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# On the existence of local quaternionic contact geometries

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Abstract. We exploit the Cartan-Kähler theory to prove the local existence of real analytic quaternionic contact structures for any prescribed values of the respective curvature functions and their covariant derivatives at a given point on a manifold. We show that, in a certain sense, the different real analytic quaternionic contact geometries in $4n+3$ dimensions depend, modulo diffeomorphisms, on $2n+2$ real analytic functions of $2n+3$ variables.

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## 1. Introduction

The quaternionic contact (briefly: qc) structures are a rather recently developed concept in the differential geometry that has proven to be a very useful tool when dealing with a certain type of analytic problems concerning the extremals and the choice of a best constant in the $L^2$ Folland-Stein inequality on the quaternionic Heisenberg group [6], [8], [7]. Originally, the concept was introduced by O. Biquard [1], who was partially motivated by a preceding result of C. LeBrun [11] concerning the existence of a large family of complete quaternionic Kähler metrics of negative scalar curvature, defined on the unit ball $B^{4n+4} \subset \mathbb{R}^{4n+4}$. By interpreting $B^{4n+4}$ as a quaternionic hyperbolic space, $B^{4n+4} \cong Sp(n+1,1)/Sp(n) \times Sp(1)$, LeBrun was able to

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construct deformations of the associated twistor space $\mathcal{Z}$—a complex manifold which, in this case, is biholomorphically equivalent to a certain open subset of the complex projective space $\mathbb{CP}^{2n+3}$—that preserve its induced contact structure and anti-holomorphic involution, and thus can be pushed down to produce deformations of the standard (hyperbolic) quaternionic Kähler metric of $B^{4n+4}$. The whole construction is parametrized by an arbitrary choice of a sufficiently small holomorphic function of $2n+3$ complex variables and the result in [11] is that the moduli space of the so arising family of complete quaternionic Kähler metrics on $B^{4n+4}$ is infinite dimensional. LeBrun also observed that, if multiplied by a function that vanishes along the boundary sphere $S^{4n+3}$ to order two, the deformed metric tensors on $B^{4n+4}$ extend smoothly across $S^{4n+3}$ but their rank drops to 4 there.

It was discovered later by Biquard [1] that the arising structure on $S^{4n+3}$ is essentially given by a certain very special type of a co-dimension 3 distribution which he introduced as a qc structure on $S^{4n+3}$ and called the conformal boundary at infinity of the corresponding quaternionic Kähler metric on $B^{4n+4}$. Biquard proved also the converse [1]: He showed that each real analytic qc structure on a manifold $M$ is the conformal boundary at infinity of a (germ) unique quaternionic Kähler metric defined in a small neighborhood of $M$. Therefore, already by the very appearance of the new concept of a qc geometry, it was clear that there exist infinitely many examples—namely, the global qc structures on the sphere $S^{4n+3}$ obtained by the LeBrun's deformations of $B^{4n+4}$.

However, the number of the explicitly known examples remains so far very restricted. There is essentially only one generic method for obtaining such structures explicitly. It is based on the existence of a certain very special type of Riemannian manifolds, the so called 3-Sasaki like spaces. These are Riemannian manifolds that admit a special triple $R_1, R_2, R_3$ of Killing vector fields, subject to some additional requirements (we refer to [7] and the references therein for more detail on the topic), which carry a natural qc structure defined by the orthogonal complement of the triple $\{R_1, R_2, R_3\}$. There are no explicit examples of qc structures (not even locally) for which it is proven that they can not be generated by the above construction.

The formal similarity with the definition of a CR (Cauchy-Riemann) manifold, considered in the complex analysis, might suggests that one should look for new examples of qc structures by studying hypersurfaces in the quaternionic coordinate space $\mathbb{H}^{n+1}$. This idea, however, turned out to be rather unproductive in the quaternionic case: In [10] it was shown that each qc hypersurface embedded in $\mathbb{H}^{n+1}$ is necessarily given by a quadratic form there and that all such hypersurfaces are locally equivalent, as qc manifolds, to the standard (3-Sasaki) sphere.

In the present paper we reformulate the problem of local existence of qc structures as a problem of existence of integral manifolds of an appropriate exterior differential system to which we apply methods from the Cartan-Kähler theory and show its integrability. The definition of the respective exterior differential system is based entirely on the formulae obtained in [12] for the associated canonical Cartan connection and its curvature. We compute explicitly the relevant character sequence $v_1, v_2, \dots$ of the system (cf. the discussion in Section 2.1) and show that it passes the so called Cartan's test, i.e., that the system is in involution. From there we obtain our main result in the paper—this is Theorem 3.3—that asserts the local existence of qc structures for any prescribed values of their respective curvatures and associated covariant derivatives at any fixed point on a manifold.

Furthermore, since the last non-zero character of the associated exterior differential system is $v_{2n+3} = 2n + 2$, we obtain a certain description for the associated moduli spaces. Namely, we have that, in a certain sense (the precise formulation requires care, cf. [2]), the real analytic qc structures in $4n + 3$ dimensions depend, modulo diffeomorphisms, on $2n + 2$ functions of $2n + 3$ variables. Comparing our result to the LeBrun's family of qc structures on the sphere $S^{4n+3}$—parametrized by a single holomorphic function of $2n + 3$ complex variables (which has the same generality as two real analytic functions of $2n + 3$ real variables)—we observe that it simply is not "big enough" in order to provide a local model for all possible qc geometries in dimension $4n + 3$.

## 2. Quaternionic contact structures as integral manifolds of exterior differential systems

Our work has been inspired and heavily influenced by the series of lectures by Robert Bryant at the Winter School Geometry and Physics in Srní, January 2015, essentially along the lines of [2]. In particular our description of the qc structures in the following paragraphs follows this source closely.

**2.1. Exterior differential systems.** In general, an exterior differential system is a graded differentially closed ideal $\mathcal{I}$ in the algebra of differential forms on a manifold $N$. Integral manifolds of such a system are immersions $f : M \to N$ such that the pullback $f^*\alpha$ of any form $\alpha \in \mathcal{I}$ vanishes on $M$. Typically, the differential ideal $\mathcal{I}$ encounters all differential consequences of a system of partial differential equations and understanding the algebraic structure of $\mathcal{I}$ helps to understand the structure of the solution set. We need a special form of exterior differential systems corresponding to the geometric structures modelled on homogeneous spaces, the so called Cartan geometries. This means our system will be generated by one-forms forming the Cartan's coframing intrinsic to a geometric structure and its differential consequences (the curvature and its derivatives).

For this paragraph, we adopt the following ranges of indices: $1 \le a, b, c, d, e \le n$, $1 \le s \le l$, where $l$ and $n$ are some fixed positive integers.

We consider the following general problem: *Given a set of real analytic functions* $C^a_{bc} : \mathbb{R}^l \to \mathbb{R}$ *with* $C^a_{bc} = -C^a_{cb}$, *find linearly independent one-forms* $\omega^a$, *defined on a domain* $\Omega \subset \mathbb{R}^n$, *and a mapping* $u = (u^s) : \Omega \to \mathbb{R}^l$ *so that the equations*

$$d\omega^a = -\frac{1}{2} C^a_{bc}(u)\, \omega^b \wedge \omega^c \tag{2.1}$$

*are satisfied everywhere on* $\Omega$.

The problem is diffeomorphism invariant in the sense that if $(\omega^a, u)$ is any solution of (2.1) defined on $\Omega \subset \mathbb{R}^n$ and $\Phi : \Omega' \to \Omega$ is a diffeomorphism, then $(\Phi^*(\omega^a), \Phi^*(u))$ is a solution of (2.1) on $\Omega'$. We regard any such two solutions as equivalent and we are interested in the following question: *How many non-equivalent solutions does a given problem of this type admit?*

Next, we reformulate this into a question on solutions to an exterior differential system. Let $N = GL(n, \mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^l$ and denote by $p = (p^a_b) : N \to GL(n, \mathbb{R})$, $x = (x^a) : N \to \mathbb{R}^n$ and $u = (u^s) : N \to \mathbb{R}^l$ the respective projections. Setting $\omega^a \overset{def}{=} p^a_b\, dx^b$, we consider the differential ideal $\mathcal{I}$ on $N$ generated by the set of two-forms

$$\Upsilon^a \overset{def}{=} d\omega^a + \frac{1}{2} C^a_{bc}(u)\, \omega^b \wedge \omega^c.$$

Then, the solutions of (2.1) are precisely the $n$-dimensional integral manifolds of $\mathcal{I}$ on which the restriction of the $n$-form $\omega^1 \wedge \cdots \wedge \omega^n$ is nowhere vanishing. The reformulation of the problem (2.1) in this setting allows for an easy access of tools from the Cartan-Kähler theory. We shall see, we may restrict our attention to a certain set of sufficient conditions for the integrability of the system, known as the Cartan's Third Theorem, and refer the reader to [2] or [3] and the references therein for a more detailed and general discussion on the topic.

Differentiating (2.1) gives

$$\begin{aligned} 0 = d^2\omega^a &= -\frac{1}{2} d(C^a_{bc}\omega^b \wedge \omega^c) \\ &= -\frac{1}{2} \frac{\partial C^a_{bc}(u)}{\partial u^s}\, du^s \wedge \omega^b \wedge \omega^c + \frac{1}{3}\big(C^a_{be}(u)\, C^e_{cd}(u) \\ &\qquad + C^a_{ce}(u)\, C^e_{db}(u) + C^a_{de}(u)\, C^e_{bc}(u)\big)\, \omega^b \wedge \omega^c \wedge \omega^d. \end{aligned} \tag{2.2}$$

If $C^a_{bc}$ were curvature functions of a Cartan connection, then these differential consequences are governed by the well known Bianchi identities, and they are then quadratic.

**2.2. Assumptions and conclusions.** In order to employ the Cartan--Kähler theory we need to replace the quadratic terms by some linear objects. Thus we posit the following two assumptions:

**Assumption I:** *Let us assume that there exist a real analytic mapping* $F = (F_a^s) : \mathbb{R}^l \to \mathbb{R}^n$ *for which*

$$d\big(C^a_{bc}\omega^b \wedge \omega^c\big) = \frac{\partial C^a_{bc}(u)}{\partial u^s}\big(du^s + F^s_d\omega^d\big) \wedge \omega^b \wedge \omega^c. \tag{2.3}$$

Of course, this assumption is equivalent to the requirement

$$\begin{aligned}\frac{1}{3}\big(C^a_{be}(u)\, C^e_{cd}(u) + C^a_{ce}(u)\, C^e_{db}(u) + C^a_{de}(u)\, C^e_{bc}(u)\big)\, \omega^b \wedge \omega^c \wedge \omega^d \\ = -\frac{1}{2}\frac{\partial C^a_{bc}(u)}{\partial u^s}\, F^s_d(u)\, \omega^b \wedge \omega^c \wedge \omega^d.\end{aligned}$$

and then, on the integral manifolds of $\mathcal{I}$, (2.2) takes the form

$$0 = d^2\omega^a = -\frac{1}{2}\frac{\partial C^a_{bc}(u)}{\partial u^s}\left(du^s + F^s_d(u)\, \omega^d\right) \wedge \omega^b \wedge \omega^c. \tag{2.4}$$

**Assumption II:** *Interpreting* (2.4) *as a system of algebraic equations for the unknown one-forms* $du^s$ *(for a fixed* $u$*), we assume that it is non-degenerate, i.e, that* (2.4) *yields* $du^s \in span\{\omega^a\}$*. As a consequence, at any* $u$*, the set of all solutions* $du^s$ *is parametrized by a certain vector space (since the system* (2.4) *is linear). We will assume that the dimension of this vector space is a constant* $D$ *(independent of* $u$*).*

Let us take the latter two assumptions as granted in the rest of this paragraph. Since $\mathcal{I}$ is a differential ideal, it is algebraically generated by the forms $\Upsilon^a$ and $d\Upsilon^a$. By (2.3), we have

$$\begin{aligned}2d\Upsilon^a &= d\Big(C^a_{bc}(u)\, \omega^b \wedge \omega^c\Big) \\ &= \frac{\partial C^a_{bc}(u)}{\partial u^s}\left(du^s + F^s_d(u)\omega^d\right) \wedge \omega^b \wedge \omega^c + 2C^a_{bc}\, \Upsilon^b \wedge \omega^c\end{aligned} \tag{2.5}$$

and therefore, $\mathcal{I}$ is algebraically generated by $\Upsilon^a$ and the three-forms

$$\Xi^a \overset{def}{=} \frac{\partial C^a_{bc}(u)}{\partial u^s}\left(du^s + F^s_d(u)\omega^d\right) \wedge \omega^b \wedge \omega^c.$$

If we take $\Omega^a$ to be some other basis of one-forms for the vector space $\operatorname{span}\{\omega^a\}$, we can express the forms $\Xi^a$ as

$$\Xi^a = \Pi^a_{bc} \wedge \Omega^b \wedge \Omega^c, \tag{2.6}$$

where $\Pi^a_{bc}$ are linear combinations of the linearly independent one-forms

$$\{du^s + F^s_d(u)\omega^d \ : \ s = 1, \dots, n\}.$$

Consider the sequence $v_1(u), v_2(u), \dots, v_n(u)$ of non-negative integers defined, for any fixed $u$, as $v_1(u) = 0$,

$$\begin{aligned}v_d(u) = \operatorname{rank}\Big\{\Pi^a_{bc}(u) \ : \ a = 1, \dots, n,\ 1 \le b < c \le d\Big\} \\ - \ \operatorname{rank}\Big\{\Pi^a_{bc} \ : \ a = 1, \dots, n,\ 1 \le b < c \le d-1\Big\};\end{aligned}$$

for $1 < d \leq n-1$, and

$$v_n(u) = l - \operatorname{rank}\Big\{\Pi^a_{bc} \ : \ a = 1 \dots n, \ 1 \leq b < c \leq n-1\Big\}.$$

If, for every $u \in \mathbb{R}^l$, one can find a basis $\Omega^a$ of $\operatorname{span}\{\omega^a\}$ for which the Cartan's Test

$$v_1(u) + 2v_2(u) + \cdots + nv_n(u) = D, \tag{2.7}$$

is satisfied ($D$ is the constant dimension from Assumption II), then the system (2.1) is said to be *in involution* (this method of computation for the Cartan's sequence of an ideal is based on [3], Proposition 1.15). It is an important result of the theory of exterior differential systems (essentially due to Cartan, cf. [2]) that if the system is in involution, then for any $u_0$, there exists a solution $(\omega^a, u)$ of (2.1) defined on a neighborhood $\Omega$ of $0 \in \mathbb{R}^n$ for which $u(0) = u_0$ and

$$du^s|_0 = F^s_d(u_0)\omega^d|_0.$$

Moreover, in certain sense (see again [2] for a more precise formulation), the *different solutions* $(\omega^a, u)$ *of* (2.1)*, modulo diffeomorphisms, depend on* $v_k(u)$ *functions of* $k$ *variables, where* $v_k(u)$ *is the last non-vanishing integer in the Cartan's sequence* $v_1(u), \dots, v_n(u)$.

The geometric significance of the above is quite clear: Assume that we are interested in a geometric structure of a certain type that can be characterized by a unique Cartan connection. Then, the structure equations of the corresponding Cartan connection are some equations of type (2.1) involving the curvature of the connection. The solutions of the so arising exterior differential system are precisely the different local geometries of the fixed type that we are considering.

**2.3. Quaternionic contact manifolds.** Let $M$ be a $(4n+3)$-dimensional manifold and $H$ be a smooth distribution on $M$ of codimension three. The pair $(M, H)$ is said to be a quaternionic contact (abbr. qc) structure if around each point of $M$ there exist 1-forms $\eta_1, \eta_2, \eta_3$ with common kernel $H$, a positive definite inner product $g$ on $H$, and endomorphisms $I_1, I_2, I_3$ of $H$, satisfying

$$\begin{aligned} &(I_1)^2 = (I_2)^2 = (I_3)^2 = -\mathrm{id}_H, \qquad I_1\, I_2 = -I_2\, I_1 = I_3, \\ &d\eta_s(X, Y) = 2g(I_s X, Y) \qquad \text{for all} \ \ X, Y \in H. \end{aligned} \tag{2.8}$$

As shown in [1], if $\dim(M) > 7$, one can always find, locally, a triple $\xi_1, \xi_2, \xi_3$ of vector fields on $M$ satisfying for all $X \in H$,

$$\eta_s(\xi_t) = \delta^s_t, \qquad d\eta_s(\xi_t, X) = -d\eta_t(\xi_s, X) \tag{2.9}$$

($\delta^s_t$ being the Kronecker delta). $\xi_1, \xi_2, \xi_3$ are called Reeb vector fields corresponding to $\eta_1, \eta_2, \eta_3$. In the seven dimensional case the existence of Reeb vector fields is an additional integrability condition on the qc structure (cf. [5]) which we will assume to be satisfied.

It is well known that the qc structures represent a very interesting instance of the so called parabolic geometries, i.e. Cartan geometries modelled on $G/P$ with $G$ semisimple and $P \subset G$ parabolic. The above definition is a description of these geometries with the additional assumption that their harmonic torsions vanish.

As the authors showed in [12], the canonical Cartan connection with the properly normalized curvature can be computed explicitly, including closed formulae for all its curvature components and their covariant derivatives. This provides the complete background for viewing the structures as integral manifolds of an appropriate exterior differential system (cf. paragraph 2.6), running the Cartan test, checking the involution of the system, and concluding the generality of the structures in question (the section 3 below).

For the convenience of the readers we are going to explain the results from [12] in detail now. This requires to introduce some notation first.

**2.4. Conventions for complex tensors and indices.** In the sequel, we use without comment the convention of summation over repeating indices; the small Greek indices $\alpha, \beta, \gamma, \dots$ will have the range $1, \dots, 2n$, whereas the indices $s, t, k, l, m$ will be running from 1 to 3.

Consider the Euclidean vector space $\mathbb{R}^{4n}$ with its standard inner product $\langle , \rangle$ and a quaternionic structure induced by the identification $\mathbb{R}^{4n} \cong \mathbb{H}^n$ with the quaternion coordinate space $\mathbb{H}^n$. The latter means that we endow $\mathbb{R}^{4n}$ with a fixed triple $J_1, J_2, J_3$ of complex structures which are Hermitian with respect to $\langle , \rangle$ and satisfy $J_1\, J_2 = -J_2\, J_1 = J_3$. The complex vector space $\mathbb{C}^{4n}$, being the complexification of $\mathbb{R}^{4n}$, splits as a direct sum of $+i$ and $-i$ eigenspaces, $\mathbb{C}^{4n} = \mathcal{W} \oplus \overline{\mathcal{W}}$, with respect to the complex structure $J_1$. The complex 2-form $\pi$,

$$\pi(u, v) \overset{def}{=} \langle J_2 u, v\rangle + i\langle J_3 u, v\rangle, \qquad u, v \in \mathbb{C}^{4n},$$

has type $(2, 0)$ with respect to $J_1$, i.e., it satisfies $\pi(J_1 u, v) = \pi(u, J_1 v) = i\pi(u, v)$. Let us fix an $\langle , \rangle$-orthonormal basis (once and for all)

$$\{\mathfrak{e}_\alpha \in \mathcal{W}, \mathfrak{e}_{\bar\alpha} \in \overline{\mathcal{W}}\}, \qquad \mathfrak{e}_{\bar\alpha} = \overline{\mathfrak{e}_\alpha}, \tag{2.10}$$

with dual basis $\{\mathfrak{e}^\alpha, \mathfrak{e}^{\bar\alpha}\}$ so that $\pi = \mathfrak{e}^1 \wedge \mathfrak{e}^{n+1} + \mathfrak{e}^2 \wedge \mathfrak{e}^{n+2} + \cdots + \mathfrak{e}^n \wedge \mathfrak{e}^{2n}$. Then, we have

$$\langle , \rangle = g_{\alpha\bar\beta}\, \mathfrak{e}^\alpha \otimes \mathfrak{e}^{\bar\beta} + g_{\bar\alpha\beta}\, \mathfrak{e}^{\bar\alpha} \otimes \mathfrak{e}^\beta, \qquad \pi = \pi_{\alpha\beta}\, \mathfrak{e}^\alpha \wedge \mathfrak{e}^\beta \tag{2.11}$$

with

$$g_{\alpha\bar\beta} = g_{\bar\beta\alpha} = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}, \qquad \pi_{\alpha\beta} = -\pi_{\beta\alpha} = \begin{cases} 1, & \text{if } \alpha + n = \beta \\ -1, & \text{if } \alpha = \beta + n \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Any array of complex numbers indexed by lower and upper Greek letters (with and without bars) corresponds to a tensor, e.g., $\{A_{\alpha\cdot\cdot}^{\ \ \beta\bar\gamma}\}$ corresponds to the tensor

$$A_{\alpha\,.\,.}^{\;\;\beta\bar\gamma}\,\mathfrak{e}^\alpha\otimes\mathfrak{e}_\beta\otimes\mathfrak{e}_{\bar\gamma}.$$

Clearly, the vertical as well as the horizontal position of an index carries information about the tensor. For two-tensors, we take $B_\beta^\alpha$ to mean $B_{\beta\,.}^{\;\;\alpha}$, i.e., the lower index is assumed to be first. We use $g_{\alpha\bar\beta}$ and $g^{\alpha\bar\beta}=g^{\bar\beta\alpha}=g_{\alpha\bar\beta}$ to lower and raise indices in the usual way, e.g.,

$$A_{\alpha\,.\,\gamma}^{\;\;\beta}=g_{\bar\sigma\gamma}\,A_{\alpha\,.\,.}^{\;\;\beta\bar\sigma},\qquad A^{\bar\alpha\beta\bar\gamma}=g^{\bar\alpha\sigma}A_{\sigma\,.\,.}^{\;\;\beta\bar\gamma}.$$

We use the following convention: Whenever an array $\{A_{\alpha\,.\,.}^{\;\;\beta\bar\gamma}\}$ appears, the array $\{A_{\bar\alpha\,.\,.}^{\;\;\bar\beta\gamma}\}$ will be assumed to be defined, by default, by the complex conjugation

$$A_{\bar\alpha\,.\,.}^{\;\;\bar\beta\gamma}=\overline{A_{\alpha\,.\,.}^{\;\;\beta\bar\gamma}}.$$

This means that we interpret $\{A_{\alpha\,.\,.}^{\;\;\beta\bar\gamma}\}$ as a representation of a real tensor, defined on $\mathbb{R}^{4n}$, with respect to the fixed complex basis (2.10); the corresponding real tensor in this case is

$$A_{\alpha\,.\,.}^{\;\;\beta\bar\gamma}\,\mathfrak{e}^\alpha\otimes\mathfrak{e}_\beta\otimes\mathfrak{e}_{\bar\gamma}+A_{\bar\alpha\,.\,.}^{\;\;\bar\beta\gamma}\,\mathfrak{e}^{\bar\alpha}\otimes\mathfrak{e}_{\bar\beta}\otimes\mathfrak{e}_\gamma.$$

Notice that we have $\pi^\alpha_{\bar\sigma}\,\pi^{\bar\sigma}_\beta=-\,\delta^\alpha_\beta$ ($\delta^\alpha_\beta$ is the Kronecker delta). We introduce a complex antilinear endomorphism $\mathfrak{j}$ of the tensor algebra of $\mathbb{R}^{4n}$, which takes a tensor with components $T_{\alpha_1\dots\alpha_k\bar\beta_1\dots\bar\beta_l\dots}$ to a tensor of the same type, with components $(\mathfrak{j}T)_{\alpha_1\dots\alpha_k\bar\beta_1\dots\bar\beta_l\dots}$, by the formula

$$(\mathfrak{j}T)_{\alpha_1\dots\alpha_k\bar\beta_1\dots\bar\beta_l\dots}=\sum_{\bar\sigma_1\dots\bar\sigma_k\tau_1\dots\tau_l\dots}\pi^{\bar\sigma_1}_{\alpha_1}\dots\pi^{\bar\sigma_k}_{\alpha_k}\,\pi^{\tau_1}_{\bar\beta_1}\dots\pi^{\tau_l}_{\bar\beta_l}\dots T_{\bar\sigma_1\dots\bar\sigma_k\tau_1\dots\tau_l\dots}.$$

By definition, the group $Sp(n)$ consists of all endomorphisms of $\mathbb{R}^{4n}$ that preserve the inner product $\langle,\rangle$ and commute with the complex structures $J_1, J_2$ and $J_3$. With the above notation, we can alternatively describe $Sp(n)$ as the set of all two-tensors $\{U^\alpha_\beta\}$ satisfying

$$g_{\sigma\bar\tau}U^\sigma_\alpha U^{\bar\tau}_{\bar\beta}=g_{\alpha\bar\beta},\qquad \pi_{\sigma\tau}U^\sigma_\alpha U^\tau_\beta=\pi_{\alpha\beta}.\tag{2.13}$$

For its Lie algebra, $sp(n)$, we have the following description:

**Lemma 2.1.** *For a tensor $\{X_{\alpha\bar\beta}\}$, the following conditions are equivalent:*
*(1)* $\{X_{\alpha\bar\beta}\}\in sp(n)$.
*(2)* $X_{\alpha\bar\beta}=-X_{\bar\beta\alpha}$, $(\mathfrak{j}X)_{\alpha\bar\beta}=X_{\alpha\bar\beta}$.
*(3)* $X^\alpha_\beta=\pi^{\alpha\sigma}Y_{\sigma\beta}$ *for some tensor* $\{Y_{\alpha\beta}\}$ *satisfying* $Y_{\alpha\beta}=Y_{\beta\alpha}$ *and* $(\mathfrak{j}Y)_{\alpha\beta}=Y_{\alpha\beta}$.

