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Twisted Alexander polynomials of twisted Whitehead links

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Abstract. The twisted Alexander polynomial has been explicitly computed for a few classes of knots and links, including twist knots [11] and genus one two-bridge knots [18]. In this paper we compute the two-variable twisted Alexander polynomial of twisted Whitehead links for $SL_2(\mathbb{C})$-representations. As an application, we verify the hyperbolic torsion conjecture in [3, 14] for twisted Whitehead links. We also obtain a formula for the Reidemeister torsion of the 3-manifold obtained from S^3 by $\frac{p}{1}$-surgery along one component of twisted Whitehead links.

Contents

1. Introduction 1241
2. Preliminaries 1244
 2.1. Twisted Alexander polynomial 1244
 2.2. Hyperbolic torsion conjecture 1245
 2.3. Reidemeister torsion 1245
 2.4. Chebyshev polynomials 1246
3. Twisted Whitehead links 1247
 3.1. A lemma 1248
 3.2. Proof of Theorem 1 for W_{2n-1} 1248
 3.3. Proof of Theorem 1 for W_{2n} 1251
4. Proof of Corollaries 1 and 2 1253
 4.1. Proof of Corollary 1 1253
 4.2. Proof of Corollary 2 1255
Acknowledgements 1257
References 1257

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1. Introduction

The twisted Alexander polynomial was introduced by Lin [9] for knots in S^3 and by Wada [21] for finitely presented groups. It is a generalization of the classical Alexander polynomial by using linear representations and has become an effective tool in topology. We refer to the survey papers [5, 10] and references therein for details on the twisted Alexander polynomial, Reidemeister torsion and their applications. Finding an explicit formula for the twisted Alexander polynomial of knots and links for all linear representations is a challenging problem. The twisted Alexander polynomial has been explicitly computed for a few classes of knots and links, including twist knots [11] and genus one two-bridge knots [18]. In this paper we compute the two-variable twisted Alexander polynomial of twisted Whitehead links for $SL_2(\mathbb{C})$-representations. As an application, we obtain a formula for the Reidemeister torsion of the 3-manifold obtained by $\frac{p}{1}$-surgery along one component of twisted Whitehead links.

For a link $L \subset S^3$ we denote by $E_L = S^3 \setminus N(L)$ its exterior, where $N(L)$ is an open tubular neighborhood of L. The link group of L is defined to be $\pi_1(E_L)$ which is the fundamental group of the link exterior. For $k \geq 0$, the k-twisted Whitehead link W_k is the two-component link depicted in Figure 1. Note that W_0 is the $(2,4)$-torus link and W_1 is the Whitehead link. Moreover, W_k is the two-bridge link $C(2,k,2)$ in the Conway notation and is $\mathfrak{b}(4k+4, 2k+1)$ in the Schubert notation. These links are all hyperbolic except for W_0, since it is known that a two-bridge link $\mathfrak{b}(2p,m)$ is non-hyperbolic if and only if $m=1$. The link group of W_k has a standard two-generator presentation of a two-bridge link group $\pi_1(E_{W_k}) = \langle a, b \mid aw = wa\rangle$, where a, b are meridians depicted in Figure 1 and w is a word in the letters a, b. More precisely, $w = (bab^{-1}a^{-1})^n a(a^{-1}b^{-1}ab)^n$ if $k = 2n-1$ and $w = (bab^{-1}a^{-1})^n bab(a^{-1}b^{-1}ab)^n$ if $k = 2n$. For a presentation $\rho : \pi_1(E_{W_k}) \to SL_2(\mathbb{C})$ we let x, y, z denote the traces of the images of a, b, ab respectively. We also let v denote the trace of the image of $bab^{-1}a^{-1}$. An explicit formula for v is given by $v = x^2+y^2+z^2-xyz-2$. A representation $\rho : \pi_1(E_{W_k}) \to SL_2(\mathbb{C})$ is called nonabelian if its image is a nonabelian subgroup of $SL_2(\mathbb{C})$.

To state a formula for the two-variable twisted Alexander polynomial of twisted Whitehead links, we first introduce the Chebyshev polynomials of the second kind. Let $S_k(v)$ be the Chebyshev polynomials defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$ for all integers k. We also let $P_k(v) = \sum_{i=0}^{k} S_i(v)$ for $k \geq 0$. The polynomial $P_k(v)$ can be expressed in terms of Chebyshev polynomials as $P_k(v) = (S_{k+1}(v) - S_k(v) - 1)/(v-2)$, see Lemma 2.7. Then $\rho : \pi_1(E_{W_k}) \to SL_2(\mathbb{C})$ is a nonabelian representation if and only if x, y, z satisfy the Riley equation $R_k(x,y,z) = 0$, where

$$\begin{aligned} R_{2n-1}(x,y,z) &= \big(xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v)\big)S_{n-1}(v), \\ R_{2n}(x,y,z) &= \big(zS_n(v) - (xy-z)S_{n-1}(v)\big)\big(S_n(v) - S_{n-1}(v)\big). \end{aligned}$$

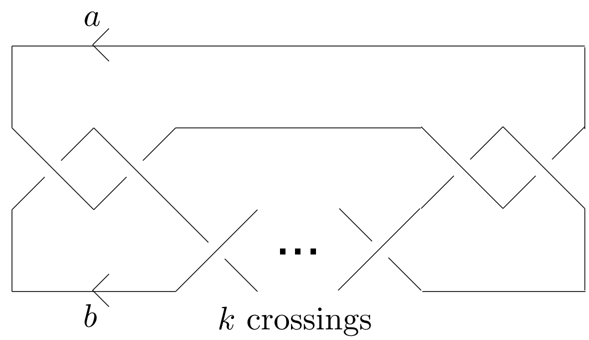

FIGURE 1. The k-twisted Whitehead link W_k.

Moreover, by [17], the defining equation for the geometric component of the $SL_2(\mathbb{C})$-character variety of W_k is $xyS_{n-1}(v)-(xy-z)S_{n-2}(v)-zS_n(v)$ if $k=2n-1$ and is $zS_n(v)-(xy-z)S_{n-1}(v)$ if $k=2n$. Here a geometric component of the $SL_2(\mathbb{C})$-character variety of a hyperbolic link is a component that contains the character of a lift of the holonomy representation of the link.

The following formula for the two-variable twisted Alexander polynomial of twisted Whitehead links is the main result of this paper.

Theorem 1. *The two-variable twisted Alexander polynomial of W_k for a nonabelian representation $\rho : \pi_1(E_{W_k}) \to SL_2(\mathbb{C})$ is given by*

$$\begin{aligned} & \Delta^{\rho}_{W_{2n-1}}(t_1,t_2) \\ = \ & \big(P_{n-1}(v)+P_{n-2}(v)\big)\big(t_1^2t_2^2 - yt_1^2t_2 - xt_1t_2^2 + t_1^2 + t_2^2 - xt_1 - yt_2 + 1\big) \\ & + 2\big(zS_{n-1}(v) + xyP_{n-2}(v)\big)t_1t_2 \end{aligned}$$

and

$$\begin{aligned} & \Delta^{\rho}_{W_{2n}}(t_1,t_2) \\ = \ & \big(P_n(v)+P_{n-1}(v)\big)(t_1^2t_2^2+1) + \big(P_{n-1}(v)+P_{n-2}(v)\big)(t_1^2+t_2^2) \\ & - 2P_{n-1}(v)(yt_1^2t_2 + xt_1t_2^2 - xyt_1t_2 + xt_1 + yt_2). \end{aligned}$$

Example 1. Consider the arithmetic two-bridge links 5^2_1 and 6^2_3 in Rolfsen's table, which are the twisted Whiteheads links $W_1 = C(2,1,2)$ and $W_2 = C(2,2,2)$ respectively.

