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Multiplication operators defined by twisted proper holomorphic maps on Bergman spaces

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Abstract. The paper studies the structure and commutative properties of von Neumann algebras induced by multiplication operators on the Bergman space of a bounded domain in the complex space $\mathbb{C}^d$. We show that there is a close interplay between operator theory, geometry, and function theory.

Contents

1. Introduction

Let Ω denote a bounded domain in the complex space $\mathbb{C}^d$ and dA be the Lebesgue measure on Ω. The Bergman space $L_a^2(\Omega)$ is the Hilbert space consisting of all holomorphic functions over Ω which are square integrable with respect to the Lebesgue measure dA. For a bounded holomorphic function ϕ on Ω, let M_ϕ denote the multiplication operator with the symbol ϕ on $L_a^2(\Omega)$, given by

$$M_\phi f = \phi f, \qquad f \in L_a^2(\Omega).$$

In general, for a tuple $\Phi = \{\phi_j : 1 \leq j \leq n\}$, let $\{M_\Phi\}'$ denote the commutant of $\{M_{\phi_j} : 1 \leq j \leq n\}$, consisting of all bounded operators commuting with each operator $M_{\phi_j} (1 \leq j \leq n)$. Here, we emphasize that

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M_Φ denotes a family of multiplication operators rather than a single vector-valued multiplication operator. Let $\mathcal{V}^*(\Phi,\Omega)$ denote the von Neumann algebra $\{M_{\phi_j}, M_{\phi_j}^* : 1 \le j \le n\}'$ which consists of all bounded operators on $L_a^2(\Omega)$ commuting with both M_{ϕ_j} and $M_{\phi_j}^*$ for each j. It is known that there is a close connection between orthogonal projections in $\mathcal{V}^*(\Phi,\Omega)$ and all joint reducing subspaces of $\{M_{\phi_j} : 1 \le j \le n\}$. Precisely, the range of an orthogonal projection in $\mathcal{V}^*(\Phi,\Omega)$ is exactly a joint reducing subspace of $\{M_{\phi_j} : 1 \le j \le n\}$, and vice versa.

In the single-variable case, commutants and reducing subspaces of multiplication operators has caught many people's interest, and steady progress has been made during the past dozen years [Cow78, Cow80a, Cow80b, DPW12, DSZ11, GH11a, GH11b, GH14, GH15, SZZ10, Tho77, Tho76]. For the multi-variable case, this seems to be a new area [DanH14, Gu18, GW16, HZ15, LZ10, SL13, WDH15].

We mention that on the Bergman space of the unit disk, the relevant topic was initiated by Zhu's conjecture in 2000 [Zhu00] on the number $k(B)$ of minimal reducing subspaces of a single multiplication operator induced by a finite Blaschke product B. As the investigations went further, a more delicate conjecture was raised by Guo, Sun, Zheng and Zhong [DSZ11, GSZZ09]. The modified conjecture establishes a direct connection between $k(B)$ and the number of connected components of the Riemann surface associated with B. Different techniques and methods are developed during the attack to this conjecture [GSZZ09, SZZ10, GH11a], and finally it was affirmatively solved by Douglas, Putinar and Wang [DPW12]. It is thus of interest to study the multi-variable case for similar phenomena that seeks to establish a link between operator theory, function theory, and geometry.

Observe that the finite Blaschke products are the only proper holomorphic maps from the unit disk onto itself [Rud69]. It is natural to consider holomorphic proper maps in several complex variables. Recently, the framework of von Neumann algebras associated with such maps has been raised in [HZ15]. Following this line, we consider the properties of the von Neumann algebras generated by multiplication operators defined by twisted holomorphic proper maps. As one will see, new phenomena emerge, and techniques of geometry, complex analysis and operator theory are intrinsic in this paper.

The paper is arranged as follows. In Section 2, we state our main theorems. Some preliminaries are given in Section 3. Section 4 provides the proofs for our main results.

2. Statement of main results

Suppose Ω_1 and Ω_2 are two bounded domains in $\mathbb{C}$, ϕ and ψ are holomorphic on Ω_1 and Ω_2, respectively. Define

$$\Upsilon_{\phi,\psi}(z_1,z_2) = \left(\phi(z_1)+\psi(z_2), \phi(z_1)^2+\psi(z_2)^2\right), \quad z_1\in\Omega_1, z_2\in\Omega_2,$$

and

$$\mathcal{S}_{\phi,\psi} = \left\{(z,w) \in \Omega_1 \times \Omega_2 : z \notin \phi^{-1}\Big(\overline{\psi(Z(\psi'))}\Big),\, w \notin \psi^{-1}\Big(\overline{\phi(Z(\phi'))}\Big)\right\},$$

where $Z(\psi')$ and $Z(\phi')$ denote the zeros of ψ' and ϕ', respectively. Under some situations, $\mathcal{S}_{\phi,\psi}$ turns out to be a Riemann surface, and then let $n(\phi,\psi)$ denote the number of components of $\mathcal{S}_{\phi,\psi}$. Our first main result is the dimension formula for $\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2}, \Omega_1 \times \Omega_2)$.

Theorem 2.1. *Suppose that ϕ_1 and ϕ_2 are holomorphic proper maps over bounded domains Ω_1 and Ω_2 in $\mathbb{C}$, respectively. If $\phi_1(\Omega_1) = \phi_2(\Omega_2)$, then*

$$\dim \mathcal{V}^*(\Upsilon_{\phi_1,\phi_2}, \Omega_1 \times \Omega_2) = n(\phi_1,\phi_1)n(\phi_2,\phi_2) + n(\phi_1,\phi_2)^2.$$

In this case, $\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2}, \Omega_1 \times \Omega_2)$ is not $*$-isomorphic to the von Neumann algebra $\mathcal{V}^*(\phi_1(z_1), \phi_2(z_2), \Omega_1 \times \Omega_2) = \mathcal{V}^*(\phi_1, \Omega_1) \otimes \mathcal{V}^*(\phi_2, \Omega_2)$.

The condition $\phi_1(\Omega_1) = \phi_2(\Omega_2)$ can not be replaced by $\overline{\phi_1(\Omega_1)} = \overline{\phi_2(\Omega_2)}$, as illustrated by Example 4.2.

If both ϕ_1 and ϕ_2 are finite Blaschke products in Theorem 2.1, the abelian property of $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ relies heavily on the connectedness of the Riemann surface $\mathcal{S}_{B_1,B_2}$ (see Subsection 2.2 for the definition of $\mathcal{S}_{B_1,B_2}$).

Theorem 2.2. *Let B_1 and B_2 be two finite Blaschke products. Then the von Neumann algebra $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ is abelian if and only if $\mathcal{S}_{B_1,B_2}$ is connected.*

Let Ω be a domain in $\mathbb{C}^2$ and

$$\Phi(z_1,z_2) = (\phi_1(z_1,z_2), \phi_2(z_1,z_2)),\ \Psi(z_1,z_2) = (\psi_1(z_1,z_2), \psi_2(z_1,z_2)),$$

where $(z_1,z_2) \in \Omega$. Write

$$P(z) = \left(\sum_{j=1}^{4} z_j, \sum_{j=1}^{4} z_j^2, \cdots, \sum_{j=1}^{4} z_j^4\right),$$

and define

$$\Upsilon_{\Phi,\Psi}(z_1,z_2,z_3,z_4) = P \circ \big(\Phi(z_1,z_2), \Psi(z_3,z_4)\big), \quad (z_1,z_2,z_3,z_4) \in \Omega^2,$$

which is called the twisted map of Φ and Ψ. The following theorem presents a comparison with Theorem 2.1.

