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A moment theoretic approach to estimate the cardinality of certain algebraic varieties

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ABSTRACT. For $n \in \mathbb{N}$, we consider the algebraic variety $\mathcal{V}$ obtained by intersecting $n+1$ algebraic curves of degree n in $\mathbb{R}^2$, when the leading terms of the associated bivariate polynomials are all different. We provide a new proof, based on the Flat Extension Theorem from the theory of truncated moment problems, that the cardinality of $\mathcal{V}$ cannot exceed $\binom{n+1}{2}$. In some instances, this provides a slightly better estimate than the one given by Bézout's Theorem. Our main result contributes to the growing literature on the interplay between linear algebra, operator theory, and real algebraic geometry.

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1. Introduction

For a positive integer $n \in \mathbb{N}$, and $n+1$ bivariate polynomials $P_i \in \mathbb{R}[x,y]$, $\deg P_i = n$ $(i = 0, \dots, n)$, we consider the algebraic variety

$$\mathcal{V}(\boldsymbol{P}) \equiv \mathcal{V}(P_0, \dots, P_n) := \bigcap_{i=0}^{n} \mathcal{Z}(P_i),$$

where $\mathcal{Z}(P_i) := \{(x,y) \in \mathbb{R}^2 : P_i(x,y) = 0\}$ denotes the zero set of the polynomial P_i $(i = 0, \dots, n)$. For finite words in the indeterminates x and y, we can easily define a linear order (called the *degree-lexicographic order*) for all monomials in $\mathbb{R}[x,y]$; for instance, (i) monomials of total degree one appear before

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those of total degree two, etc., and (ii) within the same total degree, a monomial M_1 appears before a monomial M_2 if and only if the word associated to M_1 precedes the word associated to M_2 in the dictionary order. (As is customary, we assign the monomial 1 to the empty word.)

Given a polynomial $Q \in \mathbb{R}[x, y]$, we denote by $\mathrm{LT}(Q)$ the monomial in Q of highest degree-lexicographic order; $\mathrm{LT}(Q)$ is called the *leading term* of Q. Since we are interested in $\mathcal{V}(\boldsymbol{P})$, without loss of generality we can assume that the leading term $\mathrm{LT}(P_i)$ of each polynomial P_i is monic. In this note, we provide an estimate for the cardinality of $\mathcal{V}(\boldsymbol{P})$ in terms of n, in the case when all leading terms are different; that is, when $\mathrm{LT}(P_i) \neq \mathrm{LT}(P_j)$ for all $i \neq j$.

As a special instance of the above, suppose that two conics $C_1(x, y) = 0$ and $C_2(x, y) = 0$ have four distinct points in common. Then the equation $C_1(x, y) + \lambda C_2(x, y) = 0$ $(\lambda \in \mathbb{R})$ represents a one-parameter family of conics passing through the four common points. Since demanding that a conic pass through a point requires one linear condition on the coefficients of the conic, four linear conditions will be imposed on the conic to guarantee that it passes through four points. Note that, in general, a conic allows *five* degrees of freedom; thus, we have a unique family of conics through four points parametrized only by $\lambda \in \mathbb{R}$. This means that any conic sharing the given four points must be written as $C(x, y) \equiv C_1(x, y) + \lambda C_2(x, y) = 0$ for some $\lambda \in \mathbb{R}$. The family of such conics is called a *conic pencil*.

Using the lexicographic order, we can write each conic as

$$C_1(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2, \tag{1.1}$$

$$C_2(x, y) = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2. \tag{1.2}$$

We can then summarize the required conditions in the following table, according to the leading term:

$\mathrm{LT}(C(x, y))$	Required conditions
y^2	$a_6 + \lambda b_6 \neq 0$
xy	$a_6 + \lambda b_6 = 0$, $a_5 + \lambda b_5 \neq 0$
x^2	$a_6 + \lambda b_6 = 0$, $a_5 + \lambda b_5 = 0$, $a_4 + \lambda b_4 \neq 0$