For $5^2_1 = W_1$, $\rho : \pi_1(E_{W_1}) \to SL_2(\mathbb{C})$ is a nonabelian representation if and only if x,y,z satisfy the Riley equation $R_1(x,y,z)=0$, where $R_1(x,y,z) = xy - zv = xy - (x^2+y^2-2)z + xyz^2 - z^3$. In this case we have

$$\Delta^{\rho}_{W_1}(t_1,t_2) = t_1^2t_2^2 - yt_1^2t_2 - xt_1t_2^2 + t_1^2 + t_2^2 - xt_1 - yt_2 + 1 + 2zt_1t_2.$$

For $6^2_3 = W_2$, $\rho : \pi_1(E_{W_2}) \to SL_2(\mathbb{C})$ is a nonabelian representation if and only if x,y,z satisfy the Riley equation $R_2(x,y,z)=0$, where $R_2(x,y,z) = \big(zv-(xy-z)\big)(v-1) = \big(z^3-xyz^2+(x^2+y^2-1)z-xy\big)(x^2+y^2+z^2-xyz-3)$.

In this case we have

$$\begin{aligned}\Delta^{\rho}_{W_2}(t_1,t_2) &= (x^2+y^2+z^2-xyz)(t_1^2t_2^2+1)+t_1^2+t_2^2 \\ &\quad -2(yt_1^2t_2+xt_1t_2^2-xyt_1t_2+xt_1+yt_2).\end{aligned}$$

The hyperbolic torsion conjecture of Dunfield, Friedl and Jackson [3] states that for a hyperbolic knot K in S^3, the twisted Alexander polynomial associated to a lift of the holonomy representation detects the genus and fiberedness of K. The conjecture was generalized to hyperbolic links by Morifuji and the second author [14]. This generalized conjecture states that for an oriented hyperbolic link L in S^3, the twisted Alexander polynomial associated to a lift of the holonomy representation detects the Thurston norm and fiberedness of L. The hyperbolic torsion conjecture has been verified for all hyperbolic knots with at most 15 crossings [3] and most double twist knots/links (see [12], [13], [14]). Moreover, Agol and Dunfield [1] showed that the twisted Alexander polynomial detects the genus of libroid hyperbolic knots.

As a corollary of Theorem 1 we will show the following.

Corollary 1. *The hyperbolic torsion conjecture holds true for twisted Whitehead links.*

Let $M_{k,p}$ be the 3-manifold obtained from S^3 by $\frac{p}{1}$-surgery along the component of W_k corresponding the meridian a. Note that $M_{1,p}$ is the tunnel number one once-punctured torus bundle studied by Baker and Petersen in [2]. Its character variety and twisted Alexander polynomial (and hence its Reidemeister torsion) were computed in [2]. The following gives a formula for the Reidemeister torsion of $M_{k,p}$ for all $k \geq 1$.

Corollary 2. *Suppose $\rho : \pi_1(E_{W_k}) \to SL_2(\mathbb{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M_{k,p}) \to SL_2(\mathbb{C})$. Then the Reidemeister torsion of $M_{k,p}$ for ρ is given by*

$$\begin{aligned}&\tau^{\rho}_{M_{2n-1,p}} \\ &= 2\big[(2-x-y+z)S_{n-1}(v)+(4-2x-2y+xy)P_{n-2}(v)\big]/(2-x)\end{aligned}$$

and

$$\tau^{\rho}_{M_{2n,p}} = 2\big[S_n(v)-S_{n-1}(v)+(4-2x-2y+xy)P_{n-1}(v)\big]/(2-x).$$

This paper is organized as follows. In Section 2 we review the twisted Alexander polynomial, hyperbolic torsion conjecture, Reidemeister torsion and Chebyshev polynomials. In Section 3 we compute the two-variable twisted Alexander polynomial of twisted Whitehead links and prove Theorem 1. Finally, in Section 4 we prove Corollaries 1 and 2.

2. Preliminaries

In this section we review the twisted Alexander polynomial, hyperbolic torsion conjecture, Reidemeister torsion and Chebyshev polynomials. For more details, see [9, 21, 5, 10, 20, 6, 7, 3, 14].

2.1. Twisted Alexander polynomial. Let $L = K_1 \cup \cdots \cup K_m$ be an oriented m-component link in S^3 and $E_L = S^3 \setminus N(L)$ its exterior, where $N(L)$ is an open tubular neighborhood of L. We choose a Wirtinger presentation of the link group of L:

$$\pi_1(E_L) = \langle x_1, \dots, x_l \,|\, r_1, \dots, r_{l-1} \rangle.$$

The abelianization homomorphism

$$\mathfrak{a} : \pi_1(E_L) \to H_1(E_L; \mathbb{Z}) \cong \mathbb{Z}^{\oplus m} = \langle t_1 \rangle \oplus \cdots \oplus \langle t_m \rangle$$

is given by assigning to each generator x_i a meridian element $t_j \in H_1(E_L; \mathbb{Z})$ of its corresponding component K_j. Here we denote the sum in $\mathbb{Z}$ multiplicatively.

We consider a linear representation $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$. The maps ρ and $\mathfrak{a}$ naturally induce two ring homomorphisms $\tilde{\rho} : \mathbb{Z}[\pi_1(E_L)] \to M_2(\mathbb{C})$ and $\tilde{\mathfrak{a}} : \mathbb{Z}[\pi_1(E_L)] \to \mathbb{Z}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$, where $\mathbb{Z}[\pi_1(E_L)]$ is the group ring of $\pi_1(E_L)$ and $M_2(\mathbb{C})$ is the matrix algebra of degree 2 over $\mathbb{C}$. Then the tensor product $\tilde{\rho} \otimes \tilde{\mathfrak{a}}$ defines a ring homomorphism $\mathbb{Z}[\pi_1(E_L)] \to M_2\left(\mathbb{C}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]\right)$. Let F_l denote the free group on generators $x_1, \dots, x_l$ and $\Phi : \mathbb{Z}[F_l] \to M_2\left(\mathbb{C}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]\right)$ the composition of the surjection $\tilde{\phi} : \mathbb{Z}[F_l] \to \mathbb{Z}[\pi_1(E_L)]$ induced by the presentation of $\pi_1(E_L)$ and the map $\tilde{\rho} \otimes \tilde{\mathfrak{a}} : \mathbb{Z}[\pi_1(E_L)] \to M_2\left(\mathbb{C}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]\right)$.

Let A denote the $(l-1) \times l$ matrix whose (i,j)-entry is the 2×2 matrix $\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M_2\left(\mathbb{C}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]\right)$, where $\frac{\partial}{\partial x}$ denotes the Fox differential. For $1 \le j \le l$, we denote by A_j the $(l-1) \times (l-1)$ matrix obtained from A by removing the jth column. We regard A_j as a $2(l-1) \times 2(l-1)$ matrix with coefficients in $\mathbb{C}[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$. Then Wada's twisted Alexander polynomial [21] of the link L for a representation $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$ is defined to be the rational function

$$\Delta_L^{\rho}(t_1, \cdots, t_m) = \frac{\det A_j}{\det \Phi(1 - x_j)}$$

and is well-defined up to multiplication by $t_1^{k_1} \cdots t_m^{k_m}$ $(k_i \in \mathbb{Z})$. It is known that the rational function $\Delta_L^{\rho}(t_1, \cdots, t_m)$ will be a Laurent polynomial in the variables $t_1, \cdots, t_m$ if L is a link with two or more components [21, Proposition 9], or L is a knot K and ρ is a nonabelian representation [8, Theorem 3.1].