**Proof.** The equivalence between (1) and (2) follows by differentiating (2.13) at the identity. To obtain (3), we define the tensor $\{Y_{\sigma\beta}\}$ by

$$Y_{\sigma\beta}=-\pi_{\sigma\tau}X^\tau_\beta=-\pi^{\bar\tau}_\sigma X_{\beta\bar\tau}.\qquad\square$$

**2.5. The canonical Cartan connection and its structure equations.** It is well known that to each qc manifold $(M,H)$ one can associate a unique, up to a diffeomorphism, regular, normal Cartan geometry, i.e., a certain principle bundle $P_1 \to M$ endowed with a Cartan connection that satisfies some natural normalization conditions. In [12] we have provided an explicit construction for both the bundle and the Cartan connection in terms of geometric data generated entirely by the qc structure of $M$. Here we will briefly recall this construction since it is important for the rest of the paper. The method we are using is essentially the original Cartan's method of equivalence that was applied with a great success by Chern and Moser in [4] for solving the respective equivalence problem in the CR case. It is based entirely on classical exterior calculus and does not require any preliminary knowledge concerning the theory of parabolic geometries or the related Lie algebra cohomology.

By definition, if $(M,H)$ is a qc manifold, around each point of $M$, we can find $\eta_s, I_s$ and $g$ satisfying (2.8). Moreover, if $\tilde\eta_1,\tilde\eta_2,\tilde\eta_3$ are any (other) 1-forms satisfying (2.8) for some symmetric and positive definite $\tilde g \in H^*\otimes H^*$ and endomorphisms $\tilde I_s \in End(H)$ in place of $g$ and $I_s$ respectively, then it is known (see for example the appendix of [10]) that there exists a positive real-valued function $\mu$ and an $SO(3)$-valued function $\Psi = (a_{st})_{3\times 3}$ so that

$$\tilde\eta_s = \mu\, a_{ts}\,\eta_t, \qquad \tilde g = \mu\, g, \qquad \tilde I_s = a_{ts}\, I_t.$$

Therefore, there exists a natural principle bundle $\pi_o : P_o \to M$ with structure group $CSO(3) = \mathbb{R}^+ \times SO(3)$ whose local sections are precisely the triples of 1-forms $(\eta_1,\eta_2,\eta_3)$ satisfying (2.8). Clearly, on $P_o$ we obtain a global triple of canonical one-forms which we will denote again by $(\eta_1,\eta_2,\eta_3)$. The equations (2.8) yield ([12], Lemma 3.1) the following expressions for the exterior derivatives of the canonical one-forms (using the conventions from Section 2.4)

$$\begin{cases} d\eta_1 = -\varphi_0\wedge\eta_1 - \varphi_2\wedge\eta_3 + \varphi_3\wedge\eta_2 + 2ig_{\alpha\bar\beta}\,\theta^\alpha\wedge\theta^{\bar\beta}\\ d\eta_2 = -\varphi_0\wedge\eta_2 - \varphi_3\wedge\eta_1 + \varphi_1\wedge\eta_3 + \pi_{\alpha\beta}\,\theta^\alpha\wedge\theta^\beta + \pi_{\bar\alpha\bar\beta}\,\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\\ d\eta_3 = -\varphi_0\wedge\eta_3 - \varphi_1\wedge\eta_2 + \varphi_2\wedge\eta_1 - i\pi_{\alpha\beta}\,\theta^\alpha\wedge\theta^\beta + i\pi_{\bar\alpha\bar\beta}\,\theta^{\bar\alpha}\wedge\theta^{\bar\beta},\end{cases} \tag{2.14}$$

where $\varphi_0,\varphi_1,\varphi_2,\varphi_3$ are some (local, non-unique) real one-forms on $P_o$, $\theta^\alpha$ are some (local, non-unique) complex and semibasic one-forms on $P_o$ (by semibasic we mean that the contraction of the forms with any vector field tangent to the fibers of $\pi_o$ vanishes), $g_{\alpha\bar\beta} = g_{\bar\beta\alpha}$ and $\pi_{\alpha\beta} = -\pi_{\beta\alpha}$ are the same (fixed) constants as in Section 2.4.

One can show (cf. [12], Lemma 3.2) that, if $\tilde\varphi_0,\tilde\varphi_1,\tilde\varphi_2,\tilde\varphi_3,\tilde\theta^\alpha$ are any other one-forms (with the same properties as $\varphi_0,\varphi_1,\varphi_2,\varphi_3,\theta^\alpha$) that satisfy

(2.14), then

$$\begin{cases} \tilde\theta^\alpha = U^\alpha_\beta\theta^\beta + ir^\alpha\eta_1 + \pi^\alpha_{\bar\sigma} r^{\bar\sigma}(\eta_2 + i\eta_3) \\ \tilde\varphi_0 = \varphi_0 + 2U_{\beta\bar\sigma}r^{\bar\sigma}\theta^\beta + 2U_{\bar\beta\sigma}r^\sigma\theta^{\bar\beta} + \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3 \\ \tilde\varphi_1 = \varphi_1 - 2iU_{\beta\bar\sigma}r^{\bar\sigma}\theta^\beta + 2iU_{\bar\beta\sigma}r^\sigma\theta^{\bar\beta} + 2r_\sigma r^\sigma\eta_1 - \lambda_3\eta_2 + \lambda_2\eta_3, \\ \tilde\varphi_2 = \varphi_2 - 2\pi_{\sigma\tau}U^\sigma_\beta r^\tau\theta^\beta - 2\pi_{\bar\sigma\bar\tau}U^{\bar\sigma}_{\bar\beta}r^{\bar\tau}\theta^{\bar\beta} + \lambda_3\eta_1 + 2r_\sigma r^\sigma\eta_2 - \lambda_1\eta_3, \\ \tilde\varphi_3 = \varphi_3 + 2i\pi_{\sigma\tau}U^\sigma_\beta r^\tau\theta^\beta - 2i\pi_{\bar\sigma\bar\tau}U^{\bar\sigma}_{\bar\beta}r^{\bar\tau}\theta^{\bar\beta} - \lambda_2\eta_1 + \lambda_1\eta_2 + 2r_\sigma r^\sigma\eta_3, \end{cases} \tag{2.15}$$

where $U^\alpha_\beta, r^\alpha, \lambda_s$ are some appropriate functions; $\lambda_1, \lambda_2, \lambda_3$ are real, and $\{U^\alpha_\beta\}$ satisfy (2.13), i.e., $\{U^\alpha_\beta\} \in Sp(n) \subset End(\mathbb{R}^{4n})$. Clearly, the functions $U^\alpha_\beta, r^\alpha$ and $\lambda_s$ give a parametrization of a certain Lie Group $G_1$ diffeomorphic to $Sp(n) \times \mathbb{R}^{4n+3}$. There exists a canonical principle bundle $\pi_1 : P_1 \to P_o$ whose local sections are precisely the local one-forms $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$ on $P_o$ satisfying (2.14).

We use $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \theta^\alpha$ to denote also the induced canonical (global) one-forms on the principal bundle $P_1$. Then, according to [12], Theorem 3.3, on $P_1$, there exists a unique set of complex one-forms $\Gamma_{\alpha\beta}, \phi^\alpha$ and real one-forms $\psi_1, \psi_2, \psi_3$ so that

$$\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}, \qquad (\mathrm{j}\Gamma)_{\alpha\beta} = \Gamma_{\alpha\beta}. \tag{2.16}$$

and the equations

$$\begin{cases} d\theta^\alpha = -i\phi^\alpha \wedge \eta_1 - \pi^\alpha_{\bar\sigma}\phi^{\bar\sigma} \wedge (\eta_2 + i\eta_3) - \pi^{\alpha\sigma}\Gamma_{\sigma\beta} \wedge \theta^\beta \\ \qquad\quad -\frac{1}{2}(\varphi_0 + i\varphi_1) \wedge \theta^\alpha - \frac{1}{2}\pi^\alpha_{\bar\beta}(\varphi_2 + i\varphi_3) \wedge \theta^{\bar\beta} \\ d\varphi_0 = -\psi_1 \wedge \eta_1 - \psi_2 \wedge \eta_2 - \psi_3 \wedge \eta_3 - 2\phi_\beta \wedge \theta^\beta - 2\phi_{\bar\beta} \wedge \theta^{\bar\beta} \\ d\varphi_1 = -\varphi_2 \wedge \varphi_3 - \psi_2 \wedge \eta_3 + \psi_3 \wedge \eta_2 + 2i\phi_\beta \wedge \theta^\beta - 2i\phi_{\bar\beta} \wedge \theta^{\bar\beta} \\ d\varphi_2 = -\varphi_3 \wedge \varphi_1 - \psi_3 \wedge \eta_1 + \psi_1 \wedge \eta_3 - 2\pi_{\sigma_\beta}\phi^\sigma \wedge \theta^\beta - 2\pi_{\bar\sigma\bar\beta}\phi^{\bar\sigma} \wedge \theta^{\bar\beta} \\ d\varphi_3 = -\varphi_1 \wedge \varphi_2 - \psi_1 \wedge \eta_2 + \psi_2 \wedge \eta_1 + 2i\pi_{\sigma_\beta}\phi^\sigma \wedge \theta^\beta - 2i\pi_{\bar\sigma\bar\beta}\phi^{\bar\sigma} \wedge \theta^{\bar\beta}, \end{cases} \tag{2.17}$$

are satisfied. The so obtained one-forms $\{\eta_s\}$, $\{\theta^\alpha\}$, $\{\varphi_0\}$, $\{\varphi_s\}$, $\{\Gamma_{\alpha\beta}\}$, $\{\phi^\alpha\}$, $\{\psi_s\}$ represent the components of the canonical Cartan connection (cf. [12], Section 5) corresponding to a fixed splitting of the relevant Lie algebra

$$sp(n+1,1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathbb{R} \oplus sp(1) \oplus sp(n)}_{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

The curvature of the Cartan connection may be represented (cf. [12], Proposition 4.1) by a set of globally defined complex-valued functions

$$\mathcal{S}_{\alpha\beta\gamma\delta},\ \mathcal{V}_{\alpha\beta\gamma},\ \mathcal{L}_{\alpha\beta},\ \mathcal{M}_{\alpha\beta},\ \mathcal{C}_\alpha,\ \mathcal{H}_\alpha,\ \mathcal{P},\ \mathcal{Q},\ \mathcal{R} \tag{2.18}$$

satisfying:

**(I)** Each of the arrays $\{\mathcal{S}_{\alpha\beta\gamma\delta}\}, \{\mathcal{V}_{\alpha\beta\gamma}\}, \{\mathcal{L}_{\alpha\beta}\}, \{\mathcal{M}_{\alpha\beta}\}$ is totally symmetric in its indices.

**(II)** We have

$$\begin{cases} (\mathrm{j}\mathcal{S})_{\alpha\beta\gamma\delta} = \mathcal{S}_{\alpha\beta\gamma\delta} \\ (\mathrm{j}\mathcal{L})_{\alpha\beta} = \mathcal{L}_{\alpha\beta} \\ \overline{\mathcal{R}} = \mathcal{R}. \end{cases} \tag{2.19}$$

**(III)** The exterior derivatives of the connection one-forms $\Gamma_{\alpha\beta}$, $\phi_\alpha$ and $\psi_s$ are given by

$$\begin{aligned} d\Gamma_{\alpha\beta} \;=\; & -\pi^{\sigma\tau}\Gamma_{\alpha\sigma}\wedge\Gamma_{\tau\beta} + 2\pi^{\bar\sigma}_{\alpha}(\phi_\beta\wedge\theta_{\bar\sigma} - \phi_{\bar\sigma}\wedge\theta_\beta) \\ & + 2\pi^{\bar\sigma}_{\beta}(\phi_\alpha\wedge\theta_{\bar\sigma} - \phi_{\bar\sigma}\wedge\theta_\alpha) + \pi^{\sigma}_{\bar\delta}\,\mathcal{S}_{\alpha\beta\gamma\sigma}\,\theta^\gamma\wedge\theta^{\bar\delta} \\ & + \Big(\mathcal{V}_{\alpha\beta\gamma}\,\theta^\gamma + \pi^{\bar\sigma}_{\alpha}\,\pi^{\bar\tau}_{\beta}\,\mathcal{V}_{\bar\sigma\bar\tau\bar\gamma}\,\theta^{\bar\gamma}\Big)\wedge\eta_1 - i\pi^{\sigma}_{\bar\gamma}\,\mathcal{V}_{\alpha\beta\sigma}\,\theta^{\bar\gamma}\wedge(\eta_2+i\eta_3) \\ & + i(\mathrm{j}\mathcal{V})_{\alpha\beta\gamma}\,\theta^\gamma\wedge(\eta_2-i\eta_3) - i\mathcal{L}_{\alpha\beta}\,(\eta_2+i\eta_3)\wedge(\eta_2-i\eta_3) \\ & + \mathcal{M}_{\alpha\beta}\,\eta_1\wedge(\eta_2+i\eta_3) + (\mathrm{j}M)_{\alpha\beta}\,\eta_1\wedge(\eta_2-i\eta_3), \end{aligned} \tag{2.20}$$

$$\begin{aligned} d\phi_\alpha \;=\; & \frac{1}{2}(\varphi_0+i\varphi_1)\wedge\phi_\alpha + \frac{1}{2}\pi_{\alpha\gamma}(\varphi_2-i\varphi_3)\wedge\phi^\gamma - \pi^{\bar\sigma}_{\alpha}\,\Gamma_{\bar\sigma\bar\gamma}\wedge\phi^{\bar\gamma} \\ & -\frac{i}{2}\,\psi_1\wedge\theta_\alpha - \frac{1}{2}\,\pi_{\alpha\gamma}(\psi_2-i\psi_3)\wedge\theta^\gamma - i\pi^{\sigma}_{\bar\delta}\,\mathcal{V}_{\alpha\gamma\sigma}\,\theta^\gamma\wedge\theta^{\bar\delta} \\ & +\mathcal{M}_{\alpha\gamma}\,\theta^\gamma\wedge\eta_1 + \pi^{\bar\sigma}_{\alpha}\,\mathcal{L}_{\bar\sigma\bar\gamma}\,\theta^{\bar\gamma}\wedge\eta_1 + i\mathcal{L}_{\alpha\gamma}\,\theta^\gamma\wedge(\eta_2-i\eta_3) \\ & -i\pi^{\sigma}_{\bar\gamma}\,\mathcal{M}_{\alpha\sigma}\,\theta^{\bar\gamma}\wedge(\eta_2+i\eta_3) - \mathcal{C}_\alpha(\eta_2+i\eta_3)\wedge(\eta_2-i\eta_3) \\ & +\mathcal{H}_\alpha\,\eta_1\wedge(\eta_2+i\eta_3) + i\pi_{\alpha\sigma}\,\mathcal{C}^\sigma\,\eta_1\wedge(\eta_2-i\eta_3), \end{aligned} \tag{2.21}$$

$$\begin{aligned} d\psi_1 \;=\; & \varphi_0\wedge\psi_1 - \varphi_2\wedge\psi_3 + \varphi_3\wedge\psi_2 - 4i\phi_\gamma\wedge\phi^\gamma \\ & + 4\pi^{\sigma}_{\bar\delta}\,\mathcal{L}_{\gamma\sigma}\,\theta^\gamma\wedge\theta^{\bar\delta} + 4\mathcal{C}_\gamma\,\theta^\gamma\wedge\eta_1 \\ & + 4\mathcal{C}_{\bar\gamma}\,\theta^{\bar\gamma}\wedge\eta_1 - 4i\pi_{\bar\gamma\bar\sigma}\,\mathcal{C}^{\bar\sigma}\,\theta^{\bar\gamma}\wedge(\eta_2+i\eta_3) + 4i\pi_{\gamma\sigma}\,\mathcal{C}^\sigma\,\theta^\gamma\wedge(\eta_2-i\eta_3) \\ & + \mathcal{P}\,\eta_1\wedge(\eta_2+i\eta_3) + \overline{\mathcal{P}}\,\eta_1\wedge(\eta_2-i\eta_3) \\ & + i\mathcal{R}\,(\eta_2+i\eta_3)\wedge(\eta_2-i\eta_3), \end{aligned} \tag{2.22}$$

$$\begin{aligned} d\psi_2+i\,d\psi_3 \;=\; & (\varphi_0-i\varphi_1)\wedge(\psi_2+i\psi_3) + i(\varphi_2+i\varphi_3)\wedge\psi_1 \\ & + 4\pi_{\gamma\delta}\phi^\gamma\wedge\phi^\delta + 4i\pi^{\bar\sigma}_{\gamma}\,\mathcal{M}_{\bar\sigma\bar\delta}\,\theta^\gamma\wedge\theta^{\bar\delta} + 4i\pi^{\bar\sigma}_{\gamma}\,\mathcal{C}_{\bar\sigma}\,\theta^\gamma\wedge\eta_1 \\ & - 4\mathcal{H}_{\bar\gamma}\,\theta^{\bar\gamma}\wedge\eta_1 - 4i\mathcal{C}_{\bar\gamma}\,\theta^{\bar\gamma}\wedge(\eta_2+i\eta_3) \\ & - 4i\pi^{\bar\sigma}_{\gamma}\,\mathcal{H}_{\bar\sigma}\,\theta^\gamma\wedge(\eta_2-i\eta_3) - i\mathcal{R}\,\eta_1\wedge(\eta_2+i\eta_3) \\ & + \overline{\mathcal{Q}}\,\eta_1\wedge(\eta_2-i\eta_3) - \overline{\mathcal{P}}\,(\eta_2+i\eta_3)\wedge(\eta_2-i\eta_3). \end{aligned} \tag{2.23}$$

**2.6. The qc structures as integral manifolds of an exterior differential system.** As we have seen above, each qc structure $(M,H)$ determines a principle bundle $P_1$ over $M$ with a coframing

$$\eta_s,\ \theta^\alpha,\ \varphi_0,\ \varphi_s,\ \Gamma_{\alpha\beta},\ \phi^\alpha,\ \psi_s \tag{2.24}$$

satisfying (2.16), (2.14), (2.17), together with a set of functions

$$\mathcal{S}_{\alpha\beta\gamma\delta},\ \mathcal{V}_{\alpha\beta\gamma},\ \mathcal{L}_{\alpha\beta},\ \mathcal{M}_{\alpha\beta},\ \mathcal{C}_{\alpha},\ \mathcal{H}_{\alpha},\ \mathcal{P},\ \mathcal{Q},\ \mathcal{R} \tag{2.25}$$

with the respective properties (I), (II) and (III) of Section 2.5. As it can be easily shown, the converse is also true, i.e., each manifold $P_1$ endowed with a coframing (2.24) and functions (2.25), satisfying all the respective properties, can be viewed, locally (in a unique way), as the canonical principle bundle of a (unique) qc structure. Therefore, finding local qc structures is equivalent to finding linearly independent one-forms (2.24) and functions (2.25) on an open domain in $\mathbb{R}^{\dim(P_1)}$ satisfying the above properties. This is, clearly, a problem of type (2.1) and thus it reduces—as explained in Section 2.1—to a typical problem from the theory of exterior differential systems that can be handled using the Cartan's Third Theorem.

