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On equivariant oriented cohomology of Bott-Samelson varieties

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Abstract. For any Bott-Samelson resolution of the flag variety, and any torus equivariant oriented cohomology, we compute the restriction formula of certain basis η_L of equivariant oriented cohomology of Bott-Samelson variety determined by the projective bundle formula. As an application, we show that the equivariant oriented cohomology of Bott-Samelson variety embeds into the equivariant oriented cohomology of T-fixed points, and the image can be characterized by using the Goresky-Kottwitz-MacPherson (GKM) description. Furthermore, we compute the push-forward of the basis η_L onto equivariant oriented cohomology of flag variety, and their restriction formula.

Contents

1. Introduction

Let G/B be a flag variety. For each w in the Weyl group W, and a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, one defines the variety (see Definition 3.1)

$$\hat{X}_{I_w} = P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_l}/B.$$

Here P_{i_j} is the minimal parabolic subgroup corresponding to the simple root α_{i_j}. Multiplication of all the coordinates defines a canonical map $q_{I_w} : \hat{X}_{I_w} \to G/B$, which is proper and birational over the Schubert variety $X(w)$ of w. This is called a Bott-Samelson resolution of $X(w)$. These resolutions play an important role in Schubert calculus and representation theory.

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Let T be a split maximal torus in Borel subgroup B of G. One has a natural T-action on the flag variety. We are interested in $h_T(\hat{X}_{I_w})$, where h_T is an (equivariant) oriented cohomology theory in the sense of Levine-Morel. Examples of h include the Chow group (singular cohomology) and K-theory. For any h, it is proved in [CZZ12, CZZ19, CZZ14] that, after fixing a reduced decomposition I_w for each $w \in W$, the push-forward $(q_{I_w})_*(1)$ in $h_T(G/B)$ of the fundamental class defines a basis of $h_T(G/B)$ over the base ring $h_T(\mathrm{pt})$. This enables the authors of *loc. it.* to construct the algebraic replacement of $h_T(G/B)$, and provides a standard setting for generalized Schubert calculus. For further study on equivariant oriented cohomology of T-varieties following this method, please refer to [DZ20, GZ20, LZZ16, CNZ19, Z20].

Let us consider $h_T(\hat{X}_I)$ for a general sequence $I = (i_1, ..., i_l)$. The set $\hat{X}_I^T$ of T-fixed points of $\hat{X}_I$ is in bijection with the power set of $[l] = \{1, 2, ..., l\}$. Denote by $\mathbf{j} : \hat{X}_I^T \to \hat{X}_I$ the canonical embedding. Our main result is the following:

Theorem 1.1. *(Corollary 4.4) For any sequence I, the pull-back to T-fixed points $\mathbf{j}^* : h_T(\hat{X}_I) \to h_T(\hat{X}_I^T)$ is injective.*

Furthermore, we show that elements in the image of $\mathbf{j}^*$ satisfy the Goresky-Kottwitz-MacPherson (GKM) description (see Theorem 4.5). Indeed, in the case where the sequence $I = (i_1, ..., i_l)$ consists of distinct i_j's, we prove that the GKM description uniquely characterizes the image (Theorem 4.6).

Let us mention the idea of the proof briefly. Since $\hat{X}_I$ is constructed as a tower of $\mathbb{P}^1$-bundles, there are canonically defined algebra generators $\eta_j \in h_T(\hat{X}_I)$ corresponding to each parabolic subgroup P_{i_j} in $\hat{X}_I$. Each η_j satisfies certain quadratic relation. Therefore, for each subset L of $[l]$, denoting by η_L the product of η_j with j in L, then $\{\eta_L | L \in [l]\}$ forms a basis of $h_T(\hat{X}_I)$.

We compute the restriction $\mathbf{j}^*(\eta_L)$ explicitly (Theorem 4.3). The computation uses the characteristic map $\mathbf{c} : h_T(\mathrm{pt}) \to h_T(\hat{X}_I)$ induced by the map sending a character λ of T to the first Chern class of the associated line bundle over $\hat{X}_I$. We then use the explicit formula of $\mathbf{j}^*(\eta_L)$ to prove Theorem 1.1, and use the GKM description to characterize the image of $\mathbf{j}^*$.

As another application of the computation of $\mathbf{j}^*(\eta_L)$, we also compute the push-forward of η_L via the canonical map $q_I : \hat{X}_I \to G/B$. We show that the push-forward $(q_I)_*(\eta_L)$ coincides with the Bott-Samelson class corresponding to the sequence $I \backslash L$.

For future applications, one would apply the restriction formula (Theorem 4.3) and the push-forward formula (Theorem 5.4) in the study of motivic Chern (mC) classes in K-theory. MC classes are certain K-theory classes associated to constructible subsets of T-varieties. For details, please refer [AMSS17, RTV15, RTV17]. They are closely related with the K-theoretic stable basis of Springer resolutions, defined by Maulik-Okounkov [MO12, O15] and studied in [SZZ17, SZZ19]. Indeed, Mihalcea has some recent work on the relationship between push-forward of MC classes of Bott-Samelson varieties and the Kazhdan-Lusztig basis of Hecke algebra. The authors hope to apply the computation of this paper to understanding this relationship.

The paper is organized as follows: In Section 2, we recall necessary notions of equivariant oriented cohomology theory, formal group algebra, and the characteristic map **c**. In Section 3, we recall some basic facts about Bott-Samelson varieties. In Section 4, we compute the restriction formula (Theorem 4.3) which was used to prove the injectivity of the pull-back map $\mathbf{j}^*$ and the GKM description (Theorem 4.5). In Section 5, we compute the push-forward of the basis $\{\eta_L\}$ onto $h_T(G/B)$.

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2. Equivariant oriented cohomology theory

In this section, we define some notation, and collect some basic notions and facts about equivariant oriented cohomology theory.

Let G be a split semisimple linear algebraic group over a field k, with rank n. Let T be a split maximal torus of G and $B \subset G$ be a Borel subgroup. Let Σ be the set of roots of G, and $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the set of simple roots corresponding to B. Let P_i be the minimal parabolic subgroup corresponding to the simple root α_i. The Weyl group W of G is generated by $\{s_{\alpha_1}, \dots, s_{\alpha_n}\}$ where s_{α_i} is the reflection corresponding to α_i. Note that W can be identified with $N_G(T)/T$. Sometimes we will understand $s_i = s_{\alpha_i}$ as a lifting of an element in G. We denote the group of characters of T by Λ. For each positive integer l, denote $[l] = \{1, 2, \dots, l\}$.

Let F be a formal group law over the commutative ring R. Examples include the additive formal group law $F_a = x + y$ over $\mathbb{Z}$, and the multiplicative formal group law $F = x + y - \beta xy$ over $\mathbb{Z}[\beta, \beta^{-1}]$.

Definition 2.1. Let $R[\![x_\Lambda]\!] := R[\![x_\lambda | \lambda \in \Lambda]\!]$ be the power series ring. Let J_F be the closure of the ideal generated by x_0 and $x_{\lambda+\mu} - F(x_\lambda, x_\mu)$, $\lambda, \mu \in \Lambda$. We define the formal group algebra $R[\![\Lambda]\!]_F$ to be the quotient

$$R[\![\Lambda]\!]_F = R[\![x_\Lambda]\!]/J_F.$$

It is proved in [CPZ13, Corollary 2.13] that $R[\![\Lambda]\!]_F$ is non-canonically isomorphic to the formal power series ring with n variables. For simplicity, we denote $S = R[\![\Lambda]\!]_F$. Note that by definition, $x_{-\lambda}$ is the formal inverse of x_λ, that is, $F(x_\lambda, x_{-\lambda}) = 0$. Since any formal group law F is always of the form

$$F(x, y) = x + y + a_{11}xy + \text{ higher order terms}, \quad a_{11} \in R,$$

so it is not difficult to see that $x_{-\lambda} = -x_\lambda + x_\lambda^2 f(x_\lambda)$ for some $f(t) \in R[[t]]$. Therefore, $\frac{x_\lambda}{x_{-\lambda}}$ is an invertible element in S.

Example 2.2. (1) Let F_a be an additive formal group law, then we have a ring isomorphism

$$R[\![\Lambda]\!]_{F_a} \cong S_R(\Lambda)^\wedge, \quad x_\lambda \mapsto \lambda,$$

where $S_R(\Lambda)$ is the symmetric algebra of Λ and the completion is done at the augmentation ideal.

(2) Let $R[\Lambda]$ be the group algebra $\left\{\sum_j a_j e^{\lambda_j} | a_j \in R, \lambda_j \in \Lambda\right\}$. Then we have isomorphism

$$R[\![\Lambda]\!]_{F_m} \cong R[\Lambda]^\wedge, \quad x_\lambda \mapsto \beta^{-1}(1-e^\lambda),$$

where the completion $^\wedge$ is done at the augmentation ideal.