By suitable choices of λ, it is evident that the elements C of the conic pencil generated by C_1 and C_2 can be described in terms of three conics with different leading terms. According to the Maclaurin-Bézout's Theorem [8, Theorem 5.12] (hereafter referred to as Bézout's Theorem), two nondegenerate conics without any nonconstant common factor may intersect in at most four different points, and naturally so do three conics; for example, $x^2 = 1$, $y^2 = 1$, and $x^2 + y^2 = 2$ intersect at the four points $\{(\pm 1, \pm 1)\}$ in $\mathbb{R}^2$. However, if we demand that the three conics have different leading terms, it is possible to prove, using a variation of the previous argument, that their algebraic variety cannot contain four distinct points.

In this paper, we will present an alternative approach based on the Flat Extension Theorem from the theory of truncated moment problems [2, 3, 7, 9]. We will actually establish a more general result, concerning algebraic varieties

$\mathcal{V}$ associated with $n+1$ algebraic curves of degree n; we will prove that the cardinality of $\mathcal{V}$ cannot exceed $\frac{n(n+1)}{2}$. Moreover, our proof will reveal that extending the result to d indeterminates (that is, to algebraic varieties in $\mathbb{R}^d$) is straightforward, inasmuch as the Flat Extension Theorem also holds in the multivariable setting [3].

We readily acknowledge that our two main results (Theorems 3.2 and 3.3) can be derived from well-known real algebraic geometry tools. The main novelty in our paper resides in the approach; that is, the use of tools and techniques from the theory of truncated moment problems, and especially the Flat Extension Theorem. There is an extensive literature on the applications of real algebraic geometry to the moment problem (truncated and full). Much less is known, however, about applications of truncated moment problem theory to real algebraic geometry. Moreover, our approach demonstrates that the calculation of the cardinality of a suitable collection of algebraic curves can be easily summarized in a single rank estimate for an appropriate square matrix built out of the initial data. Thus, this moment matrix approach works well whether the initial data consists of three conics, or four cubics, or $n+1$ algebraic curves of degree n, provided all leading terms are distinct; and it also works well if we consider multivariate algebraic curves with similar properties (Corollary 3.8).

To discuss our approach in detail, we first need to recall some basic facts about truncated moment problems; we do this in the next section.

2. A brief summary of the basic theory of truncated moment problems

Let

$$\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \cdots, \beta_{m,0}, \beta_{m-1,1}, \cdots, \beta_{1,m-1}, \beta_{0,m}\}$$

be a bivariate collection of real numbers, with $\beta_{00} > 0$; we say that $\beta^{(m)}$ is of order m. The *truncated moment problem* for $\beta^{(m)}$ entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ such that supp $\mu \subseteq \mathbb{R}^2$ and

$$\beta_{ij} = \int x^i y^j \, d\mu \ \ (i, j \in \mathbb{Z}_+, \ 0 \le i + j \le m).$$

In this case, we call μ a *representing measure* for β. When $m = 2n$, we define the *moment matrix* $M(n) \equiv M(n)(\beta^{(2n)})$ of β as

$$M(n) \equiv M(n)(\beta^{(2n)}) := (\beta_{\mathbf{i}+\mathbf{j}})_{\mathbf{i},\mathbf{j} \in \mathbb{Z}_+^2 : |\mathbf{i}|, |\mathbf{j}| \le n}.$$

It is well-known that for the existence of a representing measure, $M(n)$ must be necessarily positive semidefinite. To detect another necessary condition, let $\mathcal{P}_n$ denote the set of bivariate polynomials in $\mathbb{R}[x, y]$ whose degree is at most n. We label the columns in $M(n)$ with monomials $\mathit{1}, X, Y, X^2, XY, Y^2, \dots, X^n, \dots, Y^n$; that is, using the degree-lexicographic order. When $M(n)$ admits a column linear relation, it can be written as $P(X, Y) = \mathbf{0}$ for some polynomial $P(x, y) \equiv$

$\sum_{ij} a_{ij} x^i y^j \in \mathcal{P}_n$; the assignment $x^i y^j \mapsto X^i Y^j$ from monomials to matrix columns extends, by linearity, to the map $P \mapsto P(X, Y)$ from bivariate polynomials to the column space of $M(n)$; this is the so-called functional calculus for the column space of $M(n)$.

