New York Journal of Mathematics

New York J. Math. **26** (2020) 1155–1183.

A universal coefficient theorem with applications to torsion in Chow groups of Severi–Brauer varieties

Eoin Mackall

Abstract. For any variety X, and for any coefficient ring S, we define the S-topological filtration on the Grothendieck group of coherent sheaves $G(X) \otimes S$ with coefficients in S. The S-topological filtration is related to the topological filtration by means of a universal coefficient theorem. We apply this observation in the case X is a Severi–Brauer variety to obtain new examples of torsion in the Chow groups of X.

Contents

Notation and Conventions. A ring is a commutative ring. An abelian group is flat if it is a flat $\mathbb{Z}$-module; equivalently, an abelian group is flat if and only if it is torsion free. We fix an arbitrary field k, to be used as a base. For any field F, an F-variety (or simply a variety when the field F is clear) is an integral scheme separated and of finite type over F.

1. Introduction

Let X be a Severi–Brauer variety. The Chow groups $\mathrm{CH}^i(X)$ of algebraic cycles on X of codimension-i modulo rational equivalence have been the subject of a considerable amount of current research [1, 9, 10, 11, 12]. A primary focus of this research has been to determine the possible torsion subgroups of $\mathrm{CH}^i(X)$ for varying i and X. This line of study was initiated by Merkurjev [15] who used the Brown-Gersten-Quillen or BGQ spectral sequence to give the first proof that there can exist nontrivial torsion cycles in $\mathrm{CH}^i(X)$ for some X and some $i \geq 3$.

Received April 29, 2020.

2010 *Mathematics Subject Classification.* 14C35; 19E08.

Key words and phrases. Grothendieck ring; algebraic cycles.

ISSN 1076-9803/2020

Merkurjev's methods are inexplicit: although he shows that nontrivial torsion cycles exist in the Chow groups of some Severi–Brauer varieties, it's difficult to write down an explicit torsion cycle let alone to know in which codimension the cycle exists. Because of this, most modern computations in this field rely on the pioneering ideas of Karpenko [7, 8] who gave the first description of the torsion subgroups in $\mathrm{CH}^2(X)$ for a handful of Severi–Brauer varieties X including those X that are generic with associated central simple algebra A of level $\mathrm{lev}(A) \leq 1$ (see Subsection 4.3 for a definition of the level).

The torsion subgroup of $\mathrm{CH}^2(X)$ for a Severi–Brauer variety X is often useful. Karpenko's calculations [8, Example 4.15] for the torsion subgroups of $\mathrm{CH}^2(X)$ for generic Severi-Brauer varieties associated to central simple algebras A of level $\mathrm{lev}(A) \leq 1$ can be used to compute the group of indecomposable degree three cohomological invariants for the algebraic groups SL_n/μ_m, with m a divisor of n, by [9, Lemma 3.5], [16, Theorem 2.10], and [13, Lemma 5.13]; this computation also appears in [2] by different methods. Nontrivial torsion in $\mathrm{CH}^2(X)$ can also be used to show indecomposability of the associated central simple algebra in some cases [8, 1, 12].

In this paper, we develop a new technique for determining nontrivial torsion cycles in the Chow groups $\mathrm{CH}^i(X)$ of a Severi–Brauer variety X for any $i \geq 2$. We then apply this technique in the case that X is a generic Severi–Brauer variety associated to a central simple algebra A of index $\mathrm{ind}(A) = 2^n$ to determine all possible torsion subgroups of $\mathrm{CH}^2(X)$ for all $n \leq 5$; this information is compiled in Tables 1, 2, and 3 below. Some corollaries to our computations include the first examples (Corollary 4.15) of noncyclic torsion in $\mathrm{CH}^2(X)$ and the first examples (Corollaries 4.16, 4.17, 4.18) of torsion in higher codimensions for X associated to an algebra A of level $\mathrm{lev}(A) > 1$.

All of our results are based on an analog of the universal coefficient theorem of singular homology that applies to the topological filtration $\tau_\bullet(X)$ of the Grothendieck group $G(X)$ of coherent sheaves on a variety X. That is to say, we introduce an S-topological filtration $\tau_\bullet^S(X)$ on the group $G(X) \otimes S$ when S is an arbitrary coefficient ring (Definition 3.1) and we compare the topological filtration with S-coefficients to the S-topological filtration via a collection of natural maps; under some conditions, we can show that these comparison maps are isomorphisms (Proposition 3.3).

If, in the discussion above, one takes the coefficient ring S to be the finite field $\mathbb{F}_p$ of p elements for some prime p, then our results show that certain questions regarding torsion elements of order p in the associated graded groups of the topological filtration can be reduced to some computations in the $\mathbb{F}_p$-vector space $G(X) \otimes \mathbb{F}_p$. This makes it considerably easier to check some results by hand (e.g. to see that $\mathrm{CH}^2(X)$ can have nontrivial torsion) and, it suggests that these computations can most likely be done in an automated fashion.

This paper is structured as follows. Section 2 is written in a completely abstracted way; here we prove only basic results in homological algebra. The abstract results of Section 2 are applied in a geometric setting in Section 3 where we also prove Proposition 3.3 which is our most useful form of the universal coefficient theorem for the topological filtration. As an aside to this section, we would like to point out Lemma 3.4, which characterizes torsion elements in the Grothendieck ring $K(X)$ of locally free sheaves on a variety X as elements of the kernel of some Adam's operations.

Section 4 is devoted to applications of the theory developed in Sections 2 and 3. Here we settle some questions on torsion in the Chow groups of Severi–Brauer varieties. This section includes a summary (Subsections 4.1-4.5) of the notation that we use and of some results that can be obtained from the articles [8, 11, 12]. All new computations are contained in Subsection 4.6. Our proofs are highly computational and require some large amount of detail so, whenever possible, we've sorted the needed information into a table; these are provided at the end of the paper, before the references.

2. Filtered rings with coefficients

Throughout this section we let R be an abelian group equipped with an ascending filtration $F_\bullet \subset R$, i.e. for every $i \in \mathbb{Z}$ there is a group $F_i \subset R$ indexed so that

$$\cdots \subset F_{i-1} \subset F_i \subset F_{i+1} \subset \cdots \subset R.$$

Further, this filtration is assumed to satisfy the following property:

(CC) we assume that the filtration $F_\bullet$ is limiting, stable, and nonnegative, i.e. $F_{-1} = 0$ and there exists an integer $d \geq 0$ such that $F_d = R$.

We write $F_{i/i-1}$ for the associated quotient F_i/F_{i-1}. If S is an arbitrary ring, we write $F_\bullet^S$ for the ascending filtration on $R \otimes S$ whose degree-i term F_i^S is defined as the S-submodule generated by $F_i \otimes S$. Equivalently, F_i^S can be defined as the image of the map obtained by tensoring the inclusion $F_i \subset R$ by S,

$$F_i^S = \operatorname{Im}\left(F_i \otimes S \to R \otimes S\right).$$

We write $F_{i/i-1}^S$ for the quotient F_i^S/F_{i-1}^S.

Note that, for any $i \in \mathbb{Z}$, tensoring the inclusion $F_{i-1} \subset F_i$ by S induces an exact sequence

$$\begin{aligned}0 \to \operatorname{Tor}_1(S, F_{i-1}) \to \operatorname{Tor}_1(S, F_i) \to \operatorname{Tor}_1(S, F_{i/i-1}) \to \\ \to F_{i-1} \otimes S \xrightarrow{j_{i-1}} F_i \otimes S \to F_{i/i-1} \otimes S \to 0. \quad \text{(no.1)}\end{aligned}$$

The final terms of these exact sequences fit into commuting diagrams

$$\begin{tikzcd} & F_{i-1}\otimes S \arrow[r, "j_{i-1}"] \arrow[d, "h_{i-1}"] & F_i\otimes S \arrow[r] \arrow[d, "h_i"] & F_{i/i-1}\otimes S \arrow[r] \arrow[d, "f_i"] & 0 \\ 0 \arrow[r] & F^S_{i-1} \arrow[r] & F^S_i \arrow[r] & F^S_{i/i-1} \arrow[r] & 0 \end{tikzcd} \tag{no.2}$$

where the vertical maps h_i are the canonical surjections, and the maps f_i are the induced maps on the quotients. Note that because of our assumption (CC) above, we have that h_{d+k} is an isomorphism for all $k \geq 0$.

Lemma 2.1. *Fix an integer $e \leq d$. Suppose that j_i is an injection for every $e \leq i \leq d$. Then both h_i and f_i are isomorphisms for every $e \leq i \leq d$.*

Proof. Let $K_i = \ker(h_i)$. The snake lemma gives short exact sequences

$$0 \to K_i \to K_{i+1} \to \ker(f_{i+1}) \to 0.$$

From the inclusions $K_e \subset \cdots \subset K_d = 0$ we find that h_i is an isomorphism for all $e \leq i \leq d$. Applying the snake lemma again shows that f_i is an isomorphism as well. $\square$

Lemma 2.2. *The following conditions are equivalent:*

(1) *for every $i \leq d$ the map j_i is an injection;*
(2) *for every $i \leq d$ the map f_i is an isomorphism.*

Additionally, if we assume that R is flat then the above are equivalent to

(3) $\mathrm{Tor}_1(S, F_{i/i-1}) = 0$ *for all $i \leq d$.*

Proof. Let $K_i = \ker(h_i)$. Setting $e = -1$ in Lemma 2.1, we get the implication (1) $\implies$ (2).

In the other direction, applying the snake lemma to the diagrams of (no.2) gives surjections

$$0 = K_{-1} \twoheadrightarrow \cdots \twoheadrightarrow K_d,$$

so that h_i is an injection for all $i \leq d$. Since the left square of (no.2) is commutative, the map j_{i-1} is an injection whenever h_{i-1} is an injection.

Lastly, when R is flat we have that $\mathrm{Tor}_1(S, F_i) = 0$ for all $i \in \mathbb{Z}$. Thus the vanishing $\mathrm{Tor}_1(S, F_{i/i-1}) = 0$ is equivalent to the injectivity of j_{i-1} by (no.1). $\square$

The following lemma can be seen as a direct generalization of the universal coefficient theorem to the setting of filtered groups. We don't use this lemma directly but, we include it for completeness.