Theorem 2.3. *Suppose Φ and Ψ are holomorphic proper maps over Ω such that $\overline{\Phi(\Omega)} \neq \overline{\Psi(\Omega)}$. If both Φ and Ψ are holomorphic on $\overline{\Omega}$ and $\Upsilon_{\Phi,\Psi}$ has no nontrivial compatible equation, then $\mathcal{V}^*(\Upsilon_{\Phi,\Psi}, \Omega^2)$ is $*$-isomorphic to $\mathcal{V}^*(\Phi,\Omega) \otimes \mathcal{V}^*(\Psi,\Omega)$.*

The condition of $\Upsilon_{\Phi,\Psi}$ having no nontrivial compatible equation is quite geometric (see Theorem 4.7). Practically, in many cases it is easy to check whether this condition holds. In addition, an analogue of Theorem 2.3 still holds if Ω is a domain in $\mathbb{C}^d$, $d \geq 1$.

3. Some preliminaries

3.1. Proper map and zero variety. This subsection gives some preliminaries, including the notions of proper map and zero variety.

Let Ω, Ω' be domains in $\mathbb{C}^d$. A holomorphic function $\Psi : \Omega \to \Omega'$ is called *a proper map* if each compact subset K of Ω', $\Psi^{-1}(K)$ is compact. A holomorphic function Ψ on Ω is called *proper* if $\Psi(\Omega)$ is open and the map $\Psi : \Omega \to \Psi(\Omega)$ is proper. In particular, if Ψ is holomorphic on $\overline{\Omega}$, then Ψ is proper on Ω if and only if $\Psi(\Omega)$ is open and

$$\Psi(\partial\Omega) \subseteq \partial\Psi(\Omega).$$

In general, a holomorphic proper map is open, which is a direct consequence of the following [Rud80, Theorem 15.1.6].

Theorem 3.1. *Suppose $F : \Omega \to \mathbb{C}^d$ is a holomorphic function and for each $w \in \mathbb{C}^d$, $F^{-1}(w)$ is compact. Then F is an open map.*

Let $F : \Omega \to \mathbb{C}^d$ be a holomorphic map and let Z be the zero set of the determinant of the Jacobian of F. Then its image $F(Z)$ is called *the critical set* of F. Each point in $F(Z)$ is called *a critical value*, and each point in $F(\Omega) - F(Z)$ is called *a regular point*. A holomorphic proper map is always an m-folds map, and its critical set is a zero variety as follows.

Theorem 3.2. [Rud80, Theorem 15.1.9] *Given two domains Ω and Ω_0 in $\mathbb{C}^d$, suppose $F : \Omega \to \Omega_0$ is a holomorphic proper function. Let $\sharp(w)$ denote the number of points in $F^{-1}(w)$ with $w \in \Omega_0$. Then the following hold:*

(1) *There is an integer m such that $\sharp(w) = m$ for all regular values w of F and $\sharp(w') < m$ for all critical values w' of F;*
(2) *The critical set of F is a zero variety in Ω_0.*

A subset E of Ω is called *a zero variety* of Ω if there is a non-constant holomorphic function f on Ω such that $E = \{z \in \Omega | f(z) = 0\}$. A relatively closed subset V of Ω is called *an (analytic) subvariety* of Ω if for each point w in Ω there is a neighborhood $\mathcal{N}$ of w such that $V \cap \mathcal{N}$ equals the intersection of zeros of finitely many holomorphic functions over $\mathcal{N}$.

An easier version of Remmert's Proper Mapping Theorem reads as follows (see [Chi89, p. 65] or [Rem56, Rem57]).

Theorem 3.3. *If $f : \Omega_0 \to \Omega_1$ is a holomorphic proper map and $\mathcal{Z}$ is a subvariety of Ω_0, then $f(\mathcal{Z})$ is a subvariety of Ω_1.*

3.2. Analytic continuation. Some notions are needed on analytic continuation ([Rud87, Chapter 16]). A function element is an ordered pair (f, D), where D is an open ball in $\mathbb{C}^d$ and f is a holomorphic function on D. Two function elements (f_0, D_0) and (f_1, D_1) are called direct continuation if $D_0 \cap D_1$ is not empty and $f_0 = f_1$ holds on $D_0 \cap D_1$. A curve is a continuous map from $[0, 1]$ into $\mathbb{C}^d$. For a function element (f_0, D_0) and a curve γ with

$\gamma(0) \in D_0$, if there is a partition of $[0, 1]$:

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

and function elements $(f_j, D_j)(0 \le j \le n)$ such that

1. (f_j, D_j) and (f_{j+1}, D_{j+1}) are direct continuation for all j with $0 \le j \le n-1$;
2. $\gamma[s_j, s_{j+1}] \subseteq D_j (0 \le j \le n-1)$ and $\gamma(1) \in D_n$,

then (f_n, D_n) is called *an analytic continuation of* (f_0, D_0) *along* γ ; and (f_0, D_0) is called to *admit* an analytic continuation along γ. In this case, we write $f_0 \sim f_n$. Clearly, $\sim$ defines an equivalence and we write $[f]$ for *the equivalent class* of f.

3.3. Local solution. As follows, we will generalize the notion of local inverse. For convenience, assume both Φ and Ψ are holomorphic maps from $\overline{\Omega}$ to $\mathbb{C}^d$. Rewrite

$$Z(J\Phi) = \mathcal{Z}_\Phi \quad \text{and} \quad Z(J\Psi) = \mathcal{Z}_\Psi,$$

where $J\Phi$ and $J\Psi$ denote the determinants of the Jacobian of Φ and Ψ, respectively. Let

$$\mathcal{S}_{\Phi,\Psi} = \{(z, w) \in \Omega : \Psi(w) = \Phi(z), z \notin \Phi^{-1}\big(\overline{\Psi(\mathcal{Z}_\Psi)}\big)\}. \tag{3.1}$$

and

$$\mathcal{S}_{\Psi,\Phi} = \{(z, w) \in \Omega : \Phi(w) = \Psi(z), z \notin \Psi^{-1}\big(\overline{\Phi(\mathcal{Z}_\Phi)}\big)\}. \tag{3.2}$$

It can happen that $\mathcal{S}_{\Phi,\Psi}$ or $\mathcal{S}_{\Psi,\Phi}$ is empty, but in many cases they are Riemann manifolds.

Definition 3.4. If there is a subdomain Δ of Ω and a holomorphic function ρ over Δ such that

$$\Psi(\rho(z)) = \Phi(z),\ z \in \Delta,$$

then ρ is called *a local solution* for $\mathcal{S}_{\Phi,\Psi}$, denoted by

$$\rho \in \Psi^{-1} \circ \Phi.$$

In particular, if $\Phi = \Psi$, then ρ is *a local inverse* of Φ [Tho77] and we rewrite S_Φ for $\mathcal{S}_{\Phi,\Phi}$.

Following [HZ15], we give the definition of admissible local solution.