Throughout this paper, we assume that the root datum of G together with the formal group law F satisfy the regularity condition of [CZZ12, Definition 4.4]. For example, this is satisfied if 2 is regular in R. Please consult *loc.it.* for more details. In particular, x_α is regular in S, for any root α of G. The Weyl group action on Λ induces an action of W on $R[\![\Lambda]\!]_F$ by $s_\alpha(x_\lambda) = x_{s_\alpha(\lambda)}$. In particular, we have

Lemma 2.3. [CPZ13, Corollary 3.4] *For any $v, w \in W$, any root α of G and $p \in S$, we have*

$$\frac{vs_\alpha w(p) - vw(p)}{x_{v(\alpha)}} \in S.$$

Proof. According to [CPZ13, Corollary 3.4], we know that $s_\alpha w(p) - w(p)$ is uniquely divisible by x_α. In other word,

$$\frac{s_\alpha w(p) - w(p)}{x_\alpha} \in S.$$

Then

$$v(\frac{s_\alpha w(p) - w(p)}{x_\alpha}) = \frac{vs_\alpha w(p) - vw(p)}{x_{v(\alpha)}} \in S.$$

□

In particular, taking $w = v = e$, we see that $x_\alpha | (p - s_\alpha(p))$. We can then define the Demazure operator $\Delta_\alpha : S \to S$ by

$$\Delta_\alpha(p) = \frac{p - s_\alpha(p)}{x_\alpha}. \tag{1}$$

Remark 2.4. By direct calculation, we have the following formulas: for $p, q \in S$,

$$s_\alpha \Delta_\alpha(p) = -\Delta_{-\alpha}(p) \tag{2}$$

$$\Delta_\alpha(pq) = \Delta_\alpha(p)q + p\Delta_\alpha(q) - \Delta_\alpha(p)\Delta_\alpha(q)x_\alpha. \tag{3}$$

We follow [CZZ14, §2] on the assumption of equivariant oriented cohomology, however, we only consider the case when the group is fixed to be the torus T. Roughly speaking, it is an additive contravariant functor h_T from the category of smooth quasi-projective T-varieties to the category of commutative rings with units, satisfying the following axioms: existence of push-forwards for projective morphisms, existence of total equivariant characteristic class for

T-equivariant bundle, Quillen's formula, etc. [CZZ14, §2]. Moreover, there exists a formal group law F over $R = h_T(\mathrm{pt})$ such that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are locally free sheaves of rank one, then

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2)).$$

It is proved in [CZZ14, Theorem 3.3] that

$$S = R[\![\Lambda]\!]_F \cong h_T(\mathrm{pt}), \quad x_\lambda \mapsto c_1(\mathcal{L}_\lambda),$$

where $\mathcal{L}_\lambda$ is the associated line bundle. As an immediate consequence, we see that if the variety X is finite set of points of the form $\mathrm{Spec}(k)$ (with trivial T-action), then

$$h_T(X) = F(X; S),$$

where the latter is the set of all maps from $X(k)$ to S. It has a S-basis $f_x, x \in X$, and is a ring with product defined by

$$f_x \cdot f_y = \delta_{x,y} f_x, \quad \text{and unit} \sum_{x \in X} f_x.$$

By functoriality, if $p : X \to Y$ is a T-equivariant map between two finite discrete sets of points on which T acts trivially, then

$$p^*(f_y) = \sum_{x \in f^{-1}(y)} f_x, \quad p_*(f_x) = f_{p(x)}. \tag{4}$$

We recall the definition of the characteristic map. Let X be a T-variety on which B acts on the right, and the T and B actions commute. Moreover, suppose the quotient X/B exists and $X \to X/B$ is a T-equivariant principal bundle. Following [CPZ13, §10.2], we can define a ring homomorphism

$$\mathbf{c} : S = h_T(\mathrm{pt}) \to h_T(X/B), \quad x_\lambda \mapsto c_1(\mathcal{L}_\lambda).$$

It is called the characteristic map.

Let α be a simple root with corresponding minimal parabolic subgroup P_α. Consider the fiber product $X' = X \times^B P_\alpha$, then X' is a T-equivariant principal P_α-bundle over X/B. Denote $p : X'/B \to X/B$, and there is a zero section

$$\sigma : X/B \to X'/B, x \mapsto (x, 1). \tag{5}$$

As in [CPZ13, §10.5], we have

$$h_T(X'/B) \cong h_T(X/B)[\xi]/(\xi^2 - y\xi), \quad \xi = \sigma_*(1), \quad y = p^*\sigma^*\xi. \tag{6}$$

The following properties can be proved similarly as their non-equivariant versions in [CPZ13, §10].

Lemma 2.5. *Denote $\mathbf{c} : S \to h_T(X/B)$ and $\mathbf{c}' : S \to h_T(X'/B)$. For each $\lambda \in \Lambda$, denote the associated line bundles on X/B and X'/B by $\mathcal{L}_\lambda$ and $\mathcal{L}'_\lambda$, respectively.*

(1) We have $\sigma^*\xi = c_1(\mathcal{L}_{-\alpha}) = \mathbf{c}(x_{-\alpha})$.
(2) $y = p^*\sigma^*\xi = p^*\mathbf{c}(x_{-\alpha})$.
(3) For any $u \in S$, we have

$$\sigma^*\mathbf{c}'(u) = \mathbf{c}(u), \quad \mathbf{c}'(u) = p^*\mathbf{c}(s_\alpha(u)) + p^*\mathbf{c}(\Delta_{-\alpha}(u)) \cdot \xi.$$

Note that in [CPZ13, §10], the projective bundle theorem plays a key role. The equivariant projective bundle theorem for equivariant $\mathbb{P}^1$-bundle is proved in [CZZ14, Lemma 4.6], which then can be used to prove Lemma 2.5.

Lemma 2.6. *If $X = B, X' = P_\alpha$, there are two T-fixed points in P_α/B, indexed by $e, s_\alpha \in W$, whose embeddings are denoted by $\sigma_e, \sigma_\alpha : \mathrm{pt} \to P_\alpha/B$. Then $\sigma_e^*\mathbf{c}'(u) = u, \sigma_\alpha^*\mathbf{c}'(u) = s_\alpha(u)$.*

Proof. We have $\mathbf{c}(u) = u$, $p\sigma_e = \mathrm{id}$, $p\sigma_\alpha = \mathrm{id}$, and σ_e coincides with σ in (5), so the first identity of Lemma 2.5.(3) implies $\sigma_e^*\mathbf{c}'(u) = \mathbf{c}(u)$. On the other hand, applying σ_α^* on the the second identity of Lemma 2.5.(3), and using $\sigma_\alpha^*(\xi) = \sigma_\alpha^*(\sigma_e)_*(1) = 0$, we get that $\sigma_\alpha^*\mathbf{c}'(u) = s_\alpha(u)$. □

3. Bott-Samelson varieties

In this section, we collect some facts about Bott-Samelson varieties .

Definition 3.1. For any sequence $I = (i_1, i_2, \dots, i_l)$ with $1 \le i_j \le n$, we define the variety $\hat{X}_I$ to be

$$\hat{X}_I = P_{i_1} \times^B P_{i_2} \times^B \dots \times^B P_{i_l}/B,$$

which is the orbit space in $P_{i_1} \times P_{i_2} \times \dots \times P_{i_l}$ under B^l-action defined by

$$(g_1, \cdots, g_l)(b_1, \cdots, b_l) = (g_1 b_1, b_1^{-1} g_2 b_2, \cdots, b_{l-1}^{-1} g_l b_l),$$

where $b_i \in B$ and $g_j \in P_{i_j}$. Here the right B-action is given by right multiplication on the last coordinate. If $I = \emptyset$, then we set $\hat{X}_\emptyset = \mathrm{pt}$. The variety $\hat{X}_I$ is called the Bott-Samelson variety corresponding to I. It has an obvious T-action by left multiplication on the first coordinate. We denote T-fixed points on $\hat{X}_I$ by $\mathcal{E}_I$.