When $t_1 = \cdots = t_m = t$, we denote $\Delta_L^{\rho}(t_1, \cdots, t_m)$ by $\Delta_L^{\rho}(t)$ and call it the reduced twisted Alexander polynomial of L.

2.2. Hyperbolic torsion conjecture. Let M be a compact connected orientable 3-manifold and $\sigma \in H^1(M;\mathbb{Z})$. The *Thurston norm* of σ is defined as

$$||\sigma||_T = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \sigma\},$$

where for a given surface S with connected components $S_1 \cup \cdots \cup S_k$, we define $\chi_-(S) = \sum_{i=1}^{k} \max\{-\chi(S_i), 0\}$. Here $\chi(S)$ denotes the Euler characteristice of a surface S. Thurston [15] showed that $||\cdot||_T$ defines a seminorm on $H^1(M;\mathbb{Z})$.

For a hyperbolic knot K in S^3, Dunfield, Friedl and Jackson [3] studied the twisted Alexander polynomial $\Delta_{K,\rho_0}(t)$ associated to the holonomy representation $\rho_0 : \pi_1(E_K) \to SL_2(\mathbb{C})$ and called it the hyperbolic torsion polynomial of K. Based on many computations, they conjectured that $\Delta_{K,\rho_0}(t)$ determines the genus of K, in the sense that $\deg \Delta_{K,\rho_0}(t) = 4g(K) - 2$. Moreover, the knot K is fibered if and only if $\Delta_{K,\rho_0}(t)$ is monic. Here, the degree of a Laurent polynomial $f(t) \in \mathbb{C}[t^{\pm 1}]$ is the difference between the highest degree and the lowest degree of f.

Morifuji and the second author generalized the conjecture to the case of hyperbolic links. For a μ-component oriented link L, let $\omega \in H^1(E_L;\mathbb{Z})$ be given by sending each meridian of L to one. The generalized conjecture in [14] states that for an oriented hyperbolic link L in S^3, the twisted Alexander polynomial $\Delta_{L,\rho_0}(t)$ associated to the holonomy representation $\rho_0 : \pi_1(E_L) \to SL_2(\mathbb{C})$ determines the Thurston norm of ω, in the sense that $\deg \Delta_{L,\rho_0}(t) = 2||\omega||_T$. Moreover, the link L is fibered if and only if $\Delta_{L,\rho_0}(t)$ is monic. We should remark that $\deg \Delta_{L,\rho}(t) \leq 2||\omega||_T$ holds for any representation $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$, see [4].

For a knot or a μ-component alternating link L in S^3, it is known that the Thurston norm $||\omega||_T$ is equal to $2g(L) + \mu - 2$, see e.g. [14, Remark 3.4]. Hence the equality in the above conjecture becomes $\deg \Delta_{L,\rho_0}(t) = 4g(L) + 2(\mu - 2)$ for alternating links.

2.3. Reidemeister torsion.

2.3.1. Torsion of a chain complex. Let C be a chain complex of finite dimensional vector spaces over $\mathbb{C}$:

$$C = \left(0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0\right)$$

such that for each $i = 0, 1, \cdots, m$ the followings hold

- the homology group $H_i(C)$ is trivial, and
- a preferred basis c_i of C_i is given.

Let $B_i \subset C_i$ be the image of ∂_{i+1}. For each i choose a basis b_i of B_i. The short exact sequence of $\mathbb{C}$-vector spaces

$$0 \to B_i \longrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

implies that a new basis of C_i can be obtained by taking the union of the vectors of b_i and some lifts $\tilde{b}_{i-1}$ of the vectors b_{i-1}. Define $[(b_i \cup \tilde{b}_{i-1})/c_i]$ to be the determinant of the matrix expressing $(b_i \cup \tilde{b}_{i-1})$ in the basis c_i. Note that this scalar does not depend on the choice of the lift $\tilde{b}_{i-1}$ of b_{i-1}. The *torsion* of C is then defined to be

$$\tau(C) := \prod_{i=0}^{m} [(b_i \cup \tilde{b}_{i-1})/c_i]^{(-1)^{i+1}} \in \mathbb{C} \setminus \{0\}.$$

Remark 2.1. Once a preferred basis of C is given, $\tau(C)$ is independent of the choice of $b_0, \dots, b_m$. See e.g. [20].

2.3.2. Reidemeister torsion of a CW-complex. Let M be a finite CW-complex and $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ a representation. Denote by $\tilde{M}$ the universal covering of M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C(\tilde{M}; \mathbb{Z})$ has the structure of a chain complex of left $\mathbb{Z}[\pi_1(M)]$-modules.

Let V be the 2-dimensional vector space $\mathbb{C}^2$ with the canonical basis $\{e_1, e_2\}$. Using the representation ρ, V has the structure of a right $\mathbb{Z}[\pi_1(M)]$-module which we denote by V_ρ. Define the chain complex $C(M; V_\rho)$ to be $V_\rho \otimes_{\mathbb{Z}[\pi_1(M)]} C(\tilde{M}; \mathbb{Z})$, and choose a preferred basis of $C(M; V_\rho)$ as follows. Let $\{u_1^i, \cdots, u_{m_i}^i\}$ be the set of i-cells of M, and choose a lift $\tilde{u}_j^i$ of each cell. Then $\{\tilde{u}_1^i \otimes e_1, \tilde{u}_1^i \otimes e_2, \cdots, \tilde{u}_{m_i}^i \otimes e_1, \tilde{u}_{m_i}^i \otimes e_2\}$ is chosen to be the preferred basis of $C_i(M; V_\rho)$.

The Reidemeister torsion τ_M^ρ is defined as follows:

$$\tau_M^\rho = \begin{cases} \tau(C(M; V_\rho)) & \text{if } \rho \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

Here a representation ρ is called *acyclic* if all the homology groups $H_i(M; V_\rho)$ are trivial.

When M is the exterior E_L of a link $L \subset S^3$ and $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ is a representation, we also denote τ_M^ρ by τ_L^ρ and call it the Reidemeister torsion of L for ρ. In which case Johnson showed the following. See also [7].

Theorem 2.2. [6] *For any representation $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$ we have*

$$\tau_L^\rho = \Delta_L^\rho(1, \dots, 1).$$

2.4. Chebyshev polynomials. Recall that $S_k(v)$ are the Chebyshev polynomials of the second kind defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$ for all integers k. Similarly, let $T_k(v)$ be the Chebyshev polynomials of the first kind defined by $T_0(v) = 2$, $T_1(v) = v$ and $T_k(v) = vT_{k-1}(v) - T_{k-2}(v)$ for all integers k. Note that $T_k(v) = S_k(v) - S_{k-2}(v)$. For $k \geq 0$ we let $P_k(v) = \sum_{i=0}^{k} S_i(v)$.

The following four lemmas are taken from [14].

