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Brownian isometric parts of concave operators

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Abstract. We describe some invariant or reducing subspaces for a concave operator T on a complex Hilbert space which satisfies the regularity condition $\Delta_T T = \Delta_T^{1/2} T \Delta_T^{1/2}$, where $\Delta_T = T^*T - I$. We consider those subspaces on which T acts as a 2-isometry and show that T has some Brownian type properties on them. Among other, the Brownian unitary part and the Brownian isometric (reducing or invariant) parts are investigated. In the case when T is a Brownian operator or even a general 2-isometry we determine the Brownian unitary reducing parts on which T has the maximal covariance.

Contents

1. Introduction and preliminaries

In all that follows, $\mathcal{H}$ stands for a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}$. By I we mean the identity operator on a considered Hilbert space. Given $T \in \mathcal{B}(\mathcal{H})$, we write $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* for the range, the kernel and the adjoint of T, respectively. Recall that T is a contraction if $T^*T \leq I$, T is an isometry if $T^*T = I$, and T is a unitary operator if it is an isometry with $\mathcal{R}(T) = \mathcal{H}$. If $\mathcal{M} \subset \mathcal{H}$, then by $\overline{\mathcal{M}}$ we mean the closure of $\mathcal{M}$ in $\mathcal{H}$. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. We denote by $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. We say that $\mathcal{M}$ is *invariant* (resp., *reducing*) for T if $TP_{\mathcal{M}} = P_{\mathcal{M}}TP_{\mathcal{M}}$ (resp., $TP_{\mathcal{M}} = P_{\mathcal{M}}T$). If $\mathcal{M}$ is invariant for T, then $T_{\mathcal{M}} := T|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$ is the *restriction* of T to $\mathcal{M}$ and T is an *extension* for $T_{\mathcal{M}}$. The operator $P_{\mathcal{M}}T|_{\mathcal{M}}$

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is said to be the *compression* of T on $\mathcal{M}$. For an isometry $T \in \mathcal{B}(\mathcal{H})$ and a closed subspace $\mathcal{M}$ of $\mathcal{N}(T^*)$, we set $l^2_+(T, \mathcal{M}) := \bigoplus_{n=0}^{\infty} T^n\mathcal{M}$. If $\mathcal{M}_n$ ($n = 0, 1, \dots$) are subsets of $\mathcal{H}$, the closure in $\mathcal{H}$ of their linear span is denoted by $\bigvee_{n\geq 0} \mathcal{M}_n$.

Following [20], an operator $T \in \mathcal{B}(\mathcal{H})$ is called *concave* if

$$T^{*2}T^2 - 2T^*T + I \leq 0. \tag{1.1}$$

If there holds the equality in (1.1), then we say that T is a 2-*isometry*.

For $T \in \mathcal{B}(\mathcal{H})$, we put $\Delta_T := T^*T - I$. When $\Delta_T \geq 0$ the operator T is said to be *expansive*. One can check that concave operators are expansive and the inequality (1.1) can be written as $T^*\Delta_T T \leq \Delta_T$, i.e. T is a Δ_T-contraction. Also, T is a 2-isometry if and only if $T^*\Delta_T T = \Delta_T$, i.e. T is a Δ_T-isometry (see [21, 22] for other references to general A-contractions). Remark that if T is a contraction, then $\Delta_T \leq 0$ and in this case $D_T = (-\Delta_T)^{1/2}$ is the *defect operator* of T, where $Z^{1/2}$ denotes the square root of $Z \in \mathcal{B}(\mathcal{H})$. A concave operator is contractive if and only if it is an isometry. Let us recall that the concave operators also appear under the name of 2-*hyperexpansive* operators, and were well studied in the literature (see [4, 7, 9, 10, 16]). It is clear from (1.1) that if T is concave, then the subspace $\mathcal{N}(\Delta_T)$ is invariant for T and also $V = T|_{\mathcal{N}(\Delta_T)}$ is an isometry; see for instance [18, Proposition 3.1(a)]. Therefore, if T is a non-isometric concave operator, then the subspace $\overline{\mathcal{R}(\Delta_T)} \neq \{0\}$ is invariant for T^* and in this case we can obtain a usual matrix representation of T with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)}$.

In our present work we deal specifically with those concave operators T which satisfy the condition

$$\Delta_T T = \Delta_T^{1/2} T \Delta_T^{1/2}. \tag{1.2}$$

According to [13, 21, 22] a concave operator T satisfying (1.2) is called Δ_T-*regular*. Such operators are very special because in their matrix representations on $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)}$ the operators $T^*|_{\overline{\mathcal{R}(\Delta_T)}}$ are contractions which commute with $\Delta_T|_{\overline{\mathcal{R}(\Delta_T)}}$. This fact permits to use many results from the theory of contractions in the study of these concave operators (as well the Wold-type decompositions, liftings, dilations etc; see [8, 13, 20]). It is also interesting to recall that for Δ_T-regular concave operators one can completely characterize the subnormality of their associated Cauchy dual operators (see [5, 6, 8, 9]).

Following [5, 6], a 2-isometry T which is Δ_T-regular is called a *quasi-Brownian isometry*. In this case, the compression W of T to $\overline{\mathcal{R}(\Delta_T)}$ is an isometry which commutes with the operator E^*E, where $E = P_{\mathcal{N}(\Delta_T)}T|_{\overline{\mathcal{R}(\Delta_T)}}$, and also $V^*E = 0$, where $V = T|_{\mathcal{N}(\Delta_T)}$. When W is a unitary operator on $\overline{\mathcal{R}(\Delta_T)}$ with $WE^*E = E^*EW$ and $E = \delta_0 E_0$, where E_0 is a (necessarily injective) contraction while $\delta_0 = \|\Delta_T\|^{1/2}$, then T is called *Brownian isometry*. A Brownian isometry is called *Brownian unitary* if the operator E_0 is an isometry and $\mathcal{R}(E) = \mathcal{N}(V^*)$. For a 2-isometry T the above scalar $\operatorname{cov}(T) := \|\Delta_T\|^{1/2}$ is called the *covariance* of T. Clearly, if T is an isometry, then $\operatorname{cov}(T) = 0$, and conversely.

For other relevant results on concave operators and 2-isometries we refer the reader to [1, 2, 3, 4, 5, 6, 8, 9, 13, 14, 15, 16, 20] and the references therein.

If $\mathcal{Q}$ is a property for operators from $\mathcal{B}(\mathcal{H})$, then the maximum subspace in $\mathcal{H}$ which is invariant (or, resp., reducing) for an operator $T \in \mathcal{B}(\mathcal{H})$ on which T has the property $\mathcal{Q}$ is called the $\mathcal{Q}$-*invariant* (resp., $\mathcal{Q}$-*reducing*) *part* of T in $\mathcal{H}$. Also, following [17], the *reducible* part of T in an invariant subspace $\mathcal{M}$ of $\mathcal{H}$ is the maximum reducing subspace for T in $\mathcal{M}$. Below we deal with parts of concave operators which have received considerable attention in recent period in the literature, as well the Brownian unitaries (or isometries).

The organization of this paper is as follows. In Section 2 we first recall the unitary and isometric (reducing) parts of a concave operator. Next, we characterize the concave operators satisfying the condition (1.2) by their usual matrix representations. For such operators we describe the quasi-Brownian isometric invariant (resp., reducing) parts in $\mathcal{H}$, which in fact are even the corresponding 2-isometric parts. Moreover, we find the 2-isometric invariant part $\mathcal{H}_0 \subset \mathcal{H}$ with $\|\Delta_T\|^{-1}\Delta_T|_{\mathcal{H}_0}$ being an orthogonal projection, which in particular refers to the Δ_T-regular concave operators T which have Brownian extensions in the sense of McCullough [14] (i.e. with $\|T\| \leq \sqrt{2}$). In Section 3 we are concerned, on one hand, with the Brownian isometric invariant (resp., reducing) part in $\mathcal{H}$ for a concave operator T satisfying (1.2). Also, we determine the 2-isometric reducing part $\mathcal{H}_\delta \subset \mathcal{H}$ for T such that $\delta^{-2}\Delta_T|_{\mathcal{H}_\delta}$ is an orthogonal projection, where $0 < \delta \leq \|\Delta_T\|^{1/2}$. More particularly, we find the Brownian unitary (reducing) part of T in $\mathcal{H}$ on which T has the covariance δ. In Section 4, we first provide the relationship between these (invariant or reducing) parts for the Brownian operators considered in [14]. Moreover, we determine the Brownian unitary reducing part of a concave T with $\|T\| \leq \sqrt{2}$ (i.e. a subbrownian operator) on which T has the covariance 1, by using the same part of a Brownian extension of T. We remark finally that a similar argument can be used to obtain the Brownian unitary reducing part of a general 2-isometry T on which T preserves its covariance.

2. Quasi-Brownian isometric parts

Let $T \in \mathcal{B}(\mathcal{H})$ be a concave operator. Then $\mathcal{N}(\Delta_T)$ is an invariant subspace for T, and hence $\mathcal{N}(\Delta_{T^*}) \subset \mathcal{N}(\Delta_T)$. First, we take a closer look at the unitary and isometric (reducing) parts of the operator T in $\mathcal{H}$. The unitary part was described in [2, 15, 20], but we recall it below together with the description of the isometric part, which was obtained in [18, Proposition 3.1(b)] as a particular case.

Proposition 2.1. *For a concave operator $T \in \mathcal{B}(\mathcal{H})$, the unitary reducing part $\mathcal{H}_\infty$ and the isometric reducing part $\mathcal{H}^\infty$ of T are given as follows:*

$$\mathcal{H}_\infty = \bigcap_{n\geq 1} T^n\mathcal{H} = \bigcap_{n\geq 1} T^n\mathcal{N}(\Delta_T) = \bigcap_{n\geq 1} T^n\mathcal{N}(\Delta_{T^*}), \tag{2.1}$$

$$\mathcal{H}^\infty = \mathcal{H} \ominus \bigvee_{n\geq 0} T^n\overline{\mathcal{R}(\Delta_T)}. \tag{2.2}$$

Proof. It was proved in [20, Proposition 3.4] or [15, Proposition 1.1] that $\mathcal{H}_\infty = \bigcap_{n\geq 1} T^n\mathcal{H}$ is even the unitary part of T in $\mathcal{H}$. So, for every integer $n \geq 1$, we have $\mathcal{H}_\infty = T^n\mathcal{H}_\infty \subset T^n\mathcal{N}(\Delta_{T^*}) \subset T^n\mathcal{N}(\Delta_T)$, whence we obtain

$$\mathcal{H}_\infty \subset \bigcap_{n\geq 1} T^n\mathcal{N}(\Delta_{T^*}) \subset \bigcap_{n\geq 1} T^n\mathcal{N}(\Delta_T) \subset \mathcal{H}_\infty. \tag{2.3}$$

Therefore the equalities in (2.1) are true.

The equality (2.2) is known from [18, Proposition 3.1(b)], but for completeness we prove it. Clearly, the subspace $\mathcal{N} := \bigvee_{n\geq 0} T^n\overline{\mathcal{R}(\Delta_T)}$ is invariant for T, and since $\overline{\mathcal{R}(\Delta_T)}$ is invariant for T^*, we have for $n \geq 1$

$$T^*T^n|_{\overline{\mathcal{R}(\Delta_T)}} = (\Delta_T + I)T^{n-1}\overline{\mathcal{R}(\Delta_T)} \subset \mathcal{N},$$

whence it follows that $T^*\mathcal{N} \subset \mathcal{N}$. Hence the subspace $\mathcal{H}^\infty := \mathcal{H}\ominus\mathcal{N}$ reduces T, and $T|_{\mathcal{H}^\infty}$ is an isometry because $\mathcal{H}^\infty \subset \mathcal{N}(\Delta_T)$.

Now, let $\mathcal{M}$ be a subspace of $\mathcal{H}$ which reduces T to an isometry. Then $\mathcal{M} \subset \mathcal{N}(\Delta_T)$ and

$$\mathcal{H} \ominus \mathcal{M} = \bigvee_{n\geq 0} T^n(\mathcal{H} \ominus \mathcal{M}) \supset \bigvee_{n\geq 0} T^n\overline{\mathcal{R}(\Delta_T)} = \mathcal{H} \ominus \mathcal{H}^\infty.$$

This implies that $\mathcal{M} \subset \mathcal{H}^\infty$. We conclude that $\mathcal{H}^\infty$ is even the (reducing) isometric part of T in $\mathcal{H}$. □

Remark 2.2. If T is a concave operator, then

$$\mathcal{N}(\Delta_T) = \mathcal{N}(\Delta_T T - T\Delta_T). \tag{2.4}$$

Indeed, as $\mathcal{N}(\Delta_T)$ is invariant for T, we have

$$\mathcal{N}(\Delta_T) \subset \mathcal{N}(TT^*T - T^*T^2) = \mathcal{N}(T\Delta_T - \Delta_T T).$$

Conversely, if $h \in \mathcal{N}(T\Delta_T - \Delta_T T)$, then $(T^*T)^2h = T^{*2}T^2h$, which yields $\|\Delta_T h\|^2 = \langle (T^{*2}T^2 - 2T^*T + I)h, h\rangle \leq 0$ using the fact that T is concave. Hence $h \in \mathcal{N}(\Delta_T)$ and the equality (2.4) is proved.