Lemma 2.3. *Set $K_i = \ker(h_i)$. Then for any $i \in \mathbb{Z}$ there is an isomorphism*

$$K_i = \mathrm{coker}\left(\mathrm{Tor}_1(S, R) \to \mathrm{Tor}_1(S, R/F_i)\right).$$

Proof. Since F_i^S is the image of the inclusion $F_i \otimes S \to R \otimes S$, we have a short exact sequence

$$\mathrm{Tor}_1(S, R) \to \mathrm{Tor}_1(S, R/F_i) \to F_i \otimes S \to R \otimes S \to R/F_i \otimes S \to 0$$

which proves the claim. □

Corollary 2.4. $\ker(f_i) \subset F_{i/i-1} \otimes S$ *is a quotient of* $\mathrm{Tor}_1(S, R/F_i)$. □

We end with a lemma, which will be needed later.

Lemma 2.5. *In the notation above, the following statements hold.*

(1) *For every ring S, the maps h_d and f_d are isomorphisms.*
(2) *Suppose that there is a splitting $R = F_{d-1} \oplus \mathbb{Z}$. Then, for every ring S, the maps h_{d-1} and f_{d-1} are isomorphisms.*
(3) *Suppose both that the canonical map*

$$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, F_{d-1}) \to \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, F_{d-1/d-2})$$

is a surjection and that h_{d-1} is an isomorphism. If additionally R is flat then, for any ring S, the maps h_{d-2} and f_{d-2} are isomorphisms.

Proof. The proof of (1) is immediate from our assumption (CC) above. To see (2), note that a splitting $R = F_{d-1} \oplus \mathbb{Z}$ gives a splitting $R \otimes S = (F_{d-1} \otimes S) \oplus S$. The map j_{d-1} is then the inclusion $F_{d-1} \otimes S \subset R \otimes S$ and (2) follows from Lemma 2.1. To see (3), we use the assumption R is flat to find that $F_{d-1/d-2}$ is flat because of the surjection

$$0 = \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, F_{d-1}) \to \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, F_{d-1/d-2}).$$

It follows that $\mathrm{Tor}_1(S, F_{d-1/d-2}) = 0$ for every ring S. In particular the map j_{d-2} is an injection, and the maps h_{d-2} and f_{d-2} are isomorphisms again by Lemma 2.1. □

3. The topological filtration with coefficients

Let X be an arbitrary variety. In this paper, the (ascending) topological filtration $\tau_\bullet(X)$ on the Grothendieck group $G(X)$ of coherent sheaves on X is the filtration whose ith term $\tau_i(X)$ is defined as the group

$$\tau_i(X) := \sum_{Z \subset X} \ker\left(G(X) \to G(X \setminus Z)\right)$$

where the sum is indexed over all subvarieties $Z \subset X$ having dimension $\dim(Z) \leq i$ and the arrows are pullbacks along the inclusions $X \setminus Z \subset X$. The (descending) topological filtration $\tau^\bullet(X)$ is defined by setting $\tau^i(X) = \tau_{d-i}(X)$, where $d = \dim(X)$ is the dimension of X. This filtration was first considered by Grothendieck [18, Exposé 0, App., Chap. II, §3] and afterwards by others, see e.g. [4, Chapter VI, §5].

By analogy to the above, we introduce the following generalization of the topological filtration with coefficients in an arbitrary ring S.

Definition 3.1. We define the *ascending S-topological filtration* as the filtration $\tau^S_\bullet(X) \subset G(X) \otimes S$ whose ith piece $\tau^S_i(X)$ is the group

$$\tau^S_i(X) := \sum_{Z \subset X} \ker\left(G(X) \otimes S \to G(X \setminus Z) \otimes S\right)$$

where the sum is indexed over all subvarieties $Z \subset X$ with $\dim(Z) \leq i$ and the arrows are pullbacks along the inclusions $X \setminus Z \subset X$. We also define the *descending S-topological filtration* $\tau_S^\bullet(X) \subset G(X) \otimes S$ by setting $\tau_S^i(X) = \tau^S_{d-i}(X)$ where $d = \dim(X)$.

The next lemma allows us to compare the S-topological filtration of $G(X) \otimes S$ to the topological filtration of $G(X)$ tensored by S.

Lemma 3.2. *For every $i \in \mathbb{Z}$, the group $\tau^S_i(X)$ coincides with the image*

$$\tau^S_i(X) = \operatorname{Im}\left(\tau_i(X) \otimes S \to G(X) \otimes S\right)$$

induced by the inclusion $\tau_i(X) \subset G(X)$.

Proof. Let Z be a subvariety X of dimension $\dim(Z) \leq i$ with inclusion $i_Z : Z \to X$. From the exact localization sequence associated to the pair Z and $X \setminus Z$,

$$G(Z) \xrightarrow{i_{Z*}} G(X) \to G(X \setminus Z) \to 0$$

it follows that $\tau_i(X)$ is the sum of images $\operatorname{Im}(i_{Z*})$ as Z varies over all such subvarieties. Taking the tensor product with S then gives

$$\begin{aligned} \operatorname{Im}(\tau_i(X) \otimes S \to G(X) \otimes S) &= \sum_{Z \subset X} \operatorname{Im}\left(G(Z) \otimes S \xrightarrow{i_{Z*} \otimes 1} G(X) \otimes S\right) \\ &= \sum_{Z \subset X} \ker\left(G(X) \otimes S \to G(X \setminus Z) \otimes S\right) \end{aligned}$$

as claimed. $\square$

From Lemma 3.2 it follows, as in (no.2) of the previous section, that for every $i \in \mathbb{Z}$ there is a commuting diagram with exact rows

$$\begin{tikzcd} & \tau_{i-1}(X) \otimes S \arrow[r, "j_{i-1}"] \arrow[d, "h_{i-1}"] & \tau_i(X) \otimes S \arrow[r] \arrow[d, "h_i"] & \tau_{i/i-1}(X) \otimes S \arrow[r] \arrow[d, "f_i"] & 0 \\ 0 \arrow[r] & \tau^S_{i-1}(X) \arrow[r] & \tau^S_i(X) \arrow[r] & \tau^S_{i/i-1}(X) \arrow[r] & 0 \end{tikzcd} \tag{no.3}$$

where j_{i-1} is the inclusion $\tau_{i-1}(X) \subset \tau_i(X)$ tensored with S, the vertical maps h_i are the canonical surjections, and the f_i are the induced maps on the quotients. The remainder of this section is dedicated to the proof of the following proposition.

Proposition 3.3. *Let X be an arbitrary variety of dimension d. Let S be an arbitrary ring. Then, for all $i \leq 1$, the canonical surjections of* (no.3)

$$\tau_{d-i}(X) \otimes S \xrightarrow{h_{d-i}} \tau^S_{d-i}(X) \quad \textit{and} \quad \tau_{d-i/d-i-1}(X) \otimes S \xrightarrow{f_{d-i}} \tau^S_{d-i/d-i-1}(X)$$

are isomorphisms. If X is regular and $G(X)$ is torsion free, then the same holds for $i = 2$.

Recall that when X is regular, the group $G(X)$ is a ring and the multiplication of $G(X)$ is induced by that of the Grothendieck ring $K(X)$ of finite rank locally free sheaves on X. Indeed, there is a morphism

$$\varphi_X : K(X) \to G(X) \tag{no.4}$$

defined by sending the class of a locally free sheaf to the class of itself and, when X is regular, the morphism φ_X is an isomorphism.

The ring $K(X)$ is equipped with a number of operations, i.e. set maps from $K(X)$ to itself that are functorial with respect to pullbacks. We recall the ones that will be of interest to us following [4, 14]. For any $i \geq 0$, there are lambda operations

$$\lambda^i : K(X) \to K(X)$$

that are defined on the class of a locally free sheaf $\mathcal{F}$ by the formula $\lambda^i([\mathcal{F}]) = [\wedge^i \mathcal{F}]$ where $\wedge^i \mathcal{F}$ is the ith exterior power of $\mathcal{F}$. These lambda operations define a homomorphism

$$\lambda_t := \sum \lambda^i(x) t^i : K(X) \to 1 + K(X)[[t]]$$

from $K(X)$ to the group of formal power series in the variable t with coefficients in $K(X)$ and with constant term equal 1.

From the series λ_t one can construct a number of other useful operations. For any $i \geq 0$, there are gamma operations

$$\gamma^i : K(X) \to K(X)$$

whose value $\gamma^i(x)$ on an element $x \in K(X)$ is the coefficient of t^i in the formal power series

$$\gamma_t(x) := \sum \gamma^i(x) t^i := \lambda_{t/(1-t)}(x) \in 1 + K(X)[[t]].$$

For any $i \geq 0$, there are also Adams operations

$$\psi^i : K(X) \to K(X)$$

defined using the homomorphism $\mathrm{rk} : K(X) \to \mathbb{Z}$ sending the class of a locally free sheaf $\mathcal{F}$ to its rank $\mathrm{rk}(\mathcal{F})$: the value $\psi^i(x)$ is the coefficient of t^i in the formal power series

$$\psi_t(x) := \sum \psi^i(x) t^i = \mathrm{rk}(x) - t \frac{d}{dt} \log \lambda_{-t}(x) \in K(X)[[t]].$$

For the properties of these operations we refer to the references.

The (descending) gamma filtration $\gamma^\bullet(X) \subset K(X)$ is defined as the smallest multiplicative filtration (meaning $\gamma^i(X) \cdot \gamma^j(X) \subset \gamma^{i+j}(X)$ for all i, j) having the following properties:
(1) $\gamma^0(X) = K(X)$,
(2) $\gamma^1(X) = \ker(\mathrm{rk})$,
(3) $\gamma^i(x) \in \gamma^i(X)$ for every $x \in \gamma^1(X)$ and for every $i \geq 1$.
For regular varieties X, and when one identifies $K(X)$ with $G(X)$ via the map φ_X of (no.4), the gamma filtration has the property that $\gamma^i(X) \subset \tau^i(X)$ for every $i \geq 0$. When $i \leq 2$, this inclusion is even an equality, see [8, Proposition 2.14].