Lemma 2.3. *Write $v = q + q^{-1}$. Then $T_k(v) = q^k + q^{-k}$.*

We have $S_k(2) = k+1$ and $S_k(-2) = (-1)^k(k+1)$. If $q \neq \pm 1$ then $S_k(v) = \frac{q^{k+1} - q^{-(k+1)}}{q - q^{-1}}$. In particular, if $v = 2\cos\beta$, where $\frac{\beta}{\pi} \in \mathbb{R} \setminus \mathbb{Z}$, then $S_k(v) = \frac{\sin(k+1)\beta}{\sin\beta}$.

Lemma 2.4. *(i) If $k \geq 1$ is even, then*

$$T_k(v) - 2 = (v-2)(v+2) \prod_{j=1}^{\frac{k}{2}-1} \Big(v - 2\cos\frac{2j\pi}{k}\Big)^2.$$

(ii) If $k \geq 1$ is odd, then

$$T_k(v) - 2 = (v-2) \prod_{j=1}^{\frac{k-1}{2}} \Big(v - 2\cos\frac{2j\pi}{k}\Big)^2.$$

Lemma 2.5. *(i) If $k \geq 1$ is odd, then*

$$T_k(v) - v = (v-2)(v+2) \prod_{j=1}^{\frac{k-3}{2}} \Big(v - 2\cos\frac{2j\pi}{k-1}\Big) \prod_{j=1}^{\frac{k-1}{2}} \Big(v - 2\cos\frac{2j\pi}{k+1}\Big).$$

(ii) If $k \geq 1$ is even, then

$$T_k(v) - v = (v-2) \prod_{j=1}^{\frac{k-2}{2}} \Big(v - 2\cos\frac{2j\pi}{k-1}\Big) \prod_{j=1}^{\frac{k}{2}} \Big(v - 2\cos\frac{2j\pi}{k+1}\Big).$$

Lemma 2.6. *For $k \in \mathbb{Z}$ we have $S_{k+1}(v)S_{k-1}(v) + 1 = S_k^2(v)$.*

We also need the following two lemmas which are proved in [19].

Lemma 2.7. *For $k \geq 0$ we have*

(1) $P_k(v) = \big(S_{k+1}(v) - S_k(v) - 1\big)/(v-2)$.
(2) $P_{k+1}(v) + P_{k-1}(v) = vP_k(v) + 1$.
(3) $P_k^2(v) + P_{k-1}^2(v) = vP_k(v)P_{k-1}(v) + P_k(v) + P_{k-1}(v)$.

Lemma 2.8. *Suppose $V \in SL_2(\mathbb{C})$ and $v = \operatorname{tr} V$. For $k \geq 0$ we have*

$$\sum_{i=0}^{k} V^i = P_k(v)\mathbf{1} - P_{k-1}(v)V^{-1}$$

where $\mathbf{1}$ denotes the identity 2×2 matrix.

3. Twisted Whitehead links

In this section we compute the two-variable twisted Alexander polynomial of twisted Whitehead links and prove Theorem 1. We start with a linear algebra lemma which is useful for computing the twisted Alexander polynomial.

3.1. A lemma. For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we let

$$A^* = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

denote the adjugate of A. Note that

$$\operatorname{tr} AB^* = \operatorname{tr} A^*B = a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12}$$

for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Lemma 3.1. *Suppose $A, B, C \in M_2(\mathbb{C})$ and $x, y \in \mathbb{C}$. Then*

$$\begin{aligned} \det(Ax + By + C) &= (\det A)x^2 + (\det B)y^2 + (\operatorname{tr} AB^*)xy \\ &\quad + (\operatorname{tr} AC^*)x + (\operatorname{tr} BC^*)y + \det C. \end{aligned}$$

Proof. Write $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$. We have

$$\begin{aligned} & \det(Ax + By + C) \\ = \; & \det \begin{bmatrix} xa_{11} + yb_{11} + c_{11} & xa_{12} + yb_{12} + c_{12} \\ xa_{21} + yb_{21} + c_{21} & xa_{22} + yb_{22} + c_{22} \end{bmatrix} \\ = \; & (xa_{11} + yb_{11} + c_{11})(xa_{22} + yb_{22} + c_{22}) \\ & - (xa_{12} + yb_{12} + c_{12})(xa_{21} + yb_{21} + c_{21}) \\ = \; & (a_{11}a_{22} - a_{12}a_{21})x^2 + (b_{11}b_{22} - b_{12}b_{21})y^2 \\ & + (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12})xy \\ & + (a_{11}c_{22} + a_{22}c_{11} - a_{12}c_{21} - a_{21}c_{12})x \\ & + (b_{11}c_{22} + b_{22}c_{11} - b_{12}c_{21} - b_{21}c_{12})y \\ & + c_{11}c_{22} - c_{12}c_{21}. \end{aligned}$$

The lemma then follows. □

3.2. Proof of Theorem 1 for W_{2n-1}. Recall that the link group of W_{2n-1} has a presentation $\pi_1(E_{W_{2n-1}}) = \langle a, b \mid wa = aw \rangle$, where a, b are meridians depicted in Figure 1 and $w = (bab^{-1}a^{-1})^n a(a^{-1}b^{-1}ab)^n$. Suppose $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix} \tag{1}$$

where $(u, s_1, s_2) \in (\mathbb{C}^*)^3$ satisfies $\rho(aw) = \rho(wa)$. By [17] this matrix equation is equivalent to the following equation

$$\big(xyS_{n-1}(v) - (xy - z)S_{n-2}(v) - zS_n(v)\big)S_{n-1}(v) = 0. \tag{2}$$

Here x, y, z and v are the traces of the images of a, b, ab and $bab^{-1}a^{-1}$ respectively.

Let $r = awa^{-1}w^{-1}$. We have

$$\frac{\partial r}{\partial b} = a\left(\frac{\partial w}{\partial b} - wa^{-1}w^{-1}\frac{\partial w}{\partial b}\right) = a(1 - wa^{-1}w^{-1})\frac{\partial w}{\partial b}.$$

Hence $\Delta^{\rho}_{W_{2n-1}}(t_1, t_2) = \det \Phi\left(\frac{\partial r}{\partial b}\right) / \det \Phi(1-a) = \det \Phi\left(\frac{\partial w}{\partial b}\right)$.

Let $c = bab^{-1}a^{-1}$ and $d = a^{-1}b^{-1}ab$. We have $w = c^n a d^n$ and

$$\begin{aligned}\frac{\partial w}{\partial b} &= \delta_{n-1}(c)\frac{\partial c}{\partial b} + c^n a \delta_{n-1}(d)\frac{\partial d}{\partial b} \\ &= c^n\big(\delta_{n-1}(c^{-1})(aba^{-1}b^{-1} - a) + a\,\delta_{n-1}(d)(a^{-1}b^{-1}a - a^{-1}b^{-1})\big)\end{aligned}$$

where $\delta_k(g) = 1 + g + \cdots + g^k$.