Definition 3.5. A local solution ρ for $\mathcal{S}_{\Phi,\Psi}$ is called *admissible* if for each curve γ in $\Omega - \Phi^{-1}\big(\overline{\Psi(\mathcal{Z}_\Psi)}\big)$, ρ admits analytic continuation with values in Ω.

In this case, we say ρ is admissible with respect to $\Phi^{-1}\big(\overline{\Psi(\mathcal{Z}_\Psi)}\big)$. It can be shown that $\Omega - \Phi^{-1}\big(\overline{\Psi(\mathcal{Z}_\Psi)}\big)$ is connected if both Φ and Ψ are holomorphic on $\overline{\Omega}$ ([Rud80, Chapter 14]).

Remark 3.6. One can also define $\mathcal{S}_{\Phi,\Psi}$ and $\mathcal{S}_{\Psi,\Phi}$ if both Φ and Ψ are holomorphic proper maps on Ω and

$$\Phi(\Omega) = \Psi(\Omega).$$

In this case,

$$\Omega - \Phi^{-1}\big(\Psi(\mathcal{Z}_\Psi)\big)$$

is also connected. Furthermore, by Theorem 3.2(2)

$$\Psi^{-1}\big(\overline{\Phi(\mathcal{Z}_\Phi)}\big) = \Psi^{-1}\big(\Phi(\mathcal{Z}_\Phi)\big),$$

and

$$\Phi^{-1}\big(\overline{\Psi(\mathcal{Z}_\Psi)}\big) = \Phi^{-1}\big(\Psi(\mathcal{Z}_\Psi)\big).$$

To define $\mathcal{S}_{\Phi,\Psi}$ and $\mathcal{S}_{\Psi,\Phi}$ one thus can replace $\overline{\Phi(\mathcal{Z}_\Phi)}$ and $\overline{\Psi(\mathcal{Z}_\Psi)}$ with $\Phi(\mathcal{Z}_\Phi)$ and $\Psi(\mathcal{Z}_\Psi)$, respectively in (3.1) and (3.2).

Given an admissible local inverse ρ of Φ, $[\rho]$ denotes the equivalent class of ρ under analytic continuation. Set

$$\mathcal{E}_{[\rho]}h(z) = \sum_{\sigma\in[\rho]} h\circ\sigma(z)J\sigma(z), h \in L^2_a(\Omega), z \in \Omega - \Phi^{-1}\big(\overline{\Phi(\mathcal{Z}_\Phi)}\big).$$

Then we get the following [HZ15], which is the key to our results.

Theorem 3.7. *Suppose $\Phi : \Omega \to \mathbb{C}^d$ is holomorphic on $\overline{\Omega}$ and the image of Φ contains an interior point. Then $\dim \mathcal{V}^*(\Phi,\Omega) < \infty$, and $\mathcal{V}^*(\Phi,\Omega)$ is generated by $\mathcal{E}_{[\rho]}$, where ρ runs over admissible local inverses of Φ.*

Theorem 3.8. *Let Ω and Ω_0 be bounded domains in $\mathbb{C}^d$. Suppose $\Phi : \Omega \to \Omega_0$ is a holomorphic proper map. Then $\mathcal{V}^*(\Phi,\Omega)$ is generated by $\mathcal{E}_{[\rho]}$, where ρ are local inverses of Φ. In particular, the dimension of $\mathcal{V}^*(\Phi,\Omega)$ equals the number of components of $\mathcal{S}_\Phi$.*

For a domain Ω in $\mathbb{C}^d$, if both Φ and Ψ are holomorphic on $\overline{\Omega}$, then $\mathcal{S}_{\Phi,\Psi}$ is a nonempty set. For two local solutions ρ and σ for $\mathcal{S}_{\Phi,\Psi}$, if ρ is an analytic continuation of σ, then their images lie in a same component of $\mathcal{S}_{\Phi,\Psi}$, and vice versa. Therefore, *the number of equivalent classes of local solutions equals the number of components of $\mathcal{S}_{\Phi,\Psi}$.*

Now we will modify $\mathcal{S}_{\Phi,\Psi}$ a bit by setting

$$\mathcal{S}_{\Phi,\Psi} = \Big\{(z,w) \in \Omega : \Psi(w) = \Phi(z), z \notin \Phi^{-1}\big(\overline{\Psi(\mathcal{Z}_\Psi)}\big),\, z \notin \Psi^{-1}\big(\overline{\Phi(\mathcal{Z}_\Phi)}\big)\Big\}. \tag{3.3}$$

Note that the numbers of components of $\mathcal{S}_{\Phi,\Psi}$ and $\mathcal{S}_{\Psi,\Phi}$ remain invariant. Since $\mathcal{S}_{\Phi,\Psi}$ and $\mathcal{S}_{\Psi,\Phi}$ are equal up to a permutation of coordinates, they have the same number of components. Hence the numbers of equivalent classes of local solutions for $\mathcal{S}_{\Phi,\Psi}$ and $\mathcal{S}_{\Psi,\Phi}$ are exactly equal. Letting $n(\Phi,\Psi)$ denote the number of components of $\mathcal{S}_{\Phi,\Psi}$, we have the following proposition.

Proposition 3.9. *Suppose one of the following holds:*

(*i*) *both* $\Phi : \Omega \to \Phi(\Omega)$ *and* $\Psi : \Omega \to \Psi(\Omega)$ *are holomorphic proper maps and* $\Phi(\Omega) = \Psi(\Omega)$*;*

(*ii*) *both* Φ *and* Ψ *are holomorphic over* $\overline{\Omega}$ *and their images in* $\mathbb{C}^d$ *contain an interior point.*

Then $\mathcal{S}_{\Phi,\Psi}$ *and* $\mathcal{S}_{\Psi,\Phi}$ *have the same number of components; that is,* $n(\Phi, \Psi) = n(\Phi, \Psi)$*.*

Under Condition (i), the local solutions for $\mathcal{S}_{\Phi,\Psi}$ turn out to be admissible. The special case of $\Phi = \Psi$ was discussed in the proof of [HZ15, Theorem 1.4].

4. Proof of main results

In this section, we will present the proofs of main theorems. We begin with a lemma.

Lemma 4.1. *Suppose both* Φ *and* Ψ *are holomorphic proper maps on* Ω *with same images. Then each local solution* ρ *for* $\mathcal{S}_{\Phi,\Psi}$ *is admissible in* Ω*.*

Proof. Suppose both Φ and Ψ are holomorphic proper maps on Ω with the same images. Write

$$A = \Phi^{-1}\big(\Psi(\mathcal{Z}_\Psi)\big).$$

Since Ψ are proper, $\Psi(\mathcal{Z}_\Psi)$ is a zero variety by Theorem 3.2. Then $\Psi(\mathcal{Z}_\Psi)$ is relatively closed in $\Psi(\Omega)$, and thus A is relatively closed in Ω.