Consider now the zero set $\mathcal{Z}(P)$ of a polynomial P and define the *algebraic variety* of β by

$$\mathcal{V} \equiv \mathcal{V}_{M(n)} \equiv \mathcal{V}(\beta) := \bigcap_{p(X,Y)=\mathbf{0},\, \deg\, p \le n} \mathcal{Z}(p).$$

As in [2], if $M(n)$ admits a representing measure μ, then $\operatorname{supp}\ \mu \subseteq \mathcal{V}(\beta)$ and $\operatorname{rank} M(n) \le \operatorname{card} \operatorname{supp} \mu \le \operatorname{card} \mathcal{V}$; this inequality is referred to as the *variety condition*.

The first fundamental result in the truncated moment problem theory is the Flat Extension Theorem. For the reader's convenience, we provide below a statement that fits our needs.

Theorem 2.1. *(Flat Extension Theorem; cf.* [2, Theorem 5.13 and Corollary 5.14]*) Assume that $M(n) \equiv M(n)(\beta)$ admits a rank-preserving positive semi-definite extension $M(n+1)$. Then β has a* rank $M(n)$*-atomic representing measure. The extension $M(n+1)$ is called a flat extension. In particular, if* rank $M(n) =$ rank $M(n-1)$ *(that is, the case when $M(n)$ is a flat extension of the submatrix $M(n-1)$), it follows that β has a unique representing measure with moments of all orders, and it is a* rank $M(n)$*-atomic representing measure.*

Being able to build a moment matrix $M(n)$ which is a flat extension of $M(n-1)$, out of the relevant information coming from $\mathcal{V}(\boldsymbol{P})$, is the key step in the next section, and it will allow us to fully exploit Theorem 2.1. Here, as on page 357, $\mathcal{V}(\boldsymbol{P}) \equiv \mathcal{V}(P_0, \ldots, P_n) := \bigcap_{i=0}^{n} \mathcal{Z}(P_i)$ denotes the algebraic variety determined by a family $\boldsymbol{P}$ of $n+1$ bivariate polynomials with real coefficients.

3. Main results

We begin with a crucial fact about the location of the support of a representing measure.

Proposition 3.1. [2, Proposition 3.1] *Let $n \in \mathbb{N}$ and suppose that μ is a representing measure for $\beta^{(2n)}$. For $p \in \mathbb{R}[x, y]$ of total degree at most n,*

$$\operatorname{supp}\ \mu \subseteq \mathcal{Z}(p) \iff p(X, Y) = \mathbf{0}.$$

This proposition asserts that the zero set of each polynomial canonically associated to a column relation in $M(n)$ must contain the support of a representing measure. As a result, the same is true of the algebraic variety of $M(n)$; in short, if μ is a representing measure for $M(n)$, then $\operatorname{supp}\ \mu \subseteq \mathcal{V}$.

We are now ready to state and prove our first main result.

Theorem 3.2. *Any three nondegenerate conics with different leading terms can have at most three distinct points in their algebraic variety.*

Proof. Suppose we have three conics, described by quadratic polynomials C_1, C_2 and C_3, with different leading terms, and such that card $\mathcal{V} \geq 4$, where

$$\mathcal{V} := \mathcal{Z}(p_1) \cap \mathcal{Z}(p_2) \cap \mathcal{Z}(p_3).$$

Select any four points in $\mathcal{V}$ and label them $(x_1, y_1), \ldots, (x_4, y_4)$. Now let

$$\mu := \sum_{k=1}^{4} \delta_{(x_k, y_k)},$$

where $\delta_{(a,b)}$ denotes the point mass at $(a, b) \in \mathbb{R}^2$. Now use μ to build a moment matrix $M(2)$, by letting

$$\beta_{ij} := \int x^i y^j \, d\mu(x, y) \; (i, j \in \mathbb{Z}_+, \; 0 \leq i + j \leq 4).$$