Lemma 3.4. *Let X be a regular variety and let $x \in K(X)$ be a nonzero element. Then for any integer $n \geq 2$, the following statements are equivalent.*
(1) *There exists an integer $i \geq 1$ with $n^i x = 0$.*
(2) *There exists an integer $k \geq 1$ with $\psi^{n^k}(x) = 0$.*

Proof. Assume (1). Since $n^i x = 0$, we have $x \in \gamma^1(X)$. Let j be maximal with $x \in \gamma^j(X)$. By [4, Proposition 3.1] applied to the element x we have an inclusion
$$\psi^{n^i}(x) - (n^i)^j x = \psi^{n^i}(x) \in \gamma^{j+1}(X).$$
Applying [4, Proposition 3.1] to $\psi^{n^i}(x)$ and using some properties of Adams operations (specifically that they are ring homomorphisms satisfying the rule $\psi^a \circ \psi^b = \psi^{ab}$) we find
$$\psi^{n^i}(\psi^{n^i}(x)) - n^{ij+i}\psi^{n^i}(x) = \psi^{n^{2i}}(x) - \psi^{n^i}(n^{ij+i}x) = \psi^{n^{2i}}(x) \in \gamma^{j+2}(X).$$
Repeating this argument $d = \dim(X)$ times shows that there is an integer $k \geq 1$ (one can even take $k = (d+1-j)i$) such that $\psi^{n^k}(x) \in \gamma^{d+1}(X) \subset \tau^{d+1}(X) = 0$.

Conversely, assume (2). Since $\psi^{n^k}(x) = 0$, we have
$$\mathrm{rk}(\psi^{n^k}(x)) = \psi^{n^k}(\mathrm{rk}(x)) = 0$$
so that $x \in \gamma^1(X)$. Let j be maximal with $x \in \gamma^j(X)$. Applying [4, Proposition 3.1] to x we find the inclusion
$$\psi^{n^k}(x) - n^{kj}x = -n^{kj}x \in \gamma^{j+1}(X).$$
Applying [4, Proposition 3.1] to $n^{kj}x$ we get
$$\psi^{n^k}(n^{kj}x) - n^{k(j+1)}n^{kj}x = n^{kj}\psi^{n^k}(x) - n^{2jk+k}x = -n^{2jk+k}x \in \gamma^{j+2}(X).$$
Repeating this argument $d+1-j$ times we eventually find $n^i x = 0$ for some $i \geq 0$ (and one can even take $i = k(d+2-j)(j+(d+1-j)/2)$ to be precise). $\square$

Corollary 3.5. *If X is a regular variety of dimension $d = \dim(X)$, then the canonical map*
$$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \tau_{d-1}(X)) \to \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \tau_{d-1/d-2}(X))$$

is a surjection.

Proof. Since X is regular, we identify $K(X)$ and $G(X)$ using the map φ_X. In this case, we have a chain of isomorphisms

$$\tau_{d-1/d-2}(X) = \tau^{1/2}(X) = \gamma^{1/2}(X) = \mathrm{Pic}(X). \tag{no.5}$$

From left to right: the first equality is just a change of notation, the second equality follows from [8, Proposition 2.14], and the last equality is induced by the map taking the class of a locally free sheaf $\mathcal{F}$ to its determinant line bundle $\det(\mathcal{F})$, see [14, Proposition 10.6]. In particular, if $\mathcal{L}$ is a line bundle then the element $[\mathcal{L}]-1$ in $\tau_{d-1/d-2}(X)$ is mapped under (no.5) to the class of $\mathcal{L}$ in $\mathrm{Pic}(X)$.

Suppose that $[\mathcal{L}]-1$ is torsion in $\tau_{d-1/d-2}(X)$. Because of (no.5) this means there exists an integer $n>0$ with $\mathcal{L}^{\otimes n} = \mathcal{O}_X$. Hence there's an equality

$$\psi^n([\mathcal{L}]-1) = [\mathcal{L}^{\otimes n}] - 1 = 0$$

inside $K(X)$. By Lemma 3.4, the element $[\mathcal{L}]-1$ of $\tau_{d-1}(X)$ is torsion, proving the claim. □

We can now prove Proposition 3.3.

Proof of Proposition 3.3. The proof applies Lemma 2.5 above, setting $R = G(X)$ and $F_i = \tau_i(X)$. The assumptions of Lemma 2.5 (2) hold from both the existence of the rank map and the equality $\tau^1(X) = \gamma^1(X)$. The assumptions of Lemma 2.5 (3) hold by Corollary 3.5. □

We end with a definition for the S-gamma filtration of the ring $K(X)\otimes S$, for an arbitrary ring of coefficients S, keeping the spirit of this section.

Definition 3.6. We define the (*descending*) *S-gamma filtration* as the filtration $\gamma_S^{\bullet}(X) \subset K(X)\otimes S$ whose ith piece $\gamma_S^i(X)$ is the image

$$\gamma_S^i(X) := \mathrm{Im}(\gamma^i(X)\otimes S \to K(X)\otimes S)$$

induced by the inclusion $\gamma^i(X) \subset K(X)$. We define the *ith S-gamma operation* $\gamma_S^i(x)$ of an element $x \in K(X)$ as the image of $\gamma^i(x)$ in $K(X)\otimes S$.

Remark 3.7. The descending S-gamma filtration is a multiplicative filtration of $K(X)\otimes S$. If X is regular, then the descending S-topological filtration is a multiplicative filtration of $G(X)\otimes S$. When one identifies $K(X)$ with $G(X)$ via the map φ_X, there is a comparison $\gamma_S^i(X) \subset \tau_S^i(X)$ for all $i \geq 0$ with equality holding for $i \leq 2$.

Remark 3.8. Let F be a field and let X be a variety of dimension d. Assume that the F-dimension of $G(X)\otimes F$ is finite, i.e. $\dim_F(G(X)\otimes F) < \infty$. Then there are equalities

$$\dim_F(G(X)\otimes F) = \sum_{i\leq d} \big(\dim_F(\tau_i^F(X)) - \dim_F(\tau_{i-1}^F(X))\big)$$

$$= \sum_{i \le d} \dim_F \left(\tau^F_{i/i-1}(X) \right).$$

If X is a regular variety, and if $\dim_F(K(X) \otimes F) < \infty$, then an analogous argument shows $\dim_F(K(X) \otimes F) = \sum_{i \ge 0} \dim_F(\gamma_F^{i/i+1}(X))$.

4. Generic algebras of index 2, 4, 8, 16 and 32

Throughout this section, we fix a central simple algebra A over our base field k of degree $\deg(A) = d + 1$ and index $\operatorname{ind}(A) = p^n$ for a prime p (eventually we'll assume $p = 2$). Set

$$X = \mathbf{SB}(A) \subset \operatorname{Gr}(d+1, A) \tag{no.6}$$

to be the Severi–Brauer variety $\mathbf{SB}(A)$ associated with A, considered as the subvariety of the Grassmannian $\operatorname{Gr}(d+1, A)$ of $(d+1)$-dimensional subspaces of A whose R-points $X(R)$, for any finite type k-algebra R, are exactly the minimal right ideals of $A \otimes_k R$.

The primary purpose of this section is to illustrate how one can use the results above to produce nontrivial torsion cycles in the Chow ring $\operatorname{CH}(X)$ of the Severi–Brauer variety X. We do this below (Tables 1, 2, and 3; Corollaries 4.16, 4.17, and 4.18) under some additional assumptions on the algebra A and the variety X. Before doing this, however, we recall a number of results that will facilitate our computations. From now on we always identify the ring $K(X)$ with the ring $G(X)$ without mention of the canonical map φ_X of (no.4).

4.1. Structure for $K(X)$. We write ζ_X for the tautological sheaf on X. By definition, this means that ζ_X is the pullback of the universal subsheaf of $\operatorname{Gr}(d+1, A)$ under the embedding of (no.6). It follows that ζ_X is a right module under the constant sheaf A so, for any $i \ge 0$, it makes sense to define sheaves

$$\zeta_X(i) := \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i$$

for some fixed choices of simple left $A^{\otimes i}$-modules M_i. By convention we set $\zeta_X(0) = \mathcal{O}_X$.

The significance of the sheaves $\zeta_X(i)$, for the purposes of this section, is due to the following theorem of Quillen [17, §8, Theorem 4.1] describing the group $K(X)$.

Theorem 4.1. *The group homomorphism*

$$\bigoplus_{i=0}^{\deg(A)-1} K(A^{\otimes i}) \to K(X)$$

sending the class of a left $A^{\otimes i}$-module M to the class of $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

For any central simple algebra B the Grothendieck group $K(B)$, of finitely generated and projective left B-modules, is isomorphic with $\mathbb{Z}$. A canonical generator for $K(B)$ is the class of a simple left B-module. Hence Theorem 4.1 shows that $K(X)$ is free with basis the classes of the sheaves $\zeta_X(i)$ as i ranges over the interval $0 \leq i \leq d$.

It's also possible to determine the multiplication of $K(X)$ from Theorem 4.1. To do this we note that, since $K(X)$ is torsion free, the flat pullback

$$\pi^*_{F/k} : K(X) \to K(X_F),$$

along the projection $\pi_{F/k} : X_F \to X$ from any finite extension F/k, is an injection. If the extension F/k splits A, then there is an isomorphism between X_F and the projective space $\mathbb{P}^d$ so that we can identify $K(X_F)$ with the ring

$$K(X_F) = \mathbb{Z}[x]/(1-x)^{d+1} \tag{no.7}$$

where $x = [\mathcal{O}(-1)]$, see [14, Theorem 4.5]. Finally, because the equality $\mathrm{rk}(\zeta_X(i)) = \mathrm{ind}(A^{\otimes i})$ holds for every $i \geq 0$, it follows that $K(X)$ can be identified with the subring of $K(X_F)$ generated by $\pi^*_{F/k}(\zeta_X(i)) = \mathrm{ind}(A^{\otimes i})x^i$ as i ranges over the interval $0 \leq i \leq d$.