For $g \in \pi_1(E_{W_{2n-1}})$ we denote $\rho(g)$ by $\tilde{g}$. Since $\operatorname{tr}\tilde{c} = \operatorname{tr}\tilde{d} = v$, by Lemma 2.8 we have

$$\begin{aligned}\delta_{n-1}(\tilde{c}^{\,-1}) &= P_{n-1}(v)\mathbf{1} - P_{n-2}(v)\tilde{c}, \\ \delta_{n-1}(\tilde{d}) &= P_{n-1}(v)\mathbf{1} - P_{n-2}(v)\tilde{d}^{\,-1}.\end{aligned}$$

Since $\Phi(a) = t_1\tilde{a}$ and $\Phi(b) = t_2\tilde{b}$, we then obtain

$$\Phi\left(\frac{\partial w}{\partial b}\right) = \tilde{c}^{\,n}\big(P_{n-1}(v)E - P_{n-2}(v)F\big)$$

where

$$\begin{aligned}E &= \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} - t_1\tilde{a} + t_1t_2^{-1}\tilde{b}^{-1}\tilde{a} - t_2^{-1}\tilde{b}^{-1}, \\ F &= \tilde{c}(\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} - t_1\tilde{a}) + \tilde{a}\tilde{d}^{-1}(t_1t_2^{-1}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a} - t_2^{-1}\tilde{a}^{-1}\tilde{b}^{-1}) \\ &= \mathbf{1} - t_1\tilde{b}\tilde{a}\tilde{b}^{-1} + t_1t_2^{-1}\tilde{a}\tilde{b}^{-1} - t_2^{-1}\tilde{a}\tilde{b}^{-1}\tilde{a}^{-1}.\end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned}\det \Phi\left(\frac{\partial w}{\partial b}\right) &= \det(P_{n-1}(v)E - P_{n-2}(v)F) \\ &= (\det E)P^2_{n-1}(v) + (\det F)P^2_{n-2}(v) \\ &\quad - (\operatorname{tr} EF^*)P_{n-1}(v)P_{n-2}(v).\end{aligned}$$

Since $P^2_{n-1}(v) + P^2_{n-2}(v) = vP_{n-1}(v)P_{n-2}(v) + P_{n-1}(v) + P_{n-2}(v)$ (by Lemma 2.7) we get

$$\begin{aligned}\det \Phi\left(\frac{\partial w}{\partial b}\right) &= (\det F - \det E)P^2_{n-2}(v) \\ &\quad + (v\det E - \operatorname{tr} EF^*)P_{n-1}(v)P_{n-2}(v) \\ &\quad + (\det E)\big(P_{n-1}(v) + P_{n-2}(v)\big).\end{aligned} \tag{3}$$

Using the form of ρ in (1) we have $E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$ where

$$\begin{aligned}
e_{11} &= 1 - s_1t_1 - s_2^{-1}t_2^{-1} + s_1s_2^{-1}t_1t_2^{-1} + (s_1^{-1}s_2^{-1} - s_1s_2^{-1} + s_1s_2)u + u^2, \\
e_{12} &= s_1 - s_1s_2^2 - t_1 + s_2^{-1}t_1t_2^{-1} - s_2u, \\
e_{21} &= (-s_2^{-1} + s_1^{-2}s_2^{-1} + t_2^{-1} - s_1t_1t_2^{-1})u + s_1^{-1}u^2, \\
e_{22} &= 1 - s_1^{-1}t_1 - s_2t_2^{-1} + s_1^{-1}s_2t_1t_2^{-1} - (s_1^{-1}s_2 + t_1t_2^{-1})u,
\end{aligned}$$

and $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ where

$$\begin{aligned}
f_{11} &= 1 - s_1t_1 - s_2^{-1}t_2^{-1} + s_1s_2^{-1}t_1t_2^{-1} + (s_2t_1 + s_1^{-1}t_2^{-1} - t_1t_2^{-1})u, \\
f_{12} &= -s_2^2t_1 + s_1s_2^{-1}t_2^{-1} - s_1s_2t_2^{-1} + s_2t_1t_2^{-1} - t_2^{-1}u, \\
f_{21} &= (s_1^{-1}s_2^{-1}t_1 - s_1s_2^{-1}t_1 + s_1^{-2}t_2^{-1} - s_1^{-1}t_1t_2^{-1})u + t_1u^2, \\
f_{22} &= 1 - s_1^{-1}t_1 - s_2t_2^{-1} + s_1^{-1}s_2t_1t_2^{-1} - (s_2t_1 + s_1^{-1}t_2^{-1})u.
\end{aligned}$$

Then, by direct calculations, we have

$$\begin{aligned}
\det F - \det E &= (v-2)(2z - xy)t_1t_2^{-1}, && (4) \\
v \det E - \operatorname{tr} EF^* &= (v-2)(xy - vz)t_1t_2^{-1}, && (5) \\
\det E &= t_1^2t_2^{-2} - yt_1^2t_2^{-1} - xt_1t_2^{-2} + t_1^2 + t_2^{-2} && (6) \\
&\quad + \big(xy - (v-2)z\big)t_1t_2^{-1} - xt_1 - yt_2^{-1} + 1.
\end{aligned}$$

Note that $x = s_1 + s_1^{-1}$, $y = s_2 + s_2^{-1}$, $z = s_1s_2 + s_1^{-1}s_2^{-1} + u$ and $v = x^2 + y^2 + z^2 - xyz - 2$.

From equations (4) and (5) we have

$$\begin{aligned}
&(\det F - \det E)P_{n-2}^2(v) + (v \det E - \operatorname{tr} EF^*)P_{n-1}(v)P_{n-2}(v) \qquad (7) \\
= \;& \big[(2z - xy)P_{n-2}(v) + (xy - vz)P_{n-1}(v)\big](v-2)P_{n-2}(v)t_1t_2^{-1} \\
= \;& \big[(xy - 2z)\big(P_{n-1}(v) - P_{n-2}(v)\big) - z(v-2)P_{n-1}(v)\big] \\
& \times (v-2)P_{n-2}(v)t_1t_2^{-1} \\
= \;& \big[(xy - 2z)S_{n-1}(v) - z\big(S_n(v) - S_{n-1}(v) - 1\big)\big] \\
& \times \big[S_{n-1}(v) - S_{n-2}(v) - 1\big]t_1t_2^{-1} \\
= \;& \big[(xy - z)S_{n-1}(v) - zS_n(v) + z\big]\big[S_{n-1}(v) - S_{n-2}(v) - 1\big]t_1t_2^{-1}.
\end{aligned}$$

Here we use $(v-2)P_k(v) = S_{k+1}(v) - S_k(v) - 1$ (by Lemma 2.7) in the third equality.

From equations (3), (6) and (7) we have

$$\begin{aligned}
\Delta^{\rho}_{W_{2n-1}}(t_1, t_2) &= \big(t_1^2t_2^{-2} - yt_1^2t_2^{-1} - xt_1t_2^{-2} + t_1^2 + t_2^{-2} - xt_1 - yt_2^{-1} + 1\big) \\
&\quad \times \big(P_{n-1}(v) + P_{n-2}(v)\big) + \alpha t_1t_2^{-1}.
\end{aligned}$$

where

$$\begin{aligned}\alpha &= \big[(xy-z)S_{n-1}(v) - zS_n(v) + z\big]\big[S_{n-1}(v) - S_{n-2}(v) - 1\big] \\ &\quad + \big(xy - (v-2)z\big)\big(P_{n-1}(v) + P_{n-2}(v)\big).\end{aligned}$$

Finally, we simplify α using equation (2). By Lemmas 2.6 and 2.7 we have $S_n(v)S_{n-2}(v)+1 = S_{n-1}^2(v)$ and $(v-2)(P_{n-1}(v)+P_{n-2}(v)) = S_n(v) - S_{n-2}(v) - 2$. Hence

$$\begin{aligned}\alpha &= (xy-z)S_{n-1}^2(v) - S_{n-1}(v)\big[(xy-z)(S_{n-2}(v)+1) + z(S_n(v)-1)\big] \\ &\quad + z(S_n(v)-1)(S_{n-2}(v)+1) - z(S_n(v) - S_{n-2}(v) - 2) \\ &\quad + xy\big(P_{n-1}(v) + P_{n-2}(v)\big) \\ &= (xy-z)S_{n-1}^2(v) - S_{n-1}(v)\big[(xy-z)S_{n-2}(v) + zS_n(v)\big] \\ &\quad - (xy-2z)S_{n-1}(v) + xy\big(S_{n-1}(v) + 2P_{n-2}(v)\big) \\ &\quad + z(S_n(v)S_{n-2}(v) + 1) \\ &= S_{n-1}(v)\big[xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v)\big] \\ &\quad + 2zS_{n-1}(v) + 2xyP_{n-2}(v) \\ &= 2zS_{n-1}(v) + 2xyP_{n-2}(v).\end{aligned}$$

Here we use equation (2) in the last equality.