Clearly, $\beta_{ij} = \sum_{k=1}^{4} x_k^i y_k^j$. (We interpret, as is commonly done, $x_k^0 = y_k^0 = 1$.) Since μ is a positive measure, the moment matrix $M(2)$ below is positive semidefinite:

$$\begin{array}{cccccc} 1 & X & Y & X^2 & XY & Y^2 \end{array}$$
$$\begin{bmatrix}
4 & \sum_{k=1}^4 x_k & \sum_{k=1}^4 y_k & \sum_{k=1}^4 x_k^2 & \sum_{k=1}^4 x_k y_k & \sum_{k=1}^4 y_k^2 \\
\sum_{k=1}^4 x_k & \sum_{k=1}^4 x_k^2 & \sum_{k=1}^4 x_k y_k & \sum_{k=1}^4 x_k^3 & \sum_{k=1}^4 x_k^2 y_k & \sum_{k=1}^4 x_k y_k^2 \\
\sum_{k=1}^4 y_k & \sum_{k=1}^4 x_k y_k & \sum_{k=1}^4 y_k^2 & \sum_{k=1}^4 x_k^2 y_k & \sum_{k=1}^4 x_k y_k^2 & \sum_{k=1}^4 y_k^3 \\
\sum_{k=1}^4 x_k^2 & \sum_{k=1}^4 x_k^3 & \sum_{k=1}^4 x_k^2 y_k & \sum_{k=1}^4 x_k^4 & \sum_{k=1}^4 x_k^3 y_k & \sum_{k=1}^4 x_k^2 y_k^2 \\
\sum_{k=1}^4 x_k y_k & \sum_{k=1}^4 x_k^2 y_k & \sum_{k=1}^4 x_k y_k^2 & \sum_{k=1}^4 x_k^3 y_k & \sum_{k=1}^4 x_k^2 y_k^2 & \sum_{k=1}^4 x_k y_k^3 \\
\sum_{k=1}^4 y_k^2 & \sum_{k=1}^4 x_k y_k^2 & \sum_{k=1}^4 y_k^3 & \sum_{k=1}^4 x_k^2 y_k^2 & \sum_{k=1}^4 x_k y_k^3 & \sum_{k=1}^4 y_k^4
\end{bmatrix}.$$

Moreover, by construction supp $\mu \subseteq \mathcal{V}$, so Proposition 3.1 readily implies that $M(2)$ admits three quadratic column relations with different leading terms, namely $C_1(X, Y) = \mathbf{0}$, $C_2(X, Y) = \mathbf{0}$ and $C_3(X, Y) = \mathbf{0}$. Thus, we know that $M(2)$ is a flat extension of $M(1)$ and rank $M(2) =$ rank $M(1) \leq 3$. By the Flat Extension Theorem, $M(2)$ must have a unique representing measure with moments of all orders, and it is a rank $M(1)$–atomic measure. However, μ is 4–atomic, and has moments of all orders, which is a contradiction. Therefore, the intersection of three nondegenerate conics in the plane, with different leading terms, must be a set of at most three points. □

We now extend this result, and its proof, to the case of algebraic curves described by polynomials of degree n with different leading terms. First, recall that the number of monomials in the indeterminates x and y of total degree at most $n - 1$ is $\binom{n+1}{2}$.

Theorem 3.3. *The cardinality of the algebraic variety of $n+1$ bivariate polynomials of degree n, with different leading terms, is at most $\binom{n+1}{2}$.*

Proof. As in the Proof of Theorem 3.2, we build a finitely atomic measure μ by placing a point mass $\delta_{(a,b)}$ at each point $(a,b) \in \mathcal{V}$, and the associated moment matrix $M(n)$. We then use the fact that all leading terms are different to claim that $M(n)$ is a flat extension of $M(n-1)$. As before, we know that $M(n-1)$ admits a representing measure with at most rank $M(n-1)$ atoms. By the uniqueness of representing measures with moments of all orders, we know that the cardinality of the support of μ must be at most rank $M(n-1)$, which is at most $\binom{n+1}{2}$, as desired. □

We now present some examples, which help illustrate both Theorem 3.2 and Theorem 3.3.