4.2. The reduced Behavior $\boldsymbol{r\mathcal{B}eh(A)}$. Recall from [8, Definition 3.8] that the *reduced behavior of* A is the following sequence of p-adic valuations

$$r\mathcal{B}eh(A) = \Big(v_p(\mathrm{ind}(A^{\otimes p^i}))\Big)_{i=0}^{m}$$

where the index i is increasing from 0 to the p-adic valuation $m = v_p(\exp(A))$ of the exponent (or period) $\exp(A)$. The reduced behavior is a strictly decreasing sequence of length $m+1$. The first term of this sequence is always $n = v_p(\mathrm{ind}(A))$ and the last term is always 0.

Conversely, for every strictly decreasing sequence of integers S starting with n and ending with 0, there exists a central division algebra A^S such that $\mathrm{ind}(A^S) = p^n$ and $r\mathcal{B}eh(A^S) = S$. One can even choose A^S so that the gamma and topological filtrations of $K(X^S)$ coincide for the variety $X^S = \mathbf{SB}(A^S)$, see [8, Theorem 3.7 and Lemma 3.10].

Lastly, note that the ring $K(X)$ is completely determined by the reduced behavior of A because of the description of $K(X)$ given in Subsection 4.1. In fact, the gamma filtration $\gamma^\bullet(X) \subset K(X)$ is also completely determined by the reduced behavior as a consequence of the description (no.7) and the functorality of the gamma operations, [6, Corollary 1.2].

4.3. The level $\mathbf{lev}(\boldsymbol{A})$. Consider the following set of integers $i \geq 1$,

$$S_X = \{i : v_p(\mathrm{ind}(A^{\otimes p^i})) < v_p(\mathrm{ind}(A^{\otimes p^{i-1}})) - 1\}. \tag{no.8}$$

The cardinality $\#S_X$ of this set is an invariant of A called the *level of* A, i.e. $\mathrm{lev}(A) = \#S_X$. Colloquially, the integers of the set S_X are exactly the places where the reduced behavior $r\mathcal{B}eh(A)$ decreases by more than one from the previous spot. Our interest in the level of A is due to the next lemma and its subsequent corollary.

Lemma 4.2. [11, Lemma A.6] *The ring $K(X)$ is generated by the lambda operations of the classes of the sheaves $\zeta_X(p^i)$ where $i \in S_X \cup \{0\}$.*

Corollary 4.3 ([13, Lemma 5.4])**.** *The ith piece of the gamma filtration $\gamma^i(X) \subset K(X)$ is generated additively by all products*

$$\gamma^{j_1}(x_1 - \mathrm{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \mathrm{rk}(x_r))$$

with $j_1 + \cdots + j_r \geq i$ and with $x_1, ..., x_r$ classes of the sheaves $\zeta_X(p^i)$ where $i \in S_X \cup \{0\}$.

The level is also known to affect the torsion subgroups of $\mathrm{CH}(X)$.

Lemma 4.4. [8, Proposition 4.9 and Proposition 4.14] *Assume $p = 2$ and $\mathrm{lev}(A) \leq 1$. Then $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X)) = 0$ in either of the following cases:*
(1) $\mathrm{lev}(A) = 0$
(2) $\mathrm{lev}(A) = 1$ *and* $r\mathcal{B}eh(A) = (n, ..., 2, 0)$.

Moreover, if one assumes $\gamma^3(X) = \tau^3(X)$ then in the remaining case that $\mathrm{lev}(A) = 1$ *and* $r\mathcal{B}eh(A) \neq (n, ..., 2, 0)$*, one has* $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X)) = \mathbb{Z}/2^r\mathbb{Z}$ *where*

$$r = \begin{cases} \min\{i, n - n_i - i\} & \text{if } n_i > 0 \\ \min\{i, n - i - 1\} & \text{if } n_i = 0 \end{cases}$$

for the uniquely determined $i \in S_X$ and for $n_i = v_2(\mathrm{ind}(A^{\otimes 2^i}))$.

4.4. The groups $\mathbf{CT}^i(1; X)$ and $\mathbf{Q}^i(X)$. Let $\mathrm{CT}(1; X)$ be the subring of $\mathrm{CH}(X)$ generated by the Chern classes of $\zeta_X(1)$. For any $i \geq 0$, we write $\mathrm{CT}^i(1; X)$ for the subgroup of $\mathrm{CT}(1; X)$ contained in $\mathrm{CH}^i(X)$; we write $\mathrm{Q}^i(X)$ for the cokernel of the inclusion $\mathrm{CT}^i(1; X) \subset \mathrm{CH}^i(X)$. It follows from [11, Proposition A.8] that $\mathrm{CT}^i(1; X)$ is isomorphic with $\mathbb{Z}$. Consequently, for any $i \geq 0$ there is an inclusion

$$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^i(X)) \subset \mathrm{Q}^i(X). \qquad \text{(no.9)}$$

The group $\mathrm{Q}^2(X)$ has been studied in depth, e.g. in [12, Proposition 3.7]. Combined with [8, Proposition 4.7 and Proposition 4.9] one gets the next:

Lemma 4.5. *Suppose that $\gamma^3(X) = \tau^3(X)$ and $\mathrm{Q}^2(X) = \mathbb{Z}/p\mathbb{Z}$. Assume additionally that either of the following two conditions hold:*
(1) *the prime p is odd and* $\mathrm{lev}(A) > 0$
(2) $p = 2$ *and either* $\mathrm{lev}(A) > 1$ *or,* $\mathrm{lev}(A) = 1$ *and* $r\mathcal{B}eh(A) \neq (n, ..., 2, 0)$.
Then there's an equality $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X)) = \mathbb{Z}/p\mathbb{Z}$.

Proof. From [8, Corollary 2.15] there's an isomorphism $\mathrm{CH}^2(X) = \gamma^{2/3}(X)$. Under the assumption of either (1) or (2), Karpenko [8, Proposition 4.7 and Proposition 4.9] shows that $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \gamma^{2/3}(X)) \neq 0$. We conclude using the inclusion of (no.9). □

Corollary 4.6. *Suppose that $\gamma^3(X) = \tau^3(X)$. Assume that $p = 2$ and assume that $r\mathcal{B}eh(A)$ has the form of either* (1), (2), *or* (3) *below.*

(1) $r\mathcal{B}eh(A) = (4, 2, 0)$
(2) $r\mathcal{B}eh(A) = (5, 4, 2, 0)$
(3) $r\mathcal{B}eh(A) = (5, 3, 2, 0)$.

Then $\mathrm{Q}^2(X) = \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X)) = \mathbb{Z}/2\mathbb{Z}$.

Proof. In [12, Proposition 3.7], the group $\mathrm{Q}^2(X)$ is described by generators and some, but possibly not all, relations. When the reduced behavior of A has the form of (1), (2), or (3) one can check that the relations described in [12, Proposition 3.7] show that $\mathrm{Q}^2(X)$ is a quotient of $\mathbb{Z}/2\mathbb{Z}$. But, in each of these cases the group $\mathrm{CH}^2(X)$ has nontrivial torsion because of our assumption $\gamma^3(X) = \tau^3(X)$ and [8, Proposition 4.7 and Proposition 4.9]. It follows from the inclusion (no.9) that $\mathrm{Q}^2(X) = \mathbb{Z}/2\mathbb{Z}$. Now one can apply Lemma 4.5 to see that $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X)) = \mathbb{Z}/2\mathbb{Z}$. □

The group $\mathrm{Q}^2(X)$ has also been determined in the following setting.

Lemma 4.7. *Suppose that $\gamma^3(X) = \tau^3(X)$. Assume that $p = 2$ and* $\mathrm{lev}(A) \leq 1$. *Then:*

$$\mathrm{Q}^2(X) = \begin{cases} 0 & \textit{if } \mathrm{lev}(A) = 0 \\ 0 & \textit{if } \mathrm{lev}(A) = 1 \textit{ and } r\mathcal{B}eh(A) = (n, ..., 2, 0) \\ \mathbb{Z}/2^s\mathbb{Z} & \textit{if } \mathrm{lev}(A) = 1 \textit{ and } r\mathcal{B}eh(A) \neq (n, ..., 2, 0). \end{cases}$$

In the last case, the value s equals

$$s = \begin{cases} n - n_i - i & \textit{if } n_i > 0 \\ n - i - 1 & \textit{if } n_i = 0 \end{cases}$$

for the uniquely determined $i \in S_X$ and for $n_i = v_2(\mathrm{ind}(A^{\otimes 2^i}))$.

Proof. In [11, Theorem A.15], the groups $\mathrm{Q}^2(\tilde{X})$ are computed, with the values given above, for any Severi–Brauer variety $\tilde{X}$ with the property that the gamma and topological filtrations of $K(\tilde{X})$ coincide. This gives us isomorphisms

$$\mathrm{CH}^2(X) = \gamma^{2/3}(X) = \gamma^{2/3}(\tilde{X}) = \mathrm{CH}^2(\tilde{X}) \tag{no.10}$$

where, from left to right, the first is because $\gamma^3(X) = \tau^3(X)$ and [8, Corollary 2.15], the second is because the gamma filtration depends only on the reduced behavior [6, Corollary 1.2], and the last follows from [11, Theorem A.15]. One can check that the isomorphism of (no.10) commutes with the inclusions of both $\mathrm{CT}^2(1; X)$ and $\mathrm{CT}^2(1; \tilde{X})$. The claim follows since both $\mathrm{Q}^2(X)$ and $\mathrm{Q}^2(\tilde{X})$ are defined as the cokernels of these inclusions. □

4.5. A summary so far. Throughout the remainder of this section our goal is to produce nontrivial torsion cycles in the Chow ring $\mathrm{CH}(X)$ under the assumption that A is a generic division algebra (in the sense of [8, Definition 3.12]) with $\mathrm{ind}(A) = 2^n$ for some $n \leq 5$. To be precise, we recall that A is *generic* if, for every $i \geq 0$, the inclusion $\gamma^i(X) \subset \tau^i(X)$ of the gamma filtration in the topological filtration, is an equality.

