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Evolution of the first eigenvalue of weighted p-Laplacian along the Ricci-Bourguignon flow

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Abstract. Let M be an n-dimensional closed Riemannian manifold with metric g, $d\mu = e^{-\phi(x)}d\nu$ be the weighted measure and $\Delta_{p,\phi}$ be the weighted p-Laplacian. In this article we will investigate monotonicity for the first eigenvalue problem of the weighted p-Laplace operator acting on the space of functions along the Ricci-Bourguignon flow on closed Riemannian manifolds. We find the first variation formula for the eigenvalues of the weighted p-Laplacian on a closed Riemannian manifold evolving by the Ricci-Bourguignon flow and we obtain various monotonic quantities. At the end we find some applications in 2-dimensional and 3-dimensional manifolds and give an example.

Contents

1. Introduction

A smooth metric measure space is a triple $(M, g, d\mu)$, where g is a metric, $d\mu = e^{-\phi(x)}d\nu$ is the weighted volume measure on (M, g) related to function $\phi \in C^\infty(M)$ and $d\nu$ is the Riemannian volume measure. Such spaces have been used more widely in the work of mathematicians, for instance, Perelman used it in [13]. Let M be an n-dimensional closed Riemannian manifold with metric g.

Over the last few years the geometric flows as the Ricci-Bourguignon flow have been a topic of active research interest in both mathematics and physics. A geometric flow is an evolution of a geometric structure under a differential equation related to a functional on a manifold, usually associated

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with some curvature. The family $g(t)$ of Riemannian metrics on M is called a Ricci-Bourguignon flow when it satisfies the equations

$$\frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), \tag{1.1}$$

with the initial condition

$$g(0) = g_0$$

where Ric is the Ricci tensor of $g(t)$, R is the scalar curvature and ρ is a real constant. When $\rho = 0$, $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$ and $\rho = \frac{1}{2(n-1)}$, the tensor $Ric - \rho Rg$ corresponds to the Ricci tensor, Einstein tensor, the traceless Ricci tensor and Schouten tensor respectively. In fact the Ricci-Bourguignon flow is a system of partial differential equations which was introduced by Bourguignon for the first time in 1981 (see [3]). Short time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ have been shown by Catino et al. in [6] for $\rho < \frac{1}{2(n-1)}$. When $\rho = 0$, the Ricci-Bourguignon flow is the Ricci flow.
Let $f : M \to \mathbb{R}$, $f \in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. For $p \in [1, +\infty)$, the p-Laplacian of f defined as

$$\Delta_p f = div(|\nabla f|^{p-2}\nabla f) = |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}(Hessf)(\nabla f, \nabla f). \tag{1.2}$$

The Witten-Laplacian is defined by $\Delta_\phi = \Delta - \nabla\phi.\nabla$, which is a symmetric diffusion operator on $L^2(M, \mu)$ and is self-adjoint. Now, for $p \in [1, +\infty)$ and any smooth function f on M, we define the weighted p-Laplacian on M by

$$\Delta_{p,\phi} f = e^{\phi} div\left(e^{-\phi}|\nabla f|^{p-2}\nabla f\right) = \Delta_p f - |\nabla f|^{p-2}\nabla\phi.\nabla f. \tag{1.3}$$

In the weighted p-Laplacian when ϕ is a constant function, the weighted p-Laplace operator is just the p-Laplace operator and when $p = 2$, the weighted p-Laplace operator is the Witten-Laplace operator.

Let Λ satisfies in $-\Delta_{p,\phi} f = \Lambda|f|^{p-2}f$, for some $f \in W^{1,p}(M)$, in this case we say Λ is an eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ at time $t \in [0, T)$. Notice that Λ equivalently satisfies in

$$-\int_M f\Delta_{p,\phi} f d\mu = \Lambda \int_M |f|^p d\mu, \tag{1.4}$$

where $d\mu = e^{-\phi(x)}d\nu$ and $d\nu$ is the Riemannian volume measure Using integration by parts, it results that

$$\int_M |\nabla f|^p \, d\mu = \Lambda \int_M |f|^p d\mu, \tag{1.5}$$

in above equation, $f(x, t)$ called eigenfunction corresponding to eigenvalue $\Lambda(t)$. The first non-zero eigenvalue $\lambda(t) = \lambda(M, g(t), d\mu)$ is defined as follows

$$\lambda(t) = \inf_{0 \neq f \in W_0^{1,p}(M)} \left\{ \int_M |\nabla f|^p d\mu : \quad \int_M |f|^p d\mu = 1 \right\}, \tag{1.6}$$

where $W_0^{1,p}(M)$ is the completion of $C_0^\infty(M)$ with respect to the Sobolev norm

$$||f||_{W^{1,p}} = \left(\int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu\right)^{\frac{1}{p}}. \tag{1.7}$$

The eigenvalue problem for weighted p-Laplacian has been extensively studied in the literature [14, 15].

The problem of monotonicity of the eigenvalue of geometric operator is a known and an intrinsic problem. Recently many mathematicians study properties of evolution of the eigenvalue of geometric operators (for instance, Laplace, p-Laplace, Witten-Laplace) along various geometric flows (for example, Yamabe flow, Ricci flow, Ricci-Bourguignon flow, Ricci-harmonic flow and mean curvature flow). The main study of evolution of the eigenvalue of geometric operator along the geometric flow began when Perelman in [13] showed that the first eigenvalue of the geometric operator $-4\Delta + R$ is nondecreasing along the Ricci flow, where R is scalar curvature.

Then Cao [5] and Chen et al. [7] extended the geometric operator $-4\Delta+R$ to the operator $-\Delta + cR$ on closed Riemannian manifolds, and investigated the monotonicity of eigenvalues of the operator $-\Delta + cR$ under the Ricci flow and the Ricci-Bourguignon flow, respectively.

Author in [2] studied the monotonicity of the first eigenvalue of Witten-Laplace operator $-\Delta_\phi$ along the Ricci-Bourguignon flow with some assumptions and in [1] investigated the evolution for the first eigenvalue of the p-Laplacian along the Yamabe flow.

In [11] and [10] have been studied the evolution for the first eigenvalue of geometric operator $-\Delta_\phi + \frac{R}{2}$ along the Yamabe flow and the Ricci flow, respectively. For the other recent research in this subject, see [9, 8, 17].

Motivated by the described above works, in this paper, we will study the evolution of the first eigenvalue of the weighted p-Laplace operator whose metric satisfying the Ricci-Bourguignon flow (1.1) and ϕ evolves by $\frac{\partial\phi}{\partial t} = \Delta\phi$ that is $(M^n, g(t), \phi(t))$ satisfying in following system

$$\begin{cases} \frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), & g(0) = g_0, \\ \frac{\partial\phi}{\partial t} = \Delta\phi & \phi(0) = \phi_0, \end{cases} \tag{1.8}$$

where Δ is Laplace operator of metric $g(t)$.

2. Preliminaries

In this section, we will discuss the differentiable (of first nonzero eigenvalue and its corresponding eigenfunction of the weighted p-Laplacian $\Delta_{p,\phi}$ along the flow (1.8). Let M be a closed oriented Riemannian n-manifold and $(M, g(t), \phi(t))$ be a smooth solution of the evolution equations system

(1.8) for $t \in [0, T)$.