Example 3.4. Let $C_1(x,y) := y^2 + x^2 - 4$ and $C_2(x,y) := xy - 1$, and let

$$u := \sqrt{2+\sqrt{3}}, \qquad v := \sqrt{2-\sqrt{3}}.$$

The conics $C_1 \equiv 0$ and $C_2 \equiv 0$ intersect at the four-point set

$$\mathcal{V} := \{(u,v),(-u,-v),(v,u),(-v,-u)\}.$$

Assume that there is a third conic $C_3 \equiv 0$ going through these four points. Assume also that the leading terms of C_1, C_2 and C_3 are all different. Then $C_3(x,y) = x^2 + b_1 y + b_2 x + b_3$, where $b_1, b_2, b_3 \in \mathbb{R}$. Each point in $\mathcal{V}$ is of the form $(s, \frac{1}{s})$ (recall that $xy = 1$). As a result, we must have $C_3(s, \frac{1}{s}) = 0$; equivalently, $s^2 + \frac{b_1}{s} + b_2 s + b_3 = 0$, or

$$s^3 + b_1 + b_2 s^2 + b_3 s = 0. \tag{3.1}$$

Since $(-s, -\frac{1}{s})$ is also in $\mathcal{V}$, we simultaneously need to have

$$-s^3 + b_1 + b_2 s^2 - b_3 s = 0. \tag{3.2}$$

Subtracting (3.2) from (3.1), we obtain $s^3 + b_3 s = 0$, which implies that $s^2 = -b_3$. As a result, the square of s is fixed, a contradiction to the fact that, in $\mathcal{V}$, we allow s^2 to be either $2+\sqrt{3}$ or $2-\sqrt{3}$. We conclude that C_3 cannot pass through all four points in $\mathcal{V}$. This is consistent with Theorem 3.2.

Remark 3.5. *We offer a simple comparison with Bézout's Theorem.*
(i) Let P_1 and P_2 be two bivariate polynomials of degree n, without any nonconstant common factor. Bézout's Theorem establishes an upper bound for the cardinality of the intersection $\mathcal{V}(P_1, P_2)$, i.e., n^2. As a result, two nondegenerate conics without a common factor have at most four points in common. By contrast, our main result implies that three conics whose leading terms are all different (although with no assumption on the absence of a common factor) have at most three points in common.

(ii) Bézout's Theorem also implies that two cubics without a common factor have at most nine points in common. Our main result implies that four cubics with different leading terms (again, no assumption on the absence of a common factor) meet in at most six points.

Remark 3.6. *As we mentioned in the Introduction, it is possible to prove Theorem* 3.3 *using elementary algebraic geometry; for instance, via an application of Lemma 16.6 in* [6]. *The proof is definitely not short, and it entails passing to the homogenization of the polynomials defining the algebraic variety, and studying them in an appropriate projective subspace. The above mentioned lemma guarantees the existence of an element in this projective subspace passing through all points in the given algebraic variety $\mathcal{V}$ (appropriately viewed in projective space), and this establishes an upper bound for* card $\mathcal{V}$. *An alternative proof, provided by the referee, utilizes very standard and well known tools from the theory of Gröbner bases. Indeed, consider first* [1, Proposition 4, p. 250] *and let I be the ideal generated by $\boldsymbol{P}$, so that $\boldsymbol{V}(I) = \mathcal{V}(\boldsymbol{P})$. Then $\mathbb{R}[x,y]/I$ is isomorphic, as a $\mathbb{R}$–vector space, to* $\operatorname{Span}\{x^\alpha y^\beta : \alpha+\beta \le n-1\}$. *Next, recall from* [1, Proposition 7, p. 253]*(i) that the cardinality of $\boldsymbol{V}(I)$ is at most the dimension of $\mathbb{R}[x,y]/I$. Now,*

$$\dim \operatorname{Span}\{x^\alpha y^\beta : \alpha+\beta \le n-1\} = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

It follows that card $\mathcal{V}(\boldsymbol{P}) \le \binom{n+1}{2}$.