One consequence of our computations is a complete description of the torsion subgroup of $\mathrm{CH}^2(X)$ for any generic algebra A of index $\mathrm{ind}(A) = 2^n$ for any $n \leq 5$. This result still has some interest when A is not necessarily generic because the torsion subgroups that we describe below are maximal in the sense that they surject onto the torsion subgroup of $\mathrm{CH}^2(X)$ for any algebra A of the same reduced behavior [8, Theorem 3.13]. Now we summarize this result which is only completed in Subsection 4.6 below.

If $\mathrm{ind}(A) = 1$ or $\mathrm{ind}(A) = 2$, then the torsion subgroup of $\mathrm{CH}^2(X)$ is well-known to be trivial. If $\mathrm{ind}(A) = 4$, then there two cases: either $\mathrm{lev}(A) = 0$ or $r\mathcal{B}eh(A) = (2, 0)$. In both cases one has $\mathrm{CH}^2(X) = \mathbb{Z}$ by Lemma 4.4 and $\mathrm{Q}^2(X) = 0$ by Lemma 4.7.

For generic algebras A with $\mathrm{ind}(A) = 2^3$, all possible values of torsion in $\mathrm{CH}^2(X)$ are given in Table 1 below. Table 1 can be filled out with Lemma 4.4.

For generic algebras A with $\mathrm{ind}(A) = 2^4$, all possible torsion subgroups of $\mathrm{CH}^2(X)$ are given in Table 2 below. It turns out that only $\mathbb{Z}/2\mathbb{Z}$ can appear as a torsion subgroup in this case. Table 2 can be filled out with the help of Lemma 4.4, Lemma 4.7, and Corollary 4.6.

Lastly, for generic algebras A with $\mathrm{ind}(A) = 2^5$, all possible torsion subgroups of $\mathrm{CH}^2(X)$ are given in Table 3 below; the group depends on the reduced behavior of A. Rows 1-10 and 13-16 of Table 3 can be filled out using Lemma 4.4, Lemma 4.7, and Corollary 4.6. For rows 11 and 12, we note that [12, Proposition 3.7] shows $\mathrm{Q}^2(X)$ is a quotient of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In Subsection 4.6 we prove Corollary 4.15 saying that, for these two cases, we have

$$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X)) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Together with the inclusion (no.9), this completes the table.

4.6. Working with coefficients in $\mathbb{F}_2$. We write $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ for the field of two elements. We assume throughout that A is a central simple algebra of index $\mathrm{ind}(A) = 2^n$ with $n \geq 1$. We continue to use the notation $X = \mathbf{SB}(A)$ for the Severi–Brauer variety associated to A.

Because of Theorem 4.1, we have a canonical basis for the $\mathbb{F}_2$-vector space $K(X) \otimes \mathbb{F}_2$ consisting of those elements ν_i that are the classes (mod 2) of the sheaves $\zeta_X(i)$ respectively,

$$K(X) \otimes \mathbb{F}_2 = \bigoplus_{i=0}^{\deg(A)-1} \mathbb{F}_2 \cdot \nu_i. \qquad \text{(no.11)}$$

From now on F/k will be a finite extension splitting A and, under the identification (no.7), we will work in $K(X_F)$ to deduce relations in the space $K(X) \otimes \mathbb{F}_2$.

Lemma 4.8. *Let $m = v_2(\exp(A))$. Then inside of $K(X) \otimes \mathbb{F}_2$ we have the relations*

$$\nu_i \nu_j = \begin{cases} \nu_{i+j} & \text{if } 2^m \mid i \text{ or } 2^m \mid j \\ 0 & \text{otherwise.} \end{cases}$$

for any pair of integers $i, j \geq 0$.

Proof. Since $\nu_0 = 1$, it suffices to assume $i, j \geq 1$. Now, in the ring $K(X) \subset K(X_F)$ multiplication is defined so that

$$\begin{aligned} \operatorname{ind}(A^{\otimes i})x^i \cdot \operatorname{ind}(A^{\otimes j})x^j &= \frac{\operatorname{ind}(A^{\otimes j})\operatorname{ind}(A^{\otimes i})}{\operatorname{ind}(A^{\otimes i+j})}\operatorname{ind}(A^{\otimes i+j})x^{i+j} \\ &= \alpha \cdot \operatorname{ind}(A^{\otimes i+j})x^{i+j}. \end{aligned}$$

Indeed, we'll show that α is an integer.

To see this, we use the following two facts (see [5, Chapter 4 Section 5]):

(1) for any integer $t \geq 1$, we have $\operatorname{ind}(A^{\otimes t}) = \operatorname{ind}(A^{\otimes 2^{v_2(t)}})$;
(2) for any pair of integers $r \geq s \geq 0$ one has the divisibility relation $\operatorname{ind}(A^{\otimes 2^r}) \mid \operatorname{ind}(A^{\otimes 2^s})$.

Because of (1), it suffices to show the divisibility

$$\operatorname{ind}(A^{\otimes 2^{v_2(i+j)}}) \mid \operatorname{ind}(A^{\otimes 2^{v_2(j)}})\operatorname{ind}(A^{\otimes 2^{v_2(i)}}).$$

But, by properties of valuations, we have $v_2(i+j) \geq \max\{v_2(i), v_2(j)\}$ so that (2) applies.

Finally, we show that $\alpha \equiv 1 \pmod 2$ only in the suggested cases. There are only two cases: either $m \leq v_2(i)$ or $m \leq v_2(j)$; or $v_2(i) \leq v_2(j) < m$. In the former case, we get (assuming that $m \leq v_2(i)$ without loss of generality) that $\alpha \equiv 1 \pmod 2$ because

$$\operatorname{ind}(A^{\otimes j}) = \operatorname{ind}(A^{\otimes i+j})$$

whenever 2^m divides i. In the latter case, we use the inequality $v_2(i+j) \geq \max\{v_2(i), v_2(j)\}$ to find the divisibility

$$\operatorname{ind}(A^{\otimes i+j}) \mid \operatorname{ind}(A^{\otimes i}).$$

Combined with the fact $2 \mid \operatorname{ind}(A^{\otimes j})$ it follows $\alpha \equiv 0 \pmod 2$. □

Now we work towards describing the $\mathbb{F}_2$-gamma filtration (see Definition 3.6) of $K(X) \otimes \mathbb{F}_2$. In this direction, we first prove Lemma 4.10 giving an explicit description for the images of the gamma operations of the elements $\zeta_X(2^i) - \operatorname{ind}(A^{\otimes 2^i})$ for any $i \geq 1$. Together with Lemma 4.9, this provides us with an explicit description for the generators of the $\mathbb{F}_2$-gamma filtration in any given degree. Afterwards, we work by hand to determine relations between the generators that we described.

Lemma 4.9. *Let S_X be the set defined as in* (no.8). *Then the ith piece of the $\mathbb{F}_2$-gamma filtration $\gamma^i_{\mathbb{F}_2}(X) \subset K(X) \otimes \mathbb{F}_2$ is generated by all products*

$$\gamma^{j_1}_{\mathbb{F}_2}(x_1 - \mathrm{rk}(x_1)) \cdots \gamma^{j_r}_{\mathbb{F}_2}(x_r - \mathrm{rk}(x_r)) \qquad \text{(no.12)}$$

with $j_1 + \cdots + j_r \geq i$ and with $x_1, ..., x_r$ classes of the sheaves $\zeta_X(2^i)$ where $i \in S_X \cup \{0\}$.

Proof. By Corollary 4.3, the similarly defined monomials of $K(X)$ generate $\gamma^i(X) \subset K(X)$. But, the images (in $K(X) \otimes \mathbb{F}_2$) of these monomials are the products of the images of the individual factors, hence the claim. □

For the following lemma we define $S_2(r) = a_0 + \cdots + a_s$, for any integer $r \geq 1$, to be the sum of the coefficients appearing in a base-2 expansion $r = a_0 + a_1 2 + \cdots + a_s 2^s$, i.e. in such an expression with $0 \leq a_0, ..., a_s \leq 1$.

Lemma 4.10. *Fix an integer $i \geq 0$ and set $n_i = v_2(\mathrm{ind}(A^{\otimes 2^i}))$. Then, for any integer j with $1 \leq j \leq 2^{n_i}$ and for each integer k with $0 \leq k \leq j$, there is an integer $\alpha^k_{i,j}$ so that*

$$\gamma^j_{\mathbb{F}_2}(\zeta_X(2^i) - 2^{n_i}) = \sum_{0 \leq k \leq j} \alpha^k_{i,j} \nu_{2^i k}$$

when the $\alpha^k_{i,j}$ are considered in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Moreover, the integers $\alpha^k_{i,j}$ satisfy the congruences

$$\alpha^k_{i,j} \equiv \begin{cases} 0 & \text{if } n_i - v_2(j) - S_2(j) + S_2(k) + S_2(j-k) - n_{v_2(k)+i} > 0 \\ 1 & \text{if } n_i - v_2(j) - S_2(j) + S_2(k) + S_2(j-k) - n_{v_2(k)+i} = 0 \end{cases}$$

in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Proof. We claim that it suffices to consider only the case $i = 0$. To see this, choose $i > 0$ and set $Y = \mathbf{SB}(A^{\otimes 2^i})$ to be the Severi–Brauer variety associated with the tensor power $A^{\otimes 2^i}$. Then X embeds into Y via the composition

$$f : X \to X \times \cdots \times X = X^{\times 2^i} \to Y$$

of the diagonal embedding of X into the direct product $X^{\times 2^i}$ of 2^i copies of X, and the twisted Segre embedding of $X^{\times 2^i}$ into Y. The pullback f^* with coefficients in $\mathbb{F}_2$,

$$f^* : K(Y) \otimes \mathbb{F}_2 \to K(X) \otimes \mathbb{F}_2$$

sends the class of $\zeta_Y(k)$ to $f^*\zeta_Y(k) = \zeta_X(2^i k)$ and commutes with the gamma operations. Assume that the lemma holds when $i = 0$, i.e. assume that there are integers, say $\beta^k_{0,j}$, with

$$\gamma^j_{\mathbb{F}_2}(\zeta_Y(1) - 2^{n_i}) = \sum_{0 \leq k \leq j} \beta^k_{0,j} \nu_k$$

and satisfying the given congruences. Then, from the equalities

$$f^*\gamma^j_{\mathbb{F}_2}(\zeta_Y(1)-2^{n_i}) = f^*\left(\sum_{0\leq k\leq j}\beta^k_{0,j}\nu_k\right)$$
$$= \sum_{0\leq k\leq j}\beta^k_{0,j}\nu_{2^ik} = \gamma^j_{\mathbb{F}_2}(f^*\zeta_Y(1)-2^{n_i}) = \gamma^j_{\mathbb{F}_2}(\zeta_X(2^i)-2^{n_i})$$

one finds that the claim holds for this $i>0$ as well by taking $\alpha^k_{i,j}=\beta^k_{0,j}$.