Since $P_i/B \cong \mathbb{P}^1$, so we have a sequence of $\mathbb{P}^1$—bundles:

$$\hat{X}_I \underset{\sigma_l}{\rightleftarrows} \hat{X}_{(i_1,\dots,i_{l-1})} \underset{\sigma_{l-1}}{\rightleftarrows} \dots \underset{\sigma_2}{\rightleftarrows} \hat{X}_{(i_1)} \underset{\sigma_1}{\rightleftarrows} pt, \tag{7}$$

where σ_i, $1 \le i \le l$ are the zero sections. Multiplication of all factors of $\hat{X}_I$ induces a map

$$q_I : \hat{X}_I \to G/B.$$

Denote by $\pi_i : G/B \to G/P_i$ the canonical map, and denote $I' = (i_1, \dots, i_{l-1})$. We then have the following transverse Cartesian diagram:

$$\begin{array}{ccc} \hat{X}_I & \xrightarrow{q_I} & G/B \\ \downarrow p & & \downarrow \pi_{\alpha_l} \\ \hat{X}_{I'} & \xrightarrow{\pi_{\alpha_l} \circ q_{I'}} & G/P_{\alpha_l} \end{array} \quad . \tag{8}$$

So we have the base-change formula

$$(q_I)_* p^* = \pi_{\alpha_l}^* (\pi_{\alpha_l} q_{I'})_*.$$

The operator

$$\pi_{\alpha_l}^*(\pi_{\alpha_l})_* : h_T(G/B) \to h_T(G/B)$$

is called the push-pull operator.

Denote by $\mathbf{c}_I : S \to h_T(\hat{X}_I)$ the characteristic map. The following proposition describes the R-algebra structure of equivariant oriented cohomology of Bott-Samelson varieties.

Proposition 3.2. [CPZ13, §11.3] *We have the following presentation*

$$h_T(\hat{X}_I) \cong h_T(pt)[\eta_1, \eta_2, \dots, \eta_l]/(\{\eta_j{}^2 - y_j\eta_j \,|\, j = 1, \dots, l\}),$$

where

$$y_j = p_j^* \mathbf{c}_{(i_1,\dots,i_{j-1})}(x_{-\alpha_{i_j}}), \quad \eta_j = p_j^*(\sigma_j)_*(1),$$

with p_j^* the pull-back from $h_T(\hat{X}_{(i_1,\dots,i_j)})$ to $h_T(\hat{X}_I)$.

For ordinary oriented cohomology, this theorem is proved in [CPZ13]. The idea of the proof is to apply the projective bundle formula to the sequence of $\mathbb{P}^1$−bundle (7). One can check that all the arguments hold in the equivariant setting, which can be used to prove Proposition 3.2.

For each subset $L \in [l]$, define

$$\eta_L = \prod_{j \in L} \eta_j \in h_T(\hat{X}_I).$$

Since in Proposition 3.2, the y_j does not belong to the coefficient ring $h_T(\mathrm{pt})$, the presentation of $h_T(\hat{X})$ is not satisfactory. To get a polynomial presentation of it, we follow the idea in [CPZ13, Theorem 11.4].

Lemma 3.3. *For any sequence $I = (i_1, ..., i_l)$, we have*

$$\mathbf{c}_I(u) = \sum_{L \subset [l]} \theta_{I,L}(u)\eta_L, \quad u \in S,$$

where $\theta_{l,L} = \theta_1 \cdots \theta_l$ *with* $\theta_j = \begin{cases} \Delta_{-\alpha_{i_j}}, & \text{if } j \in L, \\ s_{i_j}, & \text{otherwise.} \end{cases}$

Proof. We prove it by induction on l. If $l = 1$, from Lemma 2.5, we have

$$\mathbf{c}_{i_1}(u) = p^*\mathbf{c}_\emptyset(s_{i_1}(u)) + p^*\mathbf{c}_\emptyset(\Delta_{-\alpha_{i_1}}(u)) \cdot \eta_1.$$

Note that the characteristic map $\mathbf{c}_\emptyset : S \to h_T(pt)$ is the identity map. So it holds.

Now assume the conclusion holds for $I' := (i_1, \dots, i_{l-1})$. Denote the canonical projection from $\hat{X}_I$ to $\hat{X}_{I'}$ by p. By Lemma 2.5 we have

$$\begin{aligned} \mathbf{c}_I(u) &= p^*\mathbf{c}_{I'}(s_{i_l}(u)) + p^*\mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot \eta_l \\ &= \sum_{L \subset [l-1]} \theta_{l-1,L}(s_{i_l}(u))\eta_L + \sum_{L \in [l-1]} \theta_{l-1,L}(\Delta_{-\alpha_{i_j}}(u))\eta_L \cdot \eta_l \\ &= \sum_{L \subset [l]} \theta_{l,L}(u)\eta_L. \end{aligned}$$

□

Proposition 3.4. [CPZ13, Theorem 11.4] *The ring $h_T(\hat{X}_I)$ is a quotient of the polynomial ring $S[\eta_1, \eta_2, \dots, \eta_l]$ modulo the relations*

$$\eta_j^2 = \sum_{L\subset[j-1]} \theta_{j-1,L}(x_{-\alpha_{i_j}})\eta_L\eta_j, \quad j \in [l].$$

Proof. Denote $K = (i_1, \dots, i_{j-1})$ and $p : \hat{X}_I \to \hat{X}_K$. By definition of y_j and Lemma 3.3, we have

$$y_j = p^*\mathbf{c}_K(x_{\alpha_{i_j}}) = p^*(\sum_{L\subset[j-1]} \theta_{j-1,L}(x_{-\alpha_{i_j}})\eta_L) = \sum_{L\subset[j-1]} \theta_{j-1,L}(x_{-\alpha_{i_j}})\eta_L.$$

The statement then follows from the fact that $\eta_j^2 = y_j\eta_j$. □

Corollary 3.5. *The S-module $h_T(\hat{X}_I)$ is free with basis $\{\eta_L | L \in \mathcal{P}_l\}$.*

Example 3.6. For $SL(4)$ whose simple roots are $\alpha_1, \alpha_2, \alpha_3$, let us consider Bott-Salmelson $\hat{X}_I = P_{\alpha_1} \times^B P_{\alpha_2} \times^B P_{\alpha_3}/B$. Then $h_T(\hat{X}_I)$ is a polynomial algebra generated by η_1, η_2, η_3 with following quotient relations:

$$\begin{aligned}
\eta_1^2 &= x_{-\alpha_1}\eta_1,\\
\eta_2^2 &= x_{-\alpha_1-\alpha_2}\eta_1 + \frac{x_{-\alpha_2} - x_{\alpha_1-\alpha_2}}{x_{-\alpha_1}}\eta_1\eta_2,\\
\eta_3^2 &= x_{\alpha_1-\alpha_2-\alpha_3}\eta_3 + \frac{x_{-\alpha_3-\alpha_2} - x_{2\alpha_1-\alpha_2-\alpha_3}}{x_{-\alpha_1}}\eta_1\eta_3 + \frac{x_{\alpha_3} - x_{\alpha_1+\alpha_2-\alpha_3}}{x_{-\alpha_1-\alpha_2}}\eta_2\eta_3\\
&+ (\frac{x_{-\alpha_3 - x_{\alpha_2-\alpha_3}}}{x_{-\alpha_2}x_{-\alpha_1}} - \frac{x_{-\alpha_3} - x_{\alpha_2-\alpha_1-\alpha_3}}{x_{\alpha_1-\alpha_2}x_{-\alpha_1}})\eta_1\eta_2\eta_3.
\end{aligned}$$

Let us consider some geometry information on $\hat{X}$, and its T-fixed points. We fix some notations first. For any $L \subset [l]$, define

$$(\hat{X}_I)_L = \{[g_1, g_2, \dots, g_l] \in \hat{X}_I |\ g_j \in B \text{ if } j \notin L, \text{ and } g_i \notin B \text{ if } j \in L\} \subset \hat{X}_I,$$

and

$$v_j^L = \prod_{k\in L\cap[j]} s_{i_k}, \quad v^L := v_l^L = \prod_{k\in L} s_{i_k}.$$

The following lemma will be used in the proof of Theorem 4.6.

Lemma 3.7. *Let $I = (i_1, \dots, i_l)$ be a sequence such that i_j are all distinct. Let $L \subset [l]$, then $v_{j-1}^L(\alpha_j), j \in L^c$ are all distinct. In particular, $v_{j-1}^L(x_{-\alpha_{i_j}}), j \in L^c$ are all distinct.*

Proof. Suppose $j_1, j_2 \in L^c$ and $j_1 < j_2$. Then $L \cap [j_1] \subseteq L \cap [j_2]$. There are two cases.

Case 1: $L \cap [j_1] = L \cap [j_2]$. Then

$$v_{j_1-1}^L(\alpha_{i_{j_1}}) = (\prod_{k\in L\cap[j_1]} s_{i_k})(\alpha_{i_{j_1}}),$$

and

$$v^L_{j_2-1}(\alpha_{i_{j_2}}) = (\prod_{k\in L\cap[j_2]} s_{i_k})(\alpha_{i_{j_2}}) = (\prod_{k\in L\cap[j_1]} s_{i_k})(\alpha_{i_{j_2}}).$$

They are not equal since $\alpha_{i_{j_1}} \neq \alpha_{i_{j_2}}$.