For each curve γ in $\Omega - A$, write $z_0 = \gamma(0)$. Given a local solution ρ satisfying

$$\Psi(\rho(z_0)) = \Phi(z_0),$$

it suffices to show that ρ admits analytic continuation along γ. To see this, note that Φ and Ψ have the same images. For each point w on γ,

$$\Phi(w) \in \Psi(\Omega)$$

and

$$\Phi(\gamma) \cap \Psi(\mathcal{Z}_\Psi) = \emptyset$$

since $\gamma \subseteq \Omega - A$. By Theorem 3.2, there is an integer n depending only on Ψ so that $\Psi^{-1}(\Phi(w))$ has exactly n distinct points. Furthermore, there is an open ball U_w centered at w and n holomorphic maps $\rho_1^w, \cdots, \rho_n^w$ over U_w satisfying

$$\Psi(\rho_j^w(z)) = \Phi(z),\, z \in U_w,\, 1 \le j \le n.$$

Since γ is compact, by Henie-Borel's theorem there are finitely many such balls U_w whose union contains γ. Then by rolling the balls along the curve γ, it is straightforward to prove that all $\rho_j^{z_0}$ $(1 \le j \le n)$ admit analytic continuation along γ. Since one of $\rho_j^{z_0}(1 \le j \le n)$ is the direct continuation of ρ, ρ admits analytic continuation along γ. $\square$

4.1. Dimension formulas. In this subsection, we will present the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose that both ϕ_1 and ϕ_2 are holomorphic proper maps over bounded domains Ω_1 and Ω_2 in $\mathbb{C}$, respectively, and $\phi_1(\Omega_1) = \phi_2(\Omega_2)$. Let

$$\Omega = \phi_1(\Omega_1) = \phi_2(\Omega_2).$$

By using Theorem 3.1 one can show that $(z_1 + z_2, z_1^2 + z_2^2)$ is an open map, and in fact it is a proper map on $\Omega \times \Omega$. As a composition of $(z_1+z_2, z_1^2+z_2^2)$ and $(\phi_1(z_1), \phi_2(z_2))$, Υ_{ϕ_1,ϕ_2} is a holomorphic proper map on $\Omega_1 \times \Omega_2$. Then by Theorem 3.8 the von Neumann algebra $\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2}, \Omega_1 \times \Omega_2)$ is generated by $\mathcal{E}_\rho$, where ρ runs over local inverses of Υ_{ϕ_1,ϕ_2}. By Lemma 4.1 all these ρ are necessarily admissible.

Next we will determine the local inverses of Υ_{ϕ_1,ϕ_2}. The idea is to find out the candidate of such local inverse defined first at a single point, and then to pick out the admissible local inverses as desired. As below the letters $w = (w_1, w_2)$ and $z = (z_1, z_2)$ stand for both a single point and variables, which means that they can go from a point to almost everywhere of the whole domain. Observe that

$$(\lambda_1 + \lambda_2, \lambda_1^2 + \lambda_2^2) = (\mu_1 + \mu_2, \mu_1^2 + \mu_2^2) \tag{4.1}$$

is equivalent to

$$(\lambda_1 + \lambda_2, \lambda_1\lambda_2) = (\mu_1 + \mu_2, \mu_1\mu_2),$$

Then (λ_1, λ_2) and (μ_1, μ_2) are the same zeros of the polynomial p counting multiplicity, where $p(x) = x^2 + (\lambda_1+\lambda_2)x + \lambda_1\lambda_2$. Thus the solutions of (4.1) are

$$(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$$

and

$$(\lambda_1, \lambda_2) = (\mu_2, \mu_1).$$

Hence the equation

$$\Upsilon_{\phi_1,\phi_2}(w_1, w_2) = \Upsilon_{\phi_1,\phi_2}(z_1, z_2)$$

is equivalent to

$$\begin{cases} \phi_1(w_1) = \phi_1(z_1), \\ \phi_2(w_2) = \phi_2(z_2), \end{cases}$$

or

$$\begin{cases} \phi_1(w_1) = \phi_2(z_2), \\ \phi_2(w_2) = \phi_1(z_1). \end{cases}$$

Then we get either

$$(w_1, w_2) = (\sigma_1(z_1), \sigma_2(z_2)),\ \sigma_1 \in \phi_1^{-1} \circ \phi_1,\ \sigma_2 \in \phi_2^{-1} \circ \phi_2, \tag{4.2}$$

or

$$(w_1, w_2) = (\tau_1(z_2), \tau_2(z_1)),\ \tau_1 \in \phi_1^{-1} \circ \phi_2,\ \tau_2 \in \phi_2^{-1} \circ \phi_1.$$

Since both ϕ_1 and ϕ_2 are holomorphic proper maps, by Lemma 4.1 the above local solutions σ_1, σ_2, τ_1 and τ_2 are all admissible, and then the local inverses of Υ_{ϕ_1,ϕ_2}, $(\sigma_1(z_1),\sigma_2(z_2))$ and $(\tau_1(z_2),\tau_2(z_1))$, are admissible. Hence by Proposition 3.9, Υ_{ϕ_1,ϕ_2} has exactly $n(\phi_1,\phi_1)n(\phi_2,\phi_2)+n(\phi_1,\phi_2)^2$ equivalent classes for admissible local inverses. Since $\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2},\Omega_1\times\Omega_2)$ is generated by $\mathcal{E}_\rho$ where ρ are admissible local inverses of Υ_{ϕ_1,ϕ_2}, it follows that

$$\dim\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2},\Omega_1\times\Omega_2)=n(\phi_1,\phi_1)n(\phi_2,\phi_2)+n(\phi_1,\phi_2)^2.$$

Since $\dim\mathcal{V}^*(\phi_j,\Omega_j)=n(\phi_j,\phi_j)$, $j=1,2$,

$$\dim\mathcal{V}^*(\phi_1,\Omega_1)\otimes\mathcal{V}^*(\phi_2,\Omega_2)=n(\phi_1,\phi_1)n(\phi_2,\phi_2)<\dim\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2},\Omega_1\times\Omega_2).$$

Therefore, $\mathcal{V}^*(\Upsilon_{\phi_1,\phi_2},\Omega_1\times\Omega_2)$ is not $*$-isomorphic to the von Neumann algebra $\mathcal{V}^*(\phi_1,\Omega_1)\otimes\mathcal{V}^*(\phi_2,\Omega_2)$. Besides, the map $(\phi_1(z_1),\phi_2(z_2))$ is a proper map whose local inverses are exactly of the form (4.2), and by Theorem 3.8

$$\mathcal{V}^*(\phi_1(z_1),\phi_2(z_2),\Omega_1\times\Omega_2)=\mathcal{V}^*(\phi_1,\Omega_1)\otimes\mathcal{V}^*(\phi_2,\Omega_2),$$

which immediately leads to the desired conclusion. $\square$

In Theorem 2.1, the condition $\phi_1(\Omega_1)=\phi_2(\Omega_2)$ is sharp in the sense that it can not be replaced with $\overline{\phi_1(\Omega_1)}=\overline{\phi_2(\Omega_2)}$. Here is an example.

Example 4.2. *Put* $\Omega=\mathbb{D}\backslash[-1,0]$. *Write* $f(z)=z, z\in\mathbb{D}$ *and* g *is the restriction of* f *on* Ω. *Obviously,* f*and* g *are proper maps on* $\mathbb{D}$ *and* Ω *respectively. Set*

$$\Upsilon_{f,g}(z_1,z_2)=(z_1+z_2,z_1^2+z_2^2),z_1\in\mathbb{D},z_2\in\Omega.$$

We will prove that

$$\mathcal{V}^*(\Upsilon_{f,g},\mathbb{D}\times\Omega)=\mathbb{C}I;$$

equivalently, $\mathcal{V}^*(\Upsilon_{f,g},\mathbb{D}\times\Omega)$ *is* $*$*-isomorphic to* $\mathcal{V}^*(f,\mathbb{D})\otimes\mathcal{V}^*(g,\Omega)$.