In the case $i=0$, we compute explicitly the image of $\gamma^j(\zeta_X(1)-2^n)$ in $K(X)\otimes\mathbb{F}_2$. Fix a finite field extension F/k splitting A and identify $K(X)\subset K(X_F)$ as in (no.7). Then

$$\gamma_t(\zeta_X(1)-2^n) = \gamma_t(x-1)^{2^n} = (1+(x-1)t)^{2^n}$$

and it follows

$$\gamma^j(\zeta_X(1)-2^n) = \binom{2^n}{j}(x-1)^j.$$

Expanding this again, we get

$$\gamma^j(\zeta_X(1)-2^n) = \sum_{0\leq k\leq j}(-1)^{j-k}\binom{2^n}{j}\binom{j}{k}x^k.$$

Setting

$$\beta^k_{0,j} = \frac{\binom{2^n}{j}\binom{j}{k}}{\mathrm{ind}(A^{\otimes 2^{v_2(k)}})}$$

and computing the 2-adic valuation (using Kummer's theorem) of $\beta^k_{0,j}$ gives the result, in light of the previous paragraph. □

Remark 4.11. In the case that A is a division algebra of index $\mathrm{ind}(A)=2^5$ and of reduced behavior either $r\mathcal{B}eh(A)=(5,3,1,0)$ or $r\mathcal{B}eh(A)=(5,3,0)$, the values $\alpha^k_{i,j}$ obtained from Lemma 4.10 are compiled in Table 4. To get them, one can either use the formula provided in the lemma statement or expand the polynomial $(x-1)^j$ modulo powers of 2 (e.g. to get the entry for $\gamma^3_{\mathbb{F}_2}(\zeta_X(1)-32)$ one can note that $(x-1)^3\equiv x^3+x^2+x-1 \mod 2$; now $v_2(\binom{32}{3})=5$ so that one must have $\gamma^3_{\mathbb{F}_2}(\zeta_X(1)-32)=\nu_3+4\nu_2+\nu_1+32=\nu_3+\nu_1$).

Theorem 4.12. *Suppose A is a division algebra with $\mathrm{ind}(A)=2^5$ and $r\mathcal{B}eh(A)=(5,3,1,0)$. For each $i\geq 0$, set*

$$x_i=\gamma^i_{\mathbb{F}_2}(\zeta_X(1)-32),\quad y_i=\gamma^i_{\mathbb{F}_2}(\zeta_X(2)-8),\quad \textit{and}\quad z_i=\gamma^i_{\mathbb{F}_2}(\zeta_X(4)-2).$$

Then the associated graded space for the $\mathbb{F}_2$-gamma filtration of $K(X)\otimes\mathbb{F}_2$ is determined by the information in Table 6 below.

Proof. We proceed by considering, for each degree $0\leq i\leq 31$, all of the possible monomials described in Lemma 4.9. Then we use the relations given in (no.13) below to eliminate all but the suggested generators from the associated graded space. The proof will be complete once we eliminate enough generators to prove that there's an inequality

$$\sum_{i\geq 0} \dim_{\mathbb{F}_2}\left(\gamma^{i/i+1}_{\mathbb{F}_2}(X)\right) \leq 32$$

because of Remark 3.8.

Note that by the definition of x_i, y_i, z_i we have the trivial relations: $x_0 = y_0 = z_0 = \nu_0 = 1$; $x_i = 0$ for $i \geq 32$; $y_i = 0$ for $i > 8$; $z_i = 0$ for $i > 2$. Now the following relations can be found using Lemma 4.8 and the entries of Table 4. We assume $i, j \geq 1$:

$$\begin{array}{lll}
x_{2i+1} = x_{2i+2} \text{ for } i < 15 & y_{2i+1} = y_{2i+2} \text{ for } i < 3 & y_8^2 = x_{31} \\
x_i y_j = \begin{cases} 0 & j < 8 \\ x_{i+16} & j = 8,\, i \leq 14 \\ 0 & j = 8,\, i > 16 \end{cases} & x_i z_j = \begin{cases} 0 & j = 1 \\ x_{i+8} & j = 2,\, i \leq 23 \\ 0 & j = 2,\, i > 24 \end{cases} & x_i x_j = 0 \\
y_i y_j = \begin{cases} 0 & i, j < 8 \\ y_i y_8 & i \leq 8,\, j = 8 \end{cases} & y_i z_j = \begin{cases} 0 & i < 8,\, j = 1 \\ z_1 z_2^2 & i = 8,\, j = 1 \\ y_{i+4} & i \leq 3,\, j = 2 \\ y_8 z_2 & i = 8,\, j = 2 \end{cases} & z_1^2 = 0 \\
z_2^2 = y_8 + y_7 & y_6 z_2 = y_8 y_1 &
\end{array} \tag{no.13}$$

Degree 0. The only monomial of (no.12) in degree 0 is $\nu_0 = x_0 = y_0 = z_0 = 1$.

Degree 1. There are three monomials as in (no.12) of degree 1: x_1, y_1, and z_1. Looking at Table 4, we have $x_1 = x_2$ and $y_1 = y_2$ so that $x_1 = y_1 = 0$ modulo $\gamma^2_{\mathbb{F}_2}(X)$.

Degree 2. Generators of degree 2 are $x_1^2 = 0$, $y_1^2 = 0$, $z_1^2 = 0$, x_2, y_2, and z_2. There are no relations on the x_2, y_2 and z_2 monomials.

Degree 3. Now a monomial generator like those in (no.12) of degree $l \geq 3$ will have the form

$$x_i^a y_j^b y_8^{c_0} z_1^{c_1} z_2^{c_2} \tag{no.14}$$

for some $0 \leq i < 32$ with $0 \leq j < 8$ and for some integers $a, b, c_0, c_1, c_2 \geq 0$ satisfying

$$0 \leq a, b, c_0, c_1 \leq 1 \quad \text{and} \quad 0 \leq c_2 \leq 3$$

with $ia + jb + 8c_0 + c_1 + 2c_2 = l$. Indeed, there are relations $x_r x_s = 0$ for all $r, s \geq 1$, relations $y_r y_s = 0$ whenever $1 \leq r, s < 8$, a relation $z_1^2 = 0$, and $y_8^2 = z_2^4 = x_{31} + x_{32} = x_{31}$. Note that these are some, but not all possible, restrictions on our monomial generators (e.g. no two of a, b, c_1 can be simultaneously positive).

This leaves as possible degree 3 generators: x_3, $x_1 y_2$, $x_1 z_2$, y_3, $y_1 z_2$, $z_1 z_2$. But, $x_3 = x_4$ and $y_3 = y_4$ so that both terms vanish modulo $\gamma^4_{\mathbb{F}_2}(X)$. We also have $x_1y_2 = 0$, $x_1z_2 = x_9$, $y_1z_2 = y_5$ so that these terms similarly vanish modulo $\gamma^4_{\mathbb{F}_2}(X)$. This leaves just z_1z_2 as a generator for this degree.

Degree 4. Barring the restrictions given in the previous case, possible degree 4 monomials are: x_4, x_2z_2, y_4, y_2z_2, z_2^2. But $x_2z_2 = x_{10}$, $y_2z_2 = y_6$, and $z_2^2 = y_8 + y_7$ so that these monomials must vanish modulo $\gamma^5_{\mathbb{F}_2}(X)$. This leaves only x_4 and y_4 in this degree. We note that we now exclude z_2^2 from ever being a factor of a monomial generator, i.e. we check only $0 \leq c_2 \leq 1$ in (no.14).

Degree 5. Possible monomials of degree 5 are now: x_5, x_3z_2, y_5, y_3z_2. Since $x_5 = x_6$, $y_5 = y_6$, $x_3 = x_4$, and $y_3 = y_4$ all of these generators vanish in the associated graded space.

Degree 6. Possible monomials of degree 6 are: x_6, x_4z_2, y_6, y_4z_2. But $x_4z_2 = x_{12}$ and $y_4z_2 = y_3z_2 = y_7$ both vanish modulo $\gamma^7_{\mathbb{F}_2}(X)$. This leaves x_6 and y_6.

Degree 7. Possible monomials of degree 7 are: x_7, x_5z_2, y_7, y_5z_2. Here $x_7 = x_8$, $x_5 = x_6$, and $y_5 = y_6$. Only y_7 remains.

Degree 8. Possible monomials of degree 8 are: x_8, x_6z_2, y_8, y_6z_2. Here $x_6z_2 = x_{14}$ and $y_6z_2 = y_8y_1$ so that these monomials can be eliminated. This leaves x_8 and y_8.

Degree 9. Possible monomials of degree 9 are: x_9, x_1y_8, x_7z_2, y_1y_8, y_8z_1, y_7z_2. But $x_1 = x_2$, $y_1 = y_2$, $x_7 = x_8$, and $x_9 = x_{10}$. Modding out by $\gamma^{10}_{\mathbb{F}_2}(X)$ leaves only y_8z_1 and y_7z_2. But, we have $y_7 = y_3z_2 = y_4z_2$ and $y_7z_2 = y_4z_2^2 = y_4(y_7 + y_8) = y_4y_8$ so that only y_8z_1 remains.