In what follows, we assume that $\lambda(t)$ exists and is C^1-differentiable under the flow (1.8) in the given interval $t \in [0, T)$. The first nonzero eigenvalue of weighted p-Laplacian and its corresponding eigenfunction are not known to be C^1-differentiable. For this reason, we apply techniques of Cao [4] and Wu [17] to study the evolution and monotonicity of $\lambda(t) = \lambda(t, f(t))$, where $\lambda(t, f(t))$ and $f(t)$ are assumed to be smooth. For this end, we assume that at time t_0, $f_0 = f(t_0)$ is the eigenfunction for the first eigenvalue $\lambda(t_0)$ of $\Delta_{p,\phi}$. Then we have

$$\int_M |f(t_0)|^p \, d\mu_{g(t_0)} = 1. \tag{2.1}$$

Suppose that

$$h(t) := f_0 \left[\frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))} \right]^{\frac{1}{2(p-1)}}, \tag{2.2}$$

along the Ricci-Bourguignon flow $g(t)$. We assume that

$$f(t) = \frac{h(t)}{(\int_M |h(t)|^p d\mu)^{\frac{1}{p}}}, \tag{2.3}$$

which $f(t)$ is smooth function along the Ricci-Bourguignon flow, satisfied in $\int_M |f|^p d\mu = 1$ and at time t_0, f is the eigenfunction for λ of $\Delta_{p,\phi}$. Therefore, if $\int_M |f|^p d\mu = 1$ and

$$\lambda(t, f(t)) = -\int_M f \Delta_{p,\phi} f d\mu, \tag{2.4}$$

then $\lambda(t_0, f(t_0)) = \lambda(t_0)$.

3. Variation of $\lambda(t)$

In this section, we will find some useful evolution formulas for $\lambda(t)$ along the flow (1.8). We first recall some evolution of geometric structure along the Ricci-Bourguignon flow and then give a useful proposition about the variation of eigenvalues of the weighted p-Laplacian under the flow (1.8). From [6], we have:

Lemma 3.1. *Under the Ricci-Bourguignon flow equation (1.1), we get*

(1) $\frac{\partial}{\partial t} g^{ij} = 2(R^{ij} - \rho R g^{ij})$,

(2) $\frac{\partial}{\partial t}(d\nu) = (n\rho - 1) R d\nu$,

(3) $\frac{\partial}{\partial t}(d\mu) = (-\phi_t + (n\rho - 1)R) d\mu$,

(4) $\frac{\partial}{\partial t}(\Gamma^k_{ij}) = -\nabla_j R^k_i - \nabla_i R^k_j + \nabla^k R_{ij} + \rho(\nabla_j R \delta^k_i + \nabla_i R \delta^k_j - \nabla^k R g_{ij})$,

(5) $\frac{\partial}{\partial t} R = [1 - 2(n-1)\rho] \Delta R + 2|Ric|^2 - 2\rho R^2$,

where R is scalar curvature.

Lemma 3.2. *Let $(M, g(t), \phi(t))$, $t \in [0, T)$ be a solution to the flow (1.8) on a closed oriented Riemannain manifold for $\rho < \frac{1}{2(n-1)}$. Let $f \in C^\infty(M)$ be a smooth function on $(M, g(t))$. Then we have the following evolutions:*

$$\frac{\partial}{\partial t}|\nabla f|^2 = 2R^{ij}\nabla_i f\nabla_j f - 2\rho R|\nabla f|^2 + 2g^{ij}\nabla_i f\nabla_j f_t, \tag{3.1}$$

$$\frac{\partial}{\partial t}|\nabla f|^{p-2} = (p-2)|\nabla f|^{p-4}\{R^{ij}\nabla_i f\nabla_j f - \rho R|\nabla f|^2 + g^{ij}\nabla_i f\nabla_j f_t\}, \tag{3.2}$$

$$\frac{\partial}{\partial t}(\Delta f) = 2R^{ij}\nabla_i\nabla_j f + \Delta f_t - 2\rho R\Delta f - (2-n)\rho\nabla^k R\nabla_k f, \tag{3.3}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_p f) &= 2R^{ij}\nabla_i(Z\nabla_j f) - 2\rho R\Delta_p f + g^{ij}\nabla_i(Z_t\nabla_j f) \\ &\quad + g^{ij}\nabla_i(Z\nabla_j f_t) + \rho(n-2)Zg^{ij}\nabla_i R\nabla_j f,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_{p,\phi} f) &= 2R^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) \\ &\quad - 2\rho R\Delta_{p,\phi} f + \rho(n-2)Zg^{ij}\nabla_i R\nabla_j f - Z_t\nabla\phi.\nabla f \\ &\quad - 2ZR^{ij}\nabla_i\phi\nabla_j f - Z\nabla\phi_t.\nabla f - Z\nabla\phi.\nabla f_t,\end{aligned} \tag{3.5}$$

where $Z := |\nabla f|^{p-2}$ and $f_t = \frac{\partial f}{\partial t}$.

Proof. By direct computation in local coordinates we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^2 &= \frac{\partial}{\partial t}(g^{ij}\nabla_i f\nabla_j f) \\ &= \frac{\partial g^{ij}}{\partial t}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j f_t \\ &= 2R^{ij}\nabla_i f\nabla_j f - 2\rho R|\nabla f|^2 + 2g^{ij}\nabla_i f\nabla_j f_t,\end{aligned}$$

which exactly (3.1). We prove (3.2) by using (3.1) as follows

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|^{p-2} &= \frac{\partial}{\partial t}(|\nabla f|^2)^{\frac{p-2}{2}} \\ &= \frac{p-2}{2}(|\nabla f|^2)^{\frac{p-4}{2}}\frac{\partial}{\partial t}(|\nabla f|^2) \\ &= \frac{p-2}{2}|\nabla f|^{p-4}\left\{2R^{ij}\nabla_i f\nabla_j f - 2\rho R|\nabla f|^2 + 2g^{ij}\nabla_i f\nabla_j f_t\right\} \\ &= (p-2)|\nabla f|^{p-4}\left\{R^{ij}\nabla_i f\nabla_j f - \rho R|\nabla f|^2 + g^{ij}\nabla_i f\nabla_j f_t\right\},\end{aligned}$$

which is (3.2). Now previous lemma and $2\nabla^i R_{ij} = \nabla_j R$ result that

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta f) &= \frac{\partial}{\partial t}[g^{ij}(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k})] \\
&= \frac{\partial g^{ij}}{\partial t}(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}) + g^{ij}(\frac{\partial^2 f_t}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f_t}{\partial x^k}) - g^{ij}\frac{\partial}{\partial t}(\Gamma^k_{ij})\frac{\partial f}{\partial x^k} \\
&= 2R^{ij}\nabla_i\nabla_j f - 2\rho R\Delta f + \Delta f_t - g^{ij}\left\{-\nabla_j R^k_i - \nabla_i R^k_j + \nabla^k R_{ij}\right\}\nabla_k f \\
&- g^{ij}\rho(\nabla_j R\delta^k_i + \nabla_i R\delta^k_j - \nabla^k R g_{ij})\nabla_k f \\
&= 2R^{ij}\nabla_i\nabla_j f + \Delta f_t - 2\rho R\Delta f - (2-n)\rho\nabla^k R\nabla_k f.
\end{aligned}$$