In what follows, we focus on the class of Δ_T-regular concave operators T, that is, satisfying the condition (1.2), which is weaker than the commutation relation of T with Δ_T. Obviously, (1.2) holds on $\mathcal{N}(\Delta_T)$ by (2.4), so actually the condition (1.2) plays an essential role on $\overline{\mathcal{R}(\Delta_T)}$. Now, we provide a matrix description of these operators, while in a slightly modified form it appears in the equivalence (i)$\Leftrightarrow$(ii) of [8, Theorem 2.3]. Notice also that the "only if" part of the following proposition can be deduced from [18, Proposition 3.1(c)].

Proposition 2.3. *A non-isometric operator $T \in \mathcal{B}(\mathcal{H})$ is a Δ_T-regular concave operator if and only if T has the block matrix form*

$$T = \begin{pmatrix} V & E \\ 0 & W \end{pmatrix} \tag{2.5}$$

with respect to the decomposition $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)}$, where

(a) *V is an isometry on $\mathcal{N}(\Delta_T)$,*

(b) *$E : \overline{\mathcal{R}(\Delta_T)} \to \mathcal{N}(\Delta_T)$ is an injective operator such that $V^*E = 0$, and*

(c) *W is a contraction on $\overline{\mathcal{R}(\Delta_T)}$ which commutes to $E^*E + \Delta_W$.*

In particular, a concave operator T with Δ_T being a scalar multiple of an orthogonal projection is Δ_T-regular.

Proof. Let T be a Δ_T-regular concave operator which is not an isometry, i.e. $\Delta_T \neq 0$. Then with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)}$ we have $\Delta_T = 0 \oplus \Delta_0$ with a positive injective operator $\Delta_0 = \Delta_T|_{\overline{\mathcal{R}(\Delta_T)}}$. Obviously, T has the matrix representation (2.5) with an isometry $V = T|_{\mathcal{N}(\Delta_T)}$ and $E = P_{\mathcal{N}(\Delta_T)}T|_{\overline{\mathcal{R}(\Delta_T)}}$, $W^* = T^*|_{\overline{\mathcal{R}(\Delta_T)}}$ such that $V^*E = 0$ (because $\Delta_T \geq 0$) and $W^*\Delta_0 W \leq \Delta_0$ (because $T^*\Delta_T T \leq \Delta_T$).

Since T is Δ_T-regular, W is Δ_0-regular, i.e. $\Delta_0 W = \Delta_0^{1/2} W \Delta_0^{1/2}$. As Δ_0 is injective, we have $\Delta_0 W = W\Delta_0$, which together with the above inequality involving W and Δ_0 implies that W is a contraction on $\overline{\mathcal{R}(\Delta_T)}$.

Next, expressing $\Delta_T = T^*T - I$ with the help of (2.5) we have $\Delta_0 = E^*E + \Delta_W$ and, as $\Delta_W \leq 0$, we get $\Delta_0 \leq E^*E$. This yields that E is injective (like Δ_0). Thus, (a), (b) and (c) hold for a Δ_T-regular concave T.

Conversely, if an operator T has the matrix representation (2.5) satisfying (a)-(c), then clearly $W^*\Delta_0 W = \Delta_0^{1/2} W^*W \Delta_0^{1/2} \leq \Delta_0$, where Δ_0 is as above. But this implies that $T^*\Delta_T T \leq \Delta_T$, while $\Delta_0 W = W\Delta_0$ also gives the condition $\Delta_T T = \Delta_T^{1/2} T \Delta_T^{1/2}$. Hence T is a Δ_T-regular concave operator. Thus, the first assertion of the proposition is proved.

The second assertion is immediate. Indeed, if T is concave with $\delta\Delta_T$ being an orthogonal projection for some $\delta > 0$, then, as $\mathcal{R}(\Delta_T)$ is invariant for T^*, we get

$$T^*(\delta\Delta_T) = \delta\Delta_T T^*(\delta\Delta_T) = \sqrt{\delta}\Delta_T^{1/2} T^*(\sqrt{\delta}\Delta_T^{1/2}),$$

whence by passing to the adjoint we derive the condition (1.2). The proof is complete. $\square$

Recall that the matrix representation of the form (2.5) for a quasi-Brownian isometry was obtained in [13, Proposition 5.1] and in a generalized version in [5, Theorem 4.1]. A quasi-Brownian isometry is characterized in terms of (2.5) by the fact that the operator W is an isometry which commutes to E^*E, while E is injective with $V^*E = 0$.

Now, as a completion of the last assertion in Proposition 2.3, we obtain the following proposition.

Proposition 2.4. *If T is a Δ_T-regular concave operator, then for $0 < \delta \leq \|\Delta_T\|^{1/2}$ the subspace*

$$\mathcal{M}_\delta := \mathcal{N}(\delta^2 \Delta_T - \Delta_T^2) = \mathcal{N}(\Delta_T) \oplus \mathcal{N}(\Delta_T - \delta^2 I)$$

is invariant for T and such that $\delta^{-2}\Delta_T|_{\mathcal{M}_\delta}$ is an orthogonal projection.

Proof. Indeed, take $h \in \mathcal{H}$ such that $\delta^2 \Delta_T h = \Delta_T^2 h$. Thus $\delta \Delta_T^{1/2} h = \Delta_T h$, so by (2.4),

$$\begin{aligned} \delta^2 \Delta_T T h &= \delta^2 \Delta_T^{1/2} T \Delta_T^{1/2} h = \delta \Delta_T^{1/2} T \Delta_T h \\ &= \delta \Delta_T T \Delta_T^{1/2} h = \Delta_T T \Delta_T h = \Delta_T^2 T h. \end{aligned}$$

Also, the subspace $\mathcal{N}(\Delta_T - \delta^2 I)$ is invariant for T^* (using the dual relation of (1.2)), but this is not invariant for T when it is non-null. Since both subspaces $\mathcal{N}(\Delta_T)$ and $\mathcal{N}(\Delta_T - \delta^2 I)$ are invariant for T^*T, it follows that $\mathcal{M}_\delta$ reduces Δ_T. Thus $T_\delta := T|_{\mathcal{M}_\delta}$ is a concave operator with $\Delta_{T_\delta} = \Delta_T|_{\mathcal{M}_\delta}$, while the later relation implies that T_δ is also Δ_{T_δ}-regular. Moreover, we have $\mathcal{N}(\Delta_{T_\delta}) = \mathcal{N}(\Delta_T)$ and $\mathcal{N}(\Delta_{T_\delta} - \delta^2 I) = \mathcal{N}(\Delta_T - \delta^2 I) = \mathcal{R}(\Delta_{T_\delta})$, hence $\delta^{-2}\Delta_{T_\delta}$ is even the orthogonal projection on $\mathcal{R}(\Delta_{T_\delta})$. □

Now, for a Δ_T-regular concave operator T, we determine the quasi-Brownian isometric invariant (and reducing) parts of T in $\mathcal{H}$.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a non-isometric Δ_T-regular concave operator. Then the quasi-Brownian isometric invariant part of T in $\mathcal{H}$ is the subspace*

$$\mathcal{H}_q = \mathcal{N}(\Delta_T) \oplus \mathcal{N}(I - S_W), \tag{2.6}$$

where $W^* = T^*|_{\overline{\mathcal{R}(\Delta_T)}}$ and $S_W := s - \lim_{n\to\infty} W^{*n}W^n$ for the contraction W. Moreover, $\mathcal{H}_q$ is even the 2-isometric invariant part of T in $\mathcal{H}$. In addition, if $\mathcal{N}(I - S_W) \neq \{0\}$, then $\operatorname{cov}(T|_{\mathcal{H}_q}) = \|E|_{\mathcal{N}(I-S_W)}\|$, where $E := P_{\mathcal{N}(\Delta_T)} T|_{\overline{\mathcal{R}(\Delta_T)}}$.

Proof. Consider the matrix representation (2.5) of T on $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)}$ given by the operators V, E and W. As W is a contraction on $\overline{\mathcal{R}(\Delta_T)}$, the subspace

$$\mathcal{N}(I - S_W) = \bigcap_{j \geq 1} \mathcal{N}(I - W^{*j}W^j) \tag{2.7}$$

is even the isometric invariant part of W in $\overline{\mathcal{R}(\Delta_T)}$. Since W commutes with $\Delta_0 = E^*E + \Delta_W$ (as in Proposition 2.3(c)), we have for $h \in \mathcal{N}(I - S_W)$

and $n \geq 1$ that $E^*Eh = W^{*n}W^nE^*Eh$. Therefore, $E^*Eh \in \mathcal{N}(I - S_W)$ by (2.7). Hence $\mathcal{N}(I - S_W)$ reduces E^*E and Δ_T.

Now, the matrix representation (2.5) of T on

$$\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \mathcal{N}(I - S_W) \oplus \overline{\mathcal{R}(I - S_W)}$$

has the form

$$T = \begin{pmatrix} V & E_0 & \star \\ 0 & W_0 & \star \\ 0 & 0 & \star \end{pmatrix} = \begin{pmatrix} T_0 & \star \\ 0 & \star \end{pmatrix} \quad \text{with} \quad T_0 = T|_{\mathcal{N}(\Delta_T) \oplus \mathcal{N}(I - S_W)},$$

where $E_0 = E|_{\mathcal{N}(I-S_W)}$ and $W_0 = W|_{\mathcal{N}(I-S_W)}$ is an isometry. In addition, as $\mathcal{N}(I - S_W)$ reduces E^*E, we have $E_0^*E_0 = E^*E|_{\mathcal{N}(I-S_W)}$. Therefore, E_0 is injective because so is too E, and also $V^*E_0 = 0$ (by Proposition 2.3(b)). Further, since W commutes with Δ_0 (by Proposition 2.3(c)), it follows that W_0 commutes with $E_0^*E_0$ on $\mathcal{N}(I - S_W) = \overline{\mathcal{R}(\Delta_{T_0})}$. We conclude that the subspace $\mathcal{H}_q$ from (2.6) is invariant for T and $T_0 = T|_{\mathcal{H}_q}$ is a Δ_{T_0}-regular 2-isometry, that is, a quasi-Brownian isometry. In addition, when $\mathcal{N}(I - S_W) \neq \{0\}$ we have

$$\operatorname{cov}(T_0) = \|\Delta_{T_0}\|^{1/2} = \|E_0^*E_0 + \Delta_{W_0}\|^{1/2} = \|E_0\| > 0.$$

Next, we show that $\mathcal{H}_q$ is the maximum invariant subspace for T in $\mathcal{H}$ on which T is a 2-isometry. This fact also ensures that $\mathcal{H}_q$ is even the quasi-Brownian isometric invariant part of T in $\mathcal{H}$. Indeed, let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for T such that $T' = T|_{\mathcal{M}}$ is a 2-isometry. Then with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ we can represent T as

$$T = \begin{pmatrix} T' & X \\ 0 & Y \end{pmatrix}$$

with some appropriate operators X, Y. A simple computation gives

$$T^*\Delta_T T - \Delta_T = \begin{pmatrix} T'^*\Delta_{T'}T' - \Delta_{T'} & Z \\ Z^* & Y_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & Y_0 \end{pmatrix}.$$

Here we used the property that T' is a 2-isometry, i.e. $T'^*\Delta_{T'}T' = \Delta_{T'}$, as well as the fact that T is concave, i.e. $\Delta_T - T^*\Delta_T T \geq 0$. This positivity condition in turn implies $Z = 0$. We conclude that

$$\mathcal{M} \subset \mathcal{N}(\Delta_T - T^*\Delta_T T) =: \widetilde{\mathcal{M}}.$$

Since W in (2.5) commutes with $\Delta_0 = \Delta_T|_{\overline{\mathcal{R}(\Delta_T)}}$, for $h \in \overline{\mathcal{R}(\Delta_T)} \cap \widetilde{\mathcal{M}}$ we have

$$\Delta_0 h = W^*\Delta_0 W h = \Delta_0 W^*W h,$$

and, as Δ_0 is injective, we obtain $h = W^*Wh$. It follows that

$$\mathcal{M} \subset \widetilde{\mathcal{M}} \subset \mathcal{N}(\Delta_T) \oplus \mathcal{N}(I - W^*W).$$

We consider the subspace $\mathcal{N} := \overline{\mathcal{N}(\Delta_T) + \mathcal{M}}$ which is also invariant for T, so $T_{\mathcal{N}} := T|_{\mathcal{N}}$ is a concave operator. But $\mathcal{N}(\Delta_T)$ is invariant for $T_{\mathcal{N}}$ too and

$T_{\mathcal{N}}^* T_{\mathcal{N}} h = P_{\mathcal{N}} T^* T h = h$ for $h \in \mathcal{N}(\Delta_T)$, i.e. $T_{\mathcal{N}}$ is an isometry on $\mathcal{N}(\Delta_T)$. Hence $\mathcal{N}(\Delta_T) \subset \mathcal{N}(\Delta_{T_{\mathcal{N}}})$. On the other hand, $\mathcal{N}(\Delta_{T_{\mathcal{N}}})$ is invariant for T and $\|Th\| = \|T_{\mathcal{N}} h\| = \|h\|$ for $h \in \mathcal{N}(\Delta_{T_{\mathcal{N}}})$. Thus $T|_{\mathcal{N}(\Delta_{T_{\mathcal{N}}})}$ is concave and contractive, that is, an isometry such that $\mathcal{N}(\Delta_T) = \mathcal{N}(\Delta_{T_{\mathcal{N}}})$. Therefore, $\mathcal{N} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}$, while from the above inclusion of $\mathcal{M}$ and the definition of $\mathcal{N}$ we derive that $\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} \subset \mathcal{N}(I - W^*W) \subset \overline{\mathcal{R}(\Delta_T)}$.