Remark 3.7. *In* [4], *we solved the sextic (complex) truncated moment problem with a column relation of the form $Z^3 - itZ - u\bar{Z} = 0$ where u and t are real numbers such that $0 < u < t < 2u$. The cardinality of the zero set of the associated polynomial $q_7(z,\bar{z}) := z^3 - itz - u\bar{z}$ is* 7 *(cf.* [4, Lemma 2.3]*). Alongside this polynomial, there was a second polynomial, $q_{LC}(z,\bar{z}) := i(z - i\bar{z})(z\bar{z} - u)$, whose zero set consisted of the union of the circle of radius $\sqrt{u}$ and the bisector $y = x$; it is also the case that $\mathcal{Z}(q_7) \subseteq \mathcal{Z}(q_{LC})$. As a result, there are* 7 *common zeros of these two polynomials. Now, each polynomial splits into real part and imaginary part, so a priori we have four (real) cubic polynomials, as follows:* $\operatorname{Re} q_7(x,y) = x^3 - 3xy^2 + ty - ux$, $\operatorname{Im} q_7(x,y) = -y^3 + 3x^2y - tx + uy$, $\operatorname{Re} q_{LC}(x,y) = \operatorname{Im} q_{LC}(x,y) = -ux + x^3 + uy - x^2y + xy^2 - y^3$. *Since* $\operatorname{Re} q_{LC} = \operatorname{Im} q_{LC}$, *we end up with only three polynomials. However, we can safely add a new polynomial, $q_{3PL}(x,y) := (y-x)(y-x-q+p)(y-x-p+q) = (p^2-2pq+q^2)x - (p^2-2pq+q^2)y - x^3 + 3x^2y - 3xy^2 + y^3$, which vanishes on the three parallel lines $y = x$, $y = x+q-p$ and $y = x+p-q$; here $\pm(p+iq)$ and $\pm(q+ip)$ are the four points of intersection of $\mathcal{Z}(q_7)$ with the circle of radius $\sqrt{u}$ (cf.* [4, Figure 1]*). The corresponding (monic) leading terms are xy^2, y^3, y^3, and y^3, respectively. We observe that they are not all different, so the fact that* card $\mathcal{V}(\boldsymbol{P}) = 7 > 6 = \binom{4}{2}$ *(where $\mathcal{V}(\boldsymbol{P})$ is the set of common zeros of the four polynomials), is not in contradiction with our main result.*

We conclude this section with an extension of our results to the case of multivariate polynomials. Since the Flat Extension Theorem holds for multivariate moment matrices, it is rather straightforward to formulate and prove the relevant result. We leave the fine details to the interested reader, and show here only the sketch of the proof.

Corollary 3.8. *The cardinality of an algebraic variety of $n+1$ polynomials of degree n in $x_1, \dots, x_d$ with all different leading terms is at most $\binom{d+n-1}{n-1}$.*

Proof. Suppose there are $n+1$ polynomials $p_1, \dots, p_{n+1}$ of degree n in $x_1, \dots, x_d$ with different leading terms such that card $\mathcal{V} \geq \binom{d+n-1}{n-1} + 1$, where

$$\mathcal{V} := \mathcal{Z}(p_1) \cap \cdots \cap \mathcal{Z}(p_{n+1}).$$

As before, we recall that $\binom{d+n-1}{n-1}$ is the number of all monomials in $x_1, \dots, x_d$ whose degree is at most $n-1$. Select $\tau := \binom{d+n-1}{n-1} + 1$ points