Let $Z = |\nabla f|^{p-2}$ we get

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta_p f) &= \frac{\partial}{\partial t}\big(div(|\nabla f|^{p-2}\nabla f)\big) = \frac{\partial}{\partial t}\big(g^{ij}\nabla_i(Z\nabla_j f)\big) \\
&= \frac{\partial}{\partial t}\big(g^{ij}\nabla_i Z\nabla_j f + g^{ij}Z\nabla_i\nabla_j f\big) \\
&= \frac{\partial g^{ij}}{\partial t}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f + g^{ij}\nabla_i Z\nabla_j f_t \\
&\quad + Z_t\Delta f + Z\frac{\partial}{\partial t}(\Delta f) \\
&= 2R^{ij}\nabla_i Z\nabla_j f - 2\rho R g^{ij}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Z_t\nabla_j f \\
&\quad + g^{ij}\nabla_i Z\nabla_j f_t + Z_t\Delta f \\
&\quad + Z\{2R^{ij}\nabla_i\nabla_j f + \Delta f_t - 2\rho R\Delta f - (2-n)\rho\nabla^k R\nabla_k f\} \\
&= 2R^{ij}\nabla_i(Z\nabla_j f) - 2\rho R\Delta_p f + g^{ij}\nabla_i(Z_t\nabla_j f) \\
&\quad + g^{ij}\nabla_i(Z\nabla_j f_t) + \rho(n-2)Zg^{ij}\nabla_i R\nabla_j f.
\end{aligned}$$

We have $\Delta_{p,\phi} f = \Delta_p f - |\nabla f|^{p-2}\nabla\phi.\nabla f$. Taking derivative with respect to time of both sides of last equation and (3.4) imply that

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta_{p,\phi} f) &= \frac{\partial}{\partial t}(\Delta_p f) - Z\frac{\partial g^{ij}}{\partial t}\nabla_i\phi\nabla_j f - Z_t g^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f \\
&\quad - Zg^{ij}\nabla_i\phi\nabla_j f_t \\
&= 2R^{ij}\nabla_i(Z\nabla_j f) - 2\rho R\Delta_p f + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) \\
&\quad + \rho(n-2)Zg^{ij}\nabla_i R\nabla_j f - 2ZR^{ij}\nabla_i\phi\nabla_j f + 2\rho ZRg^{ij}\nabla_i\phi\nabla_j f \\
&\quad - Z_t g^{ij}\nabla_i\phi\nabla_j f - Zg^{ij}\nabla_i\phi_t\nabla_j f - Zg^{ij}\nabla_i\phi\nabla_j f_t,
\end{aligned}$$

it results (3.5). □

Proposition 3.3. *Let $(M, g(t), \phi(t))$, $t \in [0, T)$ be a solution of the flow (1.8) on the smooth closed oriented Riemannain manifold (M^n, g_0, ϕ_0) for $\rho < \frac{1}{2(n-1)}$. If $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(t)$ under the* *flow (1.8), then*

$$\begin{aligned}\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} &= \lambda(t_0)(1-n\rho)\int_M R|f|^p\,d\mu-(1+\rho p-\rho n)\int_M R|\nabla f|^p d\mu\\ &+p\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu+\lambda(t_0)\int_M(\Delta\phi)|f|^p\,d\mu \qquad (3.6)\\ &-\int_M(\Delta\phi)|\nabla f|^p d\mu.\end{aligned}$$

Proof. Let $f(t)$ be a smooth function where $f(t_0)$ is the corresponding eigenfunction to $\lambda(t_0)=\lambda(t_0,f(t_0))$. $\lambda(t,f(t))$ is a smooth function and taking derivative of both sides $\lambda(t,f(t))=-\int_M f\Delta_{p,\phi}f\,d\mu$ with respect to time, we get

$$\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0}=-\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu. \qquad (3.7)$$

Now by applying condition $\int_M|f|^p d\mu=1$ and the time derivative, we can have

$$\begin{aligned}\frac{\partial}{\partial t}\int_M|f|^p d\mu=0&=\frac{\partial}{\partial t}\int_M|f|^{p-2}f^2 d\mu \qquad (3.8)\\ &=\int_M(p-1)|f|^{p-2}ff_t d\mu+\int_M|f|^{p-2}f\frac{\partial}{\partial t}(fd\mu),\end{aligned}$$

hence

$$\int_M|f|^{p-2}f\left[(p-1)f_t d\mu+\frac{\partial}{\partial t}(fd\mu)\right]=0. \qquad (3.9)$$

On the other hand, using (3.5), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu &= \int_M\frac{\partial}{\partial t}(\Delta_{p,\phi}f)f\,d\mu+\int_M\Delta_{p,\phi}f\,\frac{\partial}{\partial t}(f\,d\mu)\\ &=2\int_M R^{ij}\nabla_i(Z\nabla_j f)f\,d\mu-2\rho\int_M R\Delta_{p,\phi}ff\,d\mu\\ &+\int_M g^{ij}\nabla_i(Z_t\nabla_j f)f\,d\mu+\int_M g^{ij}\nabla_i(Z\nabla_j f_t)f\,d\mu \qquad (3.10)\\ &+\rho(n-2)\int_M Z\nabla R.\nabla ff\,d\mu-\int_M Z_t\nabla\phi.\nabla ff\,d\mu\\ &-\int_M Z\nabla\phi_t.\nabla ff\,d\mu-\int_M Z\nabla\phi.\nabla f_t f\,d\mu\\ &-2\int_M R^{ij}Z\nabla_i\phi\nabla_j ff\,d\mu-\int_M\lambda|f|^{p-2}f\frac{\partial}{\partial t}(fd\mu).\end{aligned}$$

By the application of integration by parts, we can conclude that

$$\int_M g^{ij}\nabla_i(Z_t\nabla_j f)f\,d\mu=-\int_M Z_t|\nabla f|^2 d\mu+\int_M Z_t\nabla f.\nabla\phi f\,d\mu. \qquad (3.11)$$

Similarly, integration by parts implies that

$$\int_M g^{ij}\nabla_i(Z\nabla_j f_t)f\,d\mu = -\int_M Z\nabla f_t.\nabla f\,d\mu + \int_M Z\nabla f_t.\nabla\phi f\,d\mu, \quad (3.12)$$

and

$$\begin{aligned}\int_M R^{ij}\nabla_i(Z\nabla_j f)f\,d\mu &= -\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu + \int_M ZR^{ij}\nabla_j f\nabla_i\phi f\,d\mu \\ &\quad -\int_M Z\nabla_i R^{ij}\nabla_j f f\,d\mu. \quad (3.13)\end{aligned}$$

But, we can write

$$\begin{aligned}2\int_M Z\nabla_i R^{ij}\nabla_j f f\,d\mu &= 2\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_i R_{kl} f\,d\mu = \int_M Zg^{jl}\nabla_j f\nabla_l R f\,d\mu \\ &= -\int_M R\Delta_{p,\phi} f\,f\,d\mu - \int_M R|\nabla f|^p d\mu. \quad (3.14)\end{aligned}$$

Putting (3.14) in (3.13), yields

$$\begin{aligned}2\int_M R^{ij}\nabla_i(Z\nabla_j f)f\,d\mu &= -2\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu + 2\int_M ZR^{ij}\nabla_j f\nabla_i\phi f\,d\mu \\ &\quad -\int_M \lambda R|f|^p\,d\mu + \int_M R|\nabla f|^p d\mu. \quad (3.15)\end{aligned}$$

Now, replacing (3.11), (3.12) and (3.15) in (3.10), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi} f\,d\mu &= -2\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu - \int_M \lambda R|f|^p\,d\mu + \int_M R|\nabla f|^p d\mu \\ &\quad + 2\rho\int_M \lambda R|f|^p d\mu + \rho(n-2)\int_M Z\nabla R.\nabla f f d\mu \\ &\quad -\int_M Z_t|\nabla f|^2 d\mu - \int_M Z\nabla f_t.\nabla f\,d\mu - \int_M Z\nabla\phi_t.\nabla f\,f\,d\mu \quad (3.16) \\ &\quad -\int_M \lambda|f|^{p-2}f\frac{\partial}{\partial t}(f d\mu).\end{aligned}$$