For the respective exterior differential system, the validity of Assumption I, Section 2.1 follows immediately from [12], Proposition 4.2 which says that the exterior differentiation of (2.20), (2.21), (2.22) and (2.23) produces equations that can be put into the form:

$$\begin{aligned}
\Big(d^2\Gamma_{\alpha\beta} \;=\Big)\ & \pi^{\sigma}_{\bar\delta}\mathcal{S}^*_{\alpha\beta\gamma\sigma}\wedge\theta^{\gamma}\wedge\theta^{\bar\delta} + \mathcal{V}^*_{\alpha\beta\gamma}\wedge\theta^{\gamma}\wedge\eta_1\\
&+\pi^{\bar\mu}_{\alpha}\pi^{\bar\nu}_{\beta}\mathcal{V}^*_{\bar\mu\bar\nu\bar\gamma}\wedge\theta^{\bar\gamma}\wedge\eta_1 - i\pi^{\sigma}_{\bar\gamma}\mathcal{V}^*_{\alpha\beta\sigma}\wedge\theta^{\bar\gamma}\wedge\big(\eta_2+i\eta_3\big)\\
&+ i\pi^{\bar\mu}_{\alpha}\pi^{\bar\nu}_{\beta}\pi^{\bar\xi}_{\gamma}\mathcal{V}^*_{\bar\mu\bar\nu\bar\xi}\wedge\theta^{\gamma}\wedge\big(\eta_2-i\eta_3\big)\\
&- i\mathcal{L}^*_{\alpha\beta}\wedge\big(\eta_2+i\eta_3\big)\wedge\big(\eta_2-i\eta_3\big)\\
&+\mathcal{M}^*_{\alpha\beta}\wedge\eta_1\wedge\big(\eta_2+i\eta_3\big)+\pi^{\bar\mu}_{\alpha}\pi^{\bar\nu}_{\beta}\mathcal{M}^*_{\bar\mu\bar\nu}\wedge\eta_1\wedge\big(\eta_2-i\eta_3\big) = 0;
\end{aligned}\tag{2.26}$$

$$\begin{aligned}
\Big(d^2\phi_{\alpha} \;=\Big)\ & - i\pi^{\nu}_{\bar\gamma}\mathcal{V}^*_{\alpha\beta\nu}\wedge\theta^{\beta}\wedge\theta^{\bar\gamma} + \pi^{\bar\mu}_{\alpha}\mathcal{L}^*_{\bar\mu\bar\beta}\wedge\theta^{\bar\beta}\wedge\eta_1 + \mathcal{M}^*_{\alpha\beta}\wedge\theta^{\beta}\wedge\eta_1\\
&- i\pi^{\nu}_{\bar\beta}\mathcal{M}^*_{\alpha\nu}\wedge\theta^{\bar\beta}\wedge\big(\eta_2+i\eta_3\big)+i\mathcal{L}^*_{\alpha\beta}\wedge\theta^{\beta}\wedge\big(\eta_2-i\eta_3\big)-\mathcal{C}^*_{\alpha}\wedge\big(\eta_2+i\eta_3\big)\wedge\big(\eta_2-i\eta_3\big)\\
&+ i\pi^{\bar\mu}_{\alpha}\mathcal{C}^*_{\bar\mu}\wedge\eta_1\wedge\big(\eta_2-i\eta_3\big)+\mathcal{H}^*_{\alpha}\wedge\eta_1\wedge\big(\eta_2+i\eta_3\big) = 0;
\end{aligned}\tag{2.27}$$

$$\begin{aligned}
\Big(d^2\psi_1 \;=\Big)\ & 4\pi^{\mu}_{\bar\gamma}\mathcal{L}^*_{\beta\mu}\wedge\theta^{\beta}\wedge\theta^{\bar\gamma}+4\mathcal{C}^*_{\beta}\wedge\theta^{\beta}\wedge\eta_1+4\mathcal{C}^*_{\bar\gamma}\wedge\theta^{\bar\gamma}\wedge\eta_1\\
&+4i\pi^{\bar\mu}_{\beta}\mathcal{C}^*_{\bar\mu}\wedge\theta^{\beta}\wedge\big(\eta_2-i\eta_3\big)-4i\pi^{\mu}_{\bar\gamma}\mathcal{C}^*_{\mu}\wedge\theta^{\bar\gamma}\wedge\big(\eta_2+i\eta_3\big)+\mathcal{P}^*\wedge\eta_1\wedge\big(\eta_2+i\eta_3\big)\\
&+\overline{\mathcal{P}^*}\wedge\eta_1\wedge\big(\eta_2-i\eta_3\big)+i\mathcal{R}^*\wedge\big(\eta_2+i\eta_3\big)\wedge\big(\eta_2-i\eta_3\big) = 0;
\end{aligned}\tag{2.28}$$

$$\begin{aligned}
\Big(d^2\big(\psi_2+i\psi_3\big) \;=\Big)\ & 4i\pi^{\bar\mu}_{\beta}\mathcal{M}^*_{\bar\mu\bar\gamma}\wedge\theta^{\beta}\wedge\theta^{\bar\gamma}+4i\pi^{\bar\mu}_{\beta}\mathcal{C}^*_{\bar\mu}\wedge\theta^{\beta}\wedge\eta_1-4\mathcal{H}^*_{\bar\gamma}\wedge\theta^{\bar\gamma}\wedge\eta_1\\
&-4\mathcal{C}^*_{\bar\gamma}\wedge\theta^{\bar\gamma}\wedge\big(\eta_2+i\eta_3\big)-4i\pi^{\bar\mu}_{\beta}\mathcal{H}^*_{\bar\mu}\wedge\theta^{\beta}\wedge\big(\eta_2-i\eta_3\big)-i\mathcal{R}^*\wedge\eta_1\wedge\big(\eta_2+i\eta_3\big)\\
&+\overline{\mathcal{Q}^*}\wedge\eta_1\wedge\big(\eta_2-i\eta_3\big)-\overline{\mathcal{P}^*}\wedge\big(\eta_2+i\eta_3\big)\wedge\big(\eta_2-i\eta_3\big) = 0,
\end{aligned}\tag{2.29}$$

where

$$\mathcal{S}^*_{\alpha\beta\gamma\delta},\ \mathcal{V}^*_{\alpha\beta\gamma},\ \mathcal{L}^*_{\alpha\beta},\ \mathcal{M}^*_{\alpha\beta},\ \mathcal{C}^*_{\alpha},\ \mathcal{H}^*_{\alpha},\ \mathcal{P}^*,\ \mathcal{Q}^*,\ \mathcal{R}^* \tag{2.30}$$

are certain (new) one-forms on $P_1$ each one of which begins with the differential of the corresponding curvature component followed by certain corrections terms. More precisely, we have

$$\begin{aligned}\mathcal{S}^*_{\alpha\beta\gamma\delta} &\overset{def}{=} d\mathcal{S}_{\alpha\beta\gamma\delta} - \pi^{\tau\nu}\Gamma_{\nu\alpha}\,\mathcal{S}_{\tau\beta\gamma\delta} - \pi^{\tau\nu}\Gamma_{\nu\beta}\,\mathcal{S}_{\alpha\tau\gamma\delta} - \pi^{\tau\nu}\Gamma_{\nu\gamma}\,\mathcal{S}_{\alpha\beta\tau\delta}\\ &-\pi^{\tau\nu}\Gamma_{\nu\delta}\,\mathcal{S}_{\alpha\beta\gamma\tau} - \varphi_0\,\mathcal{S}_{\alpha\beta\gamma\delta} - 2i\big(\pi_{\alpha\tau}\,\mathcal{V}_{\delta\beta\gamma}+\pi_{\beta\tau}\,\mathcal{V}_{\alpha\gamma\delta}+\pi_{\gamma\tau}\,\mathcal{V}_{\alpha\beta\delta}+\pi_{\delta\tau}\,\mathcal{V}_{\alpha\beta\gamma}\big)\theta^\tau\\ &- 2i\Big(g_{\alpha\bar\tau}\,(\mathrm{j}\mathcal{V})_{\delta\beta\gamma} + g_{\beta\bar\tau}\,(\mathrm{j}\mathcal{V})_{\alpha\delta\gamma} + g_{\gamma\bar\tau}\,(\mathrm{j}\mathcal{V})_{\alpha\beta\delta} + g_{\delta\bar\tau}\,(\mathrm{j}\mathcal{V})_{\alpha\beta\gamma}\Big)\theta^{\bar\tau}\end{aligned} \tag{2.31}$$

$$\begin{aligned}\mathcal{V}^*_{\alpha\beta\gamma} &\overset{def}{=} d\mathcal{V}_{\alpha\beta\gamma} - \pi^{\tau\nu}\Gamma_{\nu\alpha}\,\mathcal{V}_{\tau\beta\gamma} - \pi^{\tau\nu}\Gamma_{\nu\beta}\,\mathcal{V}_{\alpha\tau\gamma} - \pi^{\tau\nu}\Gamma_{\nu\gamma}\,\mathcal{V}_{\alpha\beta\tau}\\ &+ i\pi^{\sigma}_{\bar\tau}\,\phi^{\bar\tau}\,\mathcal{S}_{\alpha\beta\gamma\sigma} - \frac{1}{2}\big(3\varphi_0 + i\varphi_1\big)\,\mathcal{V}_{\alpha\beta\gamma} + \frac{1}{2}\big(\varphi_2 - i\varphi_3\big)\,(\mathrm{j}\mathcal{V})_{\alpha\beta\gamma}\\ &+2\big(\pi_{\alpha\tau}\,\mathcal{M}_{\beta\gamma}+\pi_{\beta\tau}\,\mathcal{M}_{\alpha\gamma}+\pi_{\gamma\tau}\,\mathcal{M}_{\alpha\beta}\big)\theta^\tau + 2\big(g_{\alpha\bar\tau}\,\mathcal{L}_{\beta\gamma}+g_{\beta\bar\tau}\,\mathcal{L}_{\alpha\gamma}+g_{\gamma\bar\tau}\,\mathcal{L}_{\alpha\beta}\big)\theta^{\bar\tau}\end{aligned} \tag{2.32}$$

$$\begin{aligned}\mathcal{L}^*_{\alpha\beta} &\overset{def}{=} d\mathcal{L}_{\alpha\beta} - \pi^{\tau\sigma}\Gamma_{\sigma\alpha}\,\mathcal{L}_{\tau\beta} - \pi^{\tau\sigma}\Gamma_{\sigma\beta}\,\mathcal{L}_{\alpha\tau} - 2\varphi_0\,\mathcal{L}_{\alpha\beta}\\ &- \frac{1}{2}\big(\varphi_2 + i\varphi_3\big)\mathcal{M}_{\alpha\beta} - \frac{1}{2}\big(\varphi_2 - i\varphi_3\big)(\mathrm{j}\mathcal{M})_{\alpha\beta} - \phi^\sigma\,\mathcal{V}_{\alpha\beta\sigma} - \pi^{\bar\mu}_{\alpha}\,\pi^{\bar\nu}_{\beta}\,\phi^{\bar\sigma}\,\mathcal{V}_{\bar\mu\bar\nu\bar\sigma}\\ &- 2i\big(\pi_{\alpha\tau}\mathcal{C}_\beta + \pi_{\beta\tau}\mathcal{C}_\alpha\big)\theta^\tau - 2i\big(g_{\alpha\bar\tau}\,\pi^{\bar\sigma}_{\beta}\,\mathcal{C}_{\bar\sigma} + g_{\beta\bar\tau}\,\pi^{\bar\sigma}_{\alpha}\,\mathcal{C}_{\bar\sigma}\big)\theta^{\bar\tau}\end{aligned} \tag{2.33}$$

$$\begin{aligned}\mathcal{M}^*_{\alpha\beta} &\overset{def}{=} d\mathcal{M}_{\alpha\beta} - \pi^{\tau\sigma}\Gamma_{\sigma\alpha}\,\mathcal{M}_{\tau\beta} - \pi^{\tau\sigma}\Gamma_{\sigma\beta}\,\mathcal{M}_{\alpha\tau} - \big(2\varphi_0 + i\varphi_1\big)\,\mathcal{M}_{\alpha\beta}\\ &+\big(\varphi_2 - i\varphi_3\big)\mathcal{L}_{\alpha\beta} + 2\pi^{\sigma}_{\bar\tau}\,\phi^{\bar\tau}\,\mathcal{V}_{\alpha\beta\sigma} + 2\big(\pi_{\alpha\tau}\mathcal{H}_\beta + \pi_{\beta\tau}\mathcal{H}_\alpha\big)\theta^\tau - 2i\big(g_{\alpha\bar\tau}\,\mathcal{C}_\beta + g_{\beta\bar\tau}\,\mathcal{C}_\alpha\big)\theta^{\bar\tau}\end{aligned} \tag{2.34}$$

$$\begin{aligned}\mathcal{C}^*_{\alpha} &\overset{def}{=} d\mathcal{C}_\alpha - \pi^{\tau\sigma}\Gamma_{\sigma\alpha}\,\mathcal{C}_\tau - \frac{1}{2}\big(5\varphi_0 + i\varphi_1\big)\,\mathcal{C}_\alpha + \pi^{\bar\sigma}_{\alpha}\big(\varphi_2 - i\varphi_3\big)\mathcal{C}_{\bar\sigma}\\ &+ 2i\pi^{\sigma}_{\bar\tau}\,\phi^{\bar\tau}\,\mathcal{L}_{\alpha\sigma} - i\phi^\tau\,\mathcal{M}_{\alpha\tau} - \frac{i}{2}\big(\varphi_2 + i\varphi_3\big)\mathcal{H}_\alpha + \frac{1}{2}\pi_{\alpha\tau}\,\mathcal{P}\,\theta^\tau - \frac{1}{2}g_{\alpha\bar\tau}\,\mathcal{R}\,\theta^{\bar\tau}\end{aligned} \tag{2.35}$$

$$\begin{aligned}\mathcal{H}^*_{\alpha} &\overset{def}{=} d\mathcal{H}_\alpha - \pi^{\tau\sigma}\Gamma_{\sigma\alpha}\,\mathcal{H}_\tau - \frac{1}{2}\big(5\varphi_0 + 3i\varphi_1\big)\,\mathcal{H}_\alpha\\ &- \frac{3i}{2}\big(\varphi_2 - i\varphi_3\big)\mathcal{C}_\alpha + 3\pi^{\sigma}_{\bar\tau}\,\phi^{\bar\tau}\,\mathcal{M}_{\alpha\sigma} - \frac{1}{2}\pi_{\alpha\tau}\,\mathcal{Q}\,\theta^\tau - \frac{i}{2}g_{\alpha\bar\tau}\,\mathcal{P}\,\theta^{\bar\tau}\end{aligned} \tag{2.36}$$

$$\mathcal{R}^* \overset{def}{=} d\mathcal{R} - 3\varphi_0\,\mathcal{R} + \big(\varphi_2 + i\varphi_3\big)\mathcal{P} + \big(\varphi_2 - i\varphi_3\big)\overline{\mathcal{P}} + 8\phi^\tau\,\mathcal{C}_\tau + 8\phi^{\bar\tau}\,\mathcal{C}_{\bar\tau} \tag{2.37}$$

$$\mathcal{P}^* \overset{def}{=} d\mathcal{P} - \big(3\varphi_0 + i\varphi_1\big)\mathcal{P} + \frac{i}{2}\big(\varphi_2 + i\varphi_3\big)\mathcal{Q} - \frac{3}{2}\big(\varphi_2 - i\varphi_3\big)\mathcal{R} - 4i\phi^\tau\mathcal{H}_\tau + 12\pi_{\bar\tau\bar\sigma}\phi^{\bar\tau}\,\mathcal{C}_{\bar\sigma} \quad (2.38)$$

$$\mathcal{Q}^* \overset{def}{=} d\mathcal{Q} - \big(3\varphi_0 + 2i\varphi_1\big)\mathcal{Q} + 2i\big(\varphi_2 - i\varphi_3\big)\mathcal{P} - 16\pi_{\bar\tau\bar\sigma}\phi^{\bar\tau}\mathcal{H}^{\bar\sigma} \quad (2.39)$$

In order to show that the Assumption II, Section 2.1 holds true for the differential system under consideration, we observe that the Bianchi identities (2.26), (2.27), (2.28) and (2.29) imply that the one-forms (2.30) belong to the linear span of $\eta_1$, $\eta_2$, $\eta_3$, $\theta^\alpha$, $\theta^{\bar\alpha}$. Furthermore, if we are considering the above Bianchi identities as a system of algebraic equations for the unknown one-forms (2.30), then—since this system is clearly linear—the solutions may be parametrized by elements of a certain vector space. In [12], Proposition 4.3 we have given an explicit description for this vector space. Namely, we have shown that, on $P_1$, there exist unique, globally defined, complex valued functions

$$\begin{gathered}\mathcal{A}_{\alpha\beta\gamma\delta\epsilon},\, \mathcal{B}_{\alpha\beta\gamma\delta},\, \mathcal{C}_{\alpha\beta\gamma\delta},\, \mathcal{D}_{\alpha\beta\gamma},\, \mathcal{E}_{\alpha\beta\gamma},\, \mathcal{F}_{\alpha\beta\gamma},\, \mathcal{G}_{\alpha\beta},\, \mathcal{X}_{\alpha\beta},\, \mathcal{Y}_{\alpha\beta},\, \mathcal{Z}_{\alpha\beta},\\ (\mathcal{N}_1)_\alpha,\, (\mathcal{N}_2)_\alpha,\, (\mathcal{N}_3)_\alpha,\, (\mathcal{N}_4)_\alpha,\, (\mathcal{N}_5)_\alpha,\, \mathcal{U}_s,\, \mathcal{W}_s\end{gathered} \quad (2.40)$$

so that:

**(I)** Each of the arrays $\{\mathcal{A}_{\alpha\beta\gamma\delta\epsilon}\}$, $\{\mathcal{B}_{\alpha\beta\gamma\delta}\}$, $\{\mathcal{C}_{\alpha\beta\gamma\delta}\}$, $\{\mathcal{D}_{\alpha\beta\gamma}\}$, $\{\mathcal{E}_{\alpha\beta\gamma}\}$, $\{\mathcal{F}_{\alpha\beta\gamma}\}$, $\{\mathcal{G}_{\alpha\beta\gamma}\}$, $\{\mathcal{X}_{\alpha\beta}\}$, $\{\mathcal{Y}_{\alpha\beta}\}$, $\{\mathcal{Z}_{\alpha\beta}\}$ is totally symmetric in its indices.

**(II)** We have

$$\begin{aligned}
\mathcal{S}^*_{\alpha\beta\gamma\delta} &= \mathcal{A}_{\alpha\beta\gamma\delta\epsilon}\,\theta^\epsilon - \pi^\sigma_{\bar\epsilon}(\mathrm{j}\mathcal{A})_{\alpha\beta\gamma\delta\sigma}\,\theta^{\bar\epsilon} + \Big(\mathcal{B}_{\alpha\beta\gamma\delta} + (\mathrm{j}\mathcal{B})_{\alpha\beta\gamma\delta}\Big)\eta_1\\
&\qquad + i\mathcal{C}_{\alpha\beta\gamma\delta}\big(\eta_2 + i\eta_3\big) - i(\mathrm{j}\mathcal{C})_{\alpha\beta\gamma\delta}\big(\eta_2 - i\eta_3\big)\\
\mathcal{V}^*_{\alpha\beta\gamma} &= \mathcal{C}_{\alpha\beta\gamma\epsilon}\,\theta^\epsilon + \pi^\sigma_{\bar\epsilon}\,\mathcal{B}_{\alpha\beta\gamma\sigma}\,\theta^{\bar\epsilon} + \mathcal{D}_{\alpha\beta\gamma}\eta_1 + \mathcal{E}_{\alpha\beta\gamma}\big(\eta_2 + i\eta_3\big)\\
&\qquad + \mathcal{F}_{\alpha\beta\gamma}\big(\eta_2 - i\eta_3\big)\\
\mathcal{L}^*_{\alpha\beta} &= -(\mathrm{j}\mathcal{F})_{\alpha\beta\epsilon}\,\theta^\epsilon - \pi^\sigma_{\bar\epsilon}\,\mathcal{F}_{\alpha\beta\sigma}\,\theta^{\bar\epsilon} + i\Big((\mathrm{j}\mathcal{Z})_{\alpha\beta} - \mathcal{Z}_{\alpha\beta}\Big)\eta_1\\
&\qquad + i\mathcal{G}_{\alpha\beta}\big(\eta_2 + i\eta_3\big) - i(\mathrm{j}\mathcal{G})_{\alpha\beta}\big(\eta_2 - i\eta_3\big)\\
\mathcal{M}^*_{\alpha\beta} &= -\mathcal{E}_{\alpha\beta\epsilon}\,\theta^\epsilon + \pi^\sigma_{\bar\epsilon}\Big((\mathrm{j}\mathcal{F})_{\alpha\beta\sigma} - i\mathcal{D}_{\alpha\beta\sigma}\Big)\,\theta^{\bar\epsilon} + \mathcal{X}_{\alpha\beta}\eta_1 + \mathcal{Y}_{\alpha\beta}\big(\eta_2 + i\eta_3\big)\\
&\qquad + \mathcal{Z}_{\alpha\beta}\big(\eta_2 - i\eta_3\big)\\
\mathcal{C}^*_\alpha &= \mathcal{G}_{\alpha\epsilon}\,\theta^\epsilon - i\pi^\sigma_{\bar\epsilon}\,\mathcal{Z}_{\alpha\sigma}\,\theta^{\bar\epsilon} + (\mathcal{N}_1)_\alpha\eta_1 + (\mathcal{N}_2)_\alpha\big(\eta_2 + i\eta_3\big)\\
&\qquad + (\mathcal{N}_3)_\alpha\big(\eta_2 - i\eta_3\big)\\
\mathcal{H}^*_\alpha &= -\mathcal{Y}_{\alpha\epsilon}\,\theta^\epsilon + i\pi^\sigma_{\bar\epsilon}\big(\mathcal{G}_{\alpha\sigma} - \mathcal{X}_{\alpha\sigma}\big)\theta^{\bar\epsilon} + (\mathcal{N}_4)_\alpha\eta_1 + (\mathcal{N}_5)_\alpha\big(\eta_2 + i\eta_3\big)\\
&\qquad + \Big((\mathcal{N}_1)_\alpha + i\pi^{\bar\sigma}_\alpha(\mathcal{N}_3)_{\bar\sigma}\Big)\big(\eta_2 - i\eta_3\big)\\
\mathcal{R}^* &= 4\pi^{\bar\sigma}_\epsilon(\mathcal{N}_3)_{\bar\sigma}\,\theta^\epsilon + 4\pi^\sigma_{\bar\epsilon}(\mathcal{N}_3)_\sigma\,\theta^{\bar\epsilon} + i\big(\mathcal{U}_3 - \overline{\mathcal{U}}_3\big)\eta_1\\
&\qquad - i\big(\mathcal{U}_1 + \mathcal{W}_3\big)\big(\eta_2 + i\eta_3\big) + i\big(\overline{\mathcal{U}}_1 + \overline{\mathcal{W}}_3\big)\big(\eta_2 - i\eta_3\big)\\
\mathcal{P}^* &= -4(\mathcal{N}_2)_\epsilon\,\theta^\epsilon - 4\Big((\mathcal{N}_3)_{\bar\epsilon} + i\pi^\sigma_{\bar\epsilon}(\mathcal{N}_1)_\sigma\Big)\,\theta^{\bar\epsilon} + \mathcal{U}_1\eta_1
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +\mathcal{U}_2\big(\eta_2+i\eta_3\big)+\mathcal{U}_3\big(\eta_2-i\eta_3\big)\\
\mathcal{Q}^* &= 4(\mathcal{N}_5)_\epsilon\,\theta^\epsilon+4i\pi^\sigma_{\bar\epsilon}\Big((\mathcal{N}_2)_\sigma+(\mathcal{N}_4)_\sigma\Big)\,\theta^{\bar\epsilon}+\mathcal{W}_1\eta_1\\
&\qquad\qquad +\mathcal{W}_2\big(\eta_2+i\eta_3\big)+\mathcal{W}_3\big(\eta_2-i\eta_3\big). \qquad (2.41)
\end{aligned}
$$