Denoting $W_{\mathcal{N}} := P_{\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}} T_{\mathcal{N}}|_{\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}}$ we obtain

$$\begin{aligned} W\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} &= P_{\overline{\mathcal{R}(\Delta_T)}} T \overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} = P_{\overline{\mathcal{R}(\Delta_T)}} T_{\mathcal{N}} \overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} \\ &\subset P_{\overline{\mathcal{R}(\Delta_T)}} (\mathcal{N}(\Delta_T) \oplus W_{\mathcal{N}} \overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}) \\ &= W_{\mathcal{N}} \overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} \subset \overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}. \end{aligned}$$

So, $\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}$ is invariant for W. Since $\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} \subset \mathcal{N}(I - W^*W)$, we infer that $W|_{\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})}}$ is an isometry. But, by (2.7), this gives $\overline{\mathcal{R}(\Delta_{T_{\mathcal{N}}})} \subset \mathcal{N}(I - S_W)$, hence

$$\mathcal{M} \subset \mathcal{N} \subset \mathcal{N}(\Delta_T) \oplus \mathcal{N}(I - S_W) = \mathcal{H}_q.$$

We conclude that $\mathcal{H}_q$ has the required maximality properties, which ends the proof. $\square$

Combining $\mathcal{H}_q$ in (2.6) with a subspace from Proposition 2.4 we obtain a particular quasi-Brownian isometric invariant part.

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a Δ_T-regular concave operator with $\delta_0 = \|\Delta_T\|^{1/2} > 0$, and let $\mathcal{M}_{\delta_0}$ and $\mathcal{H}_q$ be the subspaces given by Proposition 2.4 and Theorem 2.5, while $W = P_{\overline{\mathcal{R}(\Delta_T)}} T|_{\overline{\mathcal{R}(\Delta_T)}}$. If*

$$\mathcal{R}(\Delta_T|_{\mathcal{M}_{\delta_0}}) \cap \overline{\mathcal{R}(\Delta_T|_{\mathcal{H}_q})} \neq \{0\},$$

then

$$\mathcal{H}_0 := \mathcal{M}_{\delta_0} \cap \mathcal{H}_q = \mathcal{N}(\Delta_T) \oplus \mathcal{N}(\Delta_T - \delta_0^2 I) \cap \mathcal{N}(I - S_W) \tag{2.8}$$

is the maximum invariant subspace for T in $\mathcal{H}$ such that $T_0 := T|_{\mathcal{H}_0}$ is a 2-isometry with an orthogonal projection $\delta_0^{-2}\Delta_{T_0}$.

Proof. The subspace $\mathcal{H}_0$ in (2.8) is invariant for T and $T_0 = T|_{\mathcal{H}_0}$ is concave. So we get

$$\begin{aligned} \{0\} \neq &\mathcal{R}(\Delta_T|_{\mathcal{M}_{\delta_0}}) \cap \overline{\mathcal{R}(\Delta_T|_{\mathcal{H}_q})} = \overline{\mathcal{R}(\Delta_{T_0})} \\ = &\mathcal{N}(\Delta_T - \delta_0^2 I) \cap \mathcal{N}(I - S_W) \subset \overline{\mathcal{R}(\Delta_T)}. \end{aligned}$$

But, as in the previous proof, we see that $\overline{\mathcal{R}(\Delta_{T_0})}$ is invariant for W, while the above equality implies that $W_0 := W|_{\overline{\mathcal{R}(\Delta_{T_0})}}$ is an isometry. In addition, if $E_0 := P_{\mathcal{N}(\Delta_T)} T_0|_{\overline{\mathcal{R}(\Delta_{T_0})}}$, then using the matrix representation of T_0 on $\mathcal{H}_0 = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_{T_0})}$ we have

$$\delta_0^2 h = \Delta_T h = P_{\mathcal{H}_0} \Delta_T h = \Delta_{T_0} h = (E_0^* E_0 + \Delta_{W_0}) h = E_0^* E_0 h$$

for $h \in \overline{\mathcal{R}(\Delta_{T_0})}$. Thus $\delta_0^{-2} E_0$ is an isometry, and consequently T_0 is a 2-isometry with $\delta_0^{-2}\Delta_{T_0}$ being an orthogonal projection on $\mathcal{H}_0$.

Let us show the required maximality property of $\mathcal{H}_0$ relative to T. For this purpose, let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for T such that $T' = T|_{\mathcal{M}}$ is a 2-isometry with $\delta_0^{-2}\Delta_{T'}$ being an orthogonal projection. Obviously, $\mathcal{M} \subset \mathcal{H}_q$ by the maximality property of $\mathcal{H}_q$ established in Theorem 2.5. We represent T on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ in the form

$$T = \begin{pmatrix} T' & X \\ 0 & Y \end{pmatrix},$$

with some appropriate operators X and Y. As a consequence, we obtain

$$\Delta_T = \begin{pmatrix} \Delta_{T'} & T'^* X \\ X^* T' & X^* X + \Delta_Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & Z \\ 0 & Z^* & X^* X + \Delta_Y \end{pmatrix},$$

the second representation being on $\mathcal{H} = \mathcal{N}(\Delta_{T'}) \oplus \mathcal{N}(\Delta_{T'} - \delta_0^2 I) \oplus \mathcal{M}^\perp$, where we use that $\delta_0^{-2}\Delta_{T'}$ is an orthogonal projection so $\mathcal{R}(\Delta_{T'}) = \mathcal{N}(\Delta_{T'} - \delta_0^2 I)$, and that $\Delta_T \geq 0$ (so $\mathcal{N}(\Delta_{T'})$ is invariant for T^*T). We infer that

$$(\Delta_T - \delta_0^2 I)|_{\mathcal{R}(\Delta_{T'}) \oplus \mathcal{M}^\perp} = \begin{pmatrix} 0 & Z \\ Z^* & X^* X + \Delta_Y - I \end{pmatrix},$$

and since $\Delta_T \leq \delta_0^2 I$ (by the choice of δ_0), we get $Z = 0$. Hence $\mathcal{R}(\Delta_{T'}) \subset \mathcal{N}(\Delta_T - \delta_0^2 I)$. Also, since $\mathcal{N}(\Delta_{T'})$ is invariant for T', and so for T, we have (as above) $\mathcal{N}(\Delta_{T'}) \subset \mathcal{N}(\Delta_T)$. Thus we get $\mathcal{M} = \mathcal{N}(\Delta_{T'}) \oplus \mathcal{R}(\Delta_{T'}) \subset \mathcal{M}_{\delta_0}$ and finally $\mathcal{M} \subset \mathcal{M}_{\delta_0} \cap \mathcal{H}_q = \mathcal{H}_0$. Consequently, $\mathcal{H}_0$ has the required maximality property, which completes the proof. □

Next, we turn our attention to the quasi-Brownian isometric reducing part.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a non-isometric Δ_T-regular concave operator. Then the quasi-Brownian isometric reducing part of T in $\mathcal{H}$ is the 2-isometric reducing part of T in $\mathcal{H}$. More precisely, it is the subspace*

$$\mathcal{H}_q^* := \mathcal{H}^\infty \oplus [l_+^2(V, \overline{E\mathcal{R}^\infty}) \oplus \mathcal{R}^\infty], \tag{2.9}$$

where $\mathcal{H}^\infty$ and $\mathcal{R}^\infty$ are, respectively, the isometric reducing parts of T in $\mathcal{H}$ and of W in $\overline{\mathcal{R}(\Delta_T)}$, while V, E and W are the operators from the block matrix form (2.5) of T. In addition, if $\mathcal{R}^\infty \neq \{0\}$, then $\operatorname{cov}(T|_{\mathcal{H}_q^*}) = \|E|_{\mathcal{R}^\infty}\|$.

Proof. Consider the decomposition $\overline{\mathcal{R}(\Delta_T)} = \mathcal{R}^\infty \oplus (\overline{\mathcal{R}(\Delta_T)} \ominus \mathcal{R}^\infty)$, where $\mathcal{R}^\infty$ is the reducing isometric part of the compression W of T on $\overline{\mathcal{R}(\Delta_T)}$, that is, (by (2.2))

$$\mathcal{R}^\infty = \overline{\mathcal{R}(\Delta_T)} \ominus \bigvee_{n \geq 0} W^n \overline{\mathcal{R}(I - W^* W)}.$$

To prove that $\mathcal{R}^\infty$ is invariant for E^*E, where $E = P_{\mathcal{N}(\Delta_T)}T|_{\overline{\mathcal{R}(\Delta_T)}}$, we first express the commutation relation in Proposition 2.3(c) in the following ways:

$$E^*EW - WE^*E = WW^*W - W^*W^2,$$
$$W^*E^*E - E^*EW^* = W^*WW^* = W^{*2}W.$$

Since the right hand side of each of these relations is 0 on $\mathcal{R}^\infty$, we have $E^*EWh = WE^*Eh$ and $E^*EW^*h = W^*E^*Eh$ for $h \in \mathcal{R}^\infty$. Then for $k = (I - W^*W)h'$ ($h' \in \overline{\mathcal{R}(\Delta_T)}$), $h \in \mathcal{R}^\infty$ and any positive integer n, taking into account the previous relations and that $\mathcal{R}^\infty$ reduces W, we obtain

$$\langle E^*Eh, W^n k\rangle = \langle E^*E(I - W^*W)W^{*n}h, h'\rangle = 0.$$

But this yields $E^*Eh \in \mathcal{R}^\infty$ by the above structure of $\mathcal{R}^\infty$. Hence this subspace is invariant (and so reducing) for E^*E. This also implies that $\mathcal{R}^\infty$ reduces $\Delta_0 = E^*E + \Delta_W$.

Now, let $E = J|E|$ be the polar decomposition of E. Since E is injective and $V^*E = 0$ (by Proposition 2.3(b)), the operator J is an isometry from $\overline{\mathcal{R}(E^*E)} = \overline{\mathcal{R}(\Delta_T)}$ onto $\overline{\mathcal{R}(E)} \subset \mathcal{N}(V^*)$. In fact, $\overline{\mathcal{R}(E)} = \mathcal{N}(V_1^*)$, where $V_1 = V|_{\mathcal{N}(\Delta_T)\ominus\mathcal{H}^\infty}$, $\mathcal{H}^\infty$ being the isometric reducing part of T in $\mathcal{H}$. As $\mathcal{R}^\infty$ reduces E^*E, we have

$$\overline{\mathcal{R}(E)} = J\overline{|E|\mathcal{R}^\infty} \oplus J\overline{|E|(\overline{\mathcal{R}(\Delta_T)} \ominus \mathcal{R}^\infty)}.$$

Thus $\overline{E\mathcal{R}^\infty}$ is a wandering subspace for V, so the subspace

$$\mathcal{N}_\infty := l^2_+(V, \overline{E\mathcal{R}^\infty})$$

is well defined in $\mathcal{N}(\Delta_T)$. Furthermore, the subspace $\mathcal{K} := \mathcal{N}_\infty \oplus \mathcal{R}^\infty$ reduces T. Indeed, it is immediate that $T\mathcal{K} \subset \mathcal{K}$ by using the matrix representation (2.5) of T. On the other hand, $\mathcal{N}_\infty$ reduces V because $E^*V = 0$ and also we have

$$E^*\mathcal{N}_\infty \subset \bigvee_{n\geq 0} E^*V^n\overline{E\mathcal{R}^\infty} \subset \overline{E^*E\mathcal{R}^\infty} \subset \mathcal{R}^\infty.$$

These facts and the form of T^* given by (2.5) show that $T^*\mathcal{K} \subset \mathcal{K}$, hence $\mathcal{K}$ reduces T and so Δ_T. In addition, since $\mathcal{K} \subset \mathcal{H}_q$, it follows that $\mathcal{K}$ also reduces $T|_{\mathcal{H}_q}$ and, as this last operator is a quasi-Brownian isometry, we deduce that $T|_{\mathcal{K}}$ is a quasi-Brownian isometry. Consequently, $\mathcal{H}_q^* := \mathcal{H}^\infty \oplus \mathcal{K}$ reduces T to a quasi-Brownian isometry.