On the other hand of Lemma 3.2, we have

$$Z_t = \frac{\partial}{\partial t}(|\nabla f|^{p-2}) = (p-2)|\nabla f|^{p-4}\{R^{ij}\nabla_i f\nabla_j f - \rho R|\nabla f|^2 + g^{ij}\nabla_i f\nabla_j f_t\}. \quad (3.17)$$

Therefore, putting this into (3.16), we get

$$\begin{aligned}-\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} \quad = \quad & -p\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu+\lambda(t_0)(2\rho-1)\int_M R|f|^p\,d\mu \\ & +(1+\rho p-2\rho)\int_M R|\nabla f|^p d\mu+\rho(n-2)\int_M Z\nabla R.\nabla f f d\mu \\ & -(p-1)\int_M Z\nabla f_t.\nabla f\,d\mu-\int_M Z\nabla\phi_t.\nabla f\,f\,d\mu \\ & -\lambda(t_0)\int_M |f|^{p-2}f\frac{\partial}{\partial t}(fd\mu). \end{aligned} \tag{3.18}$$

Also,

$$\begin{aligned}-(p-1)\int_M Z\nabla f_t.\nabla f\,d\mu &= (p-1)\int_M \nabla(Z\nabla f)f_t\,d\mu-(p-1)\int_M Z\nabla f.\nabla\phi f_t\,d\mu \\ &= (p-1)\int_M f_t\Delta_{p,\phi}f\,d\mu=-(p-1)\int_M \lambda|f|^{p-2}f\,f_t\,d\mu. \end{aligned} \tag{3.19}$$

Then we arrive at

$$\begin{aligned}-\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} = & -p\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu+\lambda(t_0)(2\rho-1)\int_M R|f|^p\,d\mu \\ & +(1+\rho p-2\rho)\int_M R|\nabla f|^p d\mu \\ & +\rho(n-2)\int_M Z\nabla R.\nabla f f d\mu-\int_M Z\nabla\phi_t.\nabla f\,f d\mu \\ & -\lambda(t_0)\int_M |f|^{p-2}f\left((p-1)f_t\,d\mu+\frac{\partial}{\partial t}(fd\mu)\right). \end{aligned} \tag{3.20}$$

Hence, (3.9) yields

$$\begin{aligned}-\frac{\partial}{\partial t}\lambda(t,f(t))|_{t=t_0} \quad = \quad & -p\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu+\lambda(t_0)(2\rho-1)\int_M R|f|^p\,d\mu \\ & +(1+\rho p-2\rho)\int_M R|\nabla f|^p d\mu \\ & +\rho(n-2)\int_M Z\nabla R.\nabla f f d\mu-\int_M Z\nabla\phi_t.\nabla f\,f d\mu. \end{aligned} \tag{3.21}$$

By integration by parts, we get

$$\int_M Z\nabla\phi_t.\nabla f\,f\,d\mu=\int_M \lambda|f|^p(\Delta\phi)\,d\mu-\int_M(\Delta\phi)|\nabla f|^p d\mu \tag{3.22}$$

and

$$\int_M Z\nabla R.\nabla f\,f\,d\mu=\int_M \lambda R|f|^p\,d\mu-\int_M R|\nabla f|^p d\mu. \tag{3.23}$$

Plug in (3.22) and (3.23) into (3.21) imply that (3.6). $\square$

Corollary 3.4. *Let $(M, g(t))$, $t \in [0, T)$, be a solution of the flow (1.1) on the smooth closed oriented Riemannain manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$. If $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(x,t)$ under the Ricci-Bourguignon flow where ϕ is independent of t, then*

$$\begin{aligned}\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} &= \lambda(t_0)(1-n\rho)\int_M R|f|^p\,d\mu - (1+\rho p-\rho n)\int_M R|\nabla f|^p d\mu \\ &\quad + p\int_M ZR^{ij}\nabla_i f\nabla_j f\,d\mu. \end{aligned} \tag{3.24}$$

We can get the evolution for the first eigenvalue of the geometric operator Δ_p under the Ricci-Bourguignon flow (1.1) and along the Ricci flow, which was studied in [17]. Also, in Corollary 3.4, if $p = 2$ then we can obtain the evolution for the first eigenvalue of the Witten-Laplace operator along the the Ricci-Bourguignon flow (1.1), which was investigated in [2].

Theorem 3.5. *Let $(M, g(t), \phi(t))$, $t \in [0, T)$ be a solution of the flow (1.8) on the smooth closed oriented Riemannain manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$. Let $R_{ij} - (\beta R + \gamma\Delta\phi)g_{ij} \geq 0$, $\beta \geq \frac{1+\rho(p-n)}{p}$ and $\gamma \geq \frac{1}{p}$ along the flow (1.8) and $R < \Delta\phi$ in $M \times [0, T)$. Suppose that $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ then*

1. *If $R_{\min}(0) \geq 0$, $\lambda(t)$ is nondecreasing along the Ricci-Bourguignon flow for any $t \in [0, T)$.*
2. *If $R_{\min}(0) > 0$, then the quantity $\lambda(t)(n - 2R_{\min}(0)t)^{\frac{1}{n}}$ is nondecreasing along the Ricci-Bourguignon flow for $T \leq \frac{n}{2R_{\min}(0)}$.*
3. *If $R_{\min}(0) < 0$, then the quantity $\lambda(t)(n - 2R_{\min}(0)t)^{\frac{1}{n}}$ is nondecreasing along the Ricci-Bourguignon flow for any $t \in [0, T)$.*

Proof. According to (3.6) of Proposition 3.3, we have

$$\begin{aligned}\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} &\geq \lambda(t_0)(1-n\rho)\int_M R|f|^p\,d\mu - (1+\rho p-\rho n)\int_M R|\nabla f|^p d\mu \\ &\quad + p\beta\int_M R|\nabla f|^p\,d\mu + p\gamma\int_M (\Delta\phi)|\nabla f|^p d\mu \qquad (3.25)\\ &\quad + \lambda(t_0)\int_M R|f|^p\,d\mu - \int_M (\Delta\phi)|\nabla f|^p d\mu \\ &= \lambda(t_0)(2-n\rho)\int_M R|f|^p\,d\mu + (p\gamma-1)\int_M R|\nabla f|^p d\mu \\ &\quad + [p\beta - (1+\rho p-\rho n)]\int_M R|\nabla f|^p d\mu. \end{aligned}$$

On the other hand, the scalar curvature along the Ricci-Bourguignon flow evolves by

$$\frac{\partial R}{\partial t} = (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2. \tag{3.26}$$

The inequality $|Ric|^2 \geq \frac{R^2}{n}$ yields

$$\frac{\partial R}{\partial t} \geq (1-2(n-1)\rho)\Delta R + 2(\frac{1}{n} - \rho)R^2. \tag{3.27}$$

Since the solution to the corresponding ODE $y' = 2(\frac{1}{n} - \rho)y^2$ with initial value $c = \min_{x\in M} R(0) = R_{\min}(0)$ is

$$\sigma(t) = \frac{nc}{n - 2(1-n\rho)ct}. \tag{3.28}$$

Notice that $\sigma(t)$ defined on $[0, T')$ where $T' = \min\{T, \frac{n}{2(1-n)\rho c}\}$ when $c > 0$ and on $[0, T)$ when $c \leq 0$. Using the maximum principle to (3.27), we have $R_{g(t)} \geq \sigma(t)$. Therefore, (3.25) becomes

$$\frac{d}{dt}\lambda(t, f(t))|_{t=t_0} \geq A\lambda(t_0)\sigma(t_0),$$

where $A = p(\beta+\gamma) - \rho(p+2n)$ and this results that in any sufficiently small neighborhood of t_0 as I_0, we obtain

$$\frac{d}{dt}\lambda(t, f(t)) \geq A\lambda(f, t)\sigma(t).$$

Integrating both sides of the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln\frac{\lambda(t_0, f(t_0))}{\lambda(f(t_1), t_1)} > \ln(\frac{n-2(1-n\rho)ct_1}{n-2(1-n\rho)ct_0})^{\frac{nA}{2(1-n\rho)}}.$$