Degree 10. Possible monomials of degree 10 are: x_{10}, x_2y_8, y_2y_8, y_8z_2. Only $x_2y_8 = x_{18}$ can be eliminated so that x_{10}, y_2y_8, and y_8z_2 remain as generators.

Degree 11. Possible monomials of degree 11 are: x_{11}, x_3y_8, $x_1y_8z_2$, y_3y_8, $y_1y_8z_2$, $y_8z_1z_2$. Here the first five can be eliminated since $x_{11} = x_{12}$, $x_3 = x_4$, $x_1 = x_2$, $y_3 = y_4$, and $y_1 = y_2$. This leaves $y_8z_1z_2$.

Degree 12. Possible monomials of degree 12 are: x_{12}, x_4y_8, $x_2y_8z_2$, y_4y_8, $y_2y_8z_2$. Since we have $x_4y_8 = x_{20}$, $x_2y_8 = x_{18}$, and $y_2z_2 = y_6$ the only monomials that survive are x_{12} and y_4y_8.

Degree 13. Possible monomials of degree 13 are: x_{13}, x_5y_8, $x_3y_8z_2$, y_5y_8, $y_3y_8z_2$. There are no monomials that survive.

Degree 14. Possible monomials of degree 14 are: x_{14}, x_6y_8, $x_4y_8z_2$, y_6y_8, $y_4y_8z_2$. But we have $x_6y_8 = x_{22}$, $x_4z_2 = x_{12}$ and $y_4z_2 = y_3z_2 = y_7$. This leaves x_{14} and y_6y_8.

Degree 15. Possible monomials of degree 15 are: x_{15}, x_7y_8, $x_5y_8z_2$, y_7y_8, $y_5y_8z_2$. Here most of the odd terms are problematic. Only y_7y_8 survives.

Degree 16. Possible monomials of degree 16 are: x_{16}, x_8y_8, $x_6y_8z_2$, $y_6y_8z_2$. Note $x_6z_2 = x_{14}$, $x_8y_8 = x_{24}$ and $y_6z_2 = y_8y_1$ so that x_{16} is the only monomial left.

Degree 17. Possible monomials of degree 17 are: x_{17}, x_9y_8, $x_7y_8z_2$, $y_7y_8z_2$. Only $y_7y_8z_2$ can remain but, $y_7z_2 = y_4y_8$ as we found in degree 9 so that all monomials are eliminated.

Degree 18. Possible monomials of degree 18 are: x_{18}, $x_{10}y_8$, $x_8y_8z_2$. Here only x_{18} survives.

Degree 19. Possible monomials of degree 19 are: x_{19}, $x_{11}y_8$, $x_9y_8z_2$. This emulates the general procedure in all further degrees. There simply aren't enough large degree monomials to produce higher terms. In odd degrees (except for degree 31), all terms will vanish; in even degrees $2i$, there will be only one generator given by an x_{2i}. □

Theorem 4.13. *Suppose A is a division algebra with* $\mathrm{ind}(A) = 2^5$ *and* $r\mathcal{B}eh(A) = (5, 3, 0)$. *For each $i \geq 0$, set*

$$x_i = \gamma^i_{\mathbb{F}_2}(\zeta_X(1) - 32), \quad y_i = \gamma^i_{\mathbb{F}_2}(\zeta_X(2) - 8), \quad \text{and} \quad z_i = \gamma^i_{\mathbb{F}_2}(\zeta_X(4) - 1).$$

Then the associated graded space for the $\mathbb{F}_2$-gamma filtration of $K(X) \otimes \mathbb{F}_2$ is determined by the information in Table 7 below.

Proof. The same method of proof that works for Theorem 4.12 can be done here. By the definition of x_i, y_i, z_i we have the trivial relations: $x_0 = y_0 = z_0 = \nu_0 = 1$; $x_i = 0$ for $i \geq 32$; $y_i = 0$ for $i > 8$; $z_i = 0$ for $i > 1$. Now one can use the relations

$$\begin{array}{lll}
x_{2i+1} = x_{2i+2} \text{ for } i < 15 & y_{2i+1} = y_{2i+2} \text{ for } i < 3 & y_8^2 = z_1^8 = x_{31} \\
x_iy_j = \begin{cases} 0 & j < 8 \\ x_{i+16} & j = 8,\ i \leq 14 \\ 0 & j = 8,\ i > 16 \end{cases} & x_iz_1 = \begin{cases} x_{i+4} & i \leq 27 \\ 0 & i > 28 \end{cases} & x_ix_j = 0 \\
y_iy_j = \begin{cases} 0 & i, j < 8 \\ y_iy_8 & i \leq 8,\ j = 8 \end{cases} & y_iz_1 = \begin{cases} y_{i+2} & i \leq 5 \\ y_1y_8 & i = 7 \end{cases} & z_1^4 = y_7 + y_8
\end{array} \tag{no.15}$$

for $i, j \geq 1$ that are found using Lemma 4.8 and the entries of Table 4. $\square$

Theorem 4.14. *Suppose A is a division algebra with* $\operatorname{ind}(A) = 2^4$ *and* $r\mathcal{B}eh(A) = (4, 2, 0)$*. For each $i \geq 0$, set*

$$x_i = \gamma^i_{\mathbb{F}_2}(\zeta_X(1) - 16), \quad y_i = \gamma^i_{\mathbb{F}_2}(\zeta_X(2) - 4), \quad \text{and} \quad z_i = \gamma^i_{\mathbb{F}_2}(\zeta_X(4) - 1).$$

Then the associated graded space for the $\mathbb{F}_2$-gamma filtration of $K(X) \otimes \mathbb{F}_2$ is determined by the information in Table 5 below.

Proof. The proof follows the same lines as the proofs for Theorems 4.12 and 4.13. $\square$

Corollary 4.15. *Let A be a central simple algebra with* $\operatorname{ind}(A) = 2^5$*. Let $X = \mathbf{SB}(A)$ be the associated Severi–Brauer variety. Assume either of the following are true:*

(1) $r\mathcal{B}eh(A) = (5, 3, 1, 0)$,
(2) $r\mathcal{B}eh(A) = (5, 3, 0)$.

Then there is a surjection

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \operatorname{Tor}_1(\mathbb{Q}/\mathbb{Z}, \operatorname{CH}^2(X)).$$

Moreover, this surjection is an isomorphism if and only if $\gamma^3(X) = \tau^3(X)$.

Proof. To construct the given surjection, we let $\tilde{A}$ be a generic algebra with $\operatorname{ind}(\tilde{A}) = 2^5$ and $r\mathcal{B}eh(\tilde{A}) = r\mathcal{B}eh(A)$. Set $\tilde{X} = \mathbf{SB}(\tilde{A})$. From [8, Theorem 3.13], there is a surjection

$$\operatorname{CH}^2(\tilde{X}) \twoheadrightarrow \operatorname{CH}^2(X)$$

which is an isomorphism if and only if $\gamma^3(X) = \tau^3(X)$. Further, the kernel of this surjection is a torsion subgroup of $\operatorname{CH}^2(\tilde{X})$ so, applying the functor $\mathbb{Q}/\mathbb{Z} \otimes -$ we get a surjection

$$\operatorname{Tor}_1(\mathbb{Q}/\mathbb{Z}, \operatorname{CH}^2(\tilde{X})) \twoheadrightarrow \operatorname{Tor}_1(\mathbb{Q}/\mathbb{Z}, \operatorname{CH}^2(X)).$$

It suffices then to show $\operatorname{Tor}_1(\mathbb{Q}/\mathbb{Z}, \operatorname{CH}^2(\tilde{X})) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Since $\tilde{A}$ is generic, the topological and gamma filtration of $K(\tilde{X})$ coincide (by definition). Hence the $\mathbb{F}_2$-gamma and the descending $\mathbb{F}_2$-topological filtration of $K(X)\otimes\mathbb{F}_2$ coincide. Since $\tilde{X}$ satisfies the conditions of Proposition 3.3, the composition

$$\operatorname{CH}^2(\tilde{X}) \otimes \mathbb{F}_2 \xrightarrow{\sim} \tau^{2/3}(\tilde{X}) \otimes \mathbb{F}_2 \twoheadrightarrow \tau^{2/3}_{\mathbb{F}_2}(\tilde{X}),$$

of the canonical isomorphism [3, Example 15.3.6] and the canonical surjection of (no.3) when $S = \mathbb{F}_2$, is an isomorphism. Theorem 4.12 and Theorem 4.13 show that

$$\tau^{2/3}_{\mathbb{F}_2}(\tilde{X}) = \mathbb{Z}/2\mathbb{Z}^{\oplus 3}.$$

As $\operatorname{CH}^2(\tilde{X})$ has rank one, and its torsion subgroup is a finitely generated 2-primary group we find

$$\operatorname{Tor}_1(\mathbb{Q}/\mathbb{Z}, \operatorname{CH}^2(\tilde{X})) = \mathbb{Z}/2^r\mathbb{Z} \oplus \mathbb{Z}/2^s\mathbb{Z}$$

for some integers $r, s \geq 1$. But, it's possible to determine from [12, Proposition 3.7] that $\mathrm{Q}^2(X)$ is a quotient of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ so that $r = s = 1$ by (no.9). □

Lastly, we end with some corollaries that follow immediately from the data of the second columns of Tables 6, 7, 5 and from the existence of the canonical surjections

$$\mathrm{CH}^i(X) \otimes \mathbb{F}_2 \twoheadrightarrow \tau^{i/i+1}(X) \otimes \mathbb{F}_2 \twoheadrightarrow \tau^{i/i+1}_{\mathbb{F}_2}(X)$$

coming from the Grothendieck-Riemann-Roch without denominators ([3, Example 15.1.5]) and from (no.3) with $S = \mathbb{F}_2$.