Next, we show that $\mathcal{H}_q^*$ is the maximum subspace in $\mathcal{H}$ which reduces T to a 2-isometry. This will imply that it is even the quasi-Brownian isometric reducing part of T in $\mathcal{H}$.

Let $\mathcal{M} \subset \mathcal{H}$ be a reducing subspace for T such that $T' = T|_{\mathcal{M}}$ is a 2-isometry. Then $\Delta_{T'} = \Delta_T|_{\mathcal{M}}$ and $P_{\mathcal{M}}\Delta_T = \Delta_T P_{\mathcal{M}}$, so

$$\overline{\mathcal{R}(\Delta_{T'})} = \overline{\mathcal{R}(\Delta_T)} \cap \mathcal{M} = P_{\mathcal{M}}\overline{\mathcal{R}(\Delta_T)} = P_{\overline{\mathcal{R}(\Delta_T)}}\mathcal{M}.$$

It follows immediately that $\overline{\mathcal{R}(\Delta_{T'})}$ is invariant for T^*, and since $W^* = T^*|_{\overline{\mathcal{R}(\Delta_T)}}$, the subspace $\overline{\mathcal{R}(\Delta_{T'})}$ is invariant for W^*. This subspace is also invariant for W (as we proved in the proof of Theorem 2.5), hence $\overline{\mathcal{R}(\Delta_{T'})}$ reduces W. But T' is a quasi-Brownian isometry (T being Δ_T-regular), so $W' := W|_{\overline{\mathcal{R}(\Delta_{T'})}} = P_{\overline{\mathcal{R}(\Delta_{T'})}} T'|_{\overline{\mathcal{R}(\Delta_{T'})}}$ is an isometry. Consequently, $\overline{\mathcal{R}(\Delta_{T'})} \subset \mathcal{R}^\infty$.

On the other hand, it is clear that $\mathcal{N}(\Delta_{T'}) \subset \mathcal{N}(\Delta_T)$ and

$$\mathcal{N}(\Delta_T) = \mathcal{H}_\infty \oplus l^2_+(V, \mathcal{N}(V^*)) = \mathcal{H}^\infty \oplus l^2_+(V, \overline{\mathcal{R}(E)}),$$

where $\mathcal{H}_\infty$ is given by (2.1). But, as $\mathcal{M}$ reduces T, the 2-isometry T' has the matrix representation of the form (2.5) on $\mathcal{M} = \mathcal{N}(\Delta_{T'}) \oplus \overline{\mathcal{R}(\Delta_{T'})}$ given by the operators $V' := V|_{\mathcal{N}(\Delta_{T'})}$, W' of above and $E' = E|_{\overline{\mathcal{R}(\Delta_{T'})}}$. Thus we get

$$\mathcal{N}(\Delta_{T'}) = \mathcal{M}^\infty \oplus l^2_+(V, \overline{E\mathcal{R}(\Delta_{T'})}) \subset \mathcal{H}^\infty \oplus l^2_+(V, \overline{E\mathcal{R}^\infty}),$$

and we conclude that

$$\mathcal{M} \subset \mathcal{H}^\infty \oplus [l^2_+(V, \overline{E\mathcal{R}^\infty}) \oplus \mathcal{R}^\infty] = \mathcal{H}^*_q.$$

Hence $\mathcal{H}^*_q$ has the required maximality property. Also, from the matrix representation of $T|_{\mathcal{H}^*_q}$ on $\mathcal{H}^*_q = [\mathcal{H}^\infty \oplus l^2_+(V, \overline{E\mathcal{R}^\infty})] \oplus \mathcal{R}^\infty$ we get

$$\operatorname{cov}(T|_{\mathcal{H}^*_q}) = \|(E^*E + \Delta_W)|_{\mathcal{R}^\infty}\|^{1/2} = \|E|_{\mathcal{R}^\infty}\|$$

when $\mathcal{R}^\infty \neq \{0\}$. This ends the proof. □

From the above theorem we derive the ensuing corollary.

Corollary 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a non-isometric Δ_T-regular concave operator. Then T has a unique representation as a direct sum of the form*

$$T = T_2 \oplus T_1$$

on a reducing orthogonal decomposition $\mathcal{H} = \mathcal{H}_2 \oplus \mathcal{H}_1$ for T, such that $T_2 = T|_{\mathcal{H}_2}$ is a 2-isometry while T_1 has no non-zero 2-isometric direct summand.

3. Brownian isometric parts

We begin with a result which, for a concave operator, gives other reducing parts contained in the subspaces $\mathcal{H}^*_q$ from (2.9) and $\mathcal{H}_0$ from (2.8).

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a non-isometric Δ_T-regular concave operator having the matrix representation (2.5) given by the operators V, W, E and for $0 < \delta \leq \|\Delta_T\|^{1/2}$ we put $E_\delta = \delta^{-1}E$. Let $\mathcal{H}^\infty \subset \mathcal{H}$ and $\mathcal{R}^\infty \subset \overline{\mathcal{R}(\Delta_T)}$ be the isometric reducing parts of T and W, respectively, and we suppose that $\mathcal{R}_\delta := \mathcal{R}^\infty \cap \mathcal{N}(\Delta_{E_\delta}) \neq \{0\}$. Then the subspace*

$$\mathcal{H}_\delta := \mathcal{H}^\infty \oplus [l^2_+(V, E\mathcal{R}_\delta) \oplus \mathcal{R}_\delta] \tag{3.1}$$

is the 2-isometric reducing part of T in $\mathcal{H}$ such that $\delta^{-2}\Delta_{T|_{\mathcal{H}_\delta}}$ is an orthogonal projection.

Proof. By the block matrix representation (2.5) of T, we have

$$\Delta_T|_{\overline{\mathcal{R}(\Delta_T)}} = E^*E - D_W^2 \leq E^*E,$$

therefore $0 < \|\Delta_T\|^{1/2} \leq \|E\|$, T being non-isometric. Thus, for a fixed $\delta \in (0, \|E\|]$, we can write $E = \delta E_\delta$ with an injective (like E) operator E_δ such that $\|E_\delta\| \geq 1$.

Since W commutes to $E^*E - D_W^2$ on $\overline{\mathcal{R}(\Delta_T)}$ (by Proposition 2.3(c)), it follows that the subspace $\mathcal{R}_\delta = \mathcal{R}^\infty \cap \mathcal{N}(\Delta_{E_\delta})$ reduces W, and (using (2.2) for $\mathcal{R}^\infty$) one can see that $\mathcal{R}_\delta$ also reduces $E_\delta^* E_\delta$. Hence $W_\delta = W|_{\mathcal{R}_\delta}$ is an isometry, and E_δ isometrically maps $\mathcal{R}_\delta$ into $\mathcal{N}(V^*)$ when $\mathcal{R}_\delta \neq \{0\}$. In this case the operator

$$E : \mathcal{R}_\delta \oplus [\overline{\mathcal{R}(\Delta_T)} \ominus \mathcal{R}_\delta] \to [\mathcal{N}(\Delta_T) \ominus L_\delta] \oplus L_\delta, \qquad \text{where} \qquad L_\delta = l_+^2(V, E\mathcal{R}_\delta),$$

has the matrix representation

$$E = \begin{pmatrix} 0 & E' \\ F & 0 \end{pmatrix},$$

with $\delta^{-1}F = E_\delta|_{\mathcal{R}_\delta}$ being an isometry from $\mathcal{R}_\delta$ into L_δ. Because the subspace L_δ reduces the isometry V, from the representation (2.5) of T it follows that the subspace $L_\delta \oplus \mathcal{R}_\delta$ is reducing for T. Hence the subspace $\mathcal{H}_\delta = \mathcal{H}^\infty \oplus (L_\delta \oplus \mathcal{R}_\delta)$ from (3.1) is reducing for T, such that $T_\delta = T|_{\mathcal{H}_\delta}$ is a 2-isometry because it has on $\mathcal{H}_\delta = (\mathcal{H}^\infty \oplus L_\delta) \oplus \mathcal{R}_\delta$ the matrix representation

$$T_\delta = \begin{pmatrix} V_\delta & \delta F_\delta \\ 0 & W_\delta \end{pmatrix}.$$

Here $V_\delta = V|_{\mathcal{H}^\infty \oplus L_\delta}$, W_δ as above and $F_\delta = (0, \delta^{-1}F)^{\mathrm{tr}}$ from $\mathcal{R}_\delta$ into $\mathcal{H}^\infty \oplus L_\delta$ are isometries such that $V_\delta^* F_\delta = 0$. Hence the above matrix gives $\Delta_{T_\delta} = 0 \oplus \delta^2 I$, i.e. $\delta^{-2}\Delta_{T_\delta}$ is an orthogonal projection. But since $\mathcal{R}_\delta \neq \{0\}$ we infer that $\delta = \|\Delta_{T_\delta}\|^{1/2} = \|\Delta_T|_{\mathcal{R}_\delta}\|^{1/2} \leq \|\Delta_T\|^{1/2}$, so δ necessarily belongs to the interval $(0, \|\Delta_T\|^{1/2}]$ (comparing with the choice of δ before).

Let us prove that $\mathcal{H}_\delta$ is the maximum subspace in $\mathcal{H}$ with the required properties relative to T. Indeed, let $\mathcal{M} \subset \mathcal{H}$ be another reducing subspace for T such that $T' := T|_{\mathcal{M}}$ is a 2-isometry with $\delta^{-2}\Delta_{T'}$ an orthogonal projection. Then $\Delta_{T'} = \Delta_T|_{\mathcal{M}}$ and so T' is a quasi-Brownian isometry (T being Δ_T-regular), hence $\mathcal{M} \subset \mathcal{H}_q^*$ by Theorem 2.7 and $\mathcal{M}$ reduces $T_* := T|_{\mathcal{H}_q^*}$. This yields $\Delta_{T'} = \Delta_{T_*}|_{\mathcal{M}}$, therefore $\mathcal{R}(\Delta_{T'}) \subset \overline{\mathcal{R}(\Delta_{T_*})} = \mathcal{R}^\infty$, and also

$$P_{\mathcal{R}(\Delta_{T'})} T' \mathcal{R}(\Delta_{T'}) = P_{\mathcal{R}(\Delta_{T'})} T \mathcal{R}(\Delta_{T'}) = W\mathcal{R}(\Delta_{T'}),$$

$$T'^* \mathcal{R}(\Delta_{T'}) = T^* \mathcal{R}(\Delta_{T'}) = W^* \mathcal{R}(\Delta_{T'}).$$

Thus $\mathcal{R}(\Delta_{T'})$ reduces W to an isometry and $W|_{\mathcal{R}(\Delta_{T'})} = P_{\mathcal{R}(\Delta_{T'})} T'$.

On the other hand, by the choice of T' on $\mathcal{M}$, we have

$$\delta^2 I_{\mathcal{R}(\Delta_{T'})} = \Delta_{T'}|_{\mathcal{R}(\Delta_{T'})} = \Delta_{T_*}|_{\mathcal{R}(\Delta_{T'})} = \delta^2 E_\delta^* E_\delta|_{\mathcal{R}(\Delta_{T'})},$$

therefore $\mathcal{R}(\Delta_{T'})$ reduces $E_\delta^* E_\delta$ to an isometry. This shows that $\mathcal{R}(\Delta_{T'}) \subset \mathcal{N}(\Delta_{E_\delta})$ and finally $\mathcal{R}(\Delta_{T'}) \subset \mathcal{R}^\infty \cap \mathcal{N}(\Delta_{E_\delta}) = \mathcal{R}_\delta$.