Since $\lambda(t_0, f(t_0)) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$, we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \ln(\frac{n-2(1-n\rho)ct_1}{n-2(1-n\rho)ct_0})^{\frac{nA}{2(1-n\rho)}},$$

that is, the quantity $\lambda(t)(n-2(1-n\rho)ct)^{\frac{nA}{2(1-n\rho)}}$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)(n - 2(1-n\rho)ct)^{\frac{nA}{2(1-n\rho)}}$ is strictly increasing along the flow (1.8) on $[0, T)$. Now we have,

(1) If $R_{\min}(0) \geq 0$, by the non-negatively of $R_{g(t)}$ preserved along the Ricci-Bourguignon flow hence $\frac{d}{dt}\lambda(t, f(t)) \geq 0$, consequently $\lambda(t)$ is strictly increasing along the flow (1.1) on $[0, T)$.
(2) If $R_{\min}(0) > 0$ then $\sigma(t)$ defined on $[0, T')$, thus the quantity $\lambda(t)(n-2(1-n\rho)ct)^{\frac{nA}{2(1-n\rho)}}$ is nondecreasing along the flow (1.1) on $[0, T')$.
(3) If $R_{\min}(0) < 0$ then $\sigma(t)$ defined on $[0, T')$, thus the quantity $\lambda(t)(n-2(1-n\rho)ct)^{\frac{nA}{2(1-n\rho)}}$ is nondecreasing along the flow (1.1) on $[0, T')$.

□

Theorem 3.6. *Let $(M^n, g(t), \phi(t))$, $t \in [0, T)$ be a solution of the flow (1.8) on a closed Riemannian manifold (M^n, g_0) with $R(0) > 0$ for $\rho < \frac{1}{2(n-1)}$. Let $\lambda(t)$ be the first eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$, then $\lambda(t) \to$*

$+\infty$ *in finite time for* $p \geq 2$ *where* $Ric - \nabla\phi \otimes \nabla\phi \geq \beta R g$ *in* $M \times [0, T)$ *and* $\beta \in [0, \frac{1}{n}]$ *is a constant.*

Proof. The weighted p-Reilly formula on closed Riemannian manifolds (see [16]) as follows

$$\int_M \left[(\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} |Hess\, f|_A^2\right] d\mu$$
$$= \int_M |\nabla f|^{2p-4} (Ric + \nabla^2\phi)(\nabla f, \nabla f)\, d\mu, \quad (3.29)$$

where $f \in C^\infty(M)$ and

$$|Hess\, f|_A^2 = |Hess\, f|^2 + \frac{p-2}{2}\frac{|\nabla|\nabla f|^2 f|^2}{|\nabla f|^2} + \frac{(p-2)^2}{4}\frac{< \nabla f, \nabla|\nabla f|^2 >^2}{|\nabla f|^4}. \quad (3.30)$$

By a straightforward computation, we have the following inequality:

$$\begin{aligned} |\nabla f|^{2p-4}|Hess\, f|_A^2 &\geq \frac{1}{n}\left(\Delta_{p,\phi} f + |\nabla f|^{p-2} < \nabla\phi, \nabla f >\right)^2 \\ &\geq \frac{1}{1+n}(\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4}|\nabla\phi . \nabla f|^2. \quad (3.31)\end{aligned}$$

Recall that $\Delta_{p,\phi} f = -\lambda |f|^{p-2} f$, which implies

$$\int_M (\Delta_{p,\phi} f)^2 d\mu = \lambda^2 \int_M |f|^{2p-2} d\mu. \quad (3.32)$$

Combining (3.31) and (3.32), we can write

$$\int_M \left[(\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4}|Hess\, f|_A^2\right] d\mu$$
$$\leq (1 - \frac{1}{1+n})\lambda^2 \int_M |f|^{2p-2} d\mu + \int_M |\nabla f|^{2p-4} |\nabla\phi . \nabla f|^2 d\mu, \quad (3.33)$$

putting (3.33) in (3.29) yields

$$(1 - \frac{1}{1+n})\lambda^2 \int_M |f|^{2p-2} d\mu + \int_M |\nabla f|^{2p-4}|\nabla\phi . \nabla f|^2 d\mu \geq$$
$$\int_M |\nabla f|^{2p-4} Ric(\nabla f, \nabla f)\, d\mu + \int_M |\nabla f|^{2p-4}\nabla^2\phi(\nabla f, \nabla f)\, d\mu. \quad (3.34)$$

By identifying $\nabla\phi \otimes \nabla\phi(\nabla f, \nabla f)$ with $|\nabla\phi . \nabla f|^2$ (see [12]), we obtain

$$\int_M |\nabla f|^{2p-4}\ \nabla\phi \otimes \nabla\phi(\nabla f, \nabla f)\, d\mu = \int_M |\nabla f|^{2p-4}|\nabla\phi . \nabla f|^2 d\mu. \quad (3.35)$$

Therefore, it and $Ric - \nabla\phi \otimes \nabla\phi \geq \beta R g$ yield that

$$(1 - \frac{1}{1+n})\lambda^2 \int_M |f|^{2p-2} d\mu$$
$$\geq \beta \int_M R|\nabla f|^{2p-2} d\mu + \int_M |\nabla f|^{2p-4}\nabla^2\phi(\nabla f, \nabla f)\, d\mu. \quad (3.36)$$

Now, since ϕ satisfies in $\phi_t = \Delta\phi$, we get

$$|\nabla^2\phi| \geq \frac{1}{\sqrt{n}}|\Delta\phi| = \frac{1}{\sqrt{n}}|\phi_t|. \tag{3.37}$$

Hence,

$$\begin{aligned}(1-\frac{1}{1+n})\lambda^2\int_M |f|^{2p-2}d\mu &\geq \beta\int_M R|\nabla f|^{2p-2}d\mu + \frac{1}{\sqrt{n}}\int_M |\phi_t||\nabla f|^{2p-2}d\mu \\ &\geq (\beta R_{\min}(t) + \frac{1}{\sqrt{n}}\min_{x\in M}|\phi_t|)\int_M |\nabla f|^{2p-2}d\mu.\end{aligned} \tag{3.38}$$

Multiplying $\Delta_{p,\phi}f = -\lambda|f|^{p-2}f$ by $|f|^{p-2}f$ on both sides, we obtain

$$|f|^{p-2}f\Delta_{p,\phi}f = -\lambda|f|^{2p-2}f.$$

Then integrating by parts and using the Hölder inequality for $p > 2$, we obtain

$$\begin{aligned}\lambda\int_M |\nabla f|^{2p-2}d\mu &= -\int_M |f|^{p-2}f\Delta_{p,\phi}f\,d\mu = (p-1)\int_M |\nabla f|^p|f|^{p-2}d\mu \\ &\leq (p-1)\left[\int_M (|\nabla f|^p)^{\frac{2p-2}{p}}d\mu\right]^{\frac{p}{2p-2}}\left[\int_M (|f|^{p-2})^{\frac{2p-2}{p-2}}d\mu\right]^{\frac{p-2}{2p-2}} \\ &= (p-1)\left[\int_M |\nabla f|^{2p-2}d\mu\right]^{\frac{p}{2p-2}}\left[\int_M |f|^{2p-2}d\mu\right]^{\frac{p-2}{2p-2}}.\end{aligned}$$