Case 2. $L\cap[j_1] \subsetneq L\cap[j_2]$. Denote $M = (L\cap[j_2])\backslash(L\cap[j_1])$. Then

$$v^L_{j_1-1}(\alpha_{i_{j_1}}) = (\prod_{k\in L\cap[j_1]} s_{i_k})(\alpha_{i_{j_1}}),$$
$$v^L_{j_2-1}(\alpha_{i_{j_2}}) = (\prod_{k\in L\cap[j_1]} s_{i_k})(\prod_{k'\in M} s_{i_{k'}})(\alpha_{i_{j_2}}).$$

By definition of the Weyl group action,

$$(\prod_{k'\in M} s_{i_{k'}})(\alpha_{i_{j_2}}) = \alpha_{i_{j_2}} + \sum_{k'\in M} c_{k'}\alpha_{i_{k'}}, \quad c_{k'}\in\mathbb{Z},$$

which is different from $\alpha_{i_{j_1}}$, since the set $\{\alpha_{i_{j_1}}, \alpha_{i_{j_2}}\}\cup\{\pm\alpha_{i_{k'}}|k'\in M\}$ is linearly independent. Thus $v^L_{j_1-1}(\alpha_{i_{j_1}})$ and $v^L_{j_2-1}(\alpha_{i_{j_2}})$ are not equal to each other. □

The following lemma recalled from [W04, Proposition 2.6] provides some geometric information on the Bott-Samelson variety, which is useful for our computation.

Lemma 3.8. *(1) The set $\hat{X}_I^T$ of T-fixed points in $\hat{X}_I$, consists of 2^l points*

$$[g_1, g_2, \dots, g_l]$$

where $g_j\in\left\{e, s_{i_j}\right\}$. Here we think of s_{i_j} as in $W\cong N_G(T)/T$ and pick a preimage for s_{i_j} in $N_G(T)\subset G$. Consequently, we have bijection of sets from the power set $\mathcal{P}_l := \mathcal{P}([l])$ to $\hat{X}_I^T$,

$$L\mapsto \mathrm{pt}_L := [g_1,\dots,g_l], \quad g_j = \begin{cases} s_{i_j}, & \text{if } j\in L,\\ e, & \text{if } j\notin L.\end{cases}$$

(2) The set $(\hat{X}_I)_L$ is T-stable, contains the fixed point pt_L, and is isomorphic to the affine space of dimension $|L|$. The variety $\hat{X}_I$ has a decomposition $\coprod_{L\in\mathcal{E}_I}(\hat{X}_I)_L$.

(3) Suppose $L, L'\subset[l]$. then $\mathrm{pt}_L\in\overline{(\hat{X}_I)_{L'}}$ if and only if $L\subset L'$. The weights of the T-action on the tangent space of $\overline{(\hat{X}_I)_{L'}}$ at pt_L are

$$\{-v^L_j(\alpha_{i_j})|j\in L'\}.$$

Example 3.9. For the A_2-case, consider $\hat{X}_{(1,2)} = P_1\times^B P_2/B$. There are four T-fixed points, denoted by $\{00, 01, 10, 11\}$, corresponding to

$$\{[e,e],[e,s_2],[s_1,e],[s_1,s_2]\},$$

or

$$\emptyset, \{2\}, \{1\}, \{1,2\}$$

as subsets of [2]. The weights of the tangent spaces of $\hat{X}_{(1,2)}$ at the four points are:

$$\begin{array}{llll} 00: & -\alpha_1, -\alpha_2 & 01: & -\alpha_1, \alpha_2 \\ 10: & \alpha_1, -\alpha_1-\alpha_2 & 11: & \alpha_1, \alpha_1+\alpha_2. \end{array}$$

We denote the set of functions on $\mathcal{E}_I$ with values in S by $F(\mathcal{E}_I; S)$. It is a free S-module with basis $f_L, L \in \mathcal{E}_I$ defined by $f_L(L') = \delta_{L,L'}$, and have a ring structure given by

$$f_L \cdot f_{L'} = \delta_{L,L'} f_L.$$

Moreover, we have

$$h_T(\mathcal{E}_I) \cong F(\mathcal{E}_I; S),$$

where the total Chern class of the tangent space at the fixed point pt_L, corresponds to the basis element f_L up to a scalar.

Let $\mathbf{j}^I : \hat{X}_I^T \to \hat{X}_I$ be the embedding of fixed points. For each $L \subset [l]$, denote by $\mathbf{j}_L^I$ the embedding of pt_L into $\hat{X}_I$. Sometimes we will drop the superscript I for simplicity. Then

$$\mathbf{j}^*(f) = \sum_{L\subset[l]} \mathbf{j}_L^*(f) f_L, \quad f \in h_T(\hat{X}_I).$$

Denote

$$x_{I,L} = \prod_{1\le j\le l} v_j^L(x_{-\alpha_{i_j}}). \tag{9}$$

We have

Lemma 3.10. *For any $L \subset [l]$, we have $\mathbf{j}^*\mathbf{j}_*(f_L) = x_{I,L} f_L$.*

Proof. This follows easily from [CZZ14, §2.**A**8] and Lemma 3.8 concerning the weights of the tangent space of $\hat{X}_I$ at the point L. □

Example 3.11. Following Example 3.9, with $I = (\alpha_1, \alpha_2)$, we have

$$x_{I,00} = x_{-\alpha_1} x_{-\alpha_2}, \quad x_{I,10} = x_{\alpha_1} x_{-\alpha_1-\alpha_2},$$
$$x_{I,01} = x_{-\alpha_1} x_{\alpha_2}, \quad x_{I,11} = x_{\alpha_1} x_{\alpha_1+\alpha_2}.$$

4. Restriction to T-fixed points

In this section, we compute the restriction formula of the η_L basis. We first compute the restriction formula of the image of the characteristic map.

Lemma 4.1. *Let I be a sequence of length l, and $\mathbf{c}_I : S \to \hat{X}_I$ be the characteristic map, then*

$$\mathbf{j}^*\mathbf{c}_I(u) = \sum_{L\subset[l]} v^L(u) f_L.$$

Proof. We prove it by induction on the length l of I. If $I = (i_1)$, then it follows from Lemma 2.6.

Now assume it holds for all sequences of length $\leq l-1$, and assume $I = (i_1, ..., i_l)$. Denote $I' = (i_1, ..., i_{l-1})$ and $\sigma : \hat{X}_{I'} \to \hat{X}_I$ the zero section. By induction assumption, for each $L' \subset [l-1]$, we have

$$(\mathbf{j}^{I'}_{L'})^* \mathbf{c}_{I'}(u) = v^{L'}_{l-1}(u). \tag{10}$$

Concerning $L \subset [l]$, we have two cases:

Case 1: $l \in L$. In this case, $\mathrm{pt}_L \notin \sigma(\hat{X}_{I'})$, so

$$(\mathbf{j}^I_L)^* \circ \sigma_* = 0. \tag{11}$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{\mathbf{j}^I_L} & \hat{X}_I \\ & \searrow^{\mathbf{j}^{I'}_{L\backslash\{l\}}} & \downarrow p \\ & & \hat{X}_{I'}, \end{array}$$

that is, $p \circ \mathbf{j}^I_L = \mathbf{j}^{I'}_{L\backslash\{l\}}$, so

$$(\mathbf{j}^I_L)^* \circ p^* = (\mathbf{j}^{I'}_{L\backslash\{l\}})^*. \tag{12}$$

Denote $\xi = \sigma_*(1)$, then by Lemma 2.5, we have

$$\begin{aligned} (\mathbf{j}^I_L)^* \circ \mathbf{c}_I(u) &= (\mathbf{j}^I_L)^*[p^* \mathbf{c}_{I'}(s_{i_l}(u)) + p^* \mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot \xi] \\ &= (\mathbf{j}^I_L)^* p^* \mathbf{c}_{I'}(s_{i_l}(u)) + (\mathbf{j}^I_L)^* p^* \mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot (\mathbf{j}^I_L)^*(\sigma_*(1)) \\ &\overset{\sharp_1}{=} (\mathbf{j}^{I'}_{L\backslash\{l\}})^* \mathbf{c}_{I'}(s_{i_l}(u)) \\ &\overset{\sharp_2}{=} v^{L\backslash\{l\}}_{l-1} \circ s_{i_l}(u) = v^L_l(u). \end{aligned}$$

Here the identity $\sharp_1$ follows from (11) and (12), and $\sharp_2$ follows from (10).