For this, let $\rho(z_1,z_2)=(z_2,z_1)$, and each operator S in $\mathcal{V}^*(\Phi_{f,g},\mathbb{D}\times\Omega)$ is of the form

$$Sh(z_1,z_2)=c_1h(z_1,z_2)+c_2h\circ\rho(z_1,z_2),(z_1,z_2)\in\mathbb{D}\times\Omega.$$

If $\mathcal{V}^*(\Upsilon_{f,g},\mathbb{D}\times\Omega)\neq\mathbb{C}I$, $h\mapsto h\circ\rho$ defines a bounded linear operator on $L_a^2(\mathbb{D}\times\Omega)$, and it maps $L_a^2(\mathbb{D})\otimes L_a^2(\Omega)$ to $L_a^2(\mathbb{D}\times\Omega)$. By the form of ρ, each function in $L_a^2(\Omega)$ extends holomorphically to a function in $L_a^2(\mathbb{D})$. However, this can not be true because $\ln z_1\in L_a^2(\Omega)$ but $\ln z_1\notin L_a^2(\mathbb{D})$ as $\ln z_1$ can not be extended to an holomorphic function over $\mathbb{D}$.

4.2. Twisted finite Blaschke products.

This subsection mainly establishes Theorem 2.2. One can see that there is an interplay between operator theory and geometry of Riemann manifold.

Proof of Theorem 2.2. Let B_1 and B_2 be finite Blaschke products and

$$\Upsilon_{B_1,B_2}(z)=(B_1(z_1)+B_2(z_2),B_1(z_1)^2+B_2(z_2)^2),\ (z_1,z_2)\in\mathbb{D}^2.$$

Since Υ_{B_1,B_2} is the composition of two holomorphic proper maps $(z_1+z_2, z_1^2+z_2^2)$ and $(B_1(z_1), B_2(z_2))$ on $\mathbb{D}^2$, Υ_{B_1,B_2} is a holomorphic proper map on $\mathbb{D}^2$.

By Theorem 3.7, studying $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ reduces to studying admissible local inverses of Υ_{B_1,B_2}. For this, write

$$\Upsilon_{B_1,B_2}(w) = \Upsilon_{B_1,B_2}(z), w, z \in \mathbb{D}^2.$$

Then following the proof of Theorem 2.1, we get either

$$(w_1, w_2) = (\rho(z_1), \sigma(z_2)),\ \rho \in B_1^{-1} \circ B_1,\ \sigma \in B_2^{-1} \circ B_2, \tag{4.3}$$

or

$$(w_1, w_2) = (\zeta(z_2), \eta(z_1)),\ \zeta \in B_1^{-1} \circ B_2,\ \eta \in B_2^{-1} \circ B_1. \tag{4.4}$$

By Lemma 4.1, all of ρ, σ, ζ and η are admissible, and thus the local solutions of Υ_{B_1,B_2} defined in (4.3) and (4.4) are also admissible. In order to prove the abelian property of $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$, we need determine whether their equivalent classes commute with each other under composition. To clarify what is the composition of two equivalent classes [GH14], observe that for any local inverses $[\tau_1]$ and $[\tau_2]$, $\mathcal{E}_{[\tau_1]}\mathcal{E}_{[\tau_2]}$ has the form

$$\sum_j \mathcal{E}_{[\sigma_j]},$$

where the sum is finite, and σ_j can lie in the same class for distinct j; and we define *the composition*

$$[\tau_1] \circ [\tau_2]$$

to be the formal sum $\sum_j [\sigma_j]$. Thus

$$\mathcal{E}_{[\tau_1]}\mathcal{E}_{[\tau_2]} = \mathcal{E}_{[\tau_2]}\mathcal{E}_{[\tau_1]}$$

if and only if $[\tau_1] \circ [\tau_2] = [\tau_2] \circ [\tau_1]$. The formal sum of k same equivalent classes $[\sigma]$ is denoted by $k[\sigma]$.

Suppose order $B_1 = m$ and order $B_2 = n$. Let $a_1, \cdots, a_m$ be m distinct points on $\mathbb{T}$ and $b_1, \cdots, b_n$ be n distinct points on $\mathbb{T}$, both in anti-clockwise direction and

$$B_1(a_j) = B_2(b_k) = 1,\ 1 \le j \le m,\ 1 \le k \le n. \tag{4.5}$$

First, suppose S_{B_1,B_2} has more than one component, we will prove that $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ is not abelian. Since a finite Blaschke product has no critical point on the unit circle $\mathbb{T}$, it is conformal on $\mathbb{T}$. Thus for $j, k = 1, 2$ the local solutions for $\mathcal{S}_{B_j,B_k}$ are holomorphic on a neighborhood of each point on $\mathbb{T}$. Let $[\zeta](a_1)$ denote the set of all $\widetilde{\zeta}(a_1)$ as $\widetilde{\zeta}$ runs over all analytic continuations along loops in $\mathbb{T}$ beginning at a_1. Since S_{B_1,B_2} has more than one component, we have

$$[\zeta](a_1) \neq \{b_1, \cdots, b_n\}.$$

Thus there is at least a local solution η of S_{B_1,B_2} such that $\eta(a_1) \notin [\zeta](a_1)$. Denote

$$\eta(a_1) = b_{j_0}.$$

By conformal property of B_1 and B_2 on $\mathbb{T}$, local solutions for S_{B_1,B_2} (or S_{B_2,B_1}) admit continuation along any curve in $\mathbb{T}$. In particular, by (4.5) there is an a_j such that $\zeta^{-1}(b_{j_0}) = a_j$, forcing

$$\zeta(a_j) = b_{j_0}.$$

Let ρ be the identity map, and let σ be the local inverse of B_1 determined by $\sigma(a_1) = a_j$. Then it follows that

$$b_{j_0} \in \zeta \circ \sigma(a_1).$$

Since $\eta(a_1) = b_{j_0}$, we deduce that $[\zeta] \circ [\sigma]$ must contain $[\eta]$. But

$$[\rho] \circ [\zeta] = [\zeta] \neq [\eta].$$

Therefore, $[\rho] \circ [\zeta] \neq [\zeta] \circ [\sigma]$, forcing

$$([\rho] \circ [\zeta](z_2), [\sigma] \circ [\eta](z_1)) \neq ([\zeta] \circ [\sigma](z_2), [\eta] \circ [\rho](z_1)).$$

That is, there are two equivalent classes of admissible local inverses of Υ_{B_1,B_2}, (4.3) and (4.4), do not commute. Then by Theorem 3.7, $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ is not abelian.

Second, we conclude that $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ is abelian if S_{B_1,B_2} is connected. By Theorem 3.7, it suffices to show that all admissible local inverses of Υ_{B_1,B_2} commute with each other under composition. There are three cases to distinguish: both local solutions lie in (4.3), or both in (4.4), or one in (4.3) and another in (4.4).

