\left(1+\frac{x_1}{2}\right)^2 & \text{or}\\ P(x_1,0)=\left(1+\frac{x_1}{3}\right)^3,\ \frac{\partial P}{\partial x_2}(x_1,0)=\left(1+\frac{x_1}{3}\right)^2 & \text{when } s=1. \end{cases} \tag{17}$$

Proof. Let P be a solution of type (10) to (11). By Lemma 2.8, $P(x_1,0) = \left(1+\frac{x_1}{k}\right)^k$ and $P(0,x_2) = \left(1+\frac{x_2}{h}\right)^h$ for suitable $k, h \in \mathbb{Z}^+$. Moreover, by (15), $\frac{\partial P}{\partial x_2}(x_1,0) = \left(1+\frac{x_1}{k}\right)^{k(1-s)+2}$ and $\frac{\partial P}{\partial x_1}(0,x_2) = \left(1+\frac{x_2}{h}\right)^{h(1-s)+2}$. By computing $\frac{\partial^2 P}{\partial x_1 \partial x_2}(0,0)$, we get $k=h$. Therefore, P reads as:

$$\left(1+\frac{x_1}{k}\right)^k + \left(1+\frac{x_2}{k}\right)^k - 1 + x_1\left(1+\frac{x_2}{k}\right)^{k(1-s)+2} + x_2\left(1+\frac{x_1}{k}\right)^{k(1-s)+2} - x_1 - x_2 - \left(1-s+\frac{2}{k}\right)x_1x_2 + x_1^2x_2^2\,\eta(x_1,x_2), \quad (18)$$

where η is a polynomial. By putting (18) in (11), by differentiating both sides of the equation by $\frac{\partial^2}{\partial x_1 \partial x_2}$ and by evaluating at $(0,0)$, we straightforwardly get the Diophantine equation $s^2k^2 - 5sk + 6 = 0$. Therefore, by solving the previous equation, we easily get our statement. □

Since each solution (12) satisfies the correspondent initial condition (17), we conclude the proof of Proposition 2.6 by showing that

Lemma 2.10. *If there exists a polynomial solution to* (11) *satisfying an initial condition of type* (17)*, then it is unique.*

Proof. Let F_s be a function whose zero defines the PDE (11), i.e., $F_s := \mathcal{D}_2(P) - P^{3-s}$. Then, from a straightforward computation, we get the following formula

$$\frac{\partial^h F_s}{\partial x_2^h}(x_1,0) = \left(h\left(P\frac{\partial^2 P}{\partial x_1^2}x_1 - \left(\frac{\partial P}{\partial x_1}\right)^2 x_1 + P\frac{\partial P}{\partial x_1}\right)\frac{\partial^{h+1}P}{\partial x_2^{h+1}} + T^h\right)(x_1,0), \quad (19)$$

where $T^h(x_1,0)$ is a polynomial expression in x_1, $P(x_1,0)$ and derivatives of P up to order $h+1$ (computed in $(x_1,0)$), that does not contain $\frac{\partial^h P}{\partial x_2^h}(x_1,0)$ and $\frac{\partial^{h+1}P}{\partial x_2^{h+1}}(x_1,0)$. If P is a polynomial solution to (11) satisfying an initial condition of type (17), $P(x_1,0) = \left(1+\frac{x_1}{k}\right)^k$ for a suitable integer k, hence we have

$$\left(P\frac{\partial^2 P}{\partial x_1^2}x_1 - \left(\frac{\partial P}{\partial x_1}\right)^2 x_1 + P\frac{\partial P}{\partial x_1}\right)(x_1,0) = \left(\frac{x_1}{k}+1\right)^{2k-2} \not\equiv 0.$$

By considering formula (19) when $h=1$, we realize that initial conditions (17) uniquely determine $\frac{\partial^2 P}{\partial x_2^2}(x_1,0)$, from which one obtains $\frac{\partial^{2+h}P}{\partial x_1^h \partial x_2^2}(x_1,0)$ for every $h \in \mathbb{N}$. By iteration, we get the whole Taylor expansion of P on the line $x_2 = 0$. Therefore, we get the statement of the lemma. □

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