So, we can conclude that

$$\int_M |\nabla f|^{2p-2}d\mu \geq \left(\frac{\lambda}{p-1}\right)^{\frac{2p-2}{p}}\int_M |f|^{2p-2}d\mu$$

which implies

$$\begin{aligned}&\left(1-\frac{1}{1+n}\right)\lambda^2\int_M |f|^{2p-2}d\mu \\ &\qquad\geq \left(\beta R_{\min}(t) + \frac{1}{\sqrt{n}}\min_{x\in M}|\phi_t|\right)\left(\frac{\lambda}{p-1}\right)^{\frac{2p-2}{p}}\int_M |f|^{2p-2}d\mu,\end{aligned}$$

or, more precisely,

$$\left[(1-\frac{1}{1+n})\lambda^2 - (\beta R_{\min}(t) + \frac{1}{\sqrt{n}}\min_{x\in M}|\phi_t|)(\frac{\lambda}{p-1})^{\frac{2p-2}{p}}\right]\int_M |f|^{2p-2}d\mu \geq 0.$$

Since $\int_M |f|^{2p-2}d\mu \geq 0$, for $p > 2$ we get

$$\lambda(t) \geq \left[(\beta R_{\min}(t) + \frac{1}{\sqrt{n}}\min_{x\in M}|\phi_t|)\frac{1+n\alpha}{1+n\alpha-\alpha}\right]^{\frac{p}{2}}\frac{1}{(p-1)^{(p-1)}}.$$

Since $R_{\min}(t) \to +\infty$ (see [6]) and $\min\limits_{x\in M}|\phi_t|$ is finite, then $\lambda(t) \to +\infty$. For $p=2$, (3.38) yields that

$$(1-\frac{1}{1+n})\lambda^2\int_M |f|^2 d\mu \geq (\beta R_{\min}(t) + \frac{1}{\sqrt{n}}\min_{x\in M}|\phi_t|)\lambda\int_M |f|^2 d\mu,$$

hence,

$$\lambda(t) \geq (\beta R_{\min}(t) + \frac{1}{\sqrt{n}}\min_{x\in M}|\phi_t|)\frac{1+n\alpha}{1+n\alpha-\alpha}.$$

This implies that $\lambda(t) \to +\infty$. □

Corollary 3.7. *Let $(M, g(t))$, $t \in [0,T)$, be a solution of the flow (1.1) on the smooth closed Riemannnian manifold (M^3, g_0), ϕ is independent of t, $\frac{1}{6} < \rho < \frac{1}{4}$ and $\lambda(t)$ be the first eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$. If $R_{ij} > \frac{1+\rho p-3\rho}{p}Rg_{ij}$ on $M^n \times \{0\}$ and $c = R_{\min}(0) \geq 0$ then the quantity $\lambda(t)(3-2(1-3\rho)ct)^{\frac{3}{2}}$ is nondecreasing along the flow (1.1) for $p \geq 3$.*

Proof. The pinching inequality $R_{ij} > \frac{1+\rho p-3\rho}{p}Rg_{ij}$ for $\frac{1}{6} < \rho < \frac{1}{4}$ and $p \geq 3$ is preserved along the Ricci-Bourguignon flow. Therefore, we have

$$R_{ij} > \frac{1+\rho p-3\rho}{p}Rg_{ij}, \quad \text{on } [0,T)\times M.$$

Now according to Corollary 3.4, we get

$$\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} \geq \lambda(t_0)(1-n\rho)\int_M R|f|^p\, d\mu$$

hence, similar to the proof of Theorem 3.5, we have $R_{g(t)} \geq \sigma(t)$ on $[0,T)$ and then

$$\frac{\partial}{\partial t}\lambda(t, f(t))|_{t=t_0} \geq \lambda(t_0)(1-n\rho)\sigma(t_0)$$

thus we arrive at the the quantity $\lambda(t)(3-2(1-3\rho)ct)^{\frac{3}{2}}$ is nondecreasing. □

Theorem 3.8. *Let $(M, g(t), \phi(t))$, $t \in [0,T)$ be a solution of the flow (1.8) on the smooth closed oriented Riemannain manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$. Let $0 < R_{ij} < \frac{1+p\rho-n\rho}{p}Rg_{ij}$ on $M^n \times [0,T)$ and $R < \Delta\phi$ in $M \times [0,T)$. Suppose that $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ and $C = R_{\max}(0)$ then the quantity $\lambda(t)(1-CAt)^{\frac{n\rho-1}{A}}$ is strictly decreasing along the flow (1.8) on $[0,T')$ where $T' = \min\{T, \frac{1}{CA}\}$ and $A = 2\big(n(\frac{1-(n-p)\rho}{p})^2-\rho\big)$.*

Proof. The proof is similar to proof of Theorem 3.5 with the difference that we need to estimate the upper bound of the right hand (3.6). Notice that $R_{ij} < \frac{1+p\rho-n\rho}{p}Rg_{ij}$ implies that $|Ric|^2 < n(\frac{1+p\rho-n\rho}{p})^2R^2$. So, the evolution of the scalar curvature under the Ricci-Bourguignon flow evolve by (3.26) and it yields

$$\frac{\partial R}{\partial t} \leq (1-2(n-1)\rho)\Delta R + 2\big(n(\frac{1+p\rho-n\rho}{p})^2-\rho\big)R^2. \tag{3.39}$$

Applying the maximum principle to (3.39), we have $0 \leq R_{g(t)} \leq \gamma(t)$ where

$$\gamma(t) = \left[C^{-1} - 2\big(n(\frac{1+p\rho-n\rho}{p})^2 - \rho\big)t\right]^{-1} = \frac{C}{1-CAt} \quad \text{on } [0, T').$$

Replacing $0 \leq R_{g(t)} \leq \gamma(t)$ and $R_{ij} < \frac{1-(n-2)\rho}{2} Rg_{ij}$ into equation (3.6), we can write $\frac{d}{dt}\lambda(t, f(t)) \leq \frac{(1-n\rho)C}{1-CAt}\lambda(t, f(t))$ in any sufficiently small neighborhood of t_0. Hence, with a sequence of calculation, the quantity $\lambda(t)(1-CAt)^{\frac{n\rho-1}{A}}$ is strictly decreasing. □

Theorem 3.9. *Let $(M, g(t))$, $t \in [0, T)$ be a solution of the Ricci-Bourguignon flow (1.1) on a closed manifold M^n and $\rho < \frac{1}{2(n-1)}$. Let $\lambda(t)$ be the first nonzero eigenvalue of the weighted p-Laplacian of the metric $g(t)$ and ϕ be independent of t. If there is a non-negative constant a such that*

$$R_{ij} - \frac{1-(n-p)\rho}{p} Rg_{ij} \geq -ag_{ij} \quad \text{in } M^n \times [0, T) \tag{3.40}$$

and

$$R \geq \frac{pa}{1-n\rho} \quad \text{in } M^n \times \{0\} \tag{3.41}$$

then $\lambda(t)$ is strictly monotone increasing along the Ricci-Bourguignon flow.

Proof. By Corollary 3.4, we write evolution of first eigenvalue as follows

$$\begin{aligned} \frac{d}{dt}\lambda(t, f(t))|_{t=t_0} &= (1-n\rho)\lambda(t_0)\int_M R f^2 d\mu \\ &\quad + p\int_M (R_{ij} - \frac{1-(n-p)\rho}{p} Rg_{ij})|\nabla f|^{p-2}\nabla_i f \nabla_j f \, d\mu \\ &\geq (1-n\rho)\lambda(t_0)\int_M R f^2 d\mu - ap\int_M |\nabla f|^p d\mu \geq 0 \end{aligned} \tag{3.42}$$

combining (3.40), (3.41) and (3.42), we arrive at $\frac{d}{dt}\lambda(f(t), t) > 0$ in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. □

3.1. Variation of $\boldsymbol{\lambda(t)}$ on a surface. Now, we rewrite Proposition 3.3 and Corollary 3.4 in some remarkable particular cases.