Case 1. Both local solutions lie in (4.3). In fact, since B_1 is a finite Blaschke product, by [DPW12, Theorem 1.1] $\mathcal{V}^*(B_1, \mathbb{D})$ is abelian. Since $\mathcal{V}^*(B_1, \mathbb{D})$ is generated by $\mathcal{E}_{[\rho]}$ where ρ are local inverses of B_1, we have

$$[\rho_1] \circ [\rho_2] = [\rho_2] \circ [\rho_1],\ \rho_1, \rho_2 \in B_1^{-1} \circ B_1.$$

Similarly,

$$[\sigma_1] \circ [\sigma_2] = [\sigma_2] \circ [\sigma_1],\ \sigma_1, \sigma_2 \in B_2^{-1} \circ B_2.$$

Therefore, we have

$$([\rho_1](z_1), [\sigma_1](z_2)) \circ ([\rho_2](z_1), [\sigma_2](z_2)) = ([\rho_2](z_1), [\sigma_2](z_2)) \circ ([\rho_1](z_1), [\sigma_1](z_2)).$$

Case 2. Both local solutions lie in (4.4). In this case, the corresponding equivalent classes of local solutions commute with each other since by assumption they are exactly the same one.

Case 3. One local solution lies in (4.3) and another local solution lies in (4.4). Since S_{B_1,B_2} is connected, we assume $[\zeta]$ and $[\eta]$ are the only equivalent class for local solutions of S_{B_1,B_2} and S_{B_2,B_1}, respectively. We will prove that

$$[\rho] \circ [\zeta] = \sharp[\rho] \cdot [\zeta] \quad \text{and} \quad [\sigma] \circ [\eta] = \sharp[\sigma] \cdot [\eta].$$

In fact, for each polynomial p we have

$$\mathcal{E}_{[\zeta]} \mathcal{E}_{[\rho]} p(z) = \sum_{\widetilde{\rho} \in [\rho], \widetilde{\zeta} \in [\zeta]} p(\widetilde{\rho} \circ \widetilde{\zeta}(z))\, (\widetilde{\rho} \circ \widetilde{\zeta})'(z), z \in \mathbb{T} \tag{4.6}$$

where z is allowed in $\mathbb{T}$ since members in $[\rho]$ and $[\zeta]$ are well defined on $\mathbb{T}$ (and then in a neighborhood of $\mathbb{T}$). Since $\rho \in B_1^{-1} \circ B_1$ and $\zeta \in B_1^{-1} \circ B_2$, it follows that $\widetilde{\rho} \circ \widetilde{\zeta} \in B_1^{-1} \circ B_2$. Since $[\zeta]$ is the only equivalent class for local solutions of S_{B_1,B_2},

$$\widetilde{\rho} \circ \widetilde{\zeta} \in [\zeta],$$

and by (4.6) there is a positive integer k such that

$$\mathcal{E}_{[\zeta]}\mathcal{E}_{[\rho]} = \mathcal{E}_{[\rho]\circ[\zeta]} = k\mathcal{E}_{[\zeta]}.$$

With $z = a_1$,

$$\{\widetilde{\rho} \circ \widetilde{\zeta}(a_1) : \widetilde{\rho} \in [\rho]\}$$

has exactly $\sharp[\rho]$ points,

$$\{\widetilde{\rho} \circ \widetilde{\zeta}(a_1) : \widetilde{\rho} \in [\rho], \widetilde{\zeta} \in [\zeta]\}$$

is a sequence of $\sharp[\rho] \cdot \sharp[\zeta]$ points, and $\{\widetilde{\zeta}(a_1) : \widetilde{\zeta} \in [\zeta]\}$ has $\sharp[\zeta]$ points. Therefore by comparing (4.6) with $\mathcal{E}_{[\zeta]}p(z) = \sum_{\widetilde{\zeta}\in[\zeta]} p(\widetilde{\zeta}(z))\,\widetilde{\zeta}'(z), z \in \mathbb{T}$,

$$k = \frac{\sharp[\rho] \cdot \sharp[\zeta]}{\sharp[\zeta]} = \sharp[\rho].$$

Hence $\mathcal{E}_{[\rho]\circ[\zeta]} = \sharp[\rho] \cdot \mathcal{E}_{[\zeta]}$; that is $[\rho] \circ [\zeta] = \sharp[\rho] \cdot [\zeta]$. By similar reasoning, $[\sigma] \circ [\eta] = \sharp[\sigma] \cdot [\eta]$.

Thus,

$$([\rho] \circ [\zeta](z_2), [\sigma] \circ [\eta](z_1)) = \sharp[\rho] \cdot \sharp[\sigma]([\zeta](z_2), [\eta](z_1)).$$

Similarly,

$$([\zeta] \circ [\sigma](z_2), [\eta] \circ [\rho](z_1)) = \sharp[\rho] \cdot \sharp[\sigma]([\zeta](z_2), [\eta](z_1)),$$

which gives

$$([\rho] \circ [\zeta](z_2), [\sigma] \circ [\eta](z_1)) = ([\zeta] \circ [\sigma](z_2), [\eta] \circ [\rho](z_1)).$$

Thus, the equivalence of a local inverse (4.3) commutes with the equivalence of (4.4).

In summary, all admissible local inverses of Υ_{B_1,B_2} commute with each other under composition. Therefore, if S_{B_1,B_2} is connected, $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ is abelian. The proof is complete. $\square$

Special cases of Theorem 2.2 are of interest.

If $B_1 = B_2$, one component of $\mathcal{S}_{B_1,B_2}$ is $\{(z, z) : z \in \mathbb{D} - J\}$, where J is a finite set. Therefore, $\mathcal{S}_{B_1,B_1}$ is connected if and only if order B_1=1. This immediately gives [HZ15, Example 6.5].

If $B_1 \neq B_2$, we have the following result on abelian property of $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$.

Corollary 4.3. *Let B_1 and B_2 be two finite Blaschke products. Write $m = \text{order}\, B_1$, and $n = \text{order}\, B_2$. If $GCD(m, n) = 1$, then $\mathcal{V}^*(\Upsilon_{B_1,B_2}, \mathbb{D}^2)$ is abelian.*

Proof. Recall that $\mathcal{S}_{B_1,B_2}$ is connected if and only if all local solutions for $\mathcal{S}_{B_1,B_2}$ are equivalent in the sense of analytic continuation. By Theorem 2.2, it suffices to show that if $GCD(m,n)=1$, then all local solutions for $\mathcal{S}_{B_1,B_2}$ are equivalent. In the proof of Theorem 2.2, we have shown that a local solution for S_{B_1,B_2} admits continuation along any curve contained in $\mathbb{T}$. Without loss of generality, $m>n$. Let a_j and b_k be chosen as in the proof of Theorem 2.2. Suppose ζ is a local solution satisfying

$$\zeta(a_1)=b_1.$$

Note that for $1\leq j\leq m-1$ and $1\leq k\leq n-1$, the image of the circular arc $\widetilde{a_j a_{j+1}}$ under B_1 is the same as that of the circular arc $\widetilde{b_k b_{k+1}}$ under B_2. Then we get

$$\widetilde{\zeta}(a_j)=b_j,\ 1\leq j\leq m.$$

where $\widetilde{\zeta}$ denotes an analytic continuation along a circular curve γ in $\mathbb{T}$. Letting γ go a bit further, and noting $\widetilde{\zeta}(a_m)=b_m$, we have

$$\breve{\zeta}(a_1)=b_{m+1},$$

where $\breve{\zeta}$ is also an analytic continuation of ζ. This procedure can be repeated. Since $GCD(m,n)=1$, for each $k(1\leq k\leq n)$ there exists an analytic continuation η of ζ such that

$$\eta(a_1)=b_k.$$

Thus all local solutions for $\mathcal{S}_{B_1,B_2}$ are an analytic continuation of ζ. $\square$

Example 4.4. *Write $B_1(\lambda)=\lambda^k$ and $B_2(\lambda)=\lambda^l$, where $\lambda\in\mathbb{D}$ and k,l are positive integers. Then $\mathcal{S}_{B_1,B_2}$ is connected if and only if*

$$GCD(k,l)=1.$$

Then by Theorem 2.2 $\mathcal{V}^*(z_1^k+z_2^l,z_1^{2k}+z_2^{2l},\mathbb{D}^2)$ is abelian if and only if

$$GCD(k,l)=1.$$

Specifically, by direct computations one can check that $\mathcal{V}^*(z_1^2+z_2^4,z_1^2+z_2^4,\mathbb{D}^2)$ is not abelian ([HZ15, Example 6.5]).