**2.7. The Cartan test.** For the (real) dimension $D$ of the vector space determined by (2.40), we calculate

$$
\begin{aligned}
D = 2\binom{2n+4}{5}+4\binom{2n+3}{4}+6\binom{2n+2}{3}+8\binom{2n+1}{2}+20n+12\\
=\frac{2}{15}(2n+5)(2n+3)(n+3)(n+2)(n+1). \qquad (2.42)
\end{aligned}
$$

Following the scheme of Section 2.1, the problem of finding all possible coframings (2.24) and functions (2.25) satisfying the respective relations (i.e., the problem of finding all local qc structures) may be seen, equivalently, as the problem of solving a certain associated exterior differential system, which we describe next: Let us denote the (real) dimension of $P_1$ by $d_1$. We have

$$
d_1=\binom{2n+5}{2}=(2n+5)(n+2) \qquad (2.43)
$$

The functions (2.25), with their respective properties (I) and (II) assumed, determine a vector space for which these functions represent the coordinate components of vectors. For the dimension $d_2$ of this vector space, we easily compute

$$
\begin{aligned}
d_2 &= \binom{2n+3}{4}+2\binom{2n+2}{3}+3\binom{2n+1}{2}+8n+5\\
&=\frac{1}{6}(2n+5)(2n+3)(n+2)(n+1).
\end{aligned}
\qquad (2.44)
$$

Then, the associated exterior differential system that we are considering is defined by a differential ideal $\mathcal{I}$ on the product manifold

$$
N=GL(d_1,\mathbb{R})\times\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}. \qquad (2.45)
$$

We can interpret, in a natural way (cf. Section 2.1), (2.24) and (2.25) as one-forms and functions on $N$ respectively. Then, the ideal $\mathcal{I}$ is algebraically generated by the two-forms given by the structure equations (2.14), (2.17), (2.20), (2.21), (2.22), (2.23), and by the three-forms determined by the Bianchi identities (2.26), (2.27), (2.28) and (2.29) (these are the only non-trivial equations that we obtain by exterior differentiation of the structure equations). Since only the latter are relevant for the computation of the character sequence of the ideal (cf. Section 2.1), we will denote them by $\Delta_{\alpha\beta}$, $\Delta_\alpha$ and $\Psi_s$ respectively, i.e., we have:

$$
\begin{aligned}
\Delta_{\alpha\beta} &= \pi^\sigma_{\bar\delta}\mathcal{S}^*_{\alpha\beta\gamma\sigma}\wedge\theta^\gamma\wedge\theta^{\bar\delta}+\ \mathcal{V}^*_{\alpha\beta\gamma}\wedge\theta^\gamma\wedge\eta_1+\ \pi^{\bar\mu}_\alpha\pi^{\bar\nu}_\beta\mathcal{V}^*_{\bar\mu\bar\nu\bar\gamma}\wedge\theta^{\bar\gamma}\wedge\eta_1\\
&\quad -\ i\pi^\sigma_{\bar\gamma}\mathcal{V}^*_{\alpha\beta\sigma}\wedge\theta^{\bar\gamma}\wedge\big(\eta_2+i\eta_3\big)+\ i\pi^{\bar\mu}_\alpha\pi^{\bar\nu}_\beta\pi^{\bar\xi}_\gamma\mathcal{V}^*_{\bar\mu\bar\nu\bar\xi}\wedge\theta^\gamma\wedge\big(\eta_2-i\eta_3\big)
\end{aligned}
$$

$$- i\mathcal{L}^*_{\alpha\beta} \wedge \big(\eta_2 + i\eta_3\big) \wedge \big(\eta_2 - i\eta_3\big) + \mathcal{M}^*_{\alpha\beta} \wedge \eta_1 \wedge \big(\eta_2 + i\eta_3\big) + \pi^{\bar\mu}_{\alpha}\pi^{\bar\nu}_{\beta}\mathcal{M}^*_{\bar\mu\bar\nu} \wedge \eta_1 \wedge \big(\eta_2 - i\eta_3\big); \quad (2.46)$$

$$\Delta_\alpha = - i\pi^{\nu}_{\bar\gamma}\mathcal{V}^*_{\alpha\beta\nu} \wedge \theta^\beta \wedge \theta^{\bar\gamma} + \pi^{\bar\mu}_{\alpha}\mathcal{L}^*_{\bar\mu\bar\beta} \wedge \theta^{\bar\beta} \wedge \eta_1 + \mathcal{M}^*_{\alpha\beta} \wedge \theta^\beta \wedge \eta_1 - i\pi^{\nu}_{\bar\beta}\mathcal{M}^*_{\alpha\nu} \wedge \theta^{\bar\beta} \wedge \big(\eta_2 + i\eta_3\big) + i\mathcal{L}^*_{\alpha\beta} \wedge \theta^\beta \wedge \big(\eta_2 - i\eta_3\big) - \mathcal{C}^*_\alpha \wedge \big(\eta_2 + i\eta_3\big) \wedge \big(\eta_2 - i\eta_3\big) + i\pi^{\bar\mu}_{\alpha}\mathcal{C}^*_{\bar\mu} \wedge \eta_1 \wedge \big(\eta_2 - i\eta_3\big) + \mathcal{H}^*_\alpha \wedge \eta_1 \wedge \big(\eta_2 + i\eta_3\big); \quad (2.47)$$

$$\Psi_1 = 4\pi^{\mu}_{\bar\gamma}\mathcal{L}^*_{\beta\mu} \wedge \theta^\beta \wedge \theta^{\bar\gamma} + 4\mathcal{C}^*_\beta \wedge \theta^\beta \wedge \eta_1 + 4\mathcal{C}^*_{\bar\gamma} \wedge \theta^{\bar\gamma} \wedge \eta_1 + 4i\pi^{\bar\mu}_{\beta}\mathcal{C}^*_{\bar\mu} \wedge \theta^\beta \wedge \big(\eta_2 - i\eta_3\big) - 4i\pi^{\mu}_{\bar\gamma}\mathcal{C}^*_{\mu} \wedge \theta^{\bar\gamma} \wedge \big(\eta_2 + i\eta_3\big) + \mathcal{P}^* \wedge \eta_1 \wedge \big(\eta_2 + i\eta_3\big) + \overline{\mathcal{P}^*} \wedge \eta_1 \wedge \big(\eta_2 - i\eta_3\big) + i\mathcal{R}^* \wedge \big(\eta_2 + i\eta_3\big) \wedge \big(\eta_2 - i\eta_3\big); \quad (2.48)$$

$$\Psi_2 + i\Psi_3 = 4i\pi^{\bar\mu}_{\beta}\mathcal{M}^*_{\bar\mu\bar\gamma} \wedge \theta^\beta \wedge \theta^{\bar\gamma} + 4i\pi^{\bar\mu}_{\beta}\mathcal{C}^*_{\bar\mu} \wedge \theta^\beta \wedge \eta_1 - 4\mathcal{H}^*_{\bar\gamma} \wedge \theta^{\bar\gamma} \wedge \eta_1 - 4\mathcal{C}^*_{\bar\gamma} \wedge \theta^{\bar\gamma} \wedge \big(\eta_2 + i\eta_3\big) - 4i\pi^{\bar\mu}_{\beta}\mathcal{H}^*_{\bar\mu} \wedge \theta^\beta \wedge \big(\eta_2 - i\eta_3\big) - i\mathcal{R}^* \wedge \eta_1 \wedge \big(\eta_2 + i\eta_3\big) + \overline{\mathcal{Q}^*} \wedge \eta_1 \wedge \big(\eta_2 - i\eta_3\big) - \overline{\mathcal{P}^*} \wedge \big(\eta_2 + i\eta_3\big) \wedge \big(\eta_2 - i\eta_3\big). \quad (2.49)$$

In order to show that our exterior differential system $\mathcal{I}$ is in involution—which would allow us to apply the Cartan's Third Theorem to it—we need to compute the character sequence $v_1, v_2, v_3, \dots, v_{d_1}$ of the system and show that the Cartan's test

$$D = v_1 + 2v_2 + 3v_3 + \cdots + d_1 v_{d_1}$$

is satisfied. We will do this in the next section.

## 3. Involutivity of the associated exterior differential system

**3.1. Setting out a few more conventions.** According to our current conventions the Greek indices $\alpha, \beta, \gamma$ are running from 1 to $2n$. Here, however, we will need also indices that have the range $1, \dots, n$ for which we will use again the small Greek letters but already printed in black, e.g, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \dots$. Primed bold indices will be used to indicate a shift by $n$, e.g., $\boldsymbol{\alpha}' \overset{def}{=} \boldsymbol{\alpha} + n$, and thus they will always have the range $(n+1), \dots, 2n$. If a number in brackets is used as an index (e.g., [15]), it means that we take a index in the range $1, \dots, n$ that is congruent to the original number in the brackets modulo $n$ (so if n=6, then [15] as an index corresponds to 3). With this conventions, the constants $\pi_{\alpha\beta}$ from Section 2.4 are determined by

$$\pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = 0, \qquad \pi^{\boldsymbol{\alpha}'}_{\boldsymbol{\beta}} = \delta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = -\pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}'} \qquad (\delta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \text{ being the Kronecker delta}).$$

Furthermore, the properties of the functions $\mathcal{S}_{\alpha\beta\gamma\delta}$ and $\mathcal{L}_{\alpha\beta}$ given by (2.19) may be, equivalently, written as

$$\mathcal{S}_{\boldsymbol{\alpha}'\boldsymbol{\beta}'\boldsymbol{\gamma}'\boldsymbol{\delta}'} = \overline{\mathcal{S}_{\boldsymbol{\alpha\beta\gamma\delta}}}, \qquad \mathcal{S}_{\boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\gamma}'\boldsymbol{\delta}'} = -\overline{\mathcal{S}_{\boldsymbol{\alpha}'\boldsymbol{\beta\gamma\delta}}}, \qquad \mathcal{S}_{\boldsymbol{\alpha\beta}\boldsymbol{\gamma}'\boldsymbol{\delta}'} = \overline{\mathcal{S}_{\boldsymbol{\alpha}'\boldsymbol{\beta}'\boldsymbol{\gamma\delta}}}, \qquad (3.1)$$

$$\mathcal{L}_{\boldsymbol{\alpha}'\boldsymbol{\beta}'} = \overline{\mathcal{L}_{\boldsymbol{\alpha}\boldsymbol{\beta}}}, \qquad \mathcal{L}_{\boldsymbol{\alpha}\boldsymbol{\beta}'} = -\overline{\mathcal{L}_{\boldsymbol{\alpha}'\boldsymbol{\beta}}}. \tag{3.2}$$

Similarly, since by (2.46) it can be easily verified that $(\mathfrak{j}\Delta)_{\alpha\beta} = \Delta_{\alpha\beta}$, we have also the equations

$$\Delta_{\boldsymbol{\alpha}'\boldsymbol{\beta}'} = \overline{\Delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}}, \qquad \Delta_{\boldsymbol{\alpha}\boldsymbol{\beta}'} = -\overline{\Delta_{\boldsymbol{\alpha}'\boldsymbol{\beta}}}. \tag{3.3}$$

**3.2. Introducing an appropriate coordinate system.** Let us fix an integral element $E \subset T_oN$ at the origin $o \in N$ of the associated exterior differential system, defined by the equations

$$\mathcal{S}^*_{\alpha\beta\gamma\delta} = \mathcal{V}^*_{\alpha\beta\gamma} = \mathcal{L}^*_{\alpha\beta} = \mathcal{M}^*_{\alpha\beta} = \mathcal{C}^*_{\alpha} = \mathcal{H}^*_{\alpha} = \mathcal{P}^* = \mathcal{Q}^* = \mathcal{R}^* = 0 \tag{3.4}$$

and the structure equations (2.14), (2.17), (2.20), (2.21), (2.22) and (2.23).

In order to compute the sequence of Cartan characters of the ideal $\mathcal{I}$ (cf. Section 2.1) we need to introduce a real basis for the vector space $\operatorname{span}\{\eta_s, \theta^\alpha, \theta^{\bar{\alpha}}\}$. Let us take $\xi^\alpha, \zeta^\alpha$ to be the real one-forms defined by $\theta^\alpha = \xi^\alpha + i\zeta^\alpha$ and consider the basis $\{\eta_s, \xi^\alpha, \zeta^\alpha\}$. In general, in the terminology of [2] and [3], the choice of a bases here corresponds to a choice of an integral flag

$$\{0\} = E_1 \subset E_2 \subset \cdots \subset E_{d_1} = E$$

which we construct by dualizing the corresponding coframe of $E$. Part of the difficulty in showing the Cartan's test and computing the corresponding Cartan characters of an ideal lies in the appropriate choice of the integral flag. Unfortunately, the natural choice of real coordinates that we have suggested above does not produce a Cartan-ordinary flag (i.e., a flag for which the Cartan's test is satisfied). Therefore, we will need a slightly more complicated construction here.

Let $\mu^{\boldsymbol{\alpha}}$ and $\nu^{\boldsymbol{\alpha}}$ be (real) one-forms on $N$ determined by the equations

$$\begin{cases} \xi^{n+1} = \mu^1 + \zeta^n + \eta_3, \\ \xi^{\boldsymbol{\alpha}'} = \mu^{\boldsymbol{\alpha}} + \zeta^{[\boldsymbol{\alpha}-1]}, \ \text{if } \boldsymbol{\alpha} \neq 1, \\ \zeta^{\boldsymbol{\beta}'} = \nu^{\boldsymbol{\beta}} + \mu^{[\boldsymbol{\beta}-2]}, \ \text{for all } 1 \leq \boldsymbol{\beta} \leq n. \end{cases}$$

Then, we choose a new basis of one-forms $\{\epsilon^1, \dots, \epsilon^{4n+3}\}$ for $\operatorname{span}\{\eta_s, \theta^\alpha, \theta^{\bar{\alpha}}\}$ by setting

$$\epsilon^{\boldsymbol{\alpha}} = \xi^{\boldsymbol{\alpha}}, \quad \epsilon^{\boldsymbol{\alpha}+n} = \zeta^{\boldsymbol{\alpha}},$$
$$\epsilon^{2n+1} = \eta_1, \quad \epsilon^{2n+2} = \eta_2, \quad \epsilon^{2n+3} = \eta_3, \quad \epsilon^{\boldsymbol{\alpha}+2n+3} = \mu^{\boldsymbol{\alpha}}, \quad \epsilon^{\boldsymbol{\alpha}+3n+3} = \nu^{\boldsymbol{\alpha}}. \tag{3.5}$$

Notice that because of (3.3), we can restrict our attention only to the three-forms $\Delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}$, $\Delta_{\boldsymbol{\alpha}\boldsymbol{\beta}'}$, $\Delta_{\boldsymbol{\alpha}}$, $\Delta_{\boldsymbol{\alpha}'}$ and $\Psi_s$. Substituting (3.5) into (2.46), (2.47), (2.48) and (2.49) gives:

$$\begin{aligned}\Delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} = {} & \frac{1}{2}\Big(\mathcal{S}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}\boldsymbol{\delta}'} - \mathcal{S}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\delta}\boldsymbol{\gamma}'}\Big) \wedge \xi^{\boldsymbol{\gamma}} \wedge \xi^{\boldsymbol{\delta}} \\ & - \Big(i\mathcal{S}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}\boldsymbol{\delta}'} + i\mathcal{S}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\delta}\boldsymbol{\gamma}'} + \mathcal{S}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}[\boldsymbol{\delta}+1]} + \mathcal{S}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}'[\boldsymbol{\delta}+1]'}\Big) \wedge \xi^{\boldsymbol{\gamma}} \wedge \zeta^{\boldsymbol{\delta}}\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{1}{2}\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}}-\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta\gamma'}}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[\boldsymbol{\delta}+1]'}-i\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta'}[\boldsymbol{\gamma}+1]'}-i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]}\\
&+\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]'}-\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]'}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\zeta^{\boldsymbol{\delta}}\ +\ \Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma}}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\eta_1\\
&+\Big(i\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma}}-i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}}}+\mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]'}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'}}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\eta_1-i\Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\eta_2\\
&+\Big(-\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}}}+i\mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]}-i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]}}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\eta_2+\Big(\mathcal{M}^*_{\boldsymbol{\alpha\beta}}+\overline{\mathcal{M}^*_{\boldsymbol{\alpha'\beta'}}}\Big)\wedge\eta_1\wedge\eta_2\\
&+\ \Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}}}-\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}1}-\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}1'}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[3]}-i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[3]'}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\eta_3\\
&+\Big(-i\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}+i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]}}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}1'}-i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}1}+\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]1'}\\
&-\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]'1}-\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[3]}-\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[3]'}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][3]'}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta}[3][\boldsymbol{\gamma}+1]'}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\eta_3\\
&+\Big(i\mathcal{M}^*_{\boldsymbol{\alpha\beta}}-i\overline{\mathcal{M}^*_{\boldsymbol{\alpha'\beta'}}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta}1'}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}1'}}-i\mathcal{V}^*_{\boldsymbol{\alpha\beta}[3]'}+i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[3]'}}\Big)\wedge\eta_1\wedge\eta_3\\
&+\Big(-2\mathcal{L}^*_{\boldsymbol{\alpha\beta}}-i\mathcal{V}^*_{\boldsymbol{\alpha\beta}1}+i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}1}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta}[3]}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[3]}}\Big)\wedge\eta_2\wedge\eta_3\ +\ \dots\ ; \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\boldsymbol{\alpha\beta'}}\ =\ &\frac{1}{2}\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}}-\mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\xi^{\boldsymbol{\delta}}\\
&-\Big(i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}+\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[\boldsymbol{\delta}+1]}-\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[\boldsymbol{\delta}+1]}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\zeta^{\boldsymbol{\delta}}\\
&+\ \frac{1}{2}\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}}-\mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}-i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[\boldsymbol{\delta}+1]}}+i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\delta}[\boldsymbol{\gamma}+1]}}-i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[\boldsymbol{\delta}+1]}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta}[\boldsymbol{\gamma}+1]}\\
&+\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]'}-\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]'}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\zeta^{\boldsymbol{\delta}}\ +\ \Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma}}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\eta_1\\
&+\Big(i\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}}+i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma}}}+\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]'}}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\eta_1-i\Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma'}}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\eta_2\\
&+\Big(-\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma'}}}+i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]}+i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]}}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\eta_2+\Big(\mathcal{M}^*_{\boldsymbol{\alpha\beta'}}-\overline{\mathcal{M}^*_{\boldsymbol{\alpha'\beta}}}\Big)\wedge\eta_1\wedge\eta_2\\
&+\Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}}-\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma'}}}-\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}1}+\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}1}}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[3]}+i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[3]}}\Big)\wedge\xi^{\boldsymbol{\gamma}}\wedge\eta_3\\
&+\Big(-i\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}}-i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma'}}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]}}-i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}1}}-i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}1}+\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]1'}\\
&-\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'1}-\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[3]}+\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[3]}}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1][3]'}+i\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[3][\boldsymbol{\gamma}+1]'}\Big)\wedge\zeta^{\boldsymbol{\gamma}}\wedge\eta_3\\
&+\Big(i\mathcal{M}^*_{\boldsymbol{\alpha\beta'}}+i\overline{\mathcal{M}^*_{\boldsymbol{\alpha'\beta}}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta'}1'}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}1'}}-i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[3]'}-i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[3]'}}\Big)\wedge\eta_1\wedge\eta_3\\
&+\Big(-2\mathcal{L}^*_{\boldsymbol{\alpha\beta'}}-i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}1}-i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}1}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[3]}+\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[3]}}\Big)\wedge\eta_2\wedge\eta_3\ +\ \dots\ ; \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\boldsymbol{\alpha}}\ =\ &-\frac{i}{2}\Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}-\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\xi^{\boldsymbol{\gamma}}\\
&-\Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}+\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}}-i\mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]}-i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\zeta^{\boldsymbol{\gamma}}
\end{aligned}
$$