Corollary 4.16. *Let A be a central simple algebra with* $\mathrm{ind}(A) = 2^5$ *and reduced behavior $r\mathcal{B}eh(A) = (5,3,1,0)$. Let $X = \mathbf{SB}(A)$ be the associated Severi–Brauer variety. Finally, assume that A is generic.*

Then the group $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^i(X))$ *is:*

(1) *nonzero if $i = 2, 4, 6, 8, 10, 12, 14$*
(2) *noncyclic if $i = 2, 10$.* □

Corollary 4.17. *Let A be a central simple algebra with index* $\mathrm{ind}(A) = 2^5$ *and reduced behavior $r\mathcal{B}eh(A) = (5,3,0)$. Let $X = \mathbf{SB}(A)$ be the associated Severi–Brauer variety. Finally, assume that A is generic.*

Then the group $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^i(X))$ *is:*

(1) *nonzero if $i = 2, 4, 6, 8, 10, 12, 14$*
(2) *noncyclic if $i = 2, 10$.* □

Corollary 4.18. *Let A be a central simple algebra with index* $\mathrm{ind}(A) = 2^4$ *and reduced behavior $r\mathcal{B}eh(A) = (4,2,0)$. Let $X = \mathbf{SB}(A)$ be the associated Severi–Brauer variety. Finally, assume that A is generic. Then* $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^i(X)) \neq 0$ *if $i = 2, 4, 6$.* □

TABLE 1. For generic algebras of index 8

	$r\mathcal{B}eh(A)$	$\mathrm{lev}(A)$	$Q^2(X)$	$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X))$
1	$(3,2,1,0)$	0	0	0
2	$(3,2,0)$	1	0	0
3	$(3,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(3,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

TABLE 2. For generic algebras of index 16

	$r\mathcal{B}eh(A)$	$\mathrm{lev}(A)$	$Q^2(X)$	$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X))$
1	$(4,3,2,1,0)$	0	0	0
2	$(4,3,2,0)$	1	0	0
3	$(4,3,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(4,3,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
5	$(4,2,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
6	$(4,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
7	$(4,1,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
8	$(4,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

TABLE 3. For generic algebras of index 32

	$r\mathcal{B}eh(A)$	$\mathrm{lev}(A)$	$Q^2(X)$	$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X))$
1	$(5,4,3,2,1,0)$	0	0	0
2	$(5,4,3,2,0)$	1	0	0
3	$(5,4,3,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(5,4,3,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
5	$(5,4,2,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
6	$(5,4,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
7	$(5,4,1,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
8	$(5,4,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
9	$(5,3,2,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
10	$(5,3,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
11	$(5,3,1,0)$	2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
12	$(5,3,0)$	2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
13	$(5,2,1,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
14	$(5,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
15	$(5,1,0)$	1	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
16	$(5,0)$	1	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

TABLE 4. $\gamma^j_{\mathbb{F}_2}(-)$'s if $r\mathcal{B}eh(A) = (5,3,1,0)$ or $r\mathcal{B}eh(A) = (5,3,0)$

j	$\zeta_X(1) - 32$	$\zeta_X(2) - 8$	$\zeta_X(4) - 2$	$\zeta_X(4) - 1$
1	ν_1	ν_2	ν_4	$\nu_4 + \nu_0$
2	ν_1	ν_2	$\nu_8 + \nu_4 + \nu_0$	-
3	$\nu_3 + \nu_1$	$\nu_6 + \nu_2$	-	
4	$\nu_3 + \nu_1$	$\nu_6 + \nu_2$		
5	$\nu_5 + \nu_1$	$\nu_{10} + \nu_2$		
6	$\nu_5 + \nu_1$	$\nu_{10} + \nu_2$		
7	$\nu_7 + \nu_5 + \nu_3 + \nu_1$	$\nu_{14} + \nu_{10}$ $+\nu_6 + \nu_2$		
8	$\nu_7 + \nu_5 + \nu_3 + \nu_1$	$\nu_{16} + \nu_{14} + \nu_{10}$ $+\nu_6 + \nu_2 + \nu_0$		
9	$\nu_9 + \nu_1$	-		
10	$\nu_9 + \nu_1$			
11	$\nu_{11} + \nu_9 + \nu_3 + \nu_1$			
12	$\nu_{11} + \nu_9 + \nu_3 + \nu_1$			
13	$\nu_{13} + \nu_9 + \nu_5 + \nu_1$			
14	$\nu_{13} + \nu_9 + \nu_5 + \nu_1$			
15	$\nu_{15} + \nu_{13} + \nu_{11} + \nu_9$ $+\nu_7 + \nu_5 + \nu_3 + \nu_1$			
16	$\nu_{15} + \nu_{13} + \nu_{11} + \nu_9$ $+\nu_7 + \nu_5 + \nu_3 + \nu_1$			
17	$\nu_{17} + \nu_1$			
18	$\nu_{17} + \nu_1$			
19	$\nu_{19} + \nu_{17} + \nu_3 + \nu_1$			
20	$\nu_{19} + \nu_{17} + \nu_3 + \nu_1$			
21	$\nu_{21} + \nu_{17} + \nu_5 + \nu_1$			
22	$\nu_{21} + \nu_{17} + \nu_5 + \nu_1$			
23	$\nu_{23} + \nu_{21} + \nu_{19} + \nu_{17}$ $+\nu_7 + \nu_5 + \nu_3 + \nu_1$			
24	$\nu_{23} + \nu_{21} + \nu_{19} + \nu_{17}$ $+\nu_7 + \nu_5 + \nu_3 + \nu_1$			
25	$\nu_{25} + \nu_{17} + \nu_9 + \nu_1$			
26	$\nu_{25} + \nu_{17} + \nu_9 + \nu_1$			
27	$\nu_{27} + \nu_{25} + \nu_{19} + \nu_{17}$ $+\nu_{11} + \nu_9 + \nu_3 + \nu_1$			
28	$\nu_{27} + \nu_{25} + \nu_{19} + \nu_{17}$ $+\nu_{11} + \nu_9 + \nu_3 + \nu_1$			
29	$\nu_{29} + \nu_{25} + \nu_{21} + \nu_{17}$ $+\nu_{13} + \nu_9 + \nu_5 + \nu_1$			

30	$\nu_{29}+\nu_{25}+\nu_{21}+\nu_{17}$ $+\nu_{13}+\nu_9+\nu_5+\nu_1$			
31	$\nu_{31}+\nu_{29}+\nu_{27}+\nu_{25}$ $+\nu_{23}+\nu_{21}+\nu_{19}+\nu_{17}$ $+\nu_{15}+\nu_{13}+\nu_{11}+\nu_9$ $+\nu_7+\nu_5+\nu_3+\nu_1$			
32	$\nu_{32}+\nu_{31}+\nu_{29}+\nu_{27}$ $+\nu_{25}+\nu_{23}+\nu_{21}+\nu_{19}$ $+\nu_{17}+\nu_{15}+\nu_{13}+\nu_{11}$ $+\nu_9+\nu_7+\nu_5+\nu_3$ $+\nu_1+\nu_0$			

TABLE 5. $r\mathcal{B}eh(A)=(4,2,0)$

i	$\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{i/i+1}(X)\right)$	generators	$\sum_{j\leq i}\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{j/j+1}(X)\right)$
0	1	ν_0	1
1	1	z_1	2
2	2	x_2, y_2	4
3	1	y_3	5
4	2	x_4, y_4	7
5	1	y_4z_1	8
6	2	x_6, y_2y_4	10
7	1	y_3y_4	11
8	1	x_8	12
9	0	-	12
10	1	x_{10}	13
11	0	-	13
12	1	x_{12}	14
13	0	-	14
14	1	x_{14}	15
15	1	x_{15}	16

TABLE 6. $r\mathcal{B}eh(A) = (5, 3, 1, 0)$

i	$\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{i/i+1}(X)\right)$	generators	$\sum_{j\leq i}\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{j/j+1}(X)\right)$
0	1	ν_0	1
1	1	z_1	2
2	3	x_2, y_2, z_2	5
3	1	z_1z_2	6
4	2	x_4, y_4	8
5	0	-	8
6	2	x_6, y_6	10
7	1	y_7	11
8	2	x_8, y_8	13
9	1	y_8z_1	14
10	3	x_{10}, y_2y_8, y_8z_2	17
11	1	$y_8z_1z_2$	18
12	2	x_{12}, y_4y_8	20
13	0	-	20
14	2	x_{14}, y_6y_8	22
15	1	y_7y_8	23
16	1	x_{16}	24
17	0	-	24
18	1	x_{18}	25
19	0	-	25
20	1	x_{20}	26
21	0	-	26
22	1	x_{22}	27
23	0	-	27
24	1	x_{24}	28
25	0	-	28
26	1	x_{26}	29
27	0	-	29
28	1	x_{28}	30
29	0	-	30
30	1	x_{30}	31
31	1	x_{31}	32

TABLE 7. $r\mathcal{B}eh(A) = (5,3,0)$

i	$\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{i/i+1}(X)\right)$	generators	$\sum_{j\leq i}\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{j/j+1}(X)\right)$
0	1	ν_0	1
1	1	z_1	2
2	3	x_2, y_2, z_1^2	5
3	1	z_1^3	6
4	2	x_4, y_4	8
5	0	-	8
6	2	x_6, y_6	10
7	1	y_7	11
8	2	x_8, y_8	13
9	1	y_8z_1	14
10	3	$x_{10}, y_2y_8, y_8z_1^2$	17
11	1	$y_8z_1^3$	18
12	2	x_{12}, y_4y_8	20
13	0	-	20
14	2	x_{14}, y_6y_8	22
15	1	y_7y_8	23
16	1	x_{16}	24
17	0	-	24
18	1	x_{18}	25
19	0	-	25
20	1	x_{20}	26
21	0	-	26
22	1	x_{22}	27
23	0	-	27
24	1	x_{24}	28
25	0	-	28
26	1	x_{26}	29
27	0	-	29
28	1	x_{28}	30
29	0	-	30
30	1	x_{30}	31
31	1	x_{31}	32

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(Eoin Mackall) Department of Mathematics, University of Maryland, College Park, MD, USA
eoinmackall@gmail.com

This paper is available via http://nyjm.albany.edu/j/2020/26-48.html.