4.3. General twisted proper maps. This subsection mainly focuses on the proof of Theorem 2.3.

Suppose that both Φ and Ψ are holomorphic proper maps on Ω with the same images. The following proposition tells us that the closure of the ranges of an admissible local solution and all its continuations equal $\overline{\Omega}$.

Proposition 4.5. *If both Φ and Ψ are holomorphic proper maps on Ω with same images, then for each local solution σ for $\mathcal{S}_{\Phi,\Psi}$*

$$\overline{\text{Image}\,[\sigma]}=\overline{\Omega}. \tag{4.7}$$

In particular, in the case of $\Phi=\Psi$, (4.7) holds for each local inverse σ of a holomorphic proper map Φ over Ω.

Proof. Let σ be an admissible local solution for $\mathcal{S}_{\Phi,\Psi}$ and $[\sigma]$ denote the equivalent class of σ, Image $[\sigma]$ denotes the union of all images of local solutions in the equivalent class $[\sigma]$ of σ. By definition we have

$$\mathrm{Image}\,[\sigma] \subseteq \Omega.$$

Then the inverse σ^- of σ is a local solution of $\mathcal{S}_{\Psi,\Phi}$. By Lemma 4.1, both σ and σ^- are admissible with respect to the set $\mathcal{E}$ defined by

$$\mathcal{E} = \Phi^{-1}(\Psi(\mathcal{Z}_\Psi)) \bigcup \Psi^{-1}(\Phi(\mathcal{Z}_\Phi).$$

Since σ^- or its continuation is well defined at each given point of $\Omega - \mathcal{E}$, the union of the images of σ and all its continuation contain $\Omega - \mathcal{E}$. That is,

$$\Omega - \mathcal{E} \subseteq \mathrm{Image}\,[\sigma],$$

forcing

$$\Omega - \mathcal{E} \subseteq \mathrm{Image}\,[\sigma] \subseteq \Omega.$$

Since $\mathcal{E}$ is relatively closed subset of Ω with zero Lebesgue measure, we have $\overline{\mathrm{Image}\,[\sigma]} = \overline{\Omega}$. □

Remark 4.6. If Φ is holomorphic over $\overline{\Omega}$ and σ is an admissible local inverse of Φ, then (4.7) still holds. The reasoning is similar to the above discussion.

We propose a general setting. Let $F = (f_1, \cdots, f_d)$ be a holomorphic function over a domain on $\mathbb{C}^d$. Define

$$\Upsilon_F(z) = (\varphi_1(z), \cdots, \varphi_d(z)), \tag{4.8}$$

where

$$\varphi_k(z) = \sum_{j=1}^{d} f_j^k(z), k = 1, 2, \cdots, d.$$

Let

$$\psi_1 = \varphi_1, \psi_2(z) = \sum_{1\le j<k\le d} f_j(z) f_k(z), \cdots$$

and $\psi_d(z) = \Pi_{1\le j\le d} f_j(z)$. Consider the equation

$$\Upsilon_F(w) = \Upsilon_F(z);$$

that is,

$$(\varphi_1(w), \cdots, \varphi_d(w)) = (\varphi_1(z), \cdots, \varphi_d(z)).$$

This is equivalent to

$$(\psi_1(w), \cdots, \psi_d(w)) = (\psi_1(z), \cdots, \psi_d(z)).$$

Note that

$$x^d - \psi_1(z)x^{d-1} + \cdots + (-1)^{d-1}\psi_{d-1}(z)x + (-1)^d\psi_d(z) = \prod_{j=1}^{d}(x - f_j(z)),$$

and then the solutions for the equation $\Upsilon_F(w) = \Upsilon_F(z)$ are the solution for these equations:

$$f_{\pi(j)}(w) = f_j(z),\ 1 \le j \le d, \tag{4.9}$$

where π runs over all permutations of $\{1, \cdots, d\}$.

Let us focus on a special case. Let Ω be a bounded domain in $\mathbb{C}^2$, and let $\Phi = (\phi_1, \phi_2)$ and $\Psi = (\psi_1, \psi_2)$ be holomorphic proper maps over Ω such that

$$\Phi(\Omega) = \Psi(\Omega),$$

and both Φ and Ψ are holomorphic on $\overline{\Omega}$. Write

$$f_1(z) = \phi_1(z_1, z_2), f_2(z) = \phi_2(z_1, z_2),$$

and

$$f_3(z) = \psi_1(z_3, z_4), f_4(z) = \psi_2(z_3, z_4).$$

Put $F = (f_1, \cdots, f_4)$, and rewrite $\Upsilon_{\Phi,\Psi} = \Upsilon_F$. To investigate the structure of $\mathcal{V}^*(\Upsilon_{\Phi,\Psi}, \Omega^2)$, we must determine all admissible local inverses for $\Upsilon_{\Phi,\Psi}$ on Ω^2. It is easy to get two admissible local inverses for $\Upsilon_{\Phi,\Psi}$ on Ω^2. Precisely, let

$$(\Phi(w_1, w_2), \Psi(w_3, w_4)) = (\Phi(z_1, z_2), \Psi(z_3, z_4)),$$

and

$$(\Phi(w_1, w_2), \Psi(w_3, w_4)) = (\Psi(z_3, z_4), \Phi(z_1, z_2)).$$

Then the solutions $w = (w_1, w_2, w_3, w_4)$ are

$$w = (\sigma_1(z_1, z_2), \sigma_2(z_3, z_4)),\ \sigma_1 \in \Phi^{-1} \circ \Phi, \sigma_2 \in \Psi^{-1} \circ \Psi, \tag{4.10}$$

and

$$w = (\eta_1(z_3, z_4), \eta_2(z_1, z_2)),\ \eta_1 \in \Phi^{-1} \circ \Psi, \eta_2 \in \Psi^{-1} \circ \Phi, \tag{4.11}$$

respectively. By Lemma 4.1, both (4.10) and (4.11) give admissible local inverses of $\Upsilon_{\Phi,\Psi}$.