Case 2: $l \notin L$. In this case, we can view $L \subset [l-1]$, so we have commutative diagrams:

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{\mathbf{j}^I_L} & \hat{X}_I \\ & \searrow^{\mathbf{j}^{I'}_L} & \downarrow p \\ & & \hat{X}_{I'} \end{array}, \qquad \begin{array}{ccc} \mathrm{pt} & \xrightarrow{\mathbf{j}^I_L} & \hat{X}_I \\ & \searrow^{\mathbf{j}^{I'}_L} & \uparrow \sigma \\ & & \hat{X}_{I'} \end{array},$$

so $p \circ \mathbf{j}^I_L = \mathbf{j}^{I'}_L$ and $\sigma \circ \mathbf{j}^{I'}_L = \mathbf{j}^I_L$. The latter implies that

$$(\mathbf{j}^I_L)^* \sigma_*(1) = (\mathbf{j}^{I'}_L)^* \sigma^* \sigma_*(1) \overset{\text{Lem.2.5}}{=} (\mathbf{j}^{I'}_L)^* \mathbf{c}_{I'}(x_{-\alpha_{i_l}}). \tag{13}$$

Therefore,

$$(\mathbf{j}^I_L)^*(\mathbf{c}_I(u)) = (\mathbf{j}^I_L)^*[p^* \mathbf{c}_{I'}(s_{i_l}(u)) + p^* \mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot \xi]$$

$$
\begin{aligned}
&= (\mathbf{j}^I_L)^* p^* \mathbf{c}_{I'}(s_{i_l}(u)) + (\mathbf{j}^I_L)^* p^* \mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot (\mathbf{j}^I_L)^*(\sigma_*(1)) \\
&= (\mathbf{j}^{I'}_L)^* \mathbf{c}_{I'}(s_{i_l}(u)) + (\mathbf{j}^{I'}_L)^* \mathbf{c}_{I'}(\Delta_{-\alpha_{i_l}}(u)) \cdot (\mathbf{j}^{I'}_L)^* \mathbf{c}_{I'}(x_{-\alpha_{i_l}}) \\
&= (\mathbf{j}^{I'}_L)^* \mathbf{c}_{I'}(s_{i_l}(u) + \frac{u - s_{i_l}(u)}{x_{-\alpha_{i_l}}} x_{-\alpha_{i_l}}) \\
&= (\mathbf{j}^{I'}_L)^* \mathbf{c}_{I'}(u) \\
&= v^L_{l-1}(u) = v^L_l(u).
\end{aligned}
$$

The proof is finished. □

Before computing the restriction formula of η_L, we first consider an example.

Example 4.2. Consider the case of A_2. Let $\{\alpha_1, \alpha_2\}$ be the set of simple roots. We consider the Bott-Samelson variety $\hat{X}_I = P_1 \times^B P_2/B$ for $I = (1, 2)$. Following Example 3.9, there are four torus-fixed points, denoted by $\mathcal{P}_2 = \{00, 01, 10, 11\}$. Similarly, denote $(P_1/B)^T$ by $\mathcal{P}_1 = \{0, 1\}$. Denote $\mathbf{j}^I : \mathcal{E}_I \hookrightarrow \hat{X}_I$ and $\mathbf{j}^1 : \mathcal{P}_1 \hookrightarrow (P_1/B)^T$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
P_1 \times^B P_2/B & \xleftarrow{\ \mathbf{j}^I\ } & \mathcal{P}_2 = \{00, 01, 10, 11\}\,. \\
\sigma_2 \uparrow\downarrow p_2 & & \downarrow p_2' \\
P_1/B & \xleftarrow{\ \mathbf{j}^1\ } & \mathcal{P}_1 = \{0, 1\} \\
\sigma_1 \uparrow\downarrow p_1 & & \\
\mathrm{pt} & &
\end{array}
$$

Here σ_i are the zero sections, p_2' is induced by the projection map p_2, so it maps 00, 01 to 0, and 10 and 11 to 1. Moreover, by definition, $\mathbf{j}^1_0 = \sigma_1$ and $\sigma_2 \circ \mathbf{j}^1_i = \mathbf{j}^I_{i0}$ for $i = 0, 1$. We have

$$\eta_1 = p_2^*(\sigma_1)_*(1), \quad \eta_2 = (\sigma_2)_*(1),$$

and

$$h_T((\hat{X}_I)^T) = S\{f_{00}, f_{01}, f_{10}, f_{11}\}, \quad h_T((P_1/B)^T) = S\{f_0, f_1\}.$$

Denote $\mathbf{c}_1 : S \to h_T(P_1/B)$.

First of all, from the definition of p_2' and (4), we know

$$(p_2')^*(f_0) = f_{00} + f_{01}, \quad (p_2')^*(f_1) = f_{10} + f_{11}.$$

Moreover, since $\mathbf{j}^1_0$ coincides with σ_1 and $\mathbf{j}^1_1(\mathrm{pt}) \notin \sigma_1(\mathrm{pt})$, so $(\mathbf{j}^1_1)^*(\sigma_1)_* = 0$ and

$$(\mathbf{j}^1)^*(\sigma_1)_*(1) = (\mathbf{j}^1_0)^*(\sigma_1)_*(1) = \sigma_1^*(\sigma_1)_*(1) = x_{-\alpha_1} f_0,$$

where the last identity follows from the fact that the tangent space of P_1/B at 0 has weight $-\alpha_1$. Hence,

$$
\begin{aligned}
(\mathbf{j}^I)^*(\eta_1) &= (\mathbf{j}^I)^* p_2^*(\sigma_1)_*(1) && (14) \\
&= (p_2')^*(\mathbf{j}^1)^*(\sigma_1)_*(1) = (p_2')^*(x_{-\alpha_1} f_0) = x_{-\alpha_1}(f_{00} + f_{01}). && (15)
\end{aligned}
$$

We then compute $(\mathbf{j}^I)^*(\eta_2)$, by using the identity

$$(\mathbf{j}^I)^*(\eta_2) = \sum_{x\in\mathcal{P}_3} (\mathbf{j}^I_x)^*(\eta_2) f_x.$$

Since $01, 11 \notin \sigma_2(P_1/B)$, we have $(\mathbf{j}^I_{01})^*(\eta_2) = (\mathbf{j}^I_{11})^*(\eta_2) = 0$. From Lemma 2.5, we know that $\sigma_2^*(\sigma_2)_*(1) = \mathbf{c}_1(x_{-\alpha_2})$. So

$$(\mathbf{j}^I_{00})^*(\eta_2) = (\mathbf{j}^I_{00})^*(\sigma_2)_*(1) = (\mathbf{j}^1_0)^*\sigma_2^*(\sigma_2)_*(1) = (\mathbf{j}^1_0)^*(\mathbf{c}_1(x_{-\alpha_2})) \overset{\sharp}{=} x_{-\alpha_2},$$

where $\sharp$ follows from Lemma 4.1. Similarly, from $\mathbf{j}^I_{10} = \sigma_2\circ\mathbf{j}^1_1$, we have

$$\begin{aligned}(\mathbf{j}^I_{10})^*(\eta_2) &= (\mathbf{j}^I_{10})^*(\sigma_2)_*(1)\\ &= (\mathbf{j}^1_1)^*\sigma_2^*(\sigma_2)_*(1)\\ &= (\mathbf{j}^1_1)^*(\mathbf{c}_1(x_{-\alpha_2}))\\ &= s_1(x_{-\alpha_2}) = x_{-\alpha_1-\alpha_2}.\end{aligned}$$

Therefore,

$$(\mathbf{j}^I)^*(\eta_2) = x_{-\alpha_2} f_{00} + x_{-\alpha_1-\alpha_2} f_{10}. \tag{16}$$

Now we compute the restriction formula of η_L.