$$
\begin{aligned}
&+\ \frac{1}{2}\Big(-i\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}+i\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}}+\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'}-\mathcal{V}^*_{\boldsymbol{\alpha\gamma'}[\boldsymbol{\beta}+1]'}-\mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]}+\mathcal{V}^*_{\boldsymbol{\alpha\gamma}[\boldsymbol{\beta}+1]}\\
&-i\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1][\boldsymbol{\gamma}+1]'}+i\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\gamma}+1][\boldsymbol{\beta}+1]'}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\zeta^{\boldsymbol{\gamma}}\ +\ \Big(-\mathcal{L}^*_{\boldsymbol{\alpha\beta'}}+\mathcal{M}^*_{\boldsymbol{\alpha\beta}}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\eta_1\\
&+\Big(i\mathcal{L}^*_{\boldsymbol{\alpha\beta'}}+i\mathcal{M}^*_{\boldsymbol{\alpha\beta}}+\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}+\mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]'}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\eta_1\ +\ \Big(i\mathcal{L}^*_{\boldsymbol{\alpha\beta}}-i\mathcal{M}^*_{\boldsymbol{\alpha\beta'}}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\eta_2\\
&+\Big(-\mathcal{L}^*_{\boldsymbol{\alpha\beta}}-\mathcal{M}^*_{\boldsymbol{\alpha\beta'}}+i\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]'}+i\mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\eta_2\ +\ \Big(i\overline{\mathcal{C}^*_{\boldsymbol{\alpha'}}}+\mathcal{H}^*_{\boldsymbol{\alpha}}\Big)\wedge\eta_1\wedge\eta_2\\
&+\Big(\mathcal{L}^*_{\boldsymbol{\alpha\beta}}+\mathcal{M}^*_{\boldsymbol{\alpha\beta'}}+i\mathcal{V}^*_{\boldsymbol{\alpha\beta}1}+i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}1'}+\mathcal{V}^*_{\boldsymbol{\alpha\beta}[3]}-\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[3]'}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\eta_3\\
&+\Big(i\mathcal{L}^*_{\boldsymbol{\alpha\beta}}-i\mathcal{M}^*_{\boldsymbol{\alpha\beta'}}+\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]'}-\mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}+\mathcal{V}^*_{\boldsymbol{\alpha\beta'}1'}-\mathcal{V}^*_{\boldsymbol{\alpha\beta}1}-i\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]1'}+i\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]'1}\\
&\quad+i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[3]'}+i\mathcal{V}^*_{\boldsymbol{\alpha\beta}[3]}+\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1][3]'}+\mathcal{V}^*_{\boldsymbol{\alpha}[3][\boldsymbol{\beta}+1]'}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\eta_3\\
&+\Big(\overline{\mathcal{C}^*_{\boldsymbol{\alpha'}}}+i\mathcal{H}^*_{\boldsymbol{\alpha}}-\mathcal{L}^*_{\boldsymbol{\alpha}1}-\mathcal{M}^*_{\boldsymbol{\alpha}1'}+i\mathcal{L}^*_{\boldsymbol{\alpha}[3]}-i\mathcal{M}^*_{\boldsymbol{\alpha}[3]'}\Big)\wedge\eta_1\wedge\eta_3\\
&+\Big(2i\mathcal{C}^*_{\boldsymbol{\alpha}}-i\mathcal{L}^*_{\boldsymbol{\alpha}1'}-i\mathcal{M}^*_{\boldsymbol{\alpha}1}+\mathcal{L}^*_{\boldsymbol{\alpha}[3]'}-\mathcal{M}^*_{\boldsymbol{\alpha}[3]}\Big)\wedge\eta_2\wedge\eta_3\ +\ \dots\ ; \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{\boldsymbol{\alpha'}}\ =\ &\frac{i}{2}\Big(\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}}-\mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\xi^{\boldsymbol{\gamma}}\\
&-\Big(\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}}+\mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}}-i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'}-i\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\zeta^{\boldsymbol{\gamma}}\\
&+\ \frac{1}{2}\Big(i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}}-i\mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}}+\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'}-\mathcal{V}^*_{\boldsymbol{\alpha'\gamma'}[\boldsymbol{\beta}+1]'}-\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]}+\mathcal{V}^*_{\boldsymbol{\alpha'\gamma}[\boldsymbol{\beta}+1]}\\
&+i\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'[\boldsymbol{\gamma}+1]}-i\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\gamma}+1]'[\boldsymbol{\beta}+1]}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\zeta^{\boldsymbol{\gamma}}\ +\ \Big(-\overline{\mathcal{L}^*_{\boldsymbol{\alpha\beta}}}+\mathcal{M}^*_{\boldsymbol{\alpha'\beta}}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\eta_1\\
&+\Big(i\overline{\mathcal{L}^*_{\boldsymbol{\alpha\beta}}}+i\mathcal{M}^*_{\boldsymbol{\alpha'\beta}}+\mathcal{L}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]}+\mathcal{M}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\eta_1\ +\ \Big(i\mathcal{L}^*_{\boldsymbol{\alpha'\beta}}-i\mathcal{M}^*_{\boldsymbol{\alpha'\beta'}}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\eta_2\\
&+\Big(-\mathcal{L}^*_{\boldsymbol{\alpha'\beta}}-\mathcal{M}^*_{\boldsymbol{\alpha'\beta'}}+i\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}}+i\mathcal{M}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\eta_2\ +\ \Big(-i\overline{\mathcal{C}^*_{\boldsymbol{\alpha}}}+\mathcal{H}^*_{\boldsymbol{\alpha'}}\Big)\wedge\eta_1\wedge\eta_2\\
&+\Big(\mathcal{L}^*_{\boldsymbol{\alpha'\beta}}+\mathcal{M}^*_{\boldsymbol{\alpha'\beta'}}+i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}1'}+i\mathcal{V}^*_{\boldsymbol{\alpha'\beta}1}+\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[3]}-\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[3]'}\Big)\wedge\xi^{\boldsymbol{\beta}}\wedge\eta_3\\
&+\Big(i\mathcal{L}^*_{\boldsymbol{\alpha'\beta}}-i\mathcal{M}^*_{\boldsymbol{\alpha'\beta'}}+\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}}-\mathcal{M}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]}+\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}1'}-\mathcal{V}^*_{\boldsymbol{\alpha'\beta}1}+i\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'1}-i\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]1'}\\
&\quad+i\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[3]}+i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[3]'}+\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'[3]}+\mathcal{V}^*_{\boldsymbol{\alpha'}[3]'[\boldsymbol{\beta}+1]}\Big)\wedge\zeta^{\boldsymbol{\beta}}\wedge\eta_3\\
&+\Big(-\overline{\mathcal{C}^*_{\boldsymbol{\alpha}}}+i\mathcal{H}^*_{\boldsymbol{\alpha'}}-\mathcal{L}^*_{\boldsymbol{\alpha'}1}-\mathcal{M}^*_{\boldsymbol{\alpha'}1'}+i\mathcal{L}^*_{\boldsymbol{\alpha'}[3]}-i\mathcal{M}^*_{\boldsymbol{\alpha'}[3]'}\Big)\wedge\eta_1\wedge\eta_3\\
&+\Big(2i\mathcal{C}^*_{\boldsymbol{\alpha'}}-i\overline{\mathcal{L}^*_{\boldsymbol{\alpha}1}}-i\mathcal{M}^*_{\boldsymbol{\alpha'}1}+\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[3]}}-\mathcal{M}^*_{\boldsymbol{\alpha'}[3]}\Big)\wedge\eta_2\wedge\eta_3\ +\ \dots\ ; \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1\ =\ &2\Big(\mathcal{L}^*_{\boldsymbol{\alpha\beta'}}-\mathcal{L}^*_{\boldsymbol{\alpha'\beta}}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\xi^{\boldsymbol{\beta}}\\
&-4\Big(i\mathcal{L}^*_{\boldsymbol{\alpha\beta'}}+i\mathcal{L}^*_{\boldsymbol{\alpha'\beta}}+\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}}+\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\zeta^{\boldsymbol{\beta}}
\end{aligned}
$$

$$
\begin{aligned}
&+\ 2\Big(\mathcal{L}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}'} - \mathcal{L}^*_{\boldsymbol{\alpha}'\boldsymbol{\beta}} - i\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]} + i\mathcal{L}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]} + i\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}} - i\overline{\mathcal{L}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]}}\\
&- \mathcal{L}^*_{[\boldsymbol{\alpha}+1]'[\boldsymbol{\beta}+1]} + \mathcal{L}^*_{[\boldsymbol{\beta}+1]'[\boldsymbol{\alpha}+1]}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\zeta^{\boldsymbol{\beta}}\ +\ 4\Big(\mathcal{C}^*_{\boldsymbol{\alpha}} + \overline{\mathcal{C}^*_{\boldsymbol{\alpha}}}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\eta_1\\
&+4\Big(i\mathcal{C}^*_{\boldsymbol{\alpha}} - i\overline{\mathcal{C}^*_{\boldsymbol{\alpha}}} + \mathcal{C}^*_{[\boldsymbol{\alpha}+1]'} + \overline{\mathcal{C}^*_{[\boldsymbol{\alpha}+1]'}}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\eta_1\ -\ 4i\Big(\mathcal{C}^*_{\boldsymbol{\alpha}'} - \overline{\mathcal{C}^*_{\boldsymbol{\alpha}'}}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\eta_2\\
&-\ 4\Big(\mathcal{C}^*_{\boldsymbol{\alpha}'} + \overline{\mathcal{C}^*_{\boldsymbol{\alpha}'}} - i\mathcal{C}^*_{[\boldsymbol{\alpha}+1]} + i\overline{\mathcal{C}^*_{[\boldsymbol{\alpha}+1]}}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\eta_2\ +\ \Big(\mathcal{P}^* + \overline{\mathcal{P}}^*\Big)\wedge\eta_1\wedge\eta_2\\
&+\ 4\Big(\mathcal{C}^*_{\boldsymbol{\alpha}'} + \overline{\mathcal{C}^*_{\boldsymbol{\alpha}'}} - \mathcal{L}^*_{\boldsymbol{\alpha}1} - \overline{\mathcal{L}^*_{\boldsymbol{\alpha}1}} + i\mathcal{L}^*_{\boldsymbol{\alpha}[3]} - i\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[3]}}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\eta_3\\
&+\ 4\Big(-i\mathcal{C}^*_{\boldsymbol{\alpha}'} + i\overline{\mathcal{C}^*_{\boldsymbol{\alpha}'}} - \mathcal{C}^*_{[\boldsymbol{\alpha}+1]} - \overline{\mathcal{C}^*_{[\boldsymbol{\alpha}+1]}} - i\mathcal{L}^*_{\boldsymbol{\alpha}1} + i\overline{\mathcal{L}^*_{\boldsymbol{\alpha}1}} - \mathcal{L}^*_{[\boldsymbol{\alpha}+1]'1} + \mathcal{L}^*_{[\boldsymbol{\alpha}+1]1'}\\
&- \mathcal{L}^*_{\boldsymbol{\alpha}[3]} - \overline{\mathcal{L}^*_{\boldsymbol{\alpha}[3]}} + i\mathcal{L}^*_{[\boldsymbol{\alpha}+1]'[3]} + i\mathcal{L}^*_{[3]'[\boldsymbol{\alpha}+1]}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\eta_3\\
&+\ \Big(i\mathcal{P}^* - i\overline{\mathcal{P}}^* - 4\mathcal{C}^*_{1'} - 4\overline{\mathcal{C}^*_{1'}} - 4i\mathcal{C}^*_{[3]'} + 4i\overline{\mathcal{C}^*_{[3]'}}\Big)\wedge\eta_1\wedge\eta_3\\
&+\Big(2\mathcal{R}^* - 4i\mathcal{C}^*_1 + 4i\overline{\mathcal{C}^*_1} - 4\mathcal{C}^*_{[3]} - 4\overline{\mathcal{C}^*_{[3]}}\Big)\wedge\eta_2\wedge\eta_3\ +\ \dots\ ; \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2 - i\Psi_3 =\ &-2i\Big(\mathcal{M}^*_{\boldsymbol{\alpha}'\boldsymbol{\beta}} - \mathcal{M}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}'}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\xi^{\boldsymbol{\beta}}\\
&+4\Big(\mathcal{M}^*_{\boldsymbol{\alpha}'\boldsymbol{\beta}} + \mathcal{M}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}'} - i\mathcal{M}^*_{\boldsymbol{\alpha}'[\boldsymbol{\beta}+1]'} - i\mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\zeta^{\boldsymbol{\beta}}\\
&+\ 2\Big(-i\mathcal{M}^*_{\boldsymbol{\alpha}'\boldsymbol{\beta}} + i\mathcal{M}^*_{\boldsymbol{\alpha}\boldsymbol{\beta}'} + \mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]} - \mathcal{M}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]} - \mathcal{M}^*_{\boldsymbol{\alpha}'[\boldsymbol{\beta}+1]'} + \mathcal{M}^*_{\boldsymbol{\beta}'[\boldsymbol{\alpha}+1]'}\\
&- i\mathcal{M}^*_{[\boldsymbol{\alpha}+1]'[\boldsymbol{\beta}+1]} + i\mathcal{M}^*_{[\boldsymbol{\beta}+1]'[\boldsymbol{\alpha}+1]}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\zeta^{\boldsymbol{\beta}}\ -\ 4\Big(i\mathcal{C}^*_{\boldsymbol{\alpha}'} + \mathcal{H}^*_{\boldsymbol{\alpha}}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\eta_1\\
&+4\Big(-\mathcal{C}^*_{\boldsymbol{\alpha}'} - i\mathcal{H}^*_{\boldsymbol{\alpha}} + i\mathcal{C}^*_{[\boldsymbol{\alpha}+1]} - \mathcal{H}^*_{[\boldsymbol{\alpha}+1]'}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\eta_1\ +\ 4i\Big(i\mathcal{C}^*_{\boldsymbol{\alpha}} + \mathcal{H}^*_{\boldsymbol{\alpha}'}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\eta_2\\
&-\ 4\Big(i\mathcal{C}^*_{\boldsymbol{\alpha}} - \mathcal{H}^*_{\boldsymbol{\alpha}'} + \mathcal{C}^*_{[\boldsymbol{\alpha}+1]'} + i\mathcal{H}^*_{[\boldsymbol{\alpha}+1]}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\eta_2\ +\ \Big(i\mathcal{R}^* + \mathcal{Q}^*\Big)\wedge\eta_1\wedge\eta_2\\
&+\ 4\Big(i\mathcal{C}^*_{\boldsymbol{\alpha}} - \mathcal{H}^*_{\boldsymbol{\alpha}'} - i\mathcal{M}^*_{\boldsymbol{\alpha}'1'} - i\mathcal{M}^*_{\boldsymbol{\alpha}1} - \mathcal{M}^*_{\boldsymbol{\alpha}[3]} + \mathcal{M}^*_{\boldsymbol{\alpha}'[3]'}\Big)\wedge\xi^{\boldsymbol{\alpha}}\wedge\eta_3\\
&+\ 4\Big(-\mathcal{C}^*_{\boldsymbol{\alpha}} + i\mathcal{H}^*_{\boldsymbol{\alpha}'} + i\mathcal{C}^*_{[\boldsymbol{\alpha}+1]'} + \mathcal{H}^*_{[\boldsymbol{\alpha}+1]} + \mathcal{M}^*_{\boldsymbol{\alpha}1} - \mathcal{M}^*_{\boldsymbol{\alpha}'1'} - i\mathcal{M}^*_{[\boldsymbol{\alpha}+1]'1} + i\mathcal{M}^*_{[\boldsymbol{\alpha}+1]1'}\\
&- i\mathcal{M}^*_{\boldsymbol{\alpha}[3]} - i\mathcal{M}^*_{\boldsymbol{\alpha}'[3]'} - \mathcal{M}^*_{[\boldsymbol{\alpha}+1]'[3]} - \mathcal{M}^*_{[3]'[\boldsymbol{\alpha}+1]}\Big)\wedge\zeta^{\boldsymbol{\alpha}}\wedge\eta_3\\
&+\ \Big(\mathcal{R}^* + i\mathcal{Q}^* - 4i\mathcal{C}^*_1 + 4\mathcal{H}^*_{1'} - 4\mathcal{C}^*_{[3]} + 4i\mathcal{H}^*_{[3]'}\Big)\wedge\eta_1\wedge\eta_3\\
&+\Big(-2i\mathcal{P}^* + 4\mathcal{C}^*_{1'} + 4i\mathcal{H}^*_1 + 4i\mathcal{C}^*_{[3]'} + 4\mathcal{H}^*_{[3]}\Big)\wedge\eta_2\wedge\eta_3\ +\ \dots\ . \qquad (3.11)
\end{aligned}
$$

In the above identities we have omitted all the terms involving wedge products with basis one-forms (3.5) of index grater than $2n+3$ (and have replaced them by "$\dots$") since they will turn out to be irrelevant for our further considerations here.

For each integer $1 \le \lambda \le d_1$ ($d_1$ is given by (2.43)), we let $\mathfrak{F}_\lambda$ be the real subspace $\mathfrak{F}_\lambda \subset T^*_oN$ generated by the real and the imaginary parts of all one forms $\Phi$ for which the term $\Phi \wedge \epsilon_a \wedge \epsilon_b$ with $1 \leq a, b \leq \lambda$ appears on the RHS of (3.6), (3.7), (3.8), (3.9), (3.10) or (3.11). Then, the character sequence $v_1, v_2, \ldots, v_{d_1}$ of the ideal $\mathcal{I}$ corresponding to the fixed basis (3.5) is given by (cf. Section 2.1)

$$\begin{cases} v_1 = 0; \\ v_\lambda = \dim(\mathfrak{F}_\lambda/\mathfrak{F}_{\lambda-1}) = \dim \mathfrak{F}_\lambda - \dim \mathfrak{F}_{(\lambda-1)}, \text{ if } 2 \leq \lambda \leq d_1 \end{cases}$$

**3.3. The characters $\boldsymbol{v_2, \ldots, v_n}$.** Let us fix an integer number $\boldsymbol{\lambda}$ between 2 and $n$. By definition, the (real) vector space $\mathfrak{F}_{\boldsymbol{\lambda}}$ is generated by the real and imaginary parts of the one-forms

$$\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\delta\gamma'}}, \qquad \mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}},$$
$$\mathcal{V}^*_{\boldsymbol{\alpha\gamma\delta'}} - \mathcal{V}^*_{\boldsymbol{\alpha\gamma'\delta}}, \qquad \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\delta}} - \mathcal{V}^*_{\boldsymbol{\alpha'\delta'\gamma}}, \qquad \mathcal{L}^*_{\boldsymbol{\gamma\delta'}} - \mathcal{L}^*_{\boldsymbol{\gamma'\delta}}, \qquad \mathcal{M}^*_{\boldsymbol{\gamma'\delta}} - \mathcal{M}^*_{\boldsymbol{\gamma\delta'}},$$

where $1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta} \leq n$ and $1 \leq \boldsymbol{\gamma}, \boldsymbol{\delta} \leq \boldsymbol{\lambda}$.

Let us introduce the one-forms

$$\begin{aligned} X_{\boldsymbol{\alpha\beta\gamma\delta}} &\overset{def}{=} \frac{1}{2}\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\delta\gamma'}}\Big), \\ Y_{\boldsymbol{\alpha\beta\gamma\delta}} &\overset{def}{=} \frac{1}{4}\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} + \mathcal{S}^*_{\boldsymbol{\alpha\beta\delta\gamma'}} + \mathcal{S}^*_{\boldsymbol{\alpha\gamma\delta\beta'}} + \mathcal{S}^*_{\boldsymbol{\beta\gamma\delta\alpha'}}\Big). \end{aligned} \tag{3.12}$$

Then, as it can be easily verified, $X_{\boldsymbol{\alpha\beta\gamma\delta}}$ is symmetric in $\boldsymbol{\alpha, \beta}$ and skew-symmetric in $\boldsymbol{\gamma, \delta}$. Furthermore, it has the property

$$X_{\boldsymbol{\alpha\beta\gamma\delta}} + X_{\boldsymbol{\alpha\gamma\delta\beta}} + X_{\boldsymbol{\alpha\delta\beta\gamma}} = 0. \tag{3.13}$$

Whereas $Y_{\boldsymbol{\alpha\beta\gamma\delta}}$ is totally symmetric in $\boldsymbol{\alpha, \beta, \gamma, \delta}$. By a straightforward substitution, one can immediately verify the identity

$$\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} = \frac{1}{2}\Big(X_{\boldsymbol{\alpha\beta\gamma\delta}} + X_{\boldsymbol{\beta\gamma\alpha\delta}} + X_{\boldsymbol{\gamma\alpha\beta\delta}}\Big) + Y_{\boldsymbol{\alpha\beta\gamma\delta}}. \tag{3.14}$$

Next, we will choose a reduced set of generators for the linear space

$$\operatorname{span}\Big\{\mathbb{R}e(X_{\boldsymbol{\alpha\beta\gamma\delta}}),\ \mathbb{I}m(X_{\boldsymbol{\alpha\beta\gamma\delta}}) \ \Big|\ 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta} \leq n,\ 1 \leq \boldsymbol{\gamma}, \boldsymbol{\delta} \leq \boldsymbol{\lambda}\Big\} \tag{3.15}$$

considered as a subspace in $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$. Notice that by (3.13), for any $1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \leq \boldsymbol{\lambda} - 1$, we have

$$X_{\boldsymbol{\alpha\beta\gamma\lambda}} + \underbrace{X_{\boldsymbol{\alpha\lambda\beta\gamma}}}_{\in \mathfrak{F}_{(\boldsymbol{\lambda}-1)}} + X_{\boldsymbol{\alpha\gamma\lambda\beta}} = 0$$

and thus, modulo $\mathfrak{F}_{\boldsymbol{\lambda}-1}$ we have the relation $X_{\boldsymbol{\alpha\beta\gamma\lambda}} \equiv X_{\boldsymbol{\alpha\gamma\beta\lambda}}$, i.e., $X_{\boldsymbol{\alpha\beta\gamma\lambda}}$ is symmetric in $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, considered as an element of the quotient space $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$. Therefore,

$$\operatorname{span}\Big\{\mathbb{R}e(X_{\boldsymbol{\alpha\beta\gamma\lambda}}),\ \mathbb{I}m(X_{\boldsymbol{\alpha\beta\gamma\lambda}}) \ \Big|\ 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \leq \boldsymbol{\lambda} - 1\Big\} \subset \frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}$$

can be generated (over the real numbers) by

$$2\binom{\boldsymbol{\lambda}+1}{3} \tag{3.16}$$

elements.

If we consider the index ranges $\boldsymbol{\lambda} \le \boldsymbol{\alpha} \le n$, $1 \le \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1$, we have again the identity

$$X_{\boldsymbol{\alpha\beta\gamma\lambda}} + \underbrace{X_{\boldsymbol{\alpha\lambda\beta\gamma}}}_{\in\mathfrak{F}_{(\boldsymbol{\lambda}-1)}} + X_{\boldsymbol{\alpha\gamma\lambda\beta}} = 0,$$

and thus $X_{\boldsymbol{\alpha\beta\gamma\lambda}}$ is symmetric in $\boldsymbol{\beta}, \boldsymbol{\gamma}$ considered as an element of the quotient space $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$. In this case, the respective subspace

$$\operatorname{span}\Big\{\mathbb{R}e(X_{\boldsymbol{\alpha\beta\gamma\lambda}}),\ \mathbb{I}m(X_{\boldsymbol{\alpha\beta\gamma\lambda}})\ \Big|\ \boldsymbol{\lambda} \le \boldsymbol{\alpha} \le n,\ 1 \le \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1\Big\} \subset \frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}$$

can by generated by

$$2(n-\boldsymbol{\lambda}+1)\binom{\boldsymbol{\lambda}}{2} \tag{3.17}$$

elements.

Similarly, the subspace

$$\operatorname{span}\Big\{\mathbb{R}e(X_{\boldsymbol{\alpha\beta\gamma\lambda}}), \mathbb{I}m(X_{\boldsymbol{\alpha\beta\gamma\lambda}})\ \Big|\ \boldsymbol{\lambda} \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le n,\ 1 \le \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1\Big\} \subset \frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}$$

can by generated by

$$2(\boldsymbol{\lambda}-1)\binom{n-\boldsymbol{\lambda}+2}{2} \tag{3.18}$$

elements.

The sum of the numbers (3.16), (3.17) and (3.18) gives an upper bound for the dimension of (3.15).