The proof of Theorem 1 for W_{2n-1} is complete, since $\Delta^{\rho}_{W_{2n-1}}(t_1,t_2)$ is well-defined up to multiplication by $t_1^{k_1}t_2^{k_2}$ ($k_i \in \mathbb{Z}$).

3.3. Proof of Theorem 1 for W_{2n}. Recall that the link group of W_{2n} has a presentation $\pi_1(E_{W_{2n}}) = \langle a, b \mid wa = aw\rangle$, where a, b are meridians depicted in Figure 1 and $w = (bab^{-1}a^{-1})^n bab(a^{-1}b^{-1}ab)^n$. Suppose $\rho : \pi_1(E_{W_{2n}}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1}\end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1}\end{bmatrix}$$

where $(u, s_1, s_2) \in (\mathbb{C}^*)^3$ satisfies $\rho(aw) = \rho(wa)$. By [17] this matrix equation is equivalent to the following equation

$$\big(zS_n(v) - (xy-z)S_{n-1}(v)\big)\big(S_n(v) - S_{n-1}(v)\big) = 0. \tag{8}$$

As in the case of W_{2n-1} we have $\Delta^{\rho}_{W_{2n}}(t_1,t_2) = \det \Phi\left(\frac{\partial w}{\partial b}\right)$ where

$$\begin{aligned}\frac{\partial w}{\partial b} &= c^n\big[\delta_{n-1}(c^{-1})(aba^{-1}b^{-1} - a) + bab\,\delta_{n-1}(d)(a^{-1}b^{-1}a - a^{-1}b^{-1}) \\ &\qquad + 1 + ba\big].\end{aligned}$$

Since $\Phi(a) = t_1\tilde{a}$ and $\Phi(b) = t_2\tilde{b}$, we obtain

$$\Phi\left(\frac{\partial w}{\partial b}\right) = \tilde{c}^{\,n}\big(P_{n-1}(v)G - P_{n-2}(v)H + I\big)$$

where

$$
\begin{aligned}
G &= \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} - t_1\tilde{a} + t_1t_2\tilde{b}\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a} - t_2\tilde{b}\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1},\\
H &= \mathbf{1} - t_1\tilde{b}\tilde{a}\tilde{b}^{-1} + t_1t_2\tilde{b}\tilde{a} - t_2\tilde{b},\\
I &= \mathbf{1} + t_1t_2\tilde{b}\tilde{a}.
\end{aligned}
$$

By Lemmas 3.1 and 2.7 we obtain

$$
\begin{aligned}
\det\Phi\left(\frac{\partial w}{\partial b}\right) &= \det(P_{n-1}(v)G - P_{n-2}(v)H + I) \qquad (9)\\
&= (\det G)P_{n-1}^2(v) + (\det H)P_{n-2}^2(v) + \det I\\
&\quad - (\operatorname{tr} GH^*)P_{n-1}(v)P_{n-2}(v)\\
&\quad + (\operatorname{tr} GI^*)P_{n-1}(v) - (\operatorname{tr} HI^*)P_{n-2}(v)\\
&= (\det G - \det H)P_{n-1}^2(v) + \det I\\
&\quad + (v\det H - \operatorname{tr} GH^*)P_{n-1}(v)P_{n-2}(v)\\
&\quad + (\det H + \operatorname{tr} GI^*)P_{n-1}(v)\\
&\quad + (\det H - \operatorname{tr} HI^*)P_{n-2}(v).
\end{aligned}
$$

By direct calculations we have

$$
\begin{aligned}
\det G - \det H &= (v-2)(vz-xy)t_1t_2, \qquad (10)\\
v\det H - \operatorname{tr} GH^* &= (v-2)(xy-2z)t_1t_2, \qquad (11)\\
\det H + \operatorname{tr} GI^* &= (v+1)t_1^2t_2^2 - 2xt_1t_2^2 - 2yt_1^2t_2 + t_1^2 + t_2^2 + 1 \qquad (12)\\
&\quad + \big(xy + 2(v-1)z\big)t_1t_2 - 2xt_1 - 2yt_2 + v,\\
\det H - \operatorname{tr} HI^* &= -t_1^2t_2^2 + t_1^2 + t_2^2 + (xy-2z)t_1t_2 - 1, \qquad (13)\\
\det I &= t_1^2t_2^2 + zt_1t_2 + 1. \qquad (14)
\end{aligned}
$$

From equations (10) and (11) we have

$$
\begin{aligned}
&(\det G - \det H)P_{n-1}^2(v) + (v\det H - \operatorname{tr} GH^*)P_{n-1}(v)P_{n-2}(v) \qquad (15)\\
={}& \big[(vz-xy)P_{n-1}(v) + (xy-2z)P_{n-2}(v)\big](v-2)P_{n-1}(v)t_1t_2\\
={}& \big[z(v-2)P_{n-1}(v) - (xy-2z)(P_{n-1}(v) - P_{n-2}(v))\big](v-2)P_{n-1}(v)t_1t_2\\
={}& \big[z\big(S_n(v) - S_{n-1}(v) - 1\big) - (xy-2z)S_{n-1}(v)\big](v-2)P_{n-1}(v)t_1t_2\\
={}& \big[zS_n(v) - (xy-z)S_{n-1}(v) - z\big]\big[S_n(v) - S_{n-1}(v) - 1\big]t_1t_2.
\end{aligned}
$$

From equations (9), (15), (12), (13) and (14) we obtain

$$
\begin{aligned}
\Delta^{\rho}_{W_{2n}}(t_1,t_2) &= \big((v+1)P_{n-1}(v) - P_{n-2}(v) + 1\big)(t_1^2t_2^2+1)\\
&\quad + \big(P_{n-1}(v) + P_{n-2}(v)\big)(t_1^2+t_2^2)\\
&\quad - 2P_{n-1}(v)(yt_1^2t_2 + xt_1t_2^2 + xt_1 + yt_2) + \beta t_1t_2
\end{aligned}
$$

where

$$
\begin{aligned}
\beta &= \big[zS_n(v) - (xy-z)S_{n-1}(v) - z\big]\big[S_n(v) - S_{n-1}(v) - 1\big]\\
&\quad + \big(xy + 2(v-1)z\big)P_{n-1}(v) + (xy-2z)P_{n-2}(v) + z.
\end{aligned}
$$

Note that $(v+1)P_{n-1}(v) - P_{n-2}(v) + 1 = P_n(v) + P_{n-1}(v)$, by Lemma 2.7.