Theorem 4.3. *Let I be a sequence of length l. For any two subsets $L, M \subset [l]$ denote $L^c = [l]\backslash L$ and*

$$a_{L,M} = \prod_{k\in L} v^M_{k-1}(x_{-\alpha_{i_k}}).$$

Then

$$\mathbf{j}^*(\eta_L) = \sum_{M\subset L^c} a_{L,M} f_M.$$

Proof. We first consider $L = \{k\}$, and prove the following identity

$$\mathbf{j}^*(\eta_k) = \sum_{M\subset L^c} v^M_{k-1}(x_{-\alpha_{i_k}}) f_M.$$

Write $I_j = (i_1, ..., i_j)$ for $j = k$ and $j = k-1$. Firstly, we compute $(\mathbf{j}^{I_k}_M)^*(\sigma_k)_*(1)$ for each $M \subset [k]$, with $\sigma_k : \hat{X}_{I_{k-1}} \to \hat{X}_{I_k}$. If $k \in M$, then the point $\mathbf{j}^{I_k}_M(\mathrm{pt}) \notin \sigma_k(\hat{X}_{I_{k-1}})$, so

$$(\mathbf{j}^{I_k}_M)^*(\sigma_k)_*(1) = 0.$$

If $k \notin M$, then $M \subset [k-1]$, $v^M_k = v^M_{k-1}$, and we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{\mathbf{j}^{I_k}_M} & \hat{X}_{I_k} \\ & \searrow^{\mathbf{j}^{I_{k-1}}_M} & \uparrow \sigma_k \\ & & \hat{X}_{I_{k-1}}. \end{array}$$

Therefore

$$
\begin{aligned}
(\mathbf{j}_M^{I_k})^*(\sigma_k)_*(1) &= (\mathbf{j}_M^{I_{k-1}})^*\sigma_k^*(\sigma_k)_*(1) && (17)\\
&\overset{\text{Lem.2.5}}{=} (\mathbf{j}_M^{I_{k-1}})^*\mathbf{c}_{I_{k-1}}(x_{-\alpha_{i_k}}) && (18)\\
&\overset{\text{Lem.4.1}}{=} v_{k-1}^M(x_{-\alpha_{i_k}}). && (19)
\end{aligned}
$$

Here $\mathbf{c}_{I_{k-1}}$ is the characteristic map on $\hat{X}_{I_{k-1}}$. Then we consider the following commutative diagram

$$
\begin{array}{ccc}
(\hat{X}_I)^T & \xrightarrow{\mathbf{j}^I} & \hat{X}_I \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
(\hat{X}_{I_k})^T & \xrightarrow{\mathbf{j}^{I_k}} & \hat{X}_{I_k}.
\end{array}
$$

We have

$$
\begin{aligned}
(\mathbf{j}^I)^*(\eta_k) &= (\mathbf{j}^I)^*p^*((\sigma_k)_*(1)) = p'^*(\mathbf{j}^{I_k})^*(\sigma_k)_*(1)\\
&= p'^*[\sum_{M\subset[k]}(\mathbf{j}_M^{I_k})^*(\sigma_k)_*(1)f_M]\\
&\overset{(17)}{=} p'^*[\sum_{M\subset[k-1]} v_{k-1}^M(x_{-\alpha_{i_k}})f_M]\\
&\overset{(4)}{=} \sum_{M\subset[k-1]} v_{k-1}^M(x_{-\alpha_{i_k}}) \sum_{M'\subset\{k+1,\dots l\}} f_{M\cup M'}\\
&= \sum_{M\subset([l]\setminus\{k\})} v_{k-1}^M(x_{-\alpha_{i_k}})f_M.
\end{aligned}
$$

So the case $L=\{k\}$ is proved.

Now for a general subset $L\subset[l]$, we have

$$
\begin{aligned}
\mathbf{j}^*(\eta_L) &= \prod_{k\in L}\mathbf{j}^*(\eta_k)\\
&= \prod_{k\in L}\sum_{M\subset([l]\setminus\{k\})} v_{k-1}^M(x_{-\alpha_{i_k}})f_M\\
&= \sum_{M\subset L^c}\prod_{k\in L} v_{k-1}^M(x_{-\alpha_{i_k}})f_M.
\end{aligned}
$$

□

For I of length l and $L\subset[l]$, note the difference between

$$
a_{[l],L} = \prod_{1\le k\le l} v_{k-1}^L(x_{-\alpha_{i_k}}), \quad x_{I,L} = \prod_{1\le k\le l} v_k^L(x_{-\alpha_{i_k}}).
$$

They are only related when $L=[l]$, in which case we have

$$
a_{[l],[l]} = \prod_{1\le k\le l} v_{k-1}^{[l]}(x_{-\alpha_{i_k}}) = \prod_{1\le k\le l} v_k^{[l]}(x_{\alpha_{i_k}}), \quad x_{I,[l]} = \prod_{1\le k\le l} v_k^{[l]}(x_{-\alpha_{i_k}}).
$$

Corollary 4.4. *The map* $\mathbf{j}^* : h_T(\hat{X}_I) \to h_T(\hat{X}_I^T)$ *is an injection.*

Proof. It follows from Theorem 4.3 that

$$\mathbf{j}^*(\eta_L) = \sum_{M \subset L^c} a_{L,M} f_M.$$

So if we order $\{\mathbf{j}^*(\eta_L) | L \subset [l]\}, \{f_M | M \subset [l]\}$ by inclusion of subsets $L' \subset L$, then the transition matrix from f_M to $\mathbf{j}^*(\eta_L)$ will be skew-triangular. Moreover, the entries on the skew-diagonal will be

$$a_{L,L^c} = \prod_{k \in L} v_{k-1}^{L^c}(x_{-\alpha_{i_k}}),$$

which is regular in S. Therefore, $\mathbf{j}^*$ is injective. □

Theorem 4.5. *Let I be a sequence of length l. Then*

$$\operatorname{im} \mathbf{j}^* \subset \{\sum_{L \subset [l]} a_L f_L | \frac{a_{L_1} - a_{L_2}}{v_{k-1}^{L_1}(x_{-\alpha_{i_k}})} \in S, \ \forall L_1, L_2 \text{ such that } L_1 = L_2 \sqcup \{k\}\}.$$

Here $\sqcup$ denotes the disjoint union.

Proof. Denote the right hand side by Ψ. We first show that Ψ is a ring. It is clearly additively closed. For the multiplication, consider

$$f = \sum_{L \subset [l]} a_L f_L, \quad g = \sum_{L' \subset [l]} b_{L'} f_{L'} \in \Psi,$$

then

$$fg = \sum_{L, L' \subset [l]} \delta_{L,L'} a_L b_{L'} f_L = \sum_{L \subset [l]} a_L b_L f_L.$$

For any L_1, L_2 such that $L_1 = L_2 \sqcup \{k\}$, by definition we have $v_{k-1}^{L_1} = v_{k-1}^{L_2}$, so $v_{k-1}^{L_1}(x_{-\alpha_{i_k}}) = v_{k-1}^{L_2}(x_{-\alpha_{i_k}})$. Therefore,

$$a_{L_1} b_{L_1} - a_{L_2} b_{L_2} = (a_{L_1} - a_{L_2}) b_{L_1} - (b_{L_2} - b_{L_1}) a_{L_2},$$

is divisible by $v_{k-1}^{L_1}(x_{-\alpha_{i_k}})$. We have $fg \in \Psi$.

We then show that $\operatorname{im} \mathbf{j}^* \subset \Psi$. Since $\mathbf{j}^*$ is multiplicative, it suffices to show

$$\mathbf{j}^*(\eta_m) = \sum_{L \subset [l] \backslash \{m\}} v_{m-1}^{L}(x_{-\alpha_{i_m}}) f_L$$

belongs to the RHS. Suppose $L_1 = L_2 \sqcup \{k\}$. Clearly $k \neq m$. If $k > m$, then by definition we have $v_{m-1}^{L_1} = v_{m-1}^{L_2}$. Thus $v_{m-1}^{L_1}(x_{-\alpha_{i_m}}) = v_{m-1}^{L_2}(x_{-\alpha_{i_m}})$, which implies that $\mathbf{j}^*(\eta_m) \in \Psi$.

If $k < m$, denote

$$L_1 \cap [m-1] = \{j_1 < j_2 < \cdots j_t < k < j_{t+1} < \cdots < j_s\},$$

$$L_2 \cap [m-1] = \{j_1 < j_2 < \cdots j_t < \hat{k} < j_{t+1} < \cdots < j_s\},$$

(in other words, k is omitted in L_2). Then

$$v_{m-1}^{L_1}(x_{-\alpha_{i_m}}) - v_{m-1}^{L_2}(x_{-\alpha_{i_m}}) = s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_t}} s_{i_k} s_{i_{j_{t+1}}} \cdots s_{i_{j_s}}(x_{-\alpha_{i_m}})$$

$$
\begin{aligned}
&- s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_t}} \widehat{s_{i_k}} s_{i_{j_{t+1}}} \cdots s_{i_{j_s}} (x_{-\alpha_{i_m}}) \\
&= v^{L_1}_{k-1} s_{i_k} (s_{i_{j_{t+1}}} \cdots s_{i_{j_s}})(x_{-\alpha_{i_m}}) \\
&- v^{L_1}_{k-1} (s_{i_{j_{t+1}}} \cdots s_{i_{j_s}})(x_{-\alpha_{i_m}}).
\end{aligned}
$$

According to Lemma 2.3, this is divisible by $v^{L_1}_{k-1}(x_{-\alpha_{i_k}})$. □

We can strengthen the conclusion in some cases. The proof essentially uses the fact that the transition matrix from $f_L, L \subset [l]$ to $\mathbf{j}^*(\eta_M), M \subset [l]$ is skew-triangular, following from Theorem 4.3.