Now, since $\Delta_{T'} = \Delta_{T_*}|_{\mathcal{M}}$, we have $\mathcal{N}(\Delta_{T'}) \subset \mathcal{N}(\Delta_{T_*})$, and by (2.9) we obtain $\mathcal{N}(\Delta_{T'}) \ominus \mathcal{H}^\infty \subset l_+^2(V, \overline{E\mathcal{R}^\infty})$. But as $\mathcal{R}(\Delta_{T'})$ reduces $E_\delta^* E_\delta$, it is clear that $P_{\mathcal{N}(\Delta_{T'})} T'|_{\mathcal{R}(\Delta_{T'})} = \delta E_\delta|_{\mathcal{R}(\Delta_{T'})} = E|_{\mathcal{R}(\Delta_{T'})}$. Thus we infer that

$$\mathcal{N}(\Delta_{T'}) = \mathcal{M}^\infty \oplus l_+^2(V, E\mathcal{R}(\Delta_{T'})) \subset \mathcal{H}^\infty \oplus l_+^2(V, E\mathcal{R}_\delta) = \mathcal{H}_\delta \ominus \mathcal{R}_\delta,$$

where $\mathcal{M}^\infty$ is the isometric reducing part of T' in $\mathcal{M}$. We conclude that $\mathcal{M} = \mathcal{N}(\Delta_{T'}) \oplus \mathcal{R}(\Delta_{T'}) \subset \mathcal{H}_\delta$, which shows the desired property of $\mathcal{H}_\delta$ relative to T. This ends the proof. □

As we remarked in the above proof, the subspace $\mathcal{H}_\delta$ from (3.1) is even the quasi-Brownian isometric reducing part of T in $\mathcal{H}$ on which $\delta^{-2}\Delta_T$ is an orthogonal projection. We also observe from the given argument that if $\delta = \|E\| > 0$ and $\mathcal{R}_\delta \neq \{0\}$, then $\delta = \|\Delta_T\|^{1/2}$, while $E_\delta = \delta^{-1}E$ is a contraction (like W) with $\|E_\delta\| = \|W\| = 1$ in this case. This certainly happens when T is a quasi-Brownian isometry, where W is an isometry, so $\overline{\mathcal{R}(\Delta_T)} = \mathcal{R}^\infty$ and $\mathcal{R}_\delta = \mathcal{N}(D_{E_\delta})$. Thus we obtain the following result.

Corollary 3.2. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-Brownian isometry of covariance $\delta_0 = \|\Delta_T\|^{1/2} > 0$, then the corresponding subspace $\mathcal{H}_\delta$ from (3.1) is*

$$\mathcal{H}_{\delta_0} = \mathcal{H}^\infty \oplus [l_+^2(V, E\mathcal{N}(D_{E_{\delta_0}})) \oplus \mathcal{N}(D_{E_{\delta_0}})] \tag{3.2}$$

where the isometry V and the contraction E_{δ_0} have the meaning as in Theorem 3.1.

Notice that for a quasi-Brownian isometry this part $\mathcal{H}_{\delta_0}$ from (3.2), as well as the Brownian isometric and unitary reducing parts, respectively, were obtained in [13, Theorem 5.4]. Regarding the last two parts for the concave operators we can formulate the following theorem.

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a non-isometric Δ_T-regular concave operator and $V, W, E = \delta E_\delta$ for $0 < \delta \leq \|\Delta_T\|^{1/2}$ be as in Theorem 3.1. Let $\mathcal{H}^\infty, \mathcal{H}_\infty$ (resp., $\mathcal{R}^\infty, \mathcal{R}_\infty$) be the isometric and the unitary parts of T (resp., of W). Then the following statements hold.*

(i) *If $\mathcal{R}_\infty \neq \{0\}$, then the Brownian isometric invariant and reducing parts, respectively, of T in $\mathcal{H}$ are the subspaces*

$$\mathcal{H}_{b0} := \mathcal{N}(\Delta_T) \oplus \mathcal{R}_\infty \tag{3.3}$$

and respectively

$$\mathcal{H}_{b0}^* := \mathcal{H}^\infty \oplus [l_+^2(V, \overline{E\mathcal{R}_\infty}) \oplus \mathcal{R}_\infty]. \tag{3.4}$$

Moreover, we have $\mathrm{cov}(T|_{\mathcal{H}_{b0}^*}) = \mathrm{cov}(T|_{\mathcal{H}_{b0}}) = \|\Delta_T|_{\mathcal{R}_\infty}\|^{1/2}$.

(ii) *If $\mathcal{R}_\delta^0 := \mathcal{R}_\infty \cap \mathcal{N}(\Delta_{E_\delta}) \neq \{0\}$, then the Brownian unitary reducing part of T in $\mathcal{H}$ on which T has the covariance δ is the subspace*

$$\mathcal{H}_\delta^0 := \mathcal{H}_\infty \oplus [l_+^2(V, E\mathcal{R}_\delta^0) \oplus \mathcal{R}_\delta^0]. \tag{3.5}$$

In addition, in this case we have

$$\delta = \operatorname{cov}(T|_{\mathcal{H}_\delta^0}) = \operatorname{cov}(T|_{\mathcal{H}_\delta}) = \|\Delta_T|_{\mathcal{R}_\delta^0}\|^{1/2}, \tag{3.6}$$

where $\mathcal{H}_\delta$ is the subspace from (3.1).

Proof. (i) Since $\mathcal{R}_\infty \subset \mathcal{R}^\infty$, it follows that $\mathcal{R}_\infty$ reduces $W|_{\mathcal{R}^\infty}$ to a unitary operator, and also by (2.1), $\mathcal{R}_\infty = \bigcap_{n\geq 1} W^n\overline{\mathcal{R}(\Delta_T)}$ reduces E^*E because $WE^*E = E^*EW$. Thus the subspace $\mathcal{H}_{b0}$ from (3.3) is invariant for T and $\mathcal{H}_{b0} \subset \mathcal{H}_q$ (from (2.6)). Therefore, $\mathcal{H}_{b0}$ is invariant for $T|_{\mathcal{H}_q}$, hence $T|_{\mathcal{H}_{b0}}$ is a 2-isometry with an injective operator $E|_{\mathcal{R}_\infty} = P_{\mathcal{N}(\Delta_T)}T|_{\mathcal{R}_\infty}$ and a unitary operator $T^*|_{\mathcal{R}_\infty} = W^*|_{\mathcal{R}_\infty}$ which commutes with $E^*E|_{\mathcal{R}_\infty}$. Since $\delta_{b0} := \operatorname{cov}(T|_{\mathcal{H}_{b0}}) = \|E|_{\mathcal{R}_\infty}\|$, it follows that $\delta_{b0}^{-1}E|_{\mathcal{R}_\infty}$ is a contraction. These properties when combined imply that $T|_{\mathcal{H}_{b0}}$ is a Brownian isometry.

Next, we show that $\mathcal{H}_{b0}$ is even the maximum invariant subspace for T with this property. Indeed, let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for T such that $T' = T|_{\mathcal{M}}$ is a Brownian isometry. Obviously, $\mathcal{N}(\Delta_{T'}) \subset \mathcal{N}(\Delta_T)$ and so $\overline{\mathcal{R}(\Delta_{T'})} \subset [\mathcal{N}(\Delta_T) \ominus \mathcal{N}(\Delta_{T'})] \oplus \overline{\mathcal{R}(\Delta_T)}$. This implies

$$\begin{aligned}\Delta_{T'}\mathcal{M} &= P_{\mathcal{M}}\Delta_T\mathcal{M} = P_{\mathcal{M}}\Delta_T\overline{\mathcal{R}(\Delta_{T'})}\\ &\subset P_{\overline{\mathcal{R}(\Delta_{T'})}}\Delta_T\overline{\mathcal{R}(\Delta_T)} = P_{\overline{\mathcal{R}(\Delta_T)}}\overline{\mathcal{R}(\Delta_T)},\end{aligned}$$

hence $\overline{\mathcal{R}(\Delta_{T'})} \subset \overline{\mathcal{R}(\Delta_T)}$. This leads to the relations

$$\begin{aligned}W\overline{\mathcal{R}(\Delta_{T'})} &= P_{\overline{\mathcal{R}(\Delta_T)}}T\overline{\mathcal{R}(\Delta_{T'})} = P_{\overline{\mathcal{R}(\Delta_T)}}T'\overline{\mathcal{R}(\Delta_{T'})}\\ &\subset P_{\overline{\mathcal{R}(\Delta_T)}}(\mathcal{N}(\Delta_{T'}) \oplus \overline{\mathcal{R}(\Delta_{T'})}) = P_{\overline{\mathcal{R}(\Delta_T)}}\overline{\mathcal{R}(\Delta_{T'})} = \overline{\mathcal{R}(\Delta_{T'})},\end{aligned}$$

which means that $\overline{\mathcal{R}(\Delta_{T'})}$ is invariant for W. In fact, by Theorem 2.5 we have $\mathcal{M} \subset \mathcal{H}_q$, therefore $\mathcal{M}$ is also invariant for $T|_{\mathcal{H}_q}$. Then, as above, we have $\overline{\mathcal{R}(\Delta_{T'})} \subset \overline{\mathcal{R}(\Delta_{T|_{\mathcal{H}_q}})} = \mathcal{N}(I - S_W) =: \mathcal{N}$ and

$$\begin{aligned}W|_{\overline{\mathcal{R}(\Delta_{T'})}} &= P_{\mathcal{N}}T|_{\overline{\mathcal{R}(\Delta_{T'})}} = P_{\mathcal{N}}T'|_{\overline{\mathcal{R}(\Delta_{T'})}}\\ &= P_{\mathcal{N}}P_{\mathcal{M}}T'|_{\overline{\mathcal{R}(\Delta_{T'})}} = P_{\overline{\mathcal{R}(\Delta_{T'})}}T'|_{\overline{\mathcal{R}(\Delta_{T'})}} =: W'.\end{aligned}$$

Here we used that $\mathcal{M} = \mathcal{N}(\Delta_{T'}) \oplus \overline{\mathcal{R}(\Delta_{T'})}$, where $\mathcal{N}(\Delta_{T'}) \subset \mathcal{H} \ominus \mathcal{N}$ and $\overline{\mathcal{R}(\Delta_{T'})} \subset \mathcal{N}$. Therefore, $\overline{\mathcal{R}(\Delta_{T'})}$ is invariant for W and so for the isometry $W|_{\mathcal{N}}$. Since W' (of above) is unitary ($T|_{\mathcal{M}}$ being a Brownian isometry), it follows that $\overline{\mathcal{R}(\Delta_{T'})}$ reduces $W|_{\mathcal{N}}$ to a unitary operator. Hence, taking into account that W is a contraction (see [12]),

$$\overline{\mathcal{R}(\Delta_{T'})} \subset \bigcap_{n\geq 1} W^n\mathcal{N}(I - S_W) = \mathcal{N}(I - S_{W^*}) \cap \mathcal{N}(I - S_W) = \mathcal{R}_\infty.$$

Finally, we have $\mathcal{M} = \mathcal{N}(\Delta_{T'}) \oplus \overline{\mathcal{R}(\Delta_{T'})} \subset \mathcal{H}_{b0}$ which is the desired property. We conclude that $\mathcal{H}_{b0}$ is the Brownian isometric invariant part of T in $\mathcal{H}$.

Concerning the subspace $\mathcal{H}_{b0}^*$ from (3.4) is easy to see that it is invariant for T. Furthermore, since $\mathcal{R}_\infty$ reduces E^*E and $W|_{\mathcal{R}_\infty}$ is unitary, using the matrix representation of T^* from (2.5), we immediately obtain that $T^*\mathcal{H}_{b0}^* \subset \mathcal{H}_{b0}^*$. Thus $\mathcal{H}_{b0}^*$ reduces T to a Brownian isometry and, in fact, it is the maximum subspace of $\mathcal{H}$ with this property.