We proceed in a similar fashion with the linear subspace

$$\begin{aligned}\operatorname{span}\Big\{&\mathbb{R}e\big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}\big),\ \mathbb{I}m\big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}\big)\ \Big|\\ &\qquad 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le n,\ 1 \le \boldsymbol{\gamma}, \boldsymbol{\delta} \le \boldsymbol{\lambda}\Big\}\end{aligned} \tag{3.19}$$

of $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$. We first introduce some new real one-forms

$$\begin{cases} R_{\boldsymbol{\alpha\beta\gamma\delta}} \overset{def}{=} \frac{1}{2}\mathbb{R}e\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}\Big), \\ T_{\boldsymbol{\alpha\beta\gamma\delta}} \overset{def}{=} \mathbb{R}e\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} + \mathcal{S}^*_{\boldsymbol{\gamma\beta'\delta\alpha'}} + \mathcal{S}^*_{\boldsymbol{\delta\beta'\alpha\gamma'}}\Big), \\ U_{\boldsymbol{\alpha\beta\gamma\delta}} \overset{def}{=} \frac{1}{2}\mathbb{I}m\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}}\Big). \end{cases}$$

Then, the properties (3.1) of $\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}}$ imply that $R_{\boldsymbol{\alpha\beta\gamma\delta}}$ is skew-symmetric with respect to each of the two pairs of indices $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}, \boldsymbol{\delta}$, and satisfies the identities

$$R_{\boldsymbol{\alpha\beta\gamma\delta}} + R_{\boldsymbol{\gamma\alpha\beta\delta}} + R_{\boldsymbol{\beta\gamma\alpha\delta}} = 0, \qquad R_{\boldsymbol{\alpha\beta\gamma\delta}} = R_{\boldsymbol{\gamma\delta\alpha\beta}}, \tag{3.20}$$

i.e., it has the algebraic properties of a Riemannian curvature tensor. We have that $T_{\boldsymbol{\alpha\beta\gamma\delta}}$ is totally symmetric, whereas $U_{\boldsymbol{\alpha\beta\gamma\delta}}$ is symmetric in $\boldsymbol{\alpha}, \boldsymbol{\beta}$, skew-symmetric in $\boldsymbol{\gamma}, \boldsymbol{\delta}$ and satisfies

$$U_{\boldsymbol{\alpha\beta\gamma\delta}} + U_{\boldsymbol{\alpha\delta\beta\gamma}} + U_{\boldsymbol{\alpha\gamma\delta\beta}} = 0.$$

We have also that

$$\begin{aligned} \mathbb{R}e(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}}) &= \frac{2}{3}\Big(R_{\boldsymbol{\alpha\beta\gamma\delta}} + R_{\boldsymbol{\alpha\delta\gamma\beta}}\Big) + \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma\delta}}, \\ \mathbb{I}m(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}}) &= \frac{1}{2}\Big(U_{\boldsymbol{\alpha\beta\gamma\delta}} + U_{\boldsymbol{\gamma\beta\alpha\delta}} + U_{\boldsymbol{\alpha\delta\gamma\beta}} + U_{\boldsymbol{\gamma\delta\alpha\beta}}\Big). \end{aligned} \tag{3.21}$$

Clearly, the subspace (3.19) is generated by $\Big\{R_{\boldsymbol{\alpha\beta\gamma\delta}}, U_{\boldsymbol{\alpha\beta\gamma\delta}} \;\Big|\; 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le n,\ 1 \le \boldsymbol{\gamma}, \boldsymbol{\delta} \le \boldsymbol{\lambda}\Big\}$. We will next reduce the number of its generators by using the above symmetry properties. The dependence of the one-forms $U_{\boldsymbol{\alpha\beta\gamma\delta}}$ on their indices is a subject to the exactly same algebraic relations as that of $X_{\boldsymbol{\alpha\beta\gamma\delta}}$ . Therefore, by (3.16), (3.17) and (3.18), the subspace

$$\mathrm{span}\Big\{U_{\boldsymbol{\alpha\beta\gamma\delta}} \;\Big|\; 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le n,\ 1 \le \boldsymbol{\gamma}, \boldsymbol{\delta} \le \boldsymbol{\lambda}\Big\} \subset \frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}.$$

can be generated by

$$\binom{\boldsymbol{\lambda}+1}{3} + (n-\boldsymbol{\lambda}+1)\binom{\boldsymbol{\lambda}}{2} + (\boldsymbol{\lambda}-1)\binom{n-\boldsymbol{\lambda}+2}{2} \tag{3.22}$$

elements.

If we assume $1 \le \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1$, then the properties (3.20) easily imply that $R_{\boldsymbol{\alpha\beta\gamma\lambda}} \equiv 0$ modulo $\mathfrak{F}_{\boldsymbol{\lambda}-1}$. Let us consider the index ranges $\boldsymbol{\lambda} \le \boldsymbol{\alpha} \le n$, $1 \le \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1$. We have

$$R_{\boldsymbol{\alpha\beta\gamma\lambda}} + \underbrace{R_{\boldsymbol{\alpha\lambda\beta}\gamma}}_{\in\mathfrak{F}_{(\boldsymbol{\lambda}-1)}} + R_{\boldsymbol{\alpha\gamma\lambda\beta}} = 0,$$

and hence, modulo $\mathfrak{F}_{\boldsymbol{\lambda}-1}$, $R_{\boldsymbol{\alpha\beta\gamma\lambda}} \equiv R_{\boldsymbol{\alpha\gamma\beta\lambda}}$. Thus

$$\mathrm{span}\Big\{R_{\boldsymbol{\alpha\beta\gamma\lambda}} \;\Big|\; \boldsymbol{\lambda} \le \boldsymbol{\alpha} \le n,\ 1 \le \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1\Big\} \subset \frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}$$

can be generated by

$$(n-\boldsymbol{\lambda}+1)\binom{\boldsymbol{\lambda}}{2} \tag{3.23}$$

elements. Whereas

$$\mathrm{span}\Big\{R_{\boldsymbol{\alpha\beta\gamma\lambda}} \;\Big|\; \boldsymbol{\lambda} \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le n,\ 1 \le \boldsymbol{\gamma} \le \boldsymbol{\lambda} - 1\Big\} \subset \frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}$$

can by generated by

$$(\boldsymbol{\lambda}-1)\binom{n-\boldsymbol{\lambda}+1}{2} \tag{3.24}$$

elements, since $R_{\boldsymbol{\alpha\beta\gamma\lambda}} = -R_{\boldsymbol{\beta\alpha\gamma\lambda}}$. Therefore, the dimension of (3.19) is less or equal to the sum of (3.22), (3.23) and (3.24).

Similarly, the linear span in $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$ of the real and imaginary parts of the one-forms

$$\Big\{\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} - \mathcal{V}^*_{\boldsymbol{\alpha\gamma\beta'}},\ \mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}} - \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}}\ \Big|\ 1 \ \le\ \boldsymbol{\alpha}\ \le n,\ 1\ \le\ \boldsymbol{\beta},\boldsymbol{\gamma}\ \le \boldsymbol{\lambda}\Big\}$$

can be generated by

$$4\binom{\boldsymbol{\lambda}}{2} + 4(n-\boldsymbol{\lambda}+1)(\boldsymbol{\lambda}-1) \tag{3.25}$$

elements. Whereas for the linear span of the real and imaginary parts of

$$\Big\{\mathcal{L}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{L}^*_{\boldsymbol{\beta\alpha'}},\ \mathcal{M}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{M}^*_{\boldsymbol{\beta\alpha'}}\ \Big|\ 1\ \le\ \boldsymbol{\alpha}\ \le n,\ 1\ \le\ \boldsymbol{\beta},\boldsymbol{\gamma}\ \le \boldsymbol{\lambda}\Big\}$$

in $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$, we need only

$$3(\boldsymbol{\lambda}-1) \tag{3.26}$$

generators (notice that by (3.2), the imaginary part of $\mathcal{L}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{L}^*_{\boldsymbol{\beta\alpha'}}$ vanishes).

The sum of (3.16), (3.17), (3.18), (3.22), (3.23), (3.24), (3.25) and (3.26) gives an upper bound for the dimension of $\mathfrak{F}_{\boldsymbol{\lambda}}/\mathfrak{F}_{\boldsymbol{\lambda}-1}$, i.e., we have

$$\begin{aligned}\dim\Big(\frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}\Big)\ &\le\ 3\binom{\boldsymbol{\lambda}+1}{3}\ +\ 3(n-\boldsymbol{\lambda}+1)\binom{\boldsymbol{\lambda}}{2}\\ &+ 3(\boldsymbol{\lambda}-1)\binom{n-\boldsymbol{\lambda}+2}{2}\ +\ (n-\boldsymbol{\lambda}+1)\binom{\boldsymbol{\lambda}}{2}\ +\ (\boldsymbol{\lambda}-1)\binom{n-\boldsymbol{\lambda}+1}{2}\\ &+ 4\binom{\boldsymbol{\lambda}}{2}\ +\ 4(n-\boldsymbol{\lambda}+1)(\boldsymbol{\lambda}-1)\ +\ 3(\lambda-1)\\ &= \frac{1}{2}(\boldsymbol{\lambda}-1)(\boldsymbol{\lambda}-2n-4)(\boldsymbol{\lambda}-2n-5).\end{aligned}$$

The vector space $\mathfrak{F}_n$ is freely generated by the real and imaginary parts of all the one-forms (modulo their respective symmetries)

$$X_{\boldsymbol{\alpha\beta\gamma\delta}}, \qquad R_{\boldsymbol{\alpha\beta\gamma\delta}}, \qquad U_{\boldsymbol{\alpha\beta\gamma\delta}}$$
$$\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} - \mathcal{V}^*_{\boldsymbol{\alpha\gamma\beta'}}, \qquad \mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}} - \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}}, \qquad \mathcal{L}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{L}^*_{\boldsymbol{\beta\alpha'}}, \qquad \mathcal{M}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{M}^*_{\boldsymbol{\beta\alpha'}}$$

and hence we can easily compute its dimension,

$$\begin{aligned}\dim(\mathfrak{F}_n) = \underbrace{\left[\binom{\frac{n(n-1)}{2}+1}{2} - \binom{n}{4}\right]}_{\text{this is for } \{R_{\boldsymbol{\alpha\beta\gamma\delta}}\}} + \underbrace{3\binom{\frac{n(n+1)}{2}}{2}}_{\{X_{\boldsymbol{\alpha\beta\gamma\delta}},\, U_{\boldsymbol{\alpha\beta\gamma\delta}}\}}\\ +\ \underbrace{4\left[n\binom{n+1}{2} - \binom{n+2}{3}\right]}_{\{\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} - \mathcal{V}^*_{\boldsymbol{\alpha\gamma\beta'}},\, \mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}} - \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}}\}} + \underbrace{3\binom{n}{2}}_{\{\mathcal{L}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{L}^*_{\boldsymbol{\beta\alpha'}},\, \mathcal{M}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{M}^*_{\boldsymbol{\beta\alpha'}}\}}\\ = \frac{1}{24}n(n-1)(11n^2+61n+86).\end{aligned}$$

By construction $0 = \mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots \subset \mathfrak{F}_n$ and thus

$$\mathfrak{F}_n \cong \left(\frac{\mathfrak{F}_2}{\mathfrak{F}_1}\right) \oplus \left(\frac{\mathfrak{F}_3}{\mathfrak{F}_2}\right) \oplus \cdots \oplus \left(\frac{\mathfrak{F}_n}{\mathfrak{F}_{n-1}}\right).$$

Therefore we can calculate (by using, for example, some of the computer algebra systems) that

$$\begin{aligned}\dim(\mathfrak{F}_n) = \sum_{\boldsymbol{\lambda}=2}^{n} \dim\left(\frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}\right) \;\leq\; & \sum_{\boldsymbol{\lambda}=2}^{n} \Big(\frac{1}{2}(\boldsymbol{\lambda}-1)(\boldsymbol{\lambda}-2n-4)(\boldsymbol{\lambda}-2n-5)\Big) \\ = \;& \frac{1}{24} n(n-1)(11n^2+61n+86),\end{aligned}$$

which implies that the above inequality must actually be an equality, i.e., we have shown

$$v_{\boldsymbol{\lambda}} = \dim\left(\frac{\mathfrak{F}_{\boldsymbol{\lambda}}}{\mathfrak{F}_{\boldsymbol{\lambda}-1}}\right) = \frac{1}{2}(\boldsymbol{\lambda}-1)(\boldsymbol{\lambda}-2n-4)(\boldsymbol{\lambda}-2n-5), \qquad 2 \leq \boldsymbol{\lambda} \leq n.$$

**3.4. The characters $\boldsymbol{v_{(n+1)}, \ldots, v_{2n}}$.** Notice that, modulo $\mathfrak{F}_n$, we have (cf. (3.14), (3.21))

$$\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} \equiv Y_{\boldsymbol{\alpha\beta\gamma\delta}}, \qquad \mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} \equiv \frac{1}{3} T_{\boldsymbol{\alpha\beta\gamma\delta}}$$

and that each of the arrays

$$Y_{\boldsymbol{\alpha\beta\gamma\delta}}, \quad T_{\boldsymbol{\alpha\beta\gamma\delta}}, \quad \mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}}, \quad \mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}}, \quad \mathcal{L}^*_{\boldsymbol{\alpha\beta'}}, \quad \mathcal{M}^*_{\boldsymbol{\alpha\beta'}}$$

depends totally symmetrically on its indices.

Let us fix $\boldsymbol{\lambda}$ to be an integer number between 1 and $n$. By definition, the quotient space $\mathfrak{F}_{(n+\boldsymbol{\lambda})}/\mathfrak{F}_n$ is generated by the real and imaginary parts of the one-forms:

$$\begin{aligned} A_{\boldsymbol{\alpha\beta\gamma\delta}} &\overset{def}{=} i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta\gamma'}} + \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} + \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[\boldsymbol{\delta}+1]'} \\ &\equiv 2iY_{\boldsymbol{\alpha\beta\gamma\delta}} + \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} + \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}, \quad 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \leq n, \quad 1 \leq \boldsymbol{\delta} \leq \boldsymbol{\lambda}; \end{aligned}$$

$$\begin{aligned} B_{\boldsymbol{\alpha\beta\gamma\delta}} &\overset{def}{=} -i\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\delta\gamma'}} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[\boldsymbol{\delta}+1]'} - i\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta'}[\boldsymbol{\gamma}+1]'} - i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} \\ &\qquad + i\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]} + \mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]'} - \mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]'}\Big) \\ &\equiv \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} - \frac{1}{3}T_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} + -S^*_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]}, \\ &\qquad\qquad 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta} \leq n, \quad 1 \leq \boldsymbol{\gamma}, \boldsymbol{\delta} \leq \boldsymbol{\lambda}; \end{aligned}$$

$$\begin{aligned} C_{\boldsymbol{\alpha\beta\gamma\delta}} &\overset{def}{=} -\frac{i}{2}\Big(i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}} + \mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[\boldsymbol{\delta}+1]} - \overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[\boldsymbol{\delta}+1]}}\Big) \\ &\equiv \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma\delta}} + \mathbb{I}m\big(Y_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}\big), \quad 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \leq n, \quad 1 \leq \boldsymbol{\delta} \leq \boldsymbol{\lambda}; \end{aligned}$$

$$
\begin{aligned}
D_{\boldsymbol{\alpha\beta\gamma\delta}} &\overset{def}{=} -i\Big(\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta\gamma'}} - i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[\boldsymbol{\delta}+1]}} + i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\delta}[\boldsymbol{\gamma}+1]}} - i{-}S^*_{\boldsymbol{\alpha\beta'\gamma}[\boldsymbol{\delta}+1]} \\
&\qquad + i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\delta}[\boldsymbol{\gamma}+1]} + \mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]'} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]'}\Big) \\
&\equiv -\mathbb{R}e\big(Y_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}\big) + \mathbb{R}e\big(Y_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]}\big), \quad 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le n, \quad 1 \le \boldsymbol{\gamma}, \boldsymbol{\delta} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
A_{\boldsymbol{\alpha\beta\gamma}} &\overset{def}{=} i\Big(\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} + \mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}} - i\mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]} - i\mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'}\Big) \\
&\equiv 2i\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} + \mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]} + \mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'}, 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta}, \le n, \quad 1 \le \boldsymbol{\gamma} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
B_{\boldsymbol{\alpha\beta\gamma}} &\overset{def}{=} -i\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} + i\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma}} + \mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha\gamma'}[\boldsymbol{\beta}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]} + \mathcal{V}^*_{\boldsymbol{\alpha\gamma}[\boldsymbol{\beta}+1]} \\
&\qquad - i\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1][\boldsymbol{\gamma}+1]'} + i\mathcal{V}^*_{\boldsymbol{\alpha}[\boldsymbol{\gamma}+1][\boldsymbol{\beta}+1]'} \\
&\equiv \mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha\gamma'}[\boldsymbol{\beta}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]} + \mathcal{V}^*_{\boldsymbol{\alpha\gamma}[\boldsymbol{\beta}+1]}, \ 1 \le \boldsymbol{\alpha} \le n, \ 1 \le \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
C_{\boldsymbol{\alpha\beta\gamma}} &\overset{def}{=} \mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}} + \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}} - i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'} - i\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]} \\
&\equiv 2\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}} - i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'} - i\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]}, \quad 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta}, \le n, \quad 1 \le \boldsymbol{\gamma} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
D_{\boldsymbol{\alpha\beta\gamma}} &\overset{def}{=} i\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma}} - i\mathcal{V}^*_{\boldsymbol{\alpha'\gamma'\beta}} + \mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'}[\boldsymbol{\beta}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]} + \mathcal{V}^*_{\boldsymbol{\alpha'\gamma}[\boldsymbol{\beta}+1]} \\
&\qquad + i\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'[\boldsymbol{\gamma}+1]} - i\mathcal{V}^*_{\boldsymbol{\alpha'}[\boldsymbol{\gamma}+1]'[\boldsymbol{\beta}+1]} \\
&\equiv \mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha'\gamma'}[\boldsymbol{\beta}+1]'} - \mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]} + \mathcal{V}^*_{\boldsymbol{\alpha'\gamma}[\boldsymbol{\beta}+1]}, \ 1 \le \boldsymbol{\alpha} \le n, \ 1 \le \boldsymbol{\beta}, \boldsymbol{\gamma} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
A_{\boldsymbol{\alpha\beta}} &\overset{def}{=} \frac{1}{2}\Big(i\mathcal{L}^*_{\boldsymbol{\alpha\beta'}} + i\mathcal{L}^*_{\boldsymbol{\alpha'\beta}} + \overline{\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}} + \mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}\Big) \equiv i\mathcal{L}^*_{\boldsymbol{\alpha\beta'}} + \mathbb{R}e\big(\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}\big), \\
&\qquad 1 \le \boldsymbol{\alpha}, \le n, \quad 1 \le \boldsymbol{\beta} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
B_{\boldsymbol{\alpha\beta}} &\overset{def}{=} \frac{1}{2}\Big(\mathcal{L}^*_{\boldsymbol{\alpha\beta'}} - \mathcal{L}^*_{\boldsymbol{\alpha'\beta}} - i\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]} + i\mathcal{L}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]} + i\overline{\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}} - i\overline{\mathcal{L}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]}} \\
&\qquad - \mathcal{L}^*_{[\boldsymbol{\alpha}+1]'[\boldsymbol{\beta}+1]} + \mathcal{L}^*_{[\boldsymbol{\beta}+1]'[\boldsymbol{\alpha}+1]}\Big) \\
&\equiv \mathbb{I}m\big(\mathcal{L}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}\big) - \mathbb{I}m\big(\mathcal{L}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]}\big), \quad 1 \le \boldsymbol{\alpha}, \boldsymbol{\beta} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
\begin{aligned}
C_{\boldsymbol{\alpha\beta}} &\overset{def}{=} \mathcal{M}^*_{\boldsymbol{\alpha'\beta}} + \mathcal{M}^*_{\boldsymbol{\alpha\beta'}} - i\mathcal{M}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'} - i\mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]} \\
&\equiv 2\mathcal{M}^*_{\boldsymbol{\alpha\beta'}} - i\mathcal{M}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'} - i\mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]}, \ 1 \le \boldsymbol{\alpha}, \le n, \ 1 \le \boldsymbol{\beta} \le \boldsymbol{\lambda};
\end{aligned}
$$

$$
D_{\boldsymbol{\alpha\beta}} \overset{def}{=} -i\mathcal{M}^*_{\boldsymbol{\alpha'\beta}} + i\mathcal{M}^*_{\boldsymbol{\alpha\beta'}} + \mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]} - \mathcal{M}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]} - \mathcal{M}^*_{\boldsymbol{\alpha'}[\boldsymbol{\beta}+1]'}
$$

$$+ \mathcal{M}^*_{\boldsymbol{\beta}'[\boldsymbol{\alpha}+1]'} - i\mathcal{M}^*_{[\boldsymbol{\alpha}+1]'[\boldsymbol{\beta}+1]} + i\mathcal{M}^*_{[\boldsymbol{\beta}+1]'[\boldsymbol{\alpha}+1]}$$
$$\equiv \mathcal{M}^*_{\boldsymbol{\alpha}[\boldsymbol{\beta}+1]} - \mathcal{M}^*_{\boldsymbol{\beta}[\boldsymbol{\alpha}+1]} - \mathcal{M}^*_{\boldsymbol{\alpha}'[\boldsymbol{\beta}+1]'} + \mathcal{M}^*_{\boldsymbol{\beta}'[\boldsymbol{\alpha}+1]'},\ 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta} \leq \boldsymbol{\lambda}.$$

**Lemma 3.1.** *The linear space $\mathfrak{F}_{(n+\boldsymbol{\lambda})}/\mathfrak{F}_{(n+\boldsymbol{\lambda}-1)}$ is generated (over the real numbers) by the real and imaginary parts of the one-forms*

$$\begin{gathered}\Big\{A_{\boldsymbol{\alpha\beta\gamma\lambda}},\ A_{\boldsymbol{\alpha\beta\lambda}},\ A_{\boldsymbol{\alpha\lambda}},\ C_{\boldsymbol{\alpha\beta\gamma\lambda}},\ C_{\boldsymbol{\alpha\beta\lambda}},\ C_{\boldsymbol{\alpha\lambda}}\ \Big|\ 1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \leq n\Big\} \\ \cup \Big\{B_{\boldsymbol{\alpha\lambda}},\ D_{\boldsymbol{\alpha\lambda}}\Big|\ 1 \leq \alpha \leq \lambda\Big\}.\end{gathered} \tag{3.27}$$

Furthermore, modulo $\mathfrak{F}_{(n+\boldsymbol{\lambda})}$, for any $1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta} \leq n,\ 1 \leq \boldsymbol{\gamma}, \boldsymbol{\delta} \leq \boldsymbol{\lambda}$, we have the identities

$$\begin{gathered}Y_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv Y_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}, \qquad \mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}, \qquad T_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv T_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}, \\ \mathcal{V}^*_{\boldsymbol{\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv \mathcal{V}^*_{\boldsymbol{\beta\gamma}[\boldsymbol{\delta}+1]}, \qquad \mathcal{V}^*_{\boldsymbol{\beta}'[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv \mathcal{V}^*_{\boldsymbol{\beta}'\boldsymbol{\gamma}[\boldsymbol{\delta}+1]}, \qquad \mathcal{V}^*_{\boldsymbol{\beta}'[\boldsymbol{\gamma}+1]'\boldsymbol{\delta}} \equiv \mathcal{V}^*_{\boldsymbol{\beta}'\boldsymbol{\gamma}'[\boldsymbol{\delta}+1]} \\ \mathcal{M}^*_{[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv \mathcal{M}^*_{\boldsymbol{\gamma}[\boldsymbol{\delta}+1]}, \qquad \mathcal{M}^*_{[\boldsymbol{\gamma}+1]'\boldsymbol{\delta}} \equiv \mathcal{M}^*_{\boldsymbol{\gamma}'[\boldsymbol{\delta}+1]}, \qquad \mathcal{M}^*_{[\boldsymbol{\gamma}+1]'\boldsymbol{\delta}'} \equiv \mathcal{M}^*_{\boldsymbol{\gamma}'[\boldsymbol{\delta}+1]'}, \\ \mathcal{L}^*_{[\boldsymbol{\gamma}+1]\boldsymbol{\delta}'} \equiv \mathcal{L}^*_{\boldsymbol{\gamma}[\boldsymbol{\delta}+1]'}, \qquad \mathcal{L}^*_{[\boldsymbol{\gamma}+1]'\boldsymbol{\delta}} \equiv \mathcal{L}^*_{\boldsymbol{\gamma}'[\boldsymbol{\delta}+1]}.\end{gathered} \tag{3.28}$$