Finally, we simplify β using equation (8). Indeed, since

$$\big(zS_n(v) - (xy-z)S_{n-1}(v)\big)\big(S_n(v) - S_{n-1}(v)\big) = 0$$

we have

$$\begin{aligned}
\beta &= \big[z - (zS_n(v) - (xy-z)S_{n-1}(v)) - z(S_n(v) - S_{n-1}(v))\big] \\
&\quad + \big(xy + 2(v-1)z\big)P_{n-1}(v) + (xy-2z)(P_{n-1}(v) - S_{n-1}(v)) + z \\
&= 2z - 2zS_n(v) + 2zS_{n-1}(v) + 2\big(xy + (v-2)z\big)P_{n-1}(v) \\
&= 2xyP_{n-1}(v).
\end{aligned}$$

Here we use $(v-2)P_{n-1}(v) = S_n(v) - S_{n-1}(v) - 1$ (by Lemma 2.7) in the last equality.

This completes the proof of Theorem 1 for W_{2n}.

4. Proof of Corollaries 1 and 2

4.1. Proof of Corollary 1. We prove the hyperbolic torsion conjecture for W_{2n-1}. The proof for W_{2n} is similar. Up to mirror image, there are two orientations on W_{2n-1} which correspond to the two cases $t_1 = t_2 = t$ and $t_1 = t_2^{-1} = t$ in the reduced twisted Alexander polynomial $\Delta_{W_{2n-1}}(t)$. In both cases, the genus of $L = W_{2n-1}$ is given by $g(L) = 1$ and L is fibered if and only if $n = 1$. (These facts can be proved by computing the reduced Alexander polynomial of L and then applying [16, Theorem 1.1].) Moreover, by Theorem 1, the degree of the reduced twisted Alexander polynomial $\Delta^{\rho}_{W_{2n-1}}(t)$ is 4 and the coefficient of the highest degree term is $P_{n-1}(v) + P_{n-2}(v)$. Note that the hyperbolic torsion conjecture holds true for the fibered link W_1 since $P_{n-1}(v) + P_{n-2}(v) = 1$ when $n = 1$. We now consider the case $n \geq 2$.

For a link L in S^3, a non-abelian representation $\rho : \pi_1(E_L) \to SL_2(\mathbb{C})$ is called parabolic if the images of all the meridians of L by ρ are matrices with trace 2. When L is hyperbolic, it is known that the holonomy representation ρ_0 is a parabolic representation. Moreover, ρ_0 lies on the geometric component of the $SL_2(\mathbb{C})$-character variety of L.

To prove Corollary 1 for W_{2n-1}, with $n \geq 2$, it suffices to show that $P_{n-1}(v) + P_{n-2}(v) \neq 0$ and $P_{n-1}(v) + P_{n-2}(v) \neq 1$ for any parabolic representation on the geometric component of W_{2n-1}. By [17] the defining equation for the geometric component of W_{2n-1} is $xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0$. Hence, for a parabolic representation on the geometric component we have $x = y = 2$ and $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$, where $v = x^2 + y^2 + z^2 - xyz - 2 = (z-2)^2 + 2$. By Lemma 2.7 we have

$$P_{n-1}(v) + P_{n-2}(v) = (S_n(v) - S_{n-2}(v) - 2)/(v-2) = (T_n(v) - 2)/(v-2).$$

4.1.1. Genus. We show that $P_{n-1}(v) + P_{n-2}(v) \neq 0$ for any parabolic representation on the geometric component of W_{2n-1}, where $n \geq 2$.

By Lemma 2.4 we have $\frac{T_n(v)-2}{v-2} = 0$ if and only if $v = 2\cos\frac{2j\pi}{n}$ for some $1 \leq j \leq \frac{n}{2}$. In particular, $v \in \mathbb{R}$ and $-2 \leq v < 2$.

Suppose $v = 2\cos\frac{2j\pi}{n}$ for some $1 \leq j < \frac{n}{2}$. Then, by Lemma 2.3,

$$S_{n-2}(v) = \frac{\sin(n-1)\frac{2j\pi}{n}}{\sin\frac{2j\pi}{n}} = -1, \qquad S_{n-1}(v) = \frac{\sin n\frac{2j\pi}{n}}{\sin\frac{2j\pi}{n}} = 0,$$

$$S_n(v) = \frac{\sin(n+1)\frac{2j\pi}{n}}{\sin\frac{2j\pi}{n}} = 1.$$

Equation $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$ then implies that $z = 2$. Hence $v = (z-2)^2 + 2 = 2$. This contradicts $-2 \leq v < 2$.

Suppose $v = -2$ (in this case n must be even). Then

$$S_{n-2}(v) = (-1)^{n-2}(n-1) = n-1, \qquad S_{n-1}(v) = (-1)^{n-1}n = -n$$

and $S_n(v) = (-1)^n(n+1) = n+1$. Equation $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$ then implies that $z = 4n-2$. Hence $v = (z-2)^2 + 2 = (4n-4)^2 + 2 \geq 2$. This contradicts $-2 \leq v < 2$.

4.1.2. Fiberedness. We show that $P_{n-1}(v) + P_{n-2}(v) \neq 1$ for any parabolic representation on the geometric component of W_{2n-1}, where $n \geq 2$.

By Lemma 2.5 we have $\frac{T_n(v)-2}{v-2} = 1$ if and only if $v = 2\cos\frac{2j\pi}{n-1}$ for some $1 \leq j \leq \frac{n-1}{2}$, or $v = 2\cos\frac{2k\pi}{n+1}$ for some $1 \leq k \leq \frac{n+1}{2}$. In particular, $v \in \mathbb{R}$ and $-2 \leq v < 2$.

Suppose $v = 2\cos\frac{2j\pi}{n-1}$ for some $1 \leq j < \frac{n-1}{2}$. Then, by Lemma 2.3, we have

$$S_{n-2}(v) = \frac{\sin(n-1)\frac{2j\pi}{n-1}}{\sin\frac{2j\pi}{n-1}} = 0, \qquad S_{n-1}(v) = \frac{\sin n\frac{2j\pi}{n-1}}{\sin\frac{2j\pi}{n-1}} = 1,$$

$$S_n(v) = \frac{\sin(n+1)\frac{2j\pi}{n-1}}{\sin\frac{2j\pi}{n-1}} = v.$$

Equation $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$ then implies that $0 = 4 - zv = 4 - 6z + 4z^2 - z^3 = 0$. This is equivalent to $(2-z)(z^2-2z+2) = 0$. Then $z \in \{2, 1 \pm i\}$. Hence $v = (z-2)^2 + 2 \in \{2, 2 \mp 2i\}$. This contradicts $-2 \leq v < 2$.

Suppose $v = 2\cos\frac{2j\pi}{n+1}$ for some $1 \leq j < \frac{n+1}{2}$. Then, by Lemma 2.3, we have

$$S_{n-2}(v) = \frac{\sin(n-1)\frac{2j\pi}{n+1}}{\sin\frac{2j\pi}{n+1}} = -v, \qquad S_{n-1}(v) = \frac{\sin n\frac{2j\pi}{n+1}}{\sin\frac{2j\pi}{n+1}} = -1,$$

$$S_n(v) = \frac{\sin(n+1)\frac{2j\pi}{n+1}}{\sin\frac{2j\pi}{n+1}} = 0.$$

Equation $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$ then implies that $0 = -4 + (4-z)v = 20 - 22z + 8z^2 - z^3 = 0$. This is equivalent to $(2-z)(z^2 - 6z + 10) = 0$. Then $z \in \{2, 3 \pm i\}$. Hence $v = (z-2)^2 + 2 \in \{2, 2 \pm 2i\}$. This contradicts $-2 \leq v < 2$.