Let (g_1, g_2, g_3, g_4) be a permutation of (f_1, f_2, f_3, f_4). By (4.9), we get

$$(g_1(w), g_2(w), g_3(w), g_4(w)) = (f_1(z), f_2(z), f_3(z), f_4(z)).$$

Letting ρ be a local inverse of $\Upsilon_{\Phi,\Psi}$ gives

$$(g_1(\rho(z)), g_2(\rho(z)), g_3(\rho(z)), g_4(\rho(z))) = (f_1(z), f_2(z), f_3(z), f_4(z)), z \in \Omega - \mathcal{E},$$

where $\mathcal{E}$ is a subset of Ω with zero Lebesgue measure. Then by (4.7) we get

$$\overline{(g_1, g_2, g_3, g_4)(\Omega^2)} = \overline{(f_1, f_2, f_3, f_4)(\Omega^2)}. \tag{4.12}$$

The equation

$$(g_1(w), g_2(w), g_3(w), g_4(w)) = (f_1(z), f_2(z), f_3(z), f_4(z)).$$

is called *compatible* if (4.12) holds. If the only possible compatible equations are

$$(\Phi(w_1, w_2), \Psi(w_3, w_4)) = (\Phi(z_1, z_2), \Psi(z_3, z_4)),$$

and

$$(\Phi(w_1, w_2), \Psi(w_3, w_4)) = (\Psi(z_3, z_4), \Phi(z_1, z_2)),$$

then we call $\Upsilon_{\Phi,\Psi}$ *has no nontrivial compatible equation.*

The above discussions immediately give the following theorem.

Theorem 4.7. *Suppose Φ and Ψ are holomorphic proper maps over Ω such that $\Phi(\Omega) = \Psi(\Omega)$, and both maps are holomorphic on $\overline{\Omega}$. Assume that $\Upsilon_{\Phi,\Psi}$ has no nontrivial compatible equation. Then $\mathcal{V}^*(\Upsilon_{\Phi,\Psi}, \Omega^2)$ is generated by $\mathcal{E}_{[\rho]}$, where ρ is of the form (4.10) or (4.11).*

Note that in Theorem 4.7 $\mathcal{V}^*(\Upsilon_{\Phi,\Psi}, \Omega^2)$ is trivial if and only if $\Phi = \Psi$ and Φ is biholomorphic.

Corollary 4.8. *Under the conditions in Theorem 4.7, $\mathcal{V}^*(\Upsilon_{\Phi,\Psi}, \Omega^2)$ is not $*$-isomorphic to $\mathcal{V}^*(\Phi, \Omega) \otimes \mathcal{V}^*(\Psi, \Omega)$.*

Theorem 2.3 provides a comparison with Theorem 4.7, and we now come to its proof.

Proof of Theorem 2.3. Suppose Φ and Ψ are two holomorphic proper maps over Ω and both maps are holomorphic on $\overline{\Omega}$. We need to determine all admissible local inverses of $\Upsilon_{\Phi,\Psi}$. Since $\Upsilon_{\Phi,\Psi}$ has no nontrivial compatible equation, it reduces to two cases of (4.9):

$$(\Phi(w_1, w_2), \Psi(w_3, w_4)) = (\Phi(z_1, z_2), \Psi(z_3, z_4)) \tag{4.13}$$

and

$$(\Phi(w_1, w_2), \Psi(w_3, w_4)) = (\Psi(z_3, z_4), \Phi(z_1, z_2)) \tag{4.14}$$

If there is an admissible local solution for (4.14), then by Remark 4.6 and (4.7)

$$\overline{\Phi(\Omega) \times \Psi(\Omega)} = \overline{\Psi(\Omega) \times \Phi(\Omega)},$$

forcing $\overline{\Phi(\Omega)} = \overline{\Psi(\Omega)}$. This is a contradiction. Therefore, there is no admissible local solution for (4.14).

For (4.13), it is clear that each admissible local solution η of (4.13) is exactly of the form $(\rho(z_1, z_2), \sigma(z_3, z_4))$, where ρ and σ are admissible local inverses of Φ and Ψ in Ω, respectively. Since Φ is a holomorphic proper map over Ω, all local inverses ρ are admissible, and those associated operators $\mathcal{E}_\rho$ generate $\mathcal{V}^*(\Phi, \Omega)$ (see Theorem 3.7). The same is true for $\mathcal{V}^*(\Psi, \Omega)$. Then by putting

$$\mathcal{E}_{[\rho(z_1,z_2),\sigma(z_3,z_4)]} \mapsto \mathcal{E}_{[\rho]} \otimes \mathcal{E}_{[\sigma]},$$

we obtain a $*$-isomorphism between $\mathcal{V}^*(\Upsilon_{\Phi,\Psi}, \Omega^2)$ and $\mathcal{V}^*(\Phi, \Omega) \otimes \mathcal{V}^*(\Psi, \Omega)$ to finish the proof of Theorem 2.3. □

As an application of Theorem 2.3, the following corollary has its own interest.

Corollary 4.9. *Suppose both f and g are holomorphic maps over $\overline{\mathbb{D}}$ such that $\overline{f(\mathbb{D})} \neq \overline{g(\mathbb{D})}$, then $\mathcal{V}^*(\Upsilon_{f,g}, \mathbb{D}^2)$ is $*$-isomorphic to $\mathcal{V}^*(f, \mathbb{D}) \otimes \mathcal{V}^*(g, \mathbb{D})$. Furthermore, $\mathcal{V}^*(\Upsilon_{f,g}, \mathbb{D}^2)$ is abelian.*

Proof. Suppose f and g are holomorphic over $\overline{\mathbb{D}}$. Following the proof of Theorem 2.3, one obtains a $*$-isomorphism between $\mathcal{V}^*(\Phi_{f,g}, \mathbb{D}^2)$ and $\mathcal{V}^*(f, \mathbb{D}) \otimes \mathcal{V}^*(g, \mathbb{D})$.

Since f is holomorphic over $\overline{\mathbb{D}}$, by Thomson's theorem [Tho77] there is a finite Blaschke product B_f such that

$$\mathcal{V}^*(f, \mathbb{D}) = \mathcal{V}^*(B_f, \mathbb{D}).$$

Recall that for each finite Blaschke product B, $\mathcal{V}^*(B, \mathbb{D})$ is abelian [DPW12, Theorem 1.1]. Then so is $\mathcal{V}^*(f, \mathbb{D})$, as well as $\mathcal{V}^*(g, \mathbb{D})$. Therefore, the von Neumann algebra $\mathcal{V}^*(f, \mathbb{D}) \otimes \mathcal{V}^*(g, \mathbb{D})$ is abelian, and hence $\mathcal{V}^*(\Phi_{f,g}, \mathbb{D}^2)$ is abelian. $\square$

To conclude this section, we present an example that does not satisfy the condition in Theorem 2.3.

Example 4.10. *Put*

$$(\Phi, \Psi)(z) = (z_1 + z_2, z_1, z_3, z_3 + z_4), z = (z_1, z_2, z_3, z_4) \in \mathbb{D}^4,$$

and rewrite $w = (w_1, w_2, w_3, w_4) \in \mathbb{D}^4$,

$$(\Phi, \Psi)(w) = (w_1 + w_2, w_1, w_3, w_3 + w_4).$$

Then the equation

$$(w_1 + w_2, w_1, w_3, w_3 + w_4) = (z_3 + z_4, z_3, z_1, z_1 + z_2)$$

is compatible. This tells us that $\Upsilon_{\Phi,\Psi}$ *does have a nontrivial compatible equation.*

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