Indeed, let $\mathcal{M} \subset \mathcal{H}$ be another reducing subspace for T such that $T' = T|_{\mathcal{M}}$ is a Brownian isometry. Then $\mathcal{M}$ reduces Δ_T and $\Delta_{T'} = \Delta_T|_{\mathcal{M}}$. So $\overline{\mathcal{R}(\Delta_{T'})} = \mathcal{M} \cap \overline{\mathcal{R}(\Delta_T)}$ reduces W to a unitary operator, hence $\overline{\mathcal{R}(\Delta_T)} \subset \mathcal{R}_\infty$. Also we have

$$\mathcal{N}(\Delta_{T'}) = \mathcal{M} \cap \mathcal{N}(\Delta_T) = \mathcal{M} \cap \mathcal{H}^\infty \oplus \mathcal{E},$$

where $\mathcal{M} \cap \mathcal{H}^\infty$ is the isometric reducing part of T' in $\mathcal{M}$, while the subspace $\mathcal{E}$ reduces the isometry $T|_{\mathcal{N}(\Delta_{T'})} = V|_{\mathcal{N}(\Delta_{T'})}$ to a shift operator. Using the matrix representation of T' on $\mathcal{M} = \mathcal{N}(\Delta_{T'}) \oplus \overline{\mathcal{R}(\Delta_{T'})}$ and the inclusion $\overline{\mathcal{R}(\Delta_{T'})} \subset \mathcal{R}_\infty$, we infer that

$$\mathcal{E} = l_+^2(V, \overline{E\mathcal{R}(\Delta_{T'})}) \subset l_+^2(V, \overline{E\mathcal{R}_\infty}),$$

which leads to the inclusion $\mathcal{M} \subset \mathcal{H}_{b0}^*$. This shows that $\mathcal{H}_{b0}^*$ is the Brownian isometric reducing part of T in $\mathcal{H}$. In addition, when $\mathcal{R}_\infty \neq \{0\}$ by using the block matrices (2.5) for $T|_{\mathcal{H}_{b0}}$ and $T|_{\mathcal{H}_{b0}^*}$ we get

$$\|\Delta_{T|_{\mathcal{H}_{b0}}}\| = \|E^*E|_{\mathcal{R}_\infty}\| = \|\Delta_{T|_{\mathcal{H}_{b0}^*}}\|.$$

Therefore,

$$\operatorname{cov}(T|_{\mathcal{H}_{b0}}) = \operatorname{cov}(T|_{\mathcal{H}_{b0}^*}) = \|\Delta_T|_{\mathcal{R}_\infty}\|^{1/2},$$

where for the last equality we used that $\Delta_{T|_{\mathcal{H}_{b0}}} = P_{\mathcal{H}_{b0}}\Delta_T|_{\mathcal{H}_{b0}}$. This ends the proof of (i).

(ii) We can proceed similarly as for $\mathcal{H}_{b0}^*$ to show that the subspace $\mathcal{H}_\delta^0$ reduces T to a Brownian unitary of covariance $\delta \in (0, \|\Delta_T\|^{1/2}]$, because if $\mathcal{R}_\delta^0 = \mathcal{R}_\infty \cap \mathcal{N}(\Delta_{E_\delta}) \neq \{0\}$, then this subspace reduces W and $E_\delta^* E_\delta$, such that $W|_{\mathcal{R}_\delta^0}$ is unitary and E_δ isometrically maps $\mathcal{R}_\delta^0$ into $\mathcal{N}(V^*)$. To see that $\mathcal{H}_\delta^0$ is the maximum subspace in $\mathcal{H}$ with these properties, let $\mathcal{M} \subset \mathcal{H}$ be another subspace which reduces T to a Brownian unitary of covariance δ. Then $\mathcal{M} \subset \mathcal{H}_{b0}^* \cap \mathcal{H}_\delta$ ($\mathcal{H}_\delta$ from (3.1)), and denoting $T' = T|_{\mathcal{M}}$ we infer that $\mathcal{R}(\Delta_{T'}) \subset \mathcal{R}_\infty \cap \mathcal{N}(\Delta_{E_\delta}) = \mathcal{R}_\delta^0$. Also, by the above inclusion of $\mathcal{M}$ and the fact that $\mathcal{M}$ reduces T, we obtain

$$\mathcal{N}(\Delta_{T'}) = \mathcal{M} \cap \mathcal{N}(\Delta_T) \subset \mathcal{H}_\infty \oplus l_+^2(V, E\mathcal{R}(\Delta_{T'})) \subset \mathcal{H}_\infty \oplus l_+^2(V, E\mathcal{R}_\delta^0).$$

Thus we get $\mathcal{M} \subset \mathcal{H}_{b1}$, which shows that $\mathcal{H}_\delta^0$ is even the Brownian unitary part of T in $\mathcal{H}$ with $\operatorname{cov}(T|_{\mathcal{H}_\delta^0}) = \delta$.

Finally, it is clear that in the case $\mathcal{R}_\delta^0 \neq \{0\}$ we have $\operatorname{cov}(T|_{\mathcal{H}_\delta^0}) = \operatorname{cov}(T|_{\mathcal{H}_\delta}) = \delta$. Also, since $\mathcal{R}_\delta^0 \subset \mathcal{R}_\infty$ we obtain $\operatorname{cov}(T|_{\mathcal{H}_\delta^0}) = \|\Delta_T|_{\mathcal{R}_\delta^0}\|^{1/2}$. Therefore the relations (3.6) hold true, which completes the assertion (ii) and ends the proof of the theorem. $\square$

Remark 3.4. The Brownian isometric (unitary) parts of a Δ_T-regular concave operator T coincide with the corresponding parts of the quasi-Brownian isometric invariant part $T|_{\mathcal{H}_q}$ of T (which can be easily provided). Here by Theorem 3.3 and Theorem 3.1, for such T we get a collection of Brownian unitary parts $\{\mathcal{H}_\delta^0\}$ and, respectively, of reducing parts $\{\mathcal{H}_\delta\}$ of T to 2-isometries with $\delta^{-2}\Delta_{T|_{\mathcal{H}_\delta}}$ being an orthogonal projection, for $\delta \in (0, \|\Delta_T\|^{1/2}]$. But we cannot select those desired δ with $\mathcal{H}_\delta^0 \neq \mathcal{H}_\infty$ ($\mathcal{R}_\delta^0 \neq \{0\}$), or $\mathcal{H}_\delta \neq \mathcal{H}_\infty$ ($\mathcal{R}_\delta \neq \{0\}$), such that in (3.1) and (3.5) we meant only these cases.

However an inspection of the proof of Theorem 3.1 reveals that for every scalar $\delta > \delta_T$ one has $\mathcal{R}_\delta = \{0\}$. For $\delta > \|E\|$ even $\mathcal{N}(\Delta_{E_\delta}) = \{0\}$ because from $E = \delta E_\delta$ we have $\|E_\delta\| < 1$. But, if $\mathcal{R}_\delta \neq \{0\}$ (or, respectively, even $\mathcal{R}_\delta^0 \neq \{0\}$), then $\mathcal{R}_\delta \cap \mathcal{R}_{\delta'} = \{0\}$ (resp., $\mathcal{R}_\delta^0 \cap \mathcal{R}_{\delta'}^0 = \{0\}$) for every $\delta' \in (0, \delta_T]$, $\delta' \neq \delta$. Hence $\mathcal{H}_\delta \cap \mathcal{H}_{\delta'} = \mathcal{H}^\infty$ and $\mathcal{H}_\delta^0 \cap \mathcal{H}_{\delta'}^0 = \mathcal{H}_\infty$, in these cases.

Remark 3.5. Let the operator E in (2.5) be bounded from below and let $\delta_0 = \|\Delta_T\|^{1/2}$ and

$$\delta_1 := \min\{\delta_0, \inf_{\|h\|=1} \|Eh\|\} > 0.$$

Then for $\delta \in (0, \delta_1)$, we have $E^*E = \delta_1^2 E_{\delta_1}^* E_{\delta_1} = \delta^2 E_\delta^* E_\delta$. The first equality and the choice of δ_1 yield $E_{\delta_1}^* E_{\delta_1} \geq I$. So $E_\delta^* E_\delta - I \geq E_{\delta_1}^* E_{\delta_1} - I \geq 0$, whence it follows $\mathcal{R}_\delta \subset \mathcal{R}_{\delta_1}$ and $\mathcal{R}_\delta^0 \subset \mathcal{R}_{\delta_1}^0$. But by the last assertion in the previous remark this implies $\mathcal{R}_\delta = \mathcal{R}_\delta^0 = \{0\}$. In this case only for $\delta \in [\delta_1, \delta_0]$ we can have $\mathcal{H}_\delta \neq \mathcal{H}^\infty$ or $\mathcal{H}_\delta^0 \neq \mathcal{H}_\infty$.

Next we derive some consequences from the above theorems. Thus Theorem 3.3 yields

Corollary 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a Δ_T-regular concave operator with $\delta_0 = \|\Delta_T\|^{1/2} > 0$ such that $\mathcal{H}_{\delta_0}^0 \neq \mathcal{H}_\infty$. Then*

$$\operatorname{cov}(T|_{\mathcal{H}_{\delta_0}^0}) = \operatorname{cov}(T|_{\mathcal{H}_{\delta_0}}) = \operatorname{cov}(T|_{\mathcal{H}_{b0}^*}) = \operatorname{cov}(T|_{\mathcal{H}_{b0}}) = \delta_0.$$

Now, from Theorem 3.3 we have a Sz.-Nagy–Foias–Langer type decomposition for the concave operators satisfying (1.2). Instead of the unitary part from the decomposition of a contraction we have here Brownian unitary parts, while the completely non-Brownian unitary part can be refined by the Brownian isometric part. To simplify the expression, if $\mathcal{R}_\delta^0 \neq \{0\}$ we will call $T|_{\mathcal{H}_\delta^0}$ (resp. $\mathcal{H}_\delta^0$) the *δ-Brownian unitary part* of T (in $\mathcal{H}$).

Corollary 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a Δ_T-regular concave operator such that $\mathcal{R}_\delta^0 \neq \{0\}$ for $0 < \delta \leq \|\Delta_T\|^{1/2}$. Then T can be uniquely represented as a direct sum of the form*

$$T = T_\delta \oplus T_{0\delta} \oplus T_{00} \quad \text{on} \quad \mathcal{H} = \mathcal{H}_\delta^0 \oplus (\mathcal{H}_{b0}^* \ominus \mathcal{H}_\delta^0) \oplus (\mathcal{H} \ominus \mathcal{H}_{b0}^*) \tag{3.7}$$

such that T_δ is the δ-Brownian unitary part of T, $T_{0\delta}$ is a Brownian isometry without a δ-Brownian unitary part, while T_{00} is a $\Delta_{T_{00}}$-regular concave operator without a Brownian isometric reducing part.

According to [2, 3], we say that a concave operator T is *pure* if it has no non-zero isometric direct summand, i.e. $\mathcal{H}^\infty = \{0\}$ for T in (2.3). Obviously, this means that $T|_{\mathcal{N}(\Delta_T)}$ is a pure isometry, i.e. a shift operator. In this case the three parts of T in (3.7), as well as $T|_{\mathcal{H}_\delta}$ are pure besides.

Now from Theorem 3.1 and Theorem 3.3 we derive the following result.

Corollary 3.8. *If $T \in \mathcal{B}(\mathcal{H})$ is a pure Δ_T-regular concave operator with non-trivial δ-Brownian unitary part $\mathcal{H}_\delta^0 \subset \mathcal{H}$ for some $\delta \in (0, \delta_0]$, then*

$$\mathcal{H}_\delta^0 = \mathcal{H}_\delta \cap \mathcal{H}_{b0}^*, \tag{3.8}$$

where $\mathcal{H}_\delta$ and $\mathcal{H}_{b0}^*$ are given by (3.1) and (3.4), respectively, and $\delta_0 = \|\Delta_T\|^{1/2}$. In particular, if T is pure with $\delta_0^{-2}\Delta_T$ being an orthogonal projection, then $\mathcal{H}_{b0}^* = \mathcal{H}_{\delta_0}^0$.

We end this section with a simple example of a pure concave operator satisfying (1.2) for which all the above reducing subspaces are null. Such an operator may be slightly modified to an expansive operator also satisfying (1.2).