Suppose $v = -2$ (in this case n must be odd). Then

$$S_{n-2}(v) = (-1)^{n-2}(n-1) = -(n-1), \qquad S_{n-1}(v) = (-1)^{n-1}n = n$$

and $S_n(v) = (-1)^n(n+1) = -(n+1)$. Equation $4S_{n-1}(v) - (4-z)S_{n-2}(v) - zS_n(v) = 0$ then implies that $z = 4n - 2$. Hence $v = (z-2)^2 + 2 = (4n-4)^2 + 2 \geq 2$. This contradicts $-2 \leq v < 2$.

4.2. Proof of Corollary 2. We prove Corollary 2 for W_{2n-1}. The proof for W_{2n} is similar. Let $M_{2n-1,p}$ be the 3-manifold obtained from S^3 by $\frac{p}{1}$-surgery along the component of W_{2n-1} corresponding the meridian a. Suppose $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ is a nonabelian representation of the form

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix}$$

where

$$\big(xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v)\big)S_{n-1}(v) = 0.$$

Here $x = s_1 + s_1^{-1}$, $y = s_2 + s_2^{-1}$, $z = s_1s_2 + s_1^{-1}s_2^{-1} + u$ and $v = x^2 + y^2 + z^2 - xyz - 2$ are the traces of the images of a, b, ab and $bab^{-1}a^{-1}$ respectively.

The following determines when the representation $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ extends to a representation $\rho : \pi_1(M_{2n-1,p}) \to SL_2(\mathbb{C})$.

Lemma 4.1. *ρ extends to a representation $\rho : \pi_1(M_{2n-1,p}) \to SL_2(\mathbb{C})$ if and only if one of the following holds:*

- *$S_{n-1}(v) = 0$, $s_1^p = 1$ and $s_1 \neq \pm 1$,*
- *$xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0$, $s_1 = -1$, $\frac{2y}{y+z} = -p$ and p is odd,*
- *$xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0$, $\frac{s_1y-z}{-s_1^{-1}y+z} = s_1^{-p}$ and $s_1 \neq \pm 1$.*

In all cases we have $x \neq 2$ (i.e. $s_1 \neq 1$).

Proof. Let λ_a be the canonical longitude corresponding to the meridian a of W_{2n-1}. By [17] we have

$$\rho(\lambda_a) = \begin{bmatrix} l_a & * \\ 0 & l_a^{-1} \end{bmatrix},$$

where

$$l_a = \begin{cases} 1 & \text{if } S_{n-1}(v) = 0, \\ \frac{s_1y-z}{-s_1^{-1}y+z} & \text{if } xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0, \end{cases} \tag{16}$$

and

$$* = \begin{cases} 0 & \text{if } S_{n-1}(v) = 0, \\ \frac{y(xy-2z)}{xyz-y^2-z^2} & \text{if } xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0. \end{cases} \tag{17}$$

Note that there is a small error in [17]: the canonical longitude in [17] is actually the inverse of the canonical longitude.

The nonabelian representation $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ extends to a representation $\rho : \pi_1(M_{2n-1,p}) \to SL_2(\mathbb{C})$ if and only if $a^p \lambda_a = \mathbf{1}$. Note that $\rho(a^p) = \begin{bmatrix} s_1^p & S_{p-1}(x) \\ 0 & s_1^{-p} \end{bmatrix}$, where $S_{p-1}(x) = \frac{s_1^p - s_1^{-p}}{s_1 - s_1^{-1}}$ if $s_1 \neq \pm 1$ and $S_{p-1}(x) = (\pm 1)^{p-1} p$ if $s_1 = \pm 1$. Hence the matrix equation $a^p \lambda_a = \mathbf{1}$ is equivalent to $l_a = s_1^{-p}$ and $* = -S_{p-1}(x)$.

Case 1: If $S_{n-1}(v) = 0$, then $l_a = 1$ and $* = 0$ by (16) and (17) respectively. Hence $a^p \lambda_a = \mathbf{1}$ is equivalent to $s_1^{-p} = 1$ and $S_{p-1}(x) = 0$, which means that $s_1^p = 1$ and $s_1 \neq \pm 1$.

Case 2: If $xyS_{n-1}(v) - (xy-z)S_{n-2}(v) - zS_n(v) = 0$, then

$$l_a = \frac{s_1 y - z}{-s_1^{-1} y + z} \qquad \text{and} \qquad * = \frac{y(xy-2z)}{xyz - y^2 - z^2}$$

by (16) and (17) respectively. Hence $a^p \lambda_a = \mathbf{1}$ is equivalent to

$$\frac{s_1 y - z}{-s_1^{-1} y + z} = s_1^{-p} \qquad \text{and} \qquad \frac{y(xy-2z)}{xyz - y^2 - z^2} = -S_{p-1}(x). \tag{18}$$

If $s_1 = 1$, then $\frac{s_1 y - z}{-s_1^{-1} y + z} = s_1^{-p}$ does not hold. Hence $a^p \lambda_a \neq \mathbf{1}$.

If $s_1 = -1$, then $\frac{s_1 y - z}{-s_1^{-1} y + z} = s_1^{-p}$ holds if and only if p is odd. In which case, the second equation of (18) is equivalent to $\frac{2y}{y+z} = -p$.

If $s_1 \neq \pm 1$ then the first equation of (18) implies the second one. Indeed, if $\frac{s_1 y - z}{-s_1^{-1} y + z} = s_1^{-p}$ then

$$S_{p-1}(x) = \frac{s_1^p - s_1^{-p}}{s_1 - s_1^{-1}} = \frac{\frac{-s_1^{-1} y + z}{s_1 y - z} - \frac{s_1 y - z}{-s_1^{-1} y + z}}{s_1 - s_1^{-1}} = -\frac{y(xy-2z)}{xyz - y^2 - z^2}.$$

In this case $a^p \lambda_a = \mathbf{1}$ is equivalent to $\frac{s_1 y - z}{-s_1^{-1} y + z} = s_1^{-p}$. $\square$

We now finish the proof of Corollary 2 for W_{2n-1}. Suppose the nonabelian representation $\rho : \pi_1(E_{W_{2n-1}}) \to SL_2(\mathbb{C})$ extends to a representation $\rho : \pi_1(M_{2n-1,p}) \to SL_2(\mathbb{C})$. By Lemma 4.1, $x \neq 2$. Since the core of the attaching solid torus is homotopic to a, by the gluing formula for the Reidemeister torsion (see e.g. [6]) we have

$$\tau^{\rho}_{M_{2n-1,p}} = \frac{\tau^{\rho}_{W_{2n-1}}}{2 - \operatorname{tr} \rho(a)} = \frac{\tau^{\rho}_{W_{2n-1}}}{2 - x}. \tag{19}$$

By Theorem 2.2 we have $\tau^\rho_{W_{2n-1}} = \Delta^\rho_{W_{2n-1}}(1,1)$. Theorem 1 then implies that

$$\tau^\rho_{W_{2n-1}} = 2\big[(2-x-y+z)S_{n-1}(v) + (4-2x-2y+xy)P_{n-2}(v)\big].$$

This, together with equation (19), completes the proof of Corollary 2 for W_{2n-1}.

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