**Proof.** Let us denote by $\mathfrak{U}_{\boldsymbol{\lambda}}$ the sum of $\mathfrak{F}_{(n+\boldsymbol{\lambda}-1)}$ and the linear span of the real and imaginary parts of the one-forms (3.27), and consider the index ranges $1 \leq \boldsymbol{\alpha}, \boldsymbol{\beta} \leq n,\ 1 \leq \boldsymbol{\gamma}, \boldsymbol{\delta} \leq \boldsymbol{\lambda}$. Since $A_{\boldsymbol{\alpha\beta\gamma\delta}} \equiv C_{\boldsymbol{\alpha\beta\gamma\delta}} \equiv 0$ modulo $\mathfrak{U}_{\boldsymbol{\lambda}}$, we have

$$\begin{cases} -2iY_{\boldsymbol{\alpha\beta\gamma\delta}} \equiv \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} + \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} \\ -\frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma\delta}} \equiv \mathbb{I}m\big(Y_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}\big) \end{cases} \mod \mathfrak{U}_{\boldsymbol{\lambda}}.$$

Using this, we obtain, modulo $\mathfrak{U}_{\boldsymbol{\lambda}}$,

$$\begin{aligned} -2iY_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} &\equiv \mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]} + \frac{1}{3}T_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]} \\ &\equiv \mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]} + \frac{1}{3}T_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]} \equiv -2iY_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1]\boldsymbol{\gamma}} \end{aligned}$$

which proves the first equation in (3.28). Similarly,

$$\mathcal{S}^*_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]} + \frac{1}{3}T_{\boldsymbol{\alpha\beta\delta}[\boldsymbol{\gamma}+1]} \equiv -2iY_{\boldsymbol{\alpha\beta\delta\gamma}} \equiv -2iY_{\boldsymbol{\alpha\beta\gamma\delta}} \equiv \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]} + \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\delta}+1]}$$

and also

$$-\frac{1}{3}T_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\delta}} \equiv \mathbb{I}m\big(Y^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][\boldsymbol{\delta}+1]}\big) \equiv \mathbb{I}m\big(Y^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1][\boldsymbol{\gamma}+1]}\big) \equiv -\frac{1}{2}T_{\boldsymbol{\alpha\beta}[\boldsymbol{\delta}+1]\boldsymbol{\gamma}}, \tag{3.29}$$

which yields the second and the third equations in the first line of (3.28). The proof of the rest of (3.28) is completely analogous. Now, applying (3.28), we have that, modulo $\mathfrak{U}_{\boldsymbol{\lambda}}$,

$$B_{\boldsymbol{\alpha\beta\gamma\lambda}} \equiv \frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\lambda}+1]} - \frac{1}{3}T_{\boldsymbol{\alpha\beta\lambda}[\boldsymbol{\gamma}+1]} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\lambda}+1]} + \mathcal{S}^*_{\boldsymbol{\alpha\beta\lambda}[\boldsymbol{\gamma}+1]} \equiv 0,$$

and similarly $D_{\boldsymbol{\alpha\beta\gamma\lambda}} \equiv B_{\boldsymbol{\beta\gamma\lambda}} \equiv D_{\boldsymbol{\beta\gamma\lambda}} \equiv 0.$ $\square$

It is easy to observe that, by a repeated application of the identities in the first line of (3.28), each $A_{\boldsymbol{\alpha\beta\gamma\lambda}}$ can be made equivalent, modulo $\mathfrak{F}_{n+\boldsymbol{\lambda}-1}$, to one of the elements in the following two sets:

$$\begin{aligned}&\Big\{A_{\boldsymbol{\alpha\beta\gamma\lambda}}\ \Big|\ \boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in\{1,\boldsymbol{\lambda},\boldsymbol{\lambda}+1,\dots,n\}\Big\};\\ &\Big\{A_{\boldsymbol{\alpha\beta\gamma\lambda}}\ \Big|\ \boldsymbol{\alpha},\boldsymbol{\beta}\in\{1,\boldsymbol{\lambda},\boldsymbol{\lambda}+1,\dots,n\},\ 2\le\boldsymbol{\gamma}\le\boldsymbol{\lambda}-1\Big\}.\end{aligned}\tag{3.30}$$

Let us consider the one-forms $C_{\boldsymbol{\alpha\beta\gamma\lambda}}$ modulo

$$\mathfrak{F}_{n+\boldsymbol{\lambda}-1}\oplus\operatorname{span}\Big\{\mathbb{R}e\big(A_{\boldsymbol{\alpha\beta\gamma\lambda}}\big),\ \mathbb{I}m\big(A_{\boldsymbol{\alpha\beta\gamma\lambda}}\big),\Big\}.$$

If we suppose $1\le\boldsymbol{\gamma}\le\boldsymbol{\lambda}-1$, then

$$\begin{aligned}C_{\boldsymbol{\alpha\beta\gamma\lambda}}&\equiv\frac{1}{3}T_{\boldsymbol{\alpha\beta\gamma\lambda}}+\mathbb{I}m\big(Y_{\boldsymbol{\alpha\beta\gamma}[\boldsymbol{\lambda}+1]}\big)\\ &\equiv\frac{1}{3}T_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][\boldsymbol{\lambda}-1]}+\mathbb{I}m\big(Y_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\boldsymbol{\lambda}}\big)\equiv C_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]\lambda}\equiv 0.\end{aligned}$$

Therefore, each $C_{\boldsymbol{\alpha\beta\gamma\lambda}}$ is equivalent to one of the forms in the set

$$\Big\{C_{\boldsymbol{\alpha\beta\gamma\lambda}}\ \Big|\ \lambda\le\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\le n\}\Big\}.\tag{3.31}$$

Thus, by (3.30) and (3.31), the linear subspace

$$\operatorname{span}\Big\{\mathbb{R}e\big(A_{\boldsymbol{\alpha\beta\gamma\lambda}}\big),\ \mathbb{I}m\big(A_{\boldsymbol{\alpha\beta\gamma\lambda}}\big),C_{\boldsymbol{\alpha\beta\gamma\lambda}}\ \Big|1\le\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\le n\Big\}\subset\frac{\mathfrak{F}_{n+\boldsymbol{\lambda}}}{\mathfrak{F}_{n+\boldsymbol{\lambda}-1}}$$

can be generated by

$$2\binom{n-\boldsymbol{\lambda}+4}{3}+2\binom{\boldsymbol{\lambda}-2}{1}\binom{n-\boldsymbol{\lambda}+3}{2}+\binom{n-\boldsymbol{\lambda}+3}{3}\tag{3.32}$$

elements.

Similarly, by a repeated application of the identities in the second line of (3.28), we obtain that each of the one-forms $A_{\boldsymbol{\alpha\beta\lambda}}$, $C_{\boldsymbol{\alpha\beta\lambda}}$ can be transformed, equivalently modulo $\mathfrak{F}_{n+\boldsymbol{\lambda}-1}$, to one of the elements in the following two sets:

$$\begin{aligned}&\Big\{A_{\boldsymbol{\alpha\beta\lambda}},\ C_{\boldsymbol{\alpha\beta\lambda}}\ \Big|\ \boldsymbol{\alpha},\boldsymbol{\beta}\in\{1,\boldsymbol{\lambda},\boldsymbol{\lambda}+1,\dots,n\}\Big\};\\ &\Big\{A_{\boldsymbol{\alpha\beta\lambda}},\ C_{\boldsymbol{\alpha\beta\lambda}}\ \Big|\ \boldsymbol{\alpha}\in\{1,\boldsymbol{\lambda},\boldsymbol{\lambda}+1,\dots,n\},\ 2\le\boldsymbol{\beta}\le\boldsymbol{\lambda}-1\Big\}.\end{aligned}$$

Therefore,

$$\operatorname{span}\Big\{\mathbb{R}e\big(A_{\boldsymbol{\alpha\beta\lambda}}\big),\mathbb{I}m\big(A_{\boldsymbol{\alpha\beta\lambda}}\big),\mathbb{R}e\big(C_{\boldsymbol{\alpha\beta\lambda}}\big),\mathbb{I}m\big(C_{\boldsymbol{\alpha\beta\lambda}}\big)\ \Big|\ 1\le\boldsymbol{\alpha},\boldsymbol{\beta}\le n\Big\}\subset\frac{\mathfrak{F}_{n+\boldsymbol{\lambda}}}{\mathfrak{F}_{n+\boldsymbol{\lambda}-1}}$$

can be generated by

$$4\binom{n-\boldsymbol{\lambda}+3}{2}+4\binom{\boldsymbol{\lambda}-2}{1}\binom{n-\boldsymbol{\lambda}+2}{1}\tag{3.33}$$

elements.

Clearly,

$$\operatorname{span}\Big\{A_{\boldsymbol{\alpha\lambda}}\Big|1 \le \boldsymbol{\alpha} \le n\Big\} \oplus \operatorname{span}\Big\{B_{\boldsymbol{\alpha\lambda}}\Big|1 \le \boldsymbol{\alpha} \le \lambda - 1\Big\} \subset \frac{\mathfrak{F}_{n+\boldsymbol{\lambda}}}{\mathfrak{F}_{n+\boldsymbol{\lambda}-1}}$$

can be generated by

$$n + \boldsymbol{\lambda} - 1 \tag{3.34}$$

elements, and similarly

$$\operatorname{span}\Big\{\mathbb{R}e\big(C_{\boldsymbol{\alpha\lambda}}\big), \mathbb{I}m\big(C_{\boldsymbol{\alpha\lambda}}\big)\Big|1 \le \boldsymbol{\alpha} \le n\Big\} \oplus \operatorname{span}\Big\{D_{\boldsymbol{\alpha\lambda}}\Big|1 \le \boldsymbol{\alpha} \le \lambda - 1\Big\} \subset \frac{\mathfrak{F}_{n+\boldsymbol{\lambda}}}{\mathfrak{F}_{n+\boldsymbol{\lambda}-1}}$$

generates by

$$2(n + \boldsymbol{\lambda} - 1). \tag{3.35}$$

elements.

Therefore, the dimension of $\mathfrak{F}_{n+\boldsymbol{\lambda}}/\mathfrak{F}_{n+\boldsymbol{\lambda}-1}$ is bounded above by the sum of (3.32), (3.33),(3.34) and (3.35), i.e.,

$$v_{n+\boldsymbol{\lambda}} \;=\; \dim\Big(\frac{\mathfrak{F}_{n+\boldsymbol{\lambda}}}{\mathfrak{F}_{n+\boldsymbol{\lambda}-1}}\Big) \;\le\; \frac{1}{2}(n + \boldsymbol{\lambda} - 1)(n - \boldsymbol{\lambda} + 4)(n - \boldsymbol{\lambda} + 5). \tag{3.36}$$

Later on, we shell see that in (3.36) we have, actually, an equality.

Let us observe that equations (3.28) and (3.29) yield the identities

$$\begin{cases} \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta}} \equiv -2iY_{111[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}+\boldsymbol{\delta}-4]} + \mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}+\boldsymbol{\delta}-2]}\big) \\ \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma\delta'}} \equiv Y_{111[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}+\boldsymbol{\delta}-3]} \\ \mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma\delta'}} \equiv -\mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}+\boldsymbol{\delta}-2]}\big) \end{cases} \mod \mathfrak{F}_{2n} \tag{3.37}$$

Similarly, the vanishing of all one-forms $A_{\boldsymbol{\alpha\beta\gamma}}$, $C_{\boldsymbol{\alpha\beta\gamma}}$, modulo $\mathfrak{F}_{2n}$, implies that

$$\begin{cases} \mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma}} \equiv -2i\mathcal{V}^*_{11[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}-3]\boldsymbol{'}} - \mathcal{V}^*_{11\boldsymbol{'}[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}-2]'} \\ \mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}} \equiv -\mathcal{V}^*_{11[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}-2]\boldsymbol{'}} - 2i\mathcal{V}^*_{11'[\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}-3]'} \end{cases} \mod \mathfrak{F}_{2n} \tag{3.38}$$

and the vanishing of $A_{\boldsymbol{\alpha\beta}}$, $C_{\boldsymbol{\alpha\beta}}$ gives

$$\begin{cases} \mathcal{L}^*_{\boldsymbol{\alpha\beta'}} \equiv i\mathbb{R}e\big(\mathcal{L}^*_{1[\boldsymbol{\alpha}+\boldsymbol{\beta}]}\big) \\ \mathcal{M}^*_{\boldsymbol{\alpha\beta}} \equiv -2i\mathcal{M}^*_{1[\boldsymbol{\alpha}+\boldsymbol{\beta}-2]\boldsymbol{'}} - \mathcal{M}^*_{1\boldsymbol{'}[\boldsymbol{\alpha}+\boldsymbol{\beta}-1]'} \end{cases} \mod \mathfrak{F}_{2n} \tag{3.39}$$

**3.5. The characters $\boldsymbol{v_{(2n+1)}, v_{(2n+2)}}$ and $\boldsymbol{v_{(2n+3)}}$.** The definition of $\mathfrak{F}_{2n+1}$ together with the identities (3.37), (3.38), (3.39) and (3.28) implies that the quotient space $\mathfrak{F}_{2n+1}/\mathfrak{F}_{2n}$ is generated by the real and imaginary parts of the one-forms:

$$\begin{gathered} \mathcal{V}^*_{11[\boldsymbol{\alpha}-3]\boldsymbol{'}} + \mathbb{I}m\big(\mathcal{V}^*_{11\boldsymbol{'}[\boldsymbol{\alpha}-2]\boldsymbol{'}}\big); \\ \mathcal{V}^*_{11[\boldsymbol{\alpha}-3]\boldsymbol{'}} + \mathbb{I}m\big(\mathcal{V}^*_{11\boldsymbol{'}[\boldsymbol{\alpha}-2]\boldsymbol{'}}\big) + i\mathbb{I}m\big(\mathcal{V}^*_{11[\boldsymbol{\alpha}-1]\boldsymbol{'}}\big); \\ \mathbb{I}m\big(\mathcal{V}^*_{11[\boldsymbol{\alpha}-2]\boldsymbol{'}}\big);\ \mathbb{R}e\big(\mathcal{V}^*_{11[\boldsymbol{\alpha}-2]\boldsymbol{'}}\big) + \mathbb{I}m\big(\mathcal{V}^*_{11\boldsymbol{'}[\boldsymbol{\alpha}-1]\boldsymbol{'}}\big); \end{gathered} \tag{3.40}$$

$$\begin{gathered}2\mathcal{M}^*_{1[\boldsymbol{\alpha}-2]'}-i\mathcal{M}^*_{1'[\boldsymbol{\alpha}-1]'}+\mathbb{R}e\big(-L^*_{1\boldsymbol{\alpha}}\big);\\ 2\mathcal{M}^*_{1[\boldsymbol{\alpha}-2]'}-i\mathcal{M}^*_{1'[\boldsymbol{\alpha}-1]'}+i\mathbb{I}m\big(\mathcal{L}^*_{1\boldsymbol{\alpha}}\big)+\mathcal{M}^*_{1\boldsymbol{\alpha}'};\\ -\overline{\mathcal{L}^*_{1[\boldsymbol{\alpha}-1]}}+\mathcal{M}^*_{1[\boldsymbol{\alpha}-1]'};\ \overline{\mathcal{L}^*_{1[\boldsymbol{\alpha}-1]}}+\mathcal{M}^*_{1[\boldsymbol{\alpha}-1]'}+\mathbb{R}e\big(\mathcal{L}^*_{1[\boldsymbol{\alpha}+1]}\big)-i\mathcal{M}^*_{1'\boldsymbol{\alpha}'};\end{gathered}\tag{3.41}$$

$$\begin{gathered}\mathbb{R}e\big(\mathcal{C}^*_{\boldsymbol{\alpha}}\big);\ \mathbb{I}m\big(\mathcal{C}^*_{\boldsymbol{\alpha}}\big)-\mathbb{R}e\big(\mathcal{C}^*_{[\boldsymbol{\alpha}+1]'}\big);\ i\mathcal{C}^*_{\boldsymbol{\alpha}'}+\mathcal{H}^*_{\boldsymbol{\alpha}};\\ -i\mathcal{C}^*_{\boldsymbol{\alpha}'}+\mathcal{H}^*_{\boldsymbol{\alpha}}-\mathcal{C}^*_{[\boldsymbol{\alpha}+1]}-i\mathcal{H}^*_{[\boldsymbol{\alpha}+1]'}.\end{gathered}\tag{3.42}$$

A brief inspection of the one forms in (3.40) shows that the linear span in $\mathfrak{F}_{2n+1}/\mathfrak{F}_{2n}$ of their real and imaginary parts can be generated by using only the real and imaginary parts of the first expression there. Similarly, for the the linear span of the real and imaginary parts of the forms in (3.41), we need only the real and imaginary parts of the forms which are in the first line, i.e.,

$$\begin{gathered}2\mathcal{M}^*_{1[\boldsymbol{\alpha}-2]'}-i\mathcal{M}^*_{1'[\boldsymbol{\alpha}-1]'}+\mathbb{R}e\big(\mathcal{L}^*_{1\boldsymbol{\alpha}}\big)\\ 2\mathcal{M}^*_{1[\boldsymbol{\alpha}-2]'}-i\mathcal{M}^*_{1'[\boldsymbol{\alpha}-1]'}+i\mathbb{I}m\big(\mathcal{L}^*_{1\boldsymbol{\alpha}}\big)+\mathcal{M}^*_{1\boldsymbol{\alpha}'}.\end{gathered}$$

Observing also that the first two expressions in (3.42) correspond to one-forms that are real, we conclude that $\mathfrak{F}_{2n+1}/\mathfrak{F}_{2n}$ can be generated, over the real numbers, by using only $12n$ one-forms, and thus

$$v_{2n+1}\le 12n.\tag{3.43}$$

Furthermore, we obtain the relations

$$\begin{cases}\mathcal{V}^*_{11\boldsymbol{\alpha}'}\equiv-\mathbb{I}m\big(\mathcal{V}^*_{11'[\boldsymbol{\alpha}+1]'}\big)\\ \mathcal{V}^*_{11\boldsymbol{\alpha}}\ \equiv-\overline{\mathcal{V}^*_{11'\boldsymbol{\alpha}'}}\\ \mathcal{V}^*_{1'1'\boldsymbol{\alpha}'}\equiv-\mathbb{I}m\big(\mathcal{V}^*_{11'[\boldsymbol{\alpha}+1]'}\big)-2i\mathcal{V}^*_{11'[\boldsymbol{\alpha}-1]'}\\ \mathcal{M}^*_{1\boldsymbol{\alpha}}\equiv i\mathbb{R}e\big(\mathcal{L}^*_{1\boldsymbol{\alpha}}\big)\\ \mathcal{M}^*_{1\boldsymbol{\alpha}'}\equiv\overline{\mathcal{L}^*_{1\boldsymbol{\alpha}}}\\ \mathcal{M}^*_{1'\boldsymbol{\alpha}'}\equiv-2i\overline{\mathcal{L}^*_{1[\boldsymbol{\alpha}-1]}}-i\mathbb{R}e\big(\mathcal{L}^*_{1[\boldsymbol{\alpha}+1]}\big)\\ \mathcal{L}^*_{1\boldsymbol{\alpha}'}\equiv i\mathbb{R}e\big(\mathcal{L}^*_{1[\boldsymbol{\alpha}+1]}\big)\\ \mathcal{C}^*_{\boldsymbol{\alpha}}\equiv i\mathbb{R}e\big(\mathcal{C}^*_{[\boldsymbol{\alpha}+1]'}\big)\\ \mathcal{H}^*_{\boldsymbol{\alpha}}\equiv-i\mathcal{C}^*_{\boldsymbol{\alpha}'}\\ \mathcal{H}^*_{\boldsymbol{\alpha}'}\equiv-2\mathcal{C}^*_{[\boldsymbol{\alpha}-1]'}-\mathbb{R}e\big(\mathcal{C}^*_{[\boldsymbol{\alpha}+1]'}\big)\end{cases}\qquad \mod \mathfrak{F}_{2n+1}\tag{3.44}$$

By (3.44), the quotient space $\mathfrak{F}_{2n+2}/\mathfrak{F}_{2n+1}$ is generated by the real and imaginary parts of

$$\mathcal{V}^*_{11'\boldsymbol{\alpha}'},\qquad \mathcal{L}^*_{1\boldsymbol{\alpha}},\qquad \mathcal{C}^*_{\boldsymbol{\alpha}'},\qquad i\mathcal{R}^*+\mathcal{Q}^*,\qquad \mathcal{P}^*+\overline{\mathcal{P}}^*$$

and therefore,

$$v_{2n+2}\le 6n+3.\tag{3.45}$$

The quotient space $\mathfrak{F}_{2n+3}/\mathfrak{F}_{2n+2}$ is generated by the real and imaginary parts of the one-forms:

$$\mathcal{P}^* - \overline{\mathcal{P}}^*; \qquad \mathcal{R}^*;$$

$$\begin{aligned}\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} + \overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}1} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}1'} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[3]} - i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[3]'}&\\ \equiv 2iY_{111[\boldsymbol{\alpha}-3]} + 2Y_{111[\boldsymbol{\alpha}-1]} + 2i\mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}+1]}\big);&\end{aligned}$$

$$\begin{aligned}&-i\mathcal{V}^*_{\boldsymbol{\alpha\beta\gamma'}} + i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'\gamma'}}} - \mathcal{V}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]} - \overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta'}[\boldsymbol{\gamma}+1]}} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}1'} - i\mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}1}\\ &+\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]1'} - \mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1]'1} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma}[3]} - \mathcal{S}^*_{\boldsymbol{\alpha\beta\gamma'}[3]'} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta}[\boldsymbol{\gamma}+1][3]'} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta}[3][\boldsymbol{\gamma}+1]'}\\ &\qquad\equiv -2Y_{111[\boldsymbol{\alpha}-3]} + 2iY_{111[\boldsymbol{\alpha}-1]} - 2i\mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}-1]}\big) + 2iY_{111[\boldsymbol{\alpha}+1]};\end{aligned}$$

$$\begin{aligned}\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}} - \overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma'}}} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}1} + \overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}1}} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[3]} + i\overline{-S^*_{\boldsymbol{\alpha'\beta\gamma}[3]}}&\\ \equiv\ 2i\Big(\mathbb{R}e\big(Y_{111\boldsymbol{\alpha}}\big) - \mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}-2]}\big)\Big);&\end{aligned}$$

$$\begin{aligned}&-i\mathcal{V}^*_{\boldsymbol{\alpha\beta'\gamma'}} - i\overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta\gamma'}}} - \mathcal{V}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]} + \overline{\mathcal{V}^*_{\boldsymbol{\alpha'\beta}[\boldsymbol{\gamma}+1]}} - i\overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}1}} - i\mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}1} + \mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]1'}\\ &\quad - \mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1]'1} - \mathcal{S}^*_{\boldsymbol{\alpha\beta'\gamma}[3]} + \overline{\mathcal{S}^*_{\boldsymbol{\alpha'\beta\gamma}[3]}} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[\boldsymbol{\gamma}+1][3]'} + i\mathcal{S}^*_{\boldsymbol{\alpha\beta'}[3][\boldsymbol{\gamma}+1]'}\\ &\qquad\equiv -2i\Big(\mathbb{R}e\big(Y_{111[\boldsymbol{\alpha}-2]}\big) + \mathbb{I}m\big(Y_{111\boldsymbol{\alpha}}\big) + \mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}+2]}\big)\Big).\end{aligned}$$

It is easy to observe that we can choose as generators the $2n+2$ one-forms

$$\begin{gathered}\mathcal{P}^* - \overline{\mathcal{P}}^*, \quad \mathcal{R}^*, \quad \mathbb{R}e\big(Y_{111\boldsymbol{\alpha}}\big) - \mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}-2]}\big),\\ \mathbb{R}e\big(Y_{111[\boldsymbol{\alpha}-2]}\big) + \mathbb{I}m\big(Y_{111\boldsymbol{\alpha}}\big) + \mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}+2]}\big),\end{gathered}$$

and thus

$$v_{2n+3} \le 2n+2. \tag{3.46}$$

Since the system of equations

$$\mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}-4]}\big) + \mathbb{I}m\big(Y_{111\boldsymbol{\alpha}}\big) + \mathbb{I}m\big(Y_{111[\boldsymbol{\alpha}+2]}\big) \equiv 0 \mod \mathfrak{F}_{2n+3} \tag{3.47}$$

is non-degenerate (cf. the technical Lemma 3.2 below), we obtain that $Y_{111\boldsymbol{\alpha}} \in \mathfrak{F}_{2n+3}$ and thus $\mathfrak{F}_{2n+3}$ is just the free vector space generated by the real and imaginary parts of all the one forms $\mathcal{S}^*_{\alpha\beta\gamma\delta}$, $\mathcal{V}^*_{\alpha\beta\gamma}$, $\mathcal{L}^*_{\alpha\beta}$, $\mathcal{M}^*_{\alpha\beta}$, $\mathcal{C}^*_{\alpha}$, $\mathcal{H}^*_{\alpha}$, $\mathcal{P}^*$, $\mathcal{Q}^*$, $\mathcal{R}^*$. Therefore, $\dim \mathfrak{F}_{2n+3} = d_2$, where $d_2$ is given by (2.44)). We have also

$$\mathfrak{F}_{(2n+3)} = \mathfrak{F}_{(2n+4)} = \cdots = \mathfrak{F}_{d_1}$$

and thus $v_{2n+4} = v_{2n+5} = \ldots = v_{d_1} = 0$, i.e., non-zero characters may appear only among $v_2, \ldots, v_{2n+3}$.