$$(x_1^{(1)}, \dots, x_d^{(1)}), \dots, (x_1^{(\tau)}, \dots x_d^{(\tau)})$$

from $\mathcal{V}$ and generate a moment matrix $M_d(n) \equiv M_d(n)[\mu]$ with the τ-atomic measure $\mu := \sum_{k=1}^{\tau} \delta_{(x_1^{(k)}, \dots x_d^{(k)})}$. Since supp $\mu \subseteq \mathcal{V}$, it follows from Proposition 3.1 that there are $n+1$ columns $p_1(X_1, \dots, X_d) = \mathbf{0}, \dots, p_{n+1}(X_1, \dots, X_d) = \mathbf{0}$ in $M_d(n)$. Thus, we know $M_d(n)$ is a flat extension of $M_d(n-1)$ and rank $M_d(n) \leq \tau - 1$. The multivariable Flat Extension Theorem [3] now implies that $M_d(n)$ must have a unique rank $M_d(n)$-atomic measure. However, μ is τ-atomic, this is a contradiction. □

4. An application

We briefly investigate the structure of polynomials vanishing on V, and so we consider an ideal of $\mathbb{R}[x_1, \dots, x_d]$:

$$I(V) := \{f \in \mathbb{R}[x_1, \dots, x_d] : f|_V \equiv 0\}.$$

Also, we recall a basic result about multivariate polynomials.

Theorem 4.1. *(Division Algorithm* [1]*) Fix a monomial order on $\mathbb{Z}_+^n$, and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as*

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, r \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination of monomials with coefficients in $\mathbb{R}$, none of which is divisible by any leading terms of $f_1, \cdots, f_s$. We call r a remainder of f on division by F. Furthermore, if $a_i f_i \neq 0$, then we have

$$\text{multideg}(f) \geq \text{multideg}(a_i f_i).$$

Let V be an algebraic set in $\mathbb{R}^d$ with card $V \leq \binom{d+n-1}{n-1}$, and assume that we have $n+1$ polynomials $p_0, p_1, p_2, \dots, p_n$ of degree n vanishing on V, with different leading terms. Theorem 4.1, the proofs of Theorems 3.2 and 3.3, and the main ideas in [5, Section 4] allow us to obtain the following result about $I(V)$.

Theorem 4.2. *Let V, $I(V)$ and $p_0, \dots, p_n$ be as before, and let $q \in I(V)$ with $\deg q \geq n$. Then q belongs to the ideal generated by $p_0, \dots, p_n$.*

Proof. For simplicity, assume that $d = 2$, and let $q \in \mathbb{R}[x, y]$ be a polynomial vanishing on V, with $\deg q \geq n$. By Theorem 4.1, we can write

$$q = a_0 p_0 + \cdots + a_n p_n + r,$$

where $a_i, r \in \mathbb{R}[x, y]$, and either $r = 0$ or r is a linear combination of monomials, none of which is divisible by any leading terms of $p_0, \dots, p_n$. Since $\mathrm{LT}(p_i) \neq \mathrm{LT}(p_j)$ for all $i \neq j$, it follows that $\deg r \leq n - 1$. Moreover, since $q, p_0, \dots, p_n$ all vanish on V, so does r. As in the Proof of Theorem 3.2, we build a finitely atomic positive measure μ by placing a point mass at every point of V. We then consider the moment matrix $M(n)$ associated to μ, and its submatrix $M(n-1)$. By the functional calculus for the column space $\mathcal{C}_{M(n-1)}$ of $M(n-1)$, we know that $r(X, Y) = \mathbf{0}$, as column vectors in $\mathcal{C}_{M(n-1)}$. Let $\mathcal{B}$ be a basis for $\mathcal{C}_{M(n-1)}$. As in [5, Proof of Theorem 4.2], we know that the Vandermonde matrix $W_{V,\mathcal{B}}$ associated to V and $\mathcal{B}$ is invertible, and therefore r must be identically zero. This proves that q is in the ideal generated by $p_0, \dots, p_n$, as desired. □

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