Theorem 4.6. *If $I = (i_1, \ldots, i_l)$ with i_j all distinct, then we have equality in Theorem 4.5.*

Proof. It suffices to show that $\Psi \subset \operatorname{im} \mathbf{j}^*$. Suppose

$$
f = \sum_{L \subset [l]} a_L f_L \in \Psi, \quad \text{with } a_L = 0 \text{ unless } L = \emptyset,
$$

then for any $k \in [l]$, $a_\emptyset = a_\emptyset - a_{\{k\}}$ is divisible by $v^{\emptyset}_{k-1}(x_{-\alpha_{i_k}}) = x_{-\alpha_{i_k}}$. Since $x_{\alpha_{i_j}}, 1 \le j \le l$ are all distinct, by [CZZ19, Lemma 2.7], we see that $a_\emptyset$ is divisible by $\prod_{k \in [l]} x_{-\alpha_{i_k}}$. Note that by Theorem 4.3,

$$
\mathbf{j}^*(\eta_{[l]}) = \prod_{k \in [l]} x_{-\alpha_{i_k}} f_\emptyset,
$$

so f is a multiple of $\mathbf{j}^*(\eta_{[l]})$, i.e., $f \in \operatorname{im} \mathbf{j}^*$.

Assume the conclusion holds for any f that can be written as a linear combination of f_L with $|L| \le t-1$. Now let

$$
f = \sum_{L \subset [l]} a_L f_L \in \Psi, \quad \text{with } a_L = 0 \text{ unless } |L| \le t.
$$

Let L_0 be a subset of $[l]$ of cardinality t. For any $k \in L_0^c$, we have $a_{L_0 \sqcup \{k\}} = 0$, so $v^{L_0}_{k-1}(x_{-\alpha_{i_k}}) | a_{L_0}$ Now from Theorem 4.3, we know

$$
\mathbf{j}^*(\eta_{L_0^c}) = \sum_{M \subset L_0} a_{L_0^c, M} f_M, \quad a_{L_0^c, L_0} = \prod_{j \in L_0^c} v^{L_0}_{j-1}(x_{-\alpha_{i_j}}).
$$

By Lemma 3.7, we have that $v^{L_0}_{j-1}(x_{-\alpha_{i_j}})$ are all distinct for $j \in L_0^c$. By [CZZ19, Lemma 2.7], we know that $a_{L_0^c, L_0} | a_{L_0}$. Write $a_{L_0} = c_{L_0} a_{L_0^c, L_0}$ with $c_{L_0} \in S$. Therefore,

$$
f' := f - \sum_{|L_0| = t} c_{L_0} \mathbf{j}^*(\eta_{L_0^c}) = \sum_{|L| < t} a'_L f_L,
$$

By induction hypothesis, $f' \in \operatorname{im} \mathbf{j}^*$. Therefore, $f \in \operatorname{im} \mathbf{j}^*$. The proof is finished. □

Remark 4.7. Let T_i be the subtorus of rank 1 corresponding to α_i, i.e., $T_i = (\ker \alpha_i)^\circ$ where α_i is viewed as a character $T \to k^*$. If $I = (i_1, ..., i_l)$ is a sequence such that i_j are all distinct, it is not difficult to see that for any $1 \le k \le l$,

$$\hat{X}_I^{T_{i_k}} = \{[g_1, ..., g_l] | g_j B \in \{B, s_{i_j} B\} \,\forall\, j \neq k\},$$

and

$$\hat{X}_I^{T'} = \{[g_1, ..., g_l] |\, g_j B \in \{B, s_{i_j} B\} \,\forall j\}$$

if T' is any subtorus of corank 1 different from $T_{i_j}, j = 1, ..., l$. In other words, for any subtorus of corank 1, the irreducible component of the invariant subvariety has dimension at most one. This corresponds to the so-called Goresky-Kottwitz-MacPherson (GKM) condition. In other words, in this case, the Bott-Samelson variety is a GKM space. This corresponds to the conclusion of Theorem 4.6.

On the other hand, if P_{i_j} are not distinct, the space $\hat{X}_I$ will not be GKM. For instance, if $I = (1, 2, 1)$, the T_1-fixed subspace contains the following subset

$$\{[g_1, e, g_1'] | g_i, g_i' \in P_1\},$$

so the dimension condition is not satisfied. Indeed, it follows from the proof of Theorem 4.6 that in this case, the inclusion in Theorem 4.5 is strict. For more detailed discussion of GKM spaces, see [GKM98, GHZ06].

5. Push-forward to cohomology of flag varieties

In this section, we compute the push-forward of the basis η_L along the canonical map $q_I : \hat{X}_I \to G/B$, which generalizes the computation of Bott-Samelson classes in [CZZ14].

Recall that the set of T-fixed points of G/B is in bijection to W, so we have

$$h_T((G/B)^T) \cong \oplus_{w \in W} S.$$

Denote by $f_w \in h_T(W)$ the basis element corresponding to $w \in W$. Denote $i : W \to G/B$ to be the embedding, and denote $\mathrm{pt}_e = (i|_e)_*(1) \in h_T(G/B)$. Let $\pi_i : G/B \to G/P_i$ be the canonical map, and denote $A_i = \pi_i^* \circ (\pi_i)_* : h_T(G/B) \to h_T(G/B)$. For any sequence I, denote by I^{rev} the sequence obtained by reversing I.

Proposition 5.1. [CZZ14, Lemma 7.6] *For any sequence I, we have*

$$(q_I)_*(1) = A_{I^{\mathrm{rev}}}(\mathrm{pt}_e).$$

The following is an easy generalization of Proposition 5.1.

Theorem 5.2. *Let I be a sequence of length l and $1 \le k \le l$. Denote by I_k the subsequence of I obtained by removing the k-th term from I. Then $(q_I)_*(\eta_k) = A_{I_k^{\mathrm{rev}}}(1)$.*

Proof. Denote the sequence by $I = (i_1, ..., i_l)$. For any $k \le l$, denote

$$\begin{array}{ccc} P_{i_1} \times^B P_{i_2} \times^B \times ... \times^B P_{i_k}/B & \xrightarrow{q_k} & G/B \\ \sigma_k \uparrow \downarrow p_k & & \\ P_{i_1} \times^B P_{i_2} \times^B \times ... \times^B P_{i_{k-1}}/B & \xrightarrow{q_{k-1}} & G/B. \end{array}$$

Note that $q_I = q_l$ and $q_k \circ \sigma_k = q_{k-1}$. Denote by p the composition of $p_{k+1}, \cdots, p_l$. By using the base change formula from diagram (8), we have

$$\begin{aligned} (q_I)_*(\eta_k) &= (q_l)_* p^*((\sigma_k)_*(1)) \\ &= (q_l)_* p_l^* p_{l-1}^* \cdots p_{k+1}^*(\sigma_k)_*(1) \\ &= \pi_{\alpha_{i_l}}^*(\pi_{\alpha_{i_l}})_*(q_{l-1})_* p_{l-1}^* \cdots p_{k+1}^*(\sigma_k)_*(1) \\ &= (\pi_{\alpha_{i_l}}^*(\pi_{\alpha_{i_l}})_*)(\pi_{\alpha_{i_{l-1}}}^*(\pi_{\alpha_{i_{l-1}}})_*) \cdots (\pi_{\alpha_{i_{k+1}}}^*(\pi_{\alpha_{i_{k+1}}})_*)(q_k)_*(\sigma_k)_*(1) \\ &= A_{i_l} A_{i_{l-1}} \cdots A_{i_{k+1}}(q_{k-1})_*(1) \\ &= A_{i_l} A_{i_{l-1}} \cdots A_{i_{k+1}} A_{i_{k-1}} \cdots A_{i_1}(\mathrm{pt}_e) = A_{I_k^{\mathrm{rev}}}(\mathrm{pt}_e). \end{aligned}$$

□

To compute $(q_I)_*(\eta_L)$ for general $L \subset [l]$, we need the following lemma.

Lemma 5.3. *For any $L \subset [l]$, we have*

$$\eta_L = \sum_{L_1 \subset [l]} \frac{a_{L,L_1}}{x_{I,L_1}} \mathbf{j}_*(f_{L_1}),$$

where a_{L,L_1} are defined in Theorem 4.3. Note that the coefficients in this formula belong to $Q := S[\frac{1}{x_\alpha} | \alpha \in \Sigma]$.