Example 3.9. *We first record some facts concerning the forward (unilateral) weighted shifts as concave operators. Such an operator T on $\mathcal{H} = l_+^2(\mathbb{C})$ can be written on the canonical orthonormal basis $\{e_n : n \geq 0\} \subset \mathcal{H}$ by*

$$Te_n = w_n e_{n+1}, \quad n \geq 0,$$

where $\{w_n\}$ is a bounded sequence of nonnegative numbers. It is known that such T is a concave operator if and only if $w_n \geq 1$ and $w_n w_{n+1} \leq \sqrt{2w_n^2 - 1}$ for $n \geq 0$ and in this case $\{w_n\}$ is necessarily decreasing and $w_n \leq \sqrt{2}$ for $n \geq 1$ (see [10, Section 6]*, or [18]). Assuming that T is concave, a direct computation shows that T is Δ_T-regular if and only if*

$$(w_{n+1}^2 - 1)w_n = (w_{n+1}^2 - 1)^{1/2} w_n (w_n^2 - 1)^{1/2}, \quad n \geq 0. \tag{3.9}$$

As $w_0 > 1$ (otherwise T will be an isometry), the relation (3.9) implies that either $w_n = w_0$ for every $n \geq 1$ or there exists $n_0 \geq 1$ such that $w_0 = \ldots = w_{n_0 - 1}$ and $w_n = 1$ for $n \geq n_0$. In the first case, $w_0 = 1$ (as T is concave), which contradicts the above assumption on w_0. Therefore it remains to analyse the second case. Thus, if $n_0 > 1$, then $\operatorname{rank}(\Delta_T) \geq 2$ *and, as $w_0 > 1$, it follows that $W = P_{\mathcal{R}(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ is not a contraction so T cannot be Δ_T-regular by Proposition 2.3. We conclude that T is non-isometric Δ_T-regular concave if and only if $w_0 > 1$ and $w_n = 1$ for $n \geq 1$. In this case* $\operatorname{rank}(\Delta_T) = 1$ *and $W = 0$, while $\mathcal{H}_q = \mathcal{H}_{b0} = \mathcal{N}(\Delta_T)$, $\mathcal{H}_{b0}^* = \mathcal{H}_\delta^0 = \mathcal{H}_\delta = \{0\}$ for $0 < \delta \leq \sqrt{w_0^2 - 1}$.*

It is worth noting that if we choose the weighted shift T with the weights $w_0 = w_1 > 1$ and $w_n = 1$ for $n \geq 2$, then T is an expansive operator, *but not concave, which satisfies the condition* $\Delta_T T = \Delta_T^{1/2} T \Delta_T^{1/2}$. *Now,* $\operatorname{rank}(\Delta_T) = 2$ *and* $\mathcal{N}(\Delta_T)$ *is invariant for* T, *while* $V = T|_{\mathcal{N}(\Delta_T)}$ *is an isometry. In addition, if* $\Delta := T_1^* T_1 + T_0^* T_0$, *where* $T_1 = P_{\mathcal{N}(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ *and* $T_0^* = T^*|_{\mathcal{R}(\Delta_T)}$, *then (as* $V^* T_1 = 0$*) one has* $T^* T = I \oplus \Delta$ *on* $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T)$. *So* $\Delta - I = \Delta_T|_{\mathcal{R}(\Delta_T)}$ *and it is easy to see that* $T_0 \Delta = \Delta T_0$ *(because* $w_0 = w_1$*).*

4. Applications to subbrownian operators and 2-isometries

According to [14], $T \in \mathcal{B}(\mathcal{H})$ is said to be a *Brownian operator* if T has the matrix representation (2.5) with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where V is an isometry on $\mathcal{H}_1$, W is a coisometry on $\mathcal{H}_2$ and $E : \mathcal{H}_2 \to \mathcal{H}_1$ is such that $V^* E = 0$, $E^* E + W^* W = 2I$ and $EE^* + VV^* \geq I$. In this case $\Delta_T|_{\mathcal{R}(\Delta_T)} = E^* E + \Delta_W = I$, so $\Delta_T = 0 \oplus I$ on $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T)$ is an orthogonal projection.

Clearly, such an operator T is concave (as $W^* W \leq I$) and Δ_T-regular (by Proposition 2.3). In addition, $E^* E \geq I$, i.e. E is expansive and so injective, and $\mathcal{R}(E) = \mathcal{N}(V^*)$ by the above inequality satisfied by V and E. Hence the Brownian operators are those concave operators which have a matrix representation (2.5) on $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \overline{\mathcal{R}(\Delta_T)}$ with an isometry V, a coisometry W, and an expansive operator E such that $\mathcal{R}(E) = \mathcal{N}(V^*)$ and $E^* E + W^* W = 2I$.

Notice that a quasi-Brownian isometry T is a Brownian operator if and only if W is a unitary operator and $E/\|\Delta_T\|^{1/2}$ is an isometry in (2.5) with $\mathcal{R}(E) = \mathcal{N}(V^*)$, that is, if and only if T is a Brownian unitary. Also, a quasi-Brownian isometry T is a Brownian isometry if and only if W is unitary in (2.5).

Following [14] we say that an operator T on $\mathcal{H}$ is *subbrownian* if it has an extension to a Brownian operator on a Hilbert space containing $\mathcal{H}$. Thus, by [14, Theorem B], T is subbrownian if and only if T is concave with $\|T\| \leq \sqrt{2}$. Such an operator is not necessarily Δ_T-regular. However, we can determine a Brownian unitary reducing part for a subbrownian operator using a Brownian extension.

Firstly, we refer to the relationship between the subspaces $\mathcal{H}_q, \mathcal{H}_{b0}, \mathcal{H}_{b0}^*$ and $\mathcal{H}_b = \mathcal{H}_1^0$ of a Brownian operator.

Theorem 4.1. *If* $T \in \mathcal{B}(\mathcal{H})$ *is a Brownian operator, then the quasi-Brownian isometric and the Brownian isometric invariant parts, respectively, the Brownian unitary and the Brownian isometric reducing parts of* T *in* $\mathcal{H}$ *coincide. In fact, if* $\mathcal{R}_\infty \neq \{0\}$ *we have*

$$\mathcal{H}_q = \mathcal{H}_{b0} = l_+^2(V, E\mathcal{R}_\infty^\perp) \oplus \mathcal{H}_{b0}^*, \quad \mathcal{H}_{b0}^* = \mathcal{H}_b \tag{4.1}$$

where the operators V, E *are from the block matrix* (2.5) *of* T, $\mathcal{R}_\infty$ *is the unitary part of* $T^*|_{\mathcal{R}(\Delta_T)}$, *while* $\mathcal{H}_b = \mathcal{H}_1^0$ *is the Brownian unitary part of* T *in* $\mathcal{H}$ *with* $\operatorname{cov}(T|_{\mathcal{H}_1}) = 1$.

Proof. We know that a Brownian operator T has the representation (2.5) on $\mathcal{H} = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T)$ with isometries $V = T|_{\mathcal{N}(\Delta_T)}$ and $W^* = T^*|_{\mathcal{R}(\Delta_T)}$, an expansive operator $E = P_{\mathcal{N}(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ with $\mathcal{R}(E) = \mathcal{N}(V^*)$ and $\Delta_T|_{\mathcal{R}(\Delta_T)} = E^*E + \Delta_W = I$.

Since W is a coisometry, one has $W^*W = I \oplus 0$, and so $E^*E = I \oplus 2I$ on $\mathcal{R}(\Delta_T) = \mathcal{R}(W^*) \oplus \mathcal{N}(W)$. In addition, the unitary part of W in $\mathcal{R}(\Delta_T)$ is $\mathcal{R}_\infty = \mathcal{N}(I - S_W) \subset \mathcal{R}(W^*)$, so $W = U \oplus S^*$ on $\mathcal{R}(\Delta_T) = \mathcal{R}_\infty \oplus \mathcal{R}_\infty^\perp$, where U is unitary and S is a shift operator. But $\mathcal{R}_\infty$ reduces E^*E (using (2.1) and the fact that E^*E commutes to W), hence if we write $E = \begin{pmatrix} F & G \end{pmatrix}$ from $\mathcal{R}_\infty \oplus \mathcal{R}_\infty^\perp$ into $\mathcal{N}(\Delta_T)$, then $E^*E = F^*F \oplus G^*G = I \oplus G^*G$ whence $F^*F = I$. Thus F is an isometry and one obtains the relations

$$\mathcal{N}(V^*) = \mathcal{R}(E) = \mathcal{R}(F) \oplus \mathcal{R}(G) = E\mathcal{R}_\infty \oplus E\mathcal{R}_\infty^\perp.$$

Therefore we can write the isometry V as a direct sum

$$V = V_0 \oplus V_1 \quad \text{on} \quad \mathcal{N}(\Delta_T) = [\mathcal{H}_\infty \oplus l^2_+(V, E\mathcal{R}_\infty^\perp)] \oplus l^2_+(V, E\mathcal{R}_\infty) =: \mathcal{E}_0 \oplus \mathcal{E}_1$$

where $\mathcal{H}_\infty$ is as in (2.1).

Next with such V and E, W as above we infer the representation

$$T = \begin{pmatrix} V_0 & 0 & 0 & G \\ 0 & V_1 & F & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & S^* \end{pmatrix} \tag{4.2}$$

on the orthogonal decomposition $\mathcal{H} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{R}_\infty \oplus \mathcal{R}_\infty^\perp$. Since $\mathcal{R}_\infty = \mathcal{N}(I - S_W) = \mathcal{R}^\infty$ and that F is injective, we conclude by (2.6) and (3.3) that $\mathcal{H}_q = \mathcal{H}_{b0}$ $(= \mathcal{N}(\Delta_T)$, if $\mathcal{R}_\infty = \{0\})$, i.e. the first equality in (4.1) holds. When $\mathcal{R}_\infty \neq \{0\}$ we infer from the above matrix representation that the subspace

$$\mathcal{H}_1 := \mathcal{H}_\infty \oplus [l^2_+(V, E\mathcal{R}_\infty) \oplus \mathcal{R}_\infty]$$

reduces T to a Brownian unitary operator of covariance $1 = \|\Delta_T\|$, which is even the 1-Brownian unitary part of T in $\mathcal{H}$. Moreover, because $\mathcal{R}(E) = \mathcal{N}(V^*)$ it follows that the isometric reducing part is even the unitary part of T in $\mathcal{H}$, i.e. $\mathcal{H}^\infty = \mathcal{H}_\infty$. So, by (3.4), we have $\mathcal{H}^*_{b0} = \mathcal{H}_1$ and from the above matrix representation of T we obtain the second equality in (4.1). This completes the proof. □

Since for a Brownian operator T, Δ_T is an orthogonal projection, we have $\|\Delta_T|_\mathcal{M}\| = 1$ for any non-zero subspace $\mathcal{M} \subset \mathcal{R}(\Delta_T)$. Hence T has a Brownian unitary reducing part $\mathcal{H}_b \neq \mathcal{H}_\infty$ in $\mathcal{H}$ (with $\mathrm{cov}(T|_{\mathcal{H}_b}) = 1$) if and only if $T^*|_{\mathcal{R}(\Delta_T)}$ has a non-zero unitary part in $\mathcal{R}(\Delta_T)$. In this case we can determine the Brownian unitary reducing parts preserving the covariance 1 of the restrictions of T to non-null invariant subspaces, and we express such parts in terms of $T|_{\mathcal{H}_b}$.

Theorem 4.2. *Let $S \in \mathcal{B}(\mathcal{H})$ be a subbrownian operator such that $\{0\} \neq \mathcal{N}(\Delta_S) \neq \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{K})$ be a Brownian extension of S on $\mathcal{K} \supset \mathcal{H}$.*

Suppose that $\mathcal{K}_b \subset \mathcal{K}$ is the Brownian unitary reducing part of T with $\operatorname{cov}(T|_{\mathcal{K}_b}) = 1$ and that S has Brownian unitary reducing part $\mathcal{H}_b \subset \mathcal{H}$ with $\operatorname{cov}(S|_{\mathcal{H}_b}) = 1$. Then $\mathcal{H}_b$ is the reducible part of T in $\mathcal{H} \cap \mathcal{K}_b$.

Moreover, let $V = T|_{\mathcal{N}(\Delta_T)}$, $W = T^*|_{\mathcal{R}(\Delta_T)}$ and $E = P_{\mathcal{N}(\Delta_T)}T|_{\mathcal{R}(\Delta_T)}$. Then

$$\mathcal{H}_b = \mathcal{H}_\infty \oplus [l^2_+(V, E\mathcal{R}) \oplus \mathcal{R}], \tag{4.3}$$

where $\mathcal{R}$ is the unitary part of the compression of W to the subspace $\overline{\mathcal{R}(\Delta_S)} \cap E^*\mathcal{N}(V_1^*)$, $V_1 = V|_{\mathcal{H}_1}$, while $\mathcal{H}_1$ is the reducible part of $V|_{\mathcal{N}(\Delta_T|_{\mathcal{K}_b})}$ in $\mathcal{N}(\Delta_S) \cap \mathcal{N}(\Delta_T|_{\mathcal{K}_b})$.