Since $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \cdots \subset \mathfrak{F}_{2n} \subset \mathfrak{F}_{2n+1} \subset \mathfrak{F}_{2n+2} \subset \mathfrak{F}_{2n+1}$, we obtain, by using the inequalities (3.36), (3.43),(3.45) and (3.46), that

$$d_2 = \dim \mathfrak{F}_{2n+3} = \dim \mathfrak{F}_n + \dim \frac{\mathfrak{F}_{n+1}}{\mathfrak{F}_n} + \cdots + \dim \frac{\mathfrak{F}_{2n+3}}{\mathfrak{F}_{2n+2}}$$

$$= \dim \mathfrak{F}_n \; + \; v_{n+1} \; + \; \dots \; + \; v_{2n+3}$$

$$\leq \; \frac{1}{24} n(n-1)(11n^2 + 61n + 86)$$

$$+ \; \sum_{\boldsymbol{\lambda}=1}^{n} \frac{1}{2}(n+\boldsymbol{\lambda}-1)(n-\boldsymbol{\lambda}+4)(n-\boldsymbol{\lambda}+5) \; + \; 12n \; + \; (6n+3) \; + \; (2n+2).$$

Computing the sum of the terms on the RHS of the above inequality produces again the number $d_2$. This implies that each of the inequalities (3.36), (3.43),(3.45), (3.46) is actually an equality and thus, we have shown

$$\begin{cases} v_{\boldsymbol{\lambda}} = \frac{1}{2}(\boldsymbol{\lambda}-1)(\boldsymbol{\lambda}-2n-4)(\boldsymbol{\lambda}-2n-5) \\ v_{n+\boldsymbol{\lambda}} \; = \; \frac{1}{2}(n+\boldsymbol{\lambda}-1)(n-\boldsymbol{\lambda}+4)(n-\boldsymbol{\lambda}+5) \\ v_{2n+1} \; = \; 12n \\ v_{2n+2} \; = \; 6n+3 \\ v_{2n+3} \; = \; 2n+2 \\ v_{2n+4} \; = \; v_{2n+5} \; = \; \dots \; = \; v_{d_1} \; = \; 0. \end{cases} \tag{3.48}$$

**3.6. A technical lemma.** Here we give a proof to an algebraic lemma which is used in Section 3.5 to show that the system of equations (3.47) yields

$$\mathbb{I}m\big(Y_{111\boldsymbol{\alpha}}\big) \equiv 0 \mod \mathfrak{F}_{2n+3}.$$

**Lemma 3.2.** *Let $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ be the least residue system modulo $n$. If $f : \mathbb{Z}_n \to \mathbb{C}$ is any function satisfying*

$$f(k) + f(k+4) + f(k+6) = 0, \qquad \forall k \in \mathbb{Z}_n \tag{3.49}$$

*then, necessarily, $f = 0$.*

**Proof.** We consider the values $f(1), \dots, f(n)$ as unknown variables $x_1, \dots,$ $x_n$. For each $k \in \mathbb{N}$, we let

$$Q_{[k]} \overset{def}{=} x_{[k]} + x_{[k+4]} + x_{[k+6]},$$

where, by following the conventions adopted in 3.1, we use indices enclosed in square brackets to indicate that their values are considered modulo $n$. Then, in order to proof the lemma, we need to show that the system of linear equations

$$Q_1 = Q_2 = \cdots = Q_n = 0 \tag{3.50}$$

is non-degenerate.

Let us define the sequence of numbers $a_1, \dots, a_k, \dots$ by the recurrence relation

$$a_k + a_{k+1} + a_{k+3} = 0, \qquad a_1 = 1, \; a_2 = 0, \; a_3 = -1. \tag{3.51}$$

Then a small combinatorial calculation shows that, for each $m \in \mathbb{N}$, we have the identity

$$\sum_{k=1}^{m} a_k Q_{[2k-1]} = x_1 - a_{m+1}x_{[2m+1]} - a_{m+2}x_{[2m+3]} + a_m x_{[2m+5]} \tag{3.52}$$

and, similarly,

$$\begin{aligned} \sum_{k=1}^{m} a_k Q_{[2k+1]} &= x_3 - a_{m+1}x_{[2m+3]} - a_{m+2}x_{[2m+5]} + a_m x_{[2m+7]}, \\ \sum_{k=1}^{m} a_k Q_{[2k+3]} &= x_5 - a_{m+1}x_{[2m+5]} - a_{m+2}x_{[2m+7]} + a_m x_{[2m+9]}. \end{aligned} \tag{3.53}$$

Setting $m = n$ into (3.52), we obtain that (3.50) yields the equation

$$(1 - a_{n+1})x_1 - a_{n+2}x_3 - a_n x_5 = 0. \tag{3.54}$$

Similarly, setting $m = n-1$ into the first equation of (3.53), and $m = n-2$ into the second, we get, respectively,

$$\begin{aligned} -a_n x_1 + (1 - a_{n+1})x_3 - a_{n-1}x_5 &= 0 \\ -a_{n-1}x_1 - a_n x_3 + (1 + a_{n-2})x_5 &= 0. \end{aligned} \tag{3.55}$$

Next we show that the determinant

$$\begin{aligned} \begin{vmatrix} 1 - a_{n+1} & -a_{n+2} & -a_n \\ -a_n & 1 - a_{n+1} & -a_{n-1} \\ -a_{n-1} & -a_n & 1 + a_{n-2} \end{vmatrix} &= a_n^3 - 2a_n a_{n-1} a_{n+1} \\ -a_n a_{n-2} a_{n+2} + a_{n-1}^2 a_{n+2} &+ a_{n-2} a_{n+1}^2 + 2a_n a_{n-1} \\ -a_n a_{n+2} - 2a_{n-2} a_{n+1} &+ a_{n+1}^2 + a_{n-2} - 2a_{n+1} + 1 \end{aligned} \tag{3.56}$$

is never vanishing. Indeed, consider the three different roots $z_1$, $z_2$, $z_3$ of the polynomial $z^3 + z + 1 = 0$ and take $c_1$, $c_2$, $c_3$ to be the unique complex numbers satisfying

$$\begin{cases} c_1 z_1 + c_2 z_2 + c_3 z_3 = 1 \\ c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2 = 0 \\ c_1 z_1^3 + c_2 z_2^3 + c_3 z_3^3 = -1. \end{cases}$$

Then, the solution of the recurrence relation(3.51) has the form $a_k = c_1 z_1^k + c_2 z_2^k + c_3 z_3^k$. Substituting back into (3.56) and using the Vieta's formulae, we obtain that

$$\begin{vmatrix} 1 - a_{n+1} & -a_{n+2} & -a_n \\ -a_n & 1 - a_{n+1} & -a_{n-1} \\ -a_{n-1} & -a_n & 1 + a_{n-2} \end{vmatrix} = (z_1^n - 1)(z_2^n - 1)(z_3^n - 1),$$

which is a never vanishing number, since non of the roots of the polynomial $z^3 + z + 1 = 0$ has a unite norm. Therefore, the linear equations (3.54) and (3.55) have a unique solution $x_1 = x_3 = x_5 = 0$. Since the system (3.50) is invariant under cyclic permutations of the indices of its variables, this is enough to conclude that it is non-degenerate. $\square$

**3.7. Main theorem.** Now, we are in a position to check that the Cartan's test (cf. (2.7)) is satisfied for our exterior differential system. Indeed, using (3.48), we compute

$$\sum_{\boldsymbol{\lambda}=1}^{n}\Big(\boldsymbol{\lambda}\, v_{\boldsymbol{\lambda}}+(n+\boldsymbol{\lambda})v_{n+\boldsymbol{\lambda}}\Big)+\,(2n+1)v_{2n+1}\,+\,(2n+2)v_{2n+2}\,+\,(2n+3)v_{2n+3}$$
$$=\ \frac{2}{15}(2n+5)(2n+3)(n+3)(n+2)(n+1). \quad (3.57)$$

The number on the RHS above is equal to the constant $D$ determined by (2.42). Therefore, the system is in involution.

**Theorem 3.3.** *Assume we are given some arbitrary complex numbers*

$$\begin{gathered}\mathcal{S}^\circ_{\alpha\beta\gamma\delta}, \mathcal{V}^\circ_{\alpha\beta\gamma},\ \mathcal{L}^\circ_{\alpha\beta},\ \mathcal{M}^\circ_{\alpha\beta},\ \mathcal{C}^\circ_{\alpha},\ \mathcal{H}^\circ_{\alpha},\ \mathcal{P}^\circ,\ \mathcal{Q}^\circ,\ \mathcal{R}^\circ\\ \mathcal{A}^\circ_{\alpha\beta\gamma\delta\epsilon}, \mathcal{B}^\circ_{\alpha\beta\gamma\delta},\ \mathcal{C}^\circ_{\alpha\beta\gamma\delta},\ \mathcal{D}^\circ_{\alpha\beta\gamma},\ \mathcal{E}^\circ_{\alpha\beta\gamma},\ \mathcal{F}^\circ_{\alpha\beta\gamma},\ \mathcal{G}^\circ_{\alpha\beta},\ \mathcal{X}^\circ_{\alpha\beta},\ \mathcal{Y}^\circ_{\alpha\beta},\ \mathcal{Z}^\circ_{\alpha\beta}, \quad (3.58)\\ (\mathcal{N}^\circ_1)_\alpha,\ (\mathcal{N}^\circ_2)_\alpha,\ (\mathcal{N}^\circ_3)_\alpha,\ (\mathcal{N}^\circ_4)_\alpha,\ (\mathcal{N}^\circ_5)_\alpha,\ \mathcal{U}^\circ_s,\ \mathcal{W}^\circ_s\end{gathered}$$

*that depend totally symmetrically on the indices* $1 \le \alpha,\beta,\gamma,\delta \le 2n$ *and satisfy the relations*

$$\begin{cases}(\mathrm{j}\mathcal{S}^\circ)_{\alpha\beta\gamma\delta} = \mathcal{S}^\circ_{\alpha\beta\gamma\delta}\\ (\mathrm{j}\mathcal{L}^\circ)_{\alpha\beta} = \mathcal{L}^\circ_{\alpha\beta}\\ \overline{\mathcal{R}^\circ} = \mathcal{R}^\circ.\end{cases}.$$

*Then, there exists a real analytic qc structure defined in a neighborhood* $\Omega$ *of* $0 \in R^{4n+3}$ *such that for some point* $u \in P_1$ *with* $\pi_o(\pi_1(u)) = 0$ *(here, we keep the notation* $\pi_1 : P_1 \to P_0$ *and* $\pi_o : P_o \to \Omega$ *for the naturally associated to the qc structure of* $\Omega$ *principle bundles, as defined in section Section* 2.5*), the curvature functions* (2.18) *and their covariant derivatives* (2.40) *take at* $u$ *values given by the corresponding complex numbers* (3.58)*.*

*Furthermore, the generality of the real analytic qc structures in* $4n+3$ *dimensions is given by* $2n+2$ *real analytic functions of* $2n+3$ *variables.*

**Proof.** By the computation (3.57), we have shown that the Cartan's test is satisfied at the origin $o \in N$ (cf. (2.45)) for the chosen integral element $E \subset T_oN$ which we have determined by the equations (3.4). In order to prove the theorem, we need to extend this computation to a slightly more general situation. Namely, let us consider the point

$$p \overset{def}{=} \Big(\mathrm{id}, 0, (\mathcal{S}^\circ_{\alpha\beta\gamma\delta}, \mathcal{V}^\circ_{\alpha\beta\gamma},\ \mathcal{L}^\circ_{\alpha\beta},\ \mathcal{M}^\circ_{\alpha\beta},\ \mathcal{C}^\circ_{\alpha},\ \mathcal{H}^\circ_{\alpha},\ \mathcal{P}^\circ,\ \mathcal{Q}^\circ,\ \mathcal{R}^\circ)\Big) \in N$$

and define the one-forms

$$\begin{aligned}
\widehat{\mathcal{S}^*}_{\alpha\beta\gamma\delta} \overset{def}{=} \mathcal{S}^*_{\alpha\beta\gamma\delta} - \Big\{ & \mathcal{A}^\circ_{\alpha\beta\gamma\delta\epsilon}\, \theta^\epsilon - \pi^\sigma_{\bar\epsilon}\, (\mathrm{j}\mathcal{A}^\circ)_{\alpha\beta\gamma\delta\sigma}\, \theta^{\bar\epsilon} + \Big( \mathcal{B}^\circ_{\alpha\beta\gamma\delta} + (\mathrm{j}\mathcal{B}^\circ)_{\alpha\beta\gamma\delta} \Big) \eta_1 \\
& + i\mathcal{C}^\circ_{\alpha\beta\gamma\delta} \big( \eta_2 + i\eta_3 \big) - i(\mathrm{j}\mathcal{C}^\circ)_{\alpha\beta\gamma\delta} \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{V}^*}_{\alpha\beta\gamma} \overset{def}{=} \mathcal{V^*}_{\alpha\beta\gamma} - \Big\{ & \mathcal{C}^\circ_{\alpha\beta\gamma\epsilon}\, \theta^\epsilon + \pi^\sigma_{\bar\epsilon}\, \mathcal{B}^\circ_{\alpha\beta\gamma\sigma}\, \theta^{\bar\epsilon} + \mathcal{D}^\circ_{\alpha\beta\gamma}\eta_1 + \mathcal{E}^\circ_{\alpha\beta\gamma} \big( \eta_2 + i\eta_3 \big) \\
& + \mathcal{F}^\circ_{\alpha\beta\gamma} \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{L}^*}_{\alpha\beta} \overset{def}{=} \mathcal{L^*}_{\alpha\beta} - \Big\{ & - (\mathrm{j}\mathcal{F}^\circ)_{\alpha\beta\epsilon}\, \theta^\epsilon - \pi^\sigma_{\bar\epsilon}\, \mathcal{F}^\circ_{\alpha\beta\sigma}\, \theta^{\bar\epsilon} + i\Big( (\mathrm{j}\mathcal{Z}^\circ)_{\alpha\beta} - \mathcal{Z}^\circ_{\alpha\beta} \Big) \eta_1 \\
& + i\mathcal{G}^\circ_{\alpha\beta} \big( \eta_2 + i\eta_3 \big) - i(\mathrm{j}\mathcal{G}^\circ)_{\alpha\beta} \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{M}^*}_{\alpha\beta} \overset{def}{=} \mathcal{M^*}_{\alpha\beta} - \Big\{ & - \mathcal{E}^\circ_{\alpha\beta\epsilon}\, \theta^\epsilon + \pi^\sigma_{\bar\epsilon} \Big( (\mathrm{j}\mathcal{F}^\circ)_{\alpha\beta\sigma} - i\mathcal{D}^\circ_{\alpha\beta\sigma} \Big)\, \theta^{\bar\epsilon} + \mathcal{X}^\circ_{\alpha\beta}\eta_1 \\
& + \mathcal{Y}^\circ_{\alpha\beta} \big( \eta_2 + i\eta_3 \big) + \mathcal{Z}^\circ_{\alpha\beta} \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{C}^*}_{\alpha} \overset{def}{=} \mathcal{C^*}_{\alpha} - \Big\{ & \mathcal{G}^\circ_{\alpha\epsilon}\, \theta^\epsilon - i\pi^\sigma_{\bar\epsilon}\, \mathcal{Z}^\circ_{\alpha\sigma}\, \theta^{\bar\epsilon} \\
& + (\mathcal{N}^\circ_1)_\alpha \eta_1 + (\mathcal{N}^\circ_2)_\alpha \big( \eta_2 + i\eta_3 \big) + (\mathcal{N}^\circ_3)_\alpha \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{H}^*}_{\alpha} \overset{def}{=} \mathcal{H^*}_{\alpha} - \Big\{ & - \mathcal{Y}^\circ_{\alpha\epsilon}\, \theta^\epsilon + i\pi^\sigma_{\bar\epsilon} \big( \mathcal{G}^\circ_{\alpha\sigma} - \mathcal{X}^\circ_{\alpha\sigma} \big) \theta^{\bar\epsilon} + (\mathcal{N}^\circ_4)_\alpha \eta_1 \\
& + (\mathcal{N}^\circ_5)_\alpha \big( \eta_2 + i\eta_3 \big) + \Big( (\mathcal{N}^\circ_1)_\alpha + i\pi^{\bar\sigma}_\alpha (\mathcal{N}^\circ_3)_{\bar\sigma} \Big) \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{R}^*} \overset{def}{=} \mathcal{R}^* - \Big\{ & 4\pi^{\bar\sigma}_\epsilon (\mathcal{N}^\circ_3)_{\bar\sigma}\, \theta^\epsilon + 4\pi^\sigma_{\bar\epsilon} (\mathcal{N}^\circ_3)_\sigma\, \theta^{\bar\epsilon} + i\big( \mathcal{U}^\circ_3 - \overline{\mathcal{U}^\circ}_3 \big) \eta_1 \\
& - i\big( \mathcal{U}^\circ_1 + \mathcal{W}^\circ_3 \big) \big( \eta_2 + i\eta_3 \big) + i\big( \overline{\mathcal{U}^\circ}_1 + \overline{\mathcal{W}^\circ}_3 \big) \big( \eta_2 - i\eta_3 \big) \Big\} \\
\widehat{\mathcal{P}^*} \overset{def}{=} \mathcal{P}^* - \Big\{ & - 4(\mathcal{N}^\circ_2)_\epsilon\, \theta^\epsilon - 4\Big( (\mathcal{N}^\circ_3)_{\bar\epsilon} + i\pi^\sigma_{\bar\epsilon} (\mathcal{N}^\circ_1)_\sigma \Big)\, \theta^{\bar\epsilon} + \mathcal{U}^\circ_1 \eta_1 \\
& + \mathcal{U}^\circ_2 \big( \eta_2 + i\eta_3 \big) + \mathcal{U}^\circ_3 \big( \eta_2 - i\eta_3 \big) \Big\}
\end{aligned}$$

$$\widehat{\mathcal{Q}^*} \overset{def}{=} \mathcal{Q}^* - \Big\{ 4(\mathcal{N}_5^\circ)_\epsilon\, \theta^\epsilon + 4i\pi_\epsilon^\sigma \Big( (\mathcal{N}_2^\circ)_\sigma + (\mathcal{N}_4^\circ)_\sigma \Big)\, \theta^{\bar\epsilon} + \mathcal{W}_1^\circ \eta_1 + \mathcal{W}_2^\circ \big(\eta_2 + i\eta_3\big) + \mathcal{W}_3^\circ \big(\eta_2 - i\eta_3\big) \Big\}.$$

Since we have used here the formulae (2.41), we know (by Proposition 4.3 from [12]) that the three-forms $\Delta_{\alpha\beta}$, $\Delta_\alpha$ and $\Psi_s$ given by (2.46), (2.47), (2.48) and (2.49) on $N$ will remain the same if replacing everywhere in the respective expressions the one-forms

$$\mathcal{S}^*_{\alpha\beta\gamma\delta},\ \mathcal{V}^*_{\alpha\beta\gamma},\ \mathcal{L}^*_{\alpha\beta},\ \mathcal{M}^*_{\alpha\beta},\ \mathcal{C}^*_{\alpha},\ \mathcal{H}^*_{\alpha},\ \mathcal{P}^*,\ \mathcal{Q}^*,\ \mathcal{R}^*$$

by the one-forms

$$\widehat{\mathcal{S}^*}_{\alpha\beta\gamma\delta},\ \widehat{\mathcal{V}^*}_{\alpha\beta\gamma},\ \widehat{\mathcal{L}^*}_{\alpha\beta},\ \widehat{\mathcal{M}^*}_{\alpha\beta},\ \widehat{\mathcal{C}^*}_{\alpha},\ \widehat{\mathcal{H}^*}_{\alpha},\ \widehat{\mathcal{P}^*},\ \widehat{\mathcal{Q}^*},\ \widehat{\mathcal{R}^*}. \tag{3.59}$$

Therefore, if we repeat our computation from above for the new integral element $\widehat{E} \subset T_pN$, which is now determined by the vanishing of (3.59), we will end up with the same result: The character sequence of $\mathcal{I}$ will be given again by the formulae (3.48) and the Cartan's test will remain satisfied. Thus $\widehat{E}$ must be again a Cartan-ordinary element of $\mathcal{I}$ and the Theorem follows by the Cartan's Third Theorem, as explained in Section 2.1. □

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