Corollary 3.10. *Let $(M^2, g(t))$, $t \in [0, T)$ be a solution of the Ricci-Bourguignon flow on a closed Riemannnian surface (M^2, g_0) for $\rho < \frac{1}{2}$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of the weighted p-Laplacian under the Ricci-Bourguignon flow, then:*

(1) *If* $\frac{\partial \phi}{\partial t} = \Delta\phi$ *then*

$$\frac{d}{dt}\lambda(t,f(t))|_{t=t_0} = (1-2\rho)\lambda(t_0)\int_M R\,|f|^p d\mu + \lambda(t_0)\int_M (\Delta\phi)|f|^p d\mu - (1+\rho\phi-2\rho-\frac{p}{2})\int_M R|\nabla f|^p d\mu - \int_M (\Delta\phi)|\nabla f|^p d\mu. \tag{3.43}$$

(2) *If* ϕ *is independent of* t *then*

$$\frac{d}{dt}\lambda(t,f(t))|_{t=t_0} = (1-2\rho)\lambda(t_0)\int_M R\,|f|^p d\mu - (1+\rho\phi-2\rho-\frac{p}{2})\int_M |\nabla f|^p d\mu. \tag{3.44}$$

Proof. In dimension $n = 2$, we have $Ric = \frac{1}{2}Rg$, then (3.6) and (3.24) imply that (3.43) and (3.44) respectively. □

Lemma 3.11. *Let* $(M^2, g(t))$, $t \in [0,T)$, *be a solution of the Ricci-Bourguignon flow on a closed surface* (M^2, g_0) *with nonnegative scalar curvature for* $\rho < \frac{1}{2}$, ϕ *be independent of* t *and* $p \geq 2$. *If* $\lambda(t)$ *denotes the evolution of the first eigenvalue of the weighted* p*-Laplacian under the Ricci-Bourguignon flow, then*

$$\frac{\lambda(0)}{(1-c(1-2\rho)t)^{\frac{p}{2}}} \leq \lambda(t)$$

on $(0,T')$ *where* $c = \min_{x\in M} R(0)$ *and* $T' = \min\left\{T, \frac{1}{c(1-2\rho)}\right\}$.

Proof. On a surface, we have $Ric = \frac{1}{2}Rg$, and for the scalar curvature R on a closed surface M along the Ricci-Bourguignon flow, we get

$$\frac{c}{1-c(1-2\rho)t} \leq R, \qquad \text{on} \quad [0,T') \tag{3.45}$$

where $T' = \min\{T, \frac{1}{c(1-2\rho)}\}$. According to (3.44) and $\int_M |f|^p d\mu = 1$, we have

$$\frac{p}{2}\frac{c(1-2\rho)\lambda(t,f(t))}{1-c(1-2\rho)t} \leq \frac{d}{dt}\lambda(t,f(t)) \tag{3.46}$$

in any small enough neighborhood of t_0. After integrating the above inequality with respect to time t, this becomes

$$\frac{\lambda(0,f(0))}{(1-c(1-2\rho)t)^{\frac{p}{2}}} \leq \lambda(t_0).$$

Now, $\lambda(0,f(0)) \geq \lambda(0)$ yields that $\frac{\lambda(0)}{(1-c(1-2\rho)t)^{\frac{p}{2}}} \leq \lambda(t_0)$. Since t_0 is arbitrary, then $\frac{\lambda(0)}{(1-c(1-2\rho)t)^{\frac{p}{2}}} \leq \lambda(t)$ on $(0,T')$. □

Lemma 3.12. *Let* (M^2, g_0) *be a closed surface with nonnegative scalar curvature and* ϕ *be independent of* t, *then the eigenvalues of the weighted* p*-Laplacian are increasing under the Ricc-Bourguignon flow for* $\rho < \frac{1}{2}$.

Proof. Along the Ricci-Bourguignon flow on a surface, we have

$$\frac{\partial R}{\partial t} = (1-2\rho)(\Delta R + R^2)$$

by the scalar maximum principle, the nonnegativity of the scalar curvature is preserved along the Ricci-Bourguignon flow (see [6]). Then (3.44) implies that $\frac{d}{dt}\lambda(t, f(t))|_{t=t_0} > 0$, this results that in any sufficiently small neighborhood of t_0 as I_0, we get $\frac{d}{dt}\lambda(t, f(t)) > 0$. Hence, by integrating on the interval $[t_1, t_0] \subset I_0$, we have $\lambda(t_1, f(t_1)) \leq \lambda(t_0, f(t_0))$. Since $\lambda(t_0, f(t_0)) = \lambda(t_0)$ and $\lambda(t_1, f(t_1)) \geq \lambda(t_1)$, we conclude that $\lambda(t_1) \leq \lambda(t_0)$. Therefore, the quantity $\lambda(t)$ is strictly increasing in any sufficiently small neighborhood of t_0, but t_0 is arbitrary, then $\lambda(t)$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$. $\square$

3.2. Variation of $\boldsymbol{\lambda(t)}$ on homogeneous manifolds. In this section, we consider the behavior of the first eigenvalue when we evolve an initial homogeneous metric along the flow (1.8).

Proposition 3.13. *Let $(M^n, g(t))$ be a solution of the Ricci-Bourguignon flow on the smooth closed homogeneous manifold (M^n, g_0) for $\rho < \frac{1}{2(n-1)}$. Let $\lambda(t)$ denote the evaluation of an eigenvalue under the Ricci-Bourguignon flow, then*

(1) *If $\frac{\partial \phi}{\partial t} = \Delta\phi$ then*

$$\frac{d}{dt}\lambda(t, f(t))|_{t=t_0} = -\rho p R\lambda(t_0) + p\int_M ZR^{ij}\nabla_i f\nabla_j f\, d\mu + \lambda(t_0)\int_M (\Delta\phi)|f|^p\, d\mu - \int_M (\Delta\phi)|\nabla f|^p d\mu. \tag{3.47}$$

(2) *If ϕ is independent of t then*

$$\frac{d}{dt}\lambda(t, f(t))|_{t=t_0} = -\rho p R\lambda(t_0) + p\int_M ZR^{ij}\nabla_i f\nabla_j f\, d\mu. \tag{3.48}$$

Proof. Since the evolving metric remains homogeneous and a homogeneous manifold has constant scalar curvature. Therefore (3.6) implies that

$$\begin{aligned}\frac{d}{dt}\lambda(t, f(t))|_{t=t_0} = {} & (1-n\rho)\lambda(t_0)R\int_M f^2 d\mu + ((n-p)\rho - 1)R\int_M |\nabla f|^2 d\mu \\ & + p\int_M ZR^{ij}\nabla_i f\nabla_j f\, d\mu + \lambda(t_0)\int_M (\Delta\phi)|f|^p\, d\mu \\ & - \int_M (\Delta\phi)|\nabla f|^p d\mu.\end{aligned}$$

But $\int_M f^2 d\mu = 1$ and $\int_M |\nabla f|^2 d\mu = 1$ therefore last equation results that (3.47) and (3.48). $\square$

3.3. Variation of $\lambda(t)$ on 3-dimensional manifolds. In this section, we consider the behavior of $\lambda(t)$ on 3-dimensional manifolds.