Proof. By Corollary 4.4, we know that $\mathbf{j}^*(\eta_L)$ becomes a basis of $Q \otimes_S h_T(W)$. In other words, $\mathbf{j}^*$ induces an isomorphism

$$\mathbf{j}^* : Q \otimes_S h_T(\hat{X}_I) \to Q \otimes_S h_T(W).$$

Moreover, by Lemma 3.10, we know that $\mathbf{j}_*$ is the inverse of the $\mathbf{j}^*$ (after tensoring with Q). Therefore, $\mathbf{j}_*(f_L)$ is a Q-basis of $Q \otimes_S h_T(\hat{X}_I)$. Denote

$$\eta_L = \sum_{L_1 \subset [l]} b_{L,L_1} \mathbf{j}_*(f_{L_1}), \quad b_{L,L_1} \in Q.$$

Then by Theorem 4.3 and Lemma 3.10, we have

$$\sum_{L_2 \subset L^c} a_{L,L_2} f_{L_2} = \mathbf{j}^*(\eta_L) = \sum_{L_1 \subset [l]} b_{L,L_1} \mathbf{j}^* \mathbf{j}_*(f_{L_1}) = \sum_{L_1 \subset [l]} b_{L,L_1} x_{I,L_1} f_{L_1}.$$

Therefore, $b_{L,L_1} = \frac{a_{L,L_1}}{x_{I,L_1}}$. □

The following is the main result of this section, which computes the push-forward of η_L to the cohomology of G/B.

Theorem 5.4. *For any sequence $I = (i_1, \ldots, i_l)$, we have*

$$i^*(q_I)_*(\eta_L) = \sum_{L_1 \subset L^c} \frac{a_{L,L_1} \cdot v^{L_1}(x_\Pi)}{x_{I,L_1}} f_{v^{L_1}}, \quad x_\Pi := \prod_{\alpha<0} x_\alpha \in S.$$

Note that a priori the coefficients of $f_{v^{L_1}}$ belong to S.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \hat{X}_I^T & \xrightarrow{\mathbf{j}} & \hat{X}_I \\ \Big\downarrow{\scriptstyle q'} & & \Big\downarrow{\scriptstyle q_I} \\ W & \xrightarrow{i} & G/B. \end{array}$$

Note that by definition, q' maps the point corresponding to $L \subset [l]$ to $v^L \in W$. Therefore,

$$(q')_*(f_L) = f_{v^L} \in h_T(W).$$

Firstly, we have

$$i^*(q_I)_*\mathbf{j}_*(f_L) = i^* i_*(q')_*(f_L) = i^* i_*(f_{v^L}) = v^L(x_\Pi) f_{v^L},$$

where the last identity follows from [CZZ14, Corollary 6.4]. Consequently, by Lemma 5.3, we have

$$i^*(q_I)_*(\eta_L) = i^*(q_I)_* \sum_{L_1 \subset L^c} \frac{a_{L,L_1}}{x_{I,L_1}} \mathbf{j}^*(f_{L_1}) = \sum_{L_1 \subset L^c} \frac{a_{L,L_1} \cdot v^{L_1}(x_\Pi)}{x_{I,L_1}} f_{v^{L_1}}.$$

□

Remark 5.5. In case $\eta_L = \eta_\emptyset$ or η_k, as in Proposition 5.1 and Theorem 5.2, one can express $(q_I)_*(\eta_L)$ as the operators A_i applied on pt_e. By using the method of formal affine Demazure algebra, started in [KK86, KK90] and continued in [CZZ12, CZZ19, CZZ14], one will obtain a restriction formula of $i^*(q_I)_*(\eta_L)$. Roughly speaking, there is an algebra $\mathbf{D}_F$ generated by algebraic analogue of the push-pull operators A_i, whose dual is isomorphic to $h_T(G/B)$. The algebra $\mathbf{D}_F$ acts on $h_T(G/B)$, via two actions (denoted by $\bullet$ and $\odot$ in [LZZ16]). Both actions will give restriction formulas of $A_I(\mathrm{pt}_e)$. Indeed, by using the two actions, one will obtain two different, but equivalent formulas, one of which coincides with the one given by Theorem 5.4.

Corollary 5.6. *Let I be any sequence of length l. For any $L \subset [l]$, denote by $\hat{X}_L = (\hat{X}_I)_L$ and $q_L : \hat{X}_L \to G/B$. Then $(q_I)_*(\eta_L) = (q_{L^c})_*(1)$.*

Proof. From Theorem 5.4 we have

$$i^*(q_I)_*(\eta_L) = \sum_{L_1 \subset L^c} \frac{a_{L,L_1} \prod_{\alpha<0} v^{L_1}(x_\alpha)}{x_{I,L_1}} f_{v^{L_1}}, \tag{20}$$

$$i^*(q_{L^c})_*(1) = \sum_{L_1 \subset L^c} \frac{\prod_{\alpha<0} v^{L_1}(x_\alpha)}{x_{L^c,L_1}} f_{v^{L_1}}. \tag{21}$$

By definition

$$x_{I,L_1} = \prod_{j\in I} v_j^{L_1}(x_{-\alpha_{i_j}}),\quad x_{L^c,L_1} = \prod_{j\in L^c} v_j^{L_1}(x_{-\alpha_{i_j}}),\quad a_{L,L_1} = \prod_{j\in L} v_{j-1}^{L_1}(x_{-\alpha_{i_j}}).$$

Since $L \cap L_1 = \emptyset$, so for any $j \in L$, $v_j^{L_1} = v_{j-1}^{L_1}$, and we have

$$\begin{aligned} x_{I,L_1} &= \prod_{j\in L^c} v_j^{L_1}(x_{-\alpha_{i_j}}) \prod_{j\in L} v_j^{L_1}(x_{-\alpha_{i_j}}) \\ &= \prod_{j\in L^c} v_j^{L_1}(x_{-\alpha_{i_j}}) \prod_{j\in L} v_{j-1}^{L_1}(x_{-\alpha_{i_j}}) \\ &= x_{L^c,L_1} a_{L,L_1}. \end{aligned}$$

Therefore, $i^*(q_I)_*(\eta_L) = i^*(q_{L^c})_*(1)$. By [CZZ14, Theorem 8.2], we know i^* is injective. So $(q_I)_*(\eta_L) = (q_{L^c})_*(1)$. □

Remark 5.7. This corollary shows that for any $L \subset [l]$, the class in $h_T(G/B)$ determined by η_L coincides with the Bott-Samelson class determined by $I|_{L^c}$, in other words, for the class η_L, the minimal parabolic subgroups P_{i_j}, $j \in L$ are 'omitted'.

By using this result, we can derive the Chevalley formula for equivariant oriented cohomology. For each $w \in W$, we fix a reduced sequence I_w, then the Bott-Samelson class ζ_{I_w} is defined to be the push-forward class along the map $q_{I_w} : \hat{X}_{I_w} \to G/B$, i.e., $\zeta_{I_w} = (q_{I_w})_*(1)$. It is proved in [CZZ14, Proposition 8.1] that $\{\zeta_{I_w} | w \in W\}$ is a basis of $h_T(G/B)$. Denote the characteristic maps from $h_T(\mathrm{pt})$ to G/B and to $\hat{X}_{I_w}$ by $\mathbf{c}'$ and $\mathbf{c}_{I_w}$, respectively. By definition, $\mathbf{c}_{I_w} = q_{I_w}^* \mathbf{c}'$.

Corollary 5.8 (Chevalley Formula)**.** *For any $u \in h_T(pt)$, we have*

$$\mathbf{c}'(u) \cdot \zeta_w = \sum_{L\subset[\ell(w)]} \theta_{I,L}(u) \zeta_{L^c},$$

where $\zeta_{L^c} = (q_{L^c})_*(1)$ and $\theta_{I,L}(u)$ was defined in Lemma 3.3.

Proof. We have

$$(q_{I_w})_*(\mathbf{c}_{I_w}(u)) = (q_{I_w})_*(\mathbf{c}_{I_w}(u)\cdot 1) = (q_{I_w})_*(q_{I_w}^*(\mathbf{c}'(u))\cdot 1) = \mathbf{c}'(u)\zeta_{I_w},$$

where the last identity follows from the projection formula. Then Lemma 3.3 and Corollary 5.6 imply that

$$\begin{aligned} (q_{I_w})_*(\mathbf{c}_{I_w}(u)) &= \sum_{L\subset[\ell(w)]} \theta_{I_w,L}(u)(q_{I_w})_*(\eta_L) \\ &= \sum_{L\subset[\ell(w)]} \theta_{I_w,L}(u)(q_{L^c})_*(1) \end{aligned}$$

$$= \sum_{L\subset[\ell(w)]} \theta_{I_w,L}(u)\zeta_{L^c}.$$

The conclusion then follows. □

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