Proof. Let $S = T|_{\mathcal{H}}$ and $\mathcal{H}_b$, $\mathcal{K}_b$ be as above. Then $\mathcal{H}_b$ is invariant for T and $T|_{\mathcal{H}_b} = S_{\mathcal{H}_b}$ is Brownian unitary, hence $\mathcal{H}_b \subset \mathcal{K}_{b0}$ (the Brownian isometric invariant part in $\mathcal{K}$ of T). Next $\mathcal{H}_b$ is invariant for the 2-isometry $T|_{\mathcal{K}_{b0}}$ and $\operatorname{cov}(T|_{\mathcal{H}_b}) = \operatorname{cov}(T|_{\mathcal{K}_{b0}}) = 1$, consequently $\mathcal{H}_b$ reduces $T|_{\mathcal{K}_{b0}}$ to a Brownian unitary operator, by [3, Lemma 5.90]. Then, by Theorem 4.1, we have $\mathcal{H}_b \subset \mathcal{K}^*_{b0} = \mathcal{K}_b$, and so (as above) $\mathcal{H}_b$ reduces $T|_{\mathcal{K}_b}$, and finally $\mathcal{H}_b$ reduces T. This last conclusion together with the fact that $\mathcal{H}_b \subset \mathcal{H} \cap \mathcal{K}_b$ and that $\mathcal{H} \cap \mathcal{K}_b$ is invariant for T implies $\mathcal{H}_b \subset \mathcal{H}_r$ is the reducible part of T in $\mathcal{H} \cap \mathcal{K}_b$. But $\mathcal{H}_r$ reduces T, hence it reduces S and $T|_{\mathcal{K}_b}$ (as $\mathcal{H}_r \subset \mathcal{H} \cap \mathcal{K}_b$), and $S|_{\mathcal{H}_r} = (T|_{\mathcal{K}_b})|_{\mathcal{H}_r}$ is Brownian unitary. Since $\Delta_{S|_{\mathcal{H}_b}} = \Delta_{S|_{\mathcal{H}_r}}|_{\mathcal{H}_b} = \Delta_{T|_{\mathcal{K}_b}}|_{\mathcal{H}_b}$ and $\operatorname{cov}(S|_{\mathcal{H}_b}) = \operatorname{cov}(T|_{\mathcal{K}_b}) = 1$, we get $\operatorname{cov}(S|_{\mathcal{H}_r}) = 1$. Consequently, $\mathcal{H}_r \subset \mathcal{H}_b$ (by the meaning of $\mathcal{H}_b$) and finally $\mathcal{H}_b = \mathcal{H}_r$. This provides the first assertion of theorem.

Next we express $\mathcal{H}_b$ in the terms of isometries $V = T|_{\mathcal{N}(\Delta_T)}$, $W = T^*|_{\mathcal{R}(\Delta_T)}$ and of the expansive operator $E = P_{\mathcal{N}(\Delta_T)}T|_{\mathcal{R}(\Delta_T)}$ with $\mathcal{R}(E) = \mathcal{N}(V^*)$ from the block matrix of T on $\mathcal{K} = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T)$. Recall that Δ_T is an orthogonal projection, so $E^*E + W^*W = 2I$, and $\mathcal{R}_\infty = \mathcal{N}(I - S_{W^*})$ is the unitary part of W in $\mathcal{R}(\Delta_T)$, W being an isometry. So $F = E|_{\mathcal{R}_\infty}$ is an isometry from $\mathcal{R}_\infty$ onto $\mathcal{N}(V_b^*)$, where $V_b = V|_{\mathcal{N}(\Delta_T|_{\mathcal{K}_b})}$.

Let $\mathcal{M} \subset \mathcal{H}$ be any reducing subspace for S such that $S' = S|_{\mathcal{M}}$ is Brownian unitary with $\operatorname{cov}(S') = 1$. Then $\mathcal{M} \subset \mathcal{K}_b$ and $\mathcal{M}$ reduces $T|_{\mathcal{K}_b}$ (as in the case of $\mathcal{H}_b$ before), therefore

$$\mathcal{M} = \mathcal{M}(\Delta_{S'}) \oplus \mathcal{R}(\Delta_{S'}) \subset \mathcal{N}(\Delta_S) \cap \mathcal{N}(\Delta_T|_{\mathcal{K}_b}) \oplus \overline{\mathcal{R}(\Delta_S)} \cap \mathcal{R}(\Delta_T|_{\mathcal{K}_b}).$$

Since $\operatorname{cov}(S') > 0$, we have $\mathcal{N}(\Delta_{S'}) \neq \{0\}$, so

$$\mathcal{H}_0 = \mathcal{N}(\Delta_S) \cap \mathcal{N}(\Delta_T|_{\mathcal{K}_b}) \neq \{0\}.$$

The subspace $\mathcal{H}_0$ is invariant for T, and so for V_b and S, while $V_0 = T|_{\mathcal{H}_0} = S|_{\mathcal{H}_0}$ is an isometry and

$$V_b = \begin{pmatrix} V_0 & D \\ 0 & C \end{pmatrix} \quad \text{on} \quad \mathcal{N}(\Delta_T|_{\mathcal{K}_b}) = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp,$$

where C, D are contractions. Consider the subspace $\mathcal{H}_1 \subset \mathcal{H}_0$ given by

$$\mathcal{H}_1 := \mathcal{H}_{0\infty} \oplus l^2_+(V, \mathcal{N}(V_0^*) \ominus \overline{\mathcal{R}(D)}),$$

where $\mathcal{H}_{0\infty}$ is the unitary part of V_0. Clearly, $\mathcal{H}_1$ reduces the isometries $V_0, V, V_b, S|_{\mathcal{N}(\Delta_S)}$ and let $V_1 := V_0|_{\mathcal{H}_1}(= V|_{\mathcal{H}_1} = S|_{\mathcal{H}_1})$. Then

$$\mathcal{N}(V_1^*) \subset \mathcal{N}(V_0^*) \subset \mathcal{N}(V_b^*) = F\mathcal{R}_\infty,$$

which implies $F^*\mathcal{N}(V_1^*) \subset \mathcal{R}_\infty$. So we have

$$\mathcal{R}_1 := F^*\mathcal{N}(V_1^*) \cap \overline{\mathcal{R}(\Delta_S)} \subset \mathcal{R}_\infty \cap \overline{\mathcal{R}(\Delta_S)}.$$

But as $\mathcal{M}$ reduces S and $T|_{\mathcal{K}_b}$ to $S' = S|_{\mathcal{M}} = T|_{\mathcal{M}}$ a Brownian unitary of covariance 1, we infer that $\{0\} \neq \mathcal{R}(\Delta_{S'}) \subset \mathcal{R}_1$.

On the other hand, the above subspace $\mathcal{H}_{0\infty}$ is invariant for the isometry $S|_{\mathcal{N}(\Delta_S)}$ and $S|_{\mathcal{H}_{0\infty}} = V_0|_{\mathcal{H}_{0\infty}}$ is unitary, therefore $\mathcal{H}_{0\infty}$ reduces $S|_{\mathcal{N}(\Delta_S)}$ and so it reduces S to a unitary operator. Hence $\mathcal{H}_{0\infty} \subset \mathcal{H}_\infty$ (the unitary part of S in $\mathcal{H}$). Also, $\mathcal{H}_\infty$ is invariant for V and so it reduces V and T to a unitary operator, therefore $\mathcal{H}_\infty \subset \mathcal{H}_0$ and $\mathcal{H}_\infty$ reduces V_0 which gives $\mathcal{H}_\infty \subset \mathcal{H}_{0\infty}$. Thus $\mathcal{H}_\infty = \mathcal{H}_{0\infty}$, and as $\mathcal{N}(\Delta_{S'}) \subset \mathcal{H}_0 \subset \mathcal{N}(\Delta_{T|_{\mathcal{K}_b}})$ and $\mathcal{M}$ reduces $T|_{\mathcal{K}_b}$, we obtain $\mathcal{N}(\Delta_{S'}) \subset \mathcal{H}_1 = \mathcal{H}_\infty \oplus l^2_+(V, \mathcal{N}(V_0^*) \ominus \overline{\mathcal{R}(D)})$.

Let $W_1 = P_{\mathcal{R}_1}W|_{\mathcal{R}_1}$ which is a contraction on $\mathcal{R}_1$ and let $\mathcal{R} \subset \mathcal{R}_1$ be the unitary part of W_1 in $\mathcal{R}_1$. Clearly, $\mathcal{R}$ reduces W to a unitary operator, so $\mathcal{R}$ is even the unitary reducing part of W in $\mathcal{R}_1$. Now we remark that $\mathcal{R}(\Delta_{S'})$ is invariant for $S'^* = T^*|_{\mathcal{M}} = W|_{\mathcal{M}}$ and $S'^*|_{\mathcal{R}(\Delta_{S'})}$ is unitary (as S' is Brownian unitary). Since $\mathcal{R}(\Delta_{S'}) \subset \mathcal{R}_1$, we obtain by the previous conclusion that $\mathcal{R}(\Delta_{S'}) \subset \mathcal{R}$. In turn this, by the above inclusion $\mathcal{N}(\Delta_{S'}) \subset \mathcal{H}_1$, yields $\mathcal{N}(\Delta_{S'}) \subset \mathcal{H}_\infty \oplus l^2_+(V, F\mathcal{R})$, and finally as $F|_{\mathcal{R}} = E|_{\mathcal{R}}$ we get

$$\mathcal{M} \subset \mathcal{H}_\infty \oplus [l^2_+(V, E\mathcal{R}) \oplus \mathcal{R}] =: \mathcal{H}'.$$

In particular we have $\mathcal{H}_b \subset \mathcal{H}'$ (taking into account the choice of $\mathcal{M}$). But $\mathcal{H}' \subset \mathcal{H} \cap \mathcal{K}_b$ and obviously $\mathcal{H}'$ is invariant for $T|_{\mathcal{K}_b}$and $T|_{\mathcal{H}'} = S|_{\mathcal{H}'}$ is Brownian unitary with $\operatorname{cov}(T|_{\mathcal{H}'}) = 1$ (as $E|_{\mathcal{R}} = F|_{\mathcal{R}}$ is an isometry and $T^*|_{\mathcal{R}} = W|_{\mathcal{R}}$ is unitary). Hence $\mathcal{H}'$ reduces the 2-isometry $T|_{\mathcal{K}_b}$ and so $\mathcal{H}'$ reduces T. Then by the first assertion of theorem we infer that $\mathcal{H}' \subset \mathcal{H}_b$ and by the above converse inclusion we get $\mathcal{H}_b = \mathcal{H}'$, that is, the representation (4.3) of $\mathcal{H}_b$. We proved the second assertion and this ends the proof. □

We retain from the previous proof that a subbrownian operator S has a Brownian unitary part $\mathcal{H}_b$ on which S has covariance 1 if and only if the subspace $\mathcal{R}$ from (4.3) is non-zero.

Remark finally that the argument from this proof can be also used for any 2-isometry of positive covariance and a Brownian unitary extension of it which preserves the covariance (see [3, Theorem 5.80]). We give this result in the sequel.

Proposition 4.3. *Let $S \in \mathcal{B}(\mathcal{H})$ be a 2-isometry with $\delta = \operatorname{cov}(S) > 0$ and $\mathcal{N}(\Delta_S) \neq \{0\}$, and let $T \in \mathcal{B}(\mathcal{K})$ be a Brownian unitary extension of S on $\mathcal{K} \supset \mathcal{H}$ such that $\operatorname{cov}(T) = \delta$. Assume that the block matrix (2.5) of T on $\mathcal{K} = \mathcal{N}(\Delta_T) \oplus \mathcal{R}(\Delta_T)$ is given by the isometries $V = T|_{\mathcal{N}(\Delta_T)}$, $F = \delta^{-1}E$*

where $E = P_{\mathcal{N}(\Delta_T)} T|_{\mathcal{R}(\Delta_T)}$ and the unitary operator $W^* = T^*|_{\mathcal{R}(\Delta_T)}$. Then S has a Brownian unitary reducing part $\mathcal{H}_b \subset \mathcal{H}$ with $\mathrm{cov}(S|_{\mathcal{H}_b}) = \delta$ if and only if W has a reducible part $\mathcal{R} \neq \{0\}$ in $F^*\mathcal{N}(V_1^*) \cap \overline{\mathcal{R}(\Delta_S)}$, where $V_1 = V|_{\mathcal{H}_1}$ and $\mathcal{H}_1$ is the reducible part of V in $\mathcal{N}(\Delta_S)$. In this case $\mathcal{H}_b$ has the form (4.3) and it is the reducible part of T in $\mathcal{H}$.

The proof of this proposition follows step by step as the previous proof in a simplified form, because in this case we have $\mathcal{K}_b = \mathcal{K}$, W is unitary, so $\mathcal{R}_\infty = \mathcal{R}(\Delta_T)$, $\delta^{-1}E$ is an isometry and $\mathcal{H}_0 = \mathcal{N}(\Delta_S)$ (in the above proof). We omit other details and comments.

Proposition 4.3 leads to a generalized von Neumann–Wold decomposition for a 2-isometry T with a non-trivial part $\mathcal{H}_b$, which thus can be written as a direct sum between a Brownian unitary part having the covariance of T and a 2-isometry without such a direct summand. Recall that the von Neumann–Wold decomposition from [15, 20] for a 2-isometry refers to the unitary part (i.e. the Brownian unitary part of covariance 0) and to the analytic 2-isometric