Proposition 3.14. *Let $(M^3, g(t))$ be a solution of the Ricci-Bourguignon flow (1.1) for $\rho < \frac{1}{4}$ on a closed Riemannian manifold M^3 whose Ricci curvature is initially positive and there exists $0 \leq \epsilon \leq \frac{1}{3}$ such that*

$$Ric \geq \epsilon R g.$$

If ϕ is independent of t and $\lambda(t)$ denotes the evolution of the first eigenvalue of the weighted p-Laplacian under the Ricci-Bourguignon flow then the quantity $e^{-\int_0^t A(\tau)d\tau}\lambda(t)$ is nondecreasing along the Ricci-Bourguignon flow (1.1) for $p \leq 3$, where

$$A(t) = \frac{3c(1-3\rho)}{3-2(1-3\rho)ct} + (3\rho + p\epsilon - 1 - \rho p)\left(-2(1-\rho)t + \frac{1}{C}\right)^{-1},$$

$C = R_{\max}(0)$ and $c = R_{\min}(0)$.

Proof. In [6], it has been shown that the pinching inequality $Ric \geq \epsilon R g$ and nonnegative scalar curvature are preserved along the Ricci-Bourguignon flow (1.1) on closed manifold M^3. Then using (3.24), we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(f,t)|_{t=t_0} &\geq (1-3\rho)\lambda(t_0)\int_M R f^2 d\mu + (3\rho - 1 - \rho p)\int_M R|\nabla f|^2 d\mu \\ &\quad + p\epsilon \int_M R|\nabla f|^2 d\mu \\ &= (1-3\rho)\lambda(t_0)\int_M R f^2 d\mu + (3\rho + p\epsilon - 1 - \rho p)\int_M R|\nabla f|^2 d\mu.\end{aligned}$$

On the other hand, the scalar curvature under the Ricci-Bourguignon flow evolves by (3.26) for $n = 3$. By $|Ric|^2 \leq R^2$ we have

$$\frac{\partial R}{\partial t} \leq (1-4\rho)\Delta R + 2(1-\rho)R^2.$$

Let $\gamma(t)$ be the solution to the ODE $y' = 2(1-\rho)y^2$ with initial value $C = R_{\max}(0)$. By the maximum principle, we have

$$R(t) \leq \gamma(t) = \left(-2(1-\rho)t + \frac{1}{C}\right)^{-1} \tag{3.49}$$

on $[0, T')$, where $T' = \min\{T, \frac{1}{2(1-\rho)C}\}$. Also, similar to proof of Theorem 3.5, we have

$$R(t) \geq \sigma(t) = \frac{3c}{3-2(1-3\rho)ct} \quad \text{on} \quad [0, T). \tag{3.50}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\lambda(t,f(t))|_{t=t_0} &\geq (1-3\rho)\lambda(t_0)\frac{3c}{3-2(1-3\rho)ct_0} \\ &\quad +(\rho-1+2\epsilon)\lambda(t_0)\left(-2(1-\rho)t_0+\frac{1}{C}\right)^{-1} \\ &= \lambda(t_0)A(t_0).\end{aligned}$$

This yields that in any sufficiently small neighborhood of t_0 as I_0, we obtain

$$\frac{d}{dt}\lambda(t,f(t)) \geq \lambda(f,t)A(t).$$

Integrating both sides of the last inequality with respect to t on $[t_1,t_0] \subset I_0$, we can write

$$\ln\frac{\lambda(t_0,f(t_0))}{\lambda(t_1,f(t_1))} > \int_{t_1}^{t_0} A(\tau)d\tau.$$

Since $\lambda(t_0,f(t_0)) = \lambda(t_0)$ and $\lambda(t_1,f(t_1)) \geq \lambda(t_1)$, we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \int_{t_1}^{t_0} A(\tau)d\tau.$$

That is, the quantity $\lambda(t)e^{-\int_0^t A(\tau)d\tau}$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then $\lambda(t)e^{-\int_0^t A(\tau)d\tau}$ is strictly increasing along the Ricci-Bourguignon flow on $[0,T)$. □

Proposition 3.15. *Let $(M^3, g(t))$ be a solution to the Ricci-Bourguignon flow for $\rho < 0$ on a closed homogeneous 3-manifold whose Ricci curvature is initially nonnegative and ϕ be independent of t then the first eigenvalues of the weighted p-Laplacian is increasing.*

Proof. In dimension three, the Ricci-Bourguignon flow preseves the non-negativity of the Ricci curvature is preserved. From (3.48), its implies that $\lambda(t)$ is increasing. □

4. Example

In this section, we consider the initial Riemannian manifold (M^n, g_0) is Einstein manifold and then find evolving first eigenvalue of the weighted p-Laplace operator along the Ricci-Bourguignon flow.

Example 4.1. Let (M^n, g_0) be an Einstein manifold i.e. there exists a constant a such that $Ric(g_0) = ag_0$. Assume that a solution to the Ricci-Bourguignon flow is of the form

$$g(t) = u(t)g_0, \quad u(0) = 1$$

where $u(t)$ is a positive function. By a straightforward computation, we have

$$\frac{\partial g}{\partial t} = u'(t)g_0, \;\; Ric(g(t)) = Ric(g_0) = ag_0 = \frac{a}{u(t)}g(t), \;\; R_{g(t)} = \frac{an}{u(t)},$$

for this to be a solution of the Ricci-Bourguignon flow, we require

$$u'(t)g_0 = -2Ric(g(t)) + 2\rho R_{g(t)}g(t) = (-2a + 2\rho an)g_0.$$

This shows that

$$u(t) = (-2a + 2\rho an)t + 1,$$

so $g(t)$ is an Einstein metric. Using formula (3.24) for evolution of first eigenvalue along the Ricci-Bourguignon flow, we obtain the following relation

$$\begin{aligned} \frac{d}{dt}\lambda(t, f(t))|_{t=t_0} &= (1-n\rho)\frac{an}{u(t_0)}\lambda(t_0)\int_M |f|^p d\mu + 2\frac{a}{u(t_0)}\int_M |\nabla f|^p d\mu \\ &\quad -((p-n)\rho - 1)\frac{an}{u(t_0)}\int_M |\nabla f|^p d\mu = \frac{pa(1-n\rho)\lambda(t_0)}{u(t_0)}, \end{aligned}$$

. This yields that in any sufficiently small neighborhood of t_0 as I_0, we get

$$\frac{d}{dt}\lambda(t, f(t)) = \frac{pa(1-n\rho)\lambda(t, f(t))}{(-2a + 2\rho an)t + 1}.$$

Integrating the last inequality with respect to t on $[t_1, t_0] \subset I_0$, we have

$$\ln\frac{\lambda(t_0, f(t_0))}{\lambda(t_1, f(t_1))} = \int_{t_1}^{t_0} \frac{pa(1-n\rho)}{(-2a + 2\rho an)\tau + 1} d\tau = \ln\left(\frac{-2a(1-n\rho)t_1 + 1}{-2a(1-n\rho)t_0 + 1}\right)^{\frac{p}{2}}.$$

Since $\lambda(t_0, f(t_0)) = \lambda(t_0)$ and $\lambda(t_1, f(t_1)) \geq \lambda(t_1)$, we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \ln\left(\frac{-2a(1-n\rho)t_1 + 1}{-2a(1-n\rho)t_0 + 1}\right)^{\frac{p}{2}}.$$

That is, the quantity $\lambda(t)[-2a(1-n\rho)t+1]^{\frac{p}{2}}$ is strictly increasing along the Ricci-Bourguignon flow on $[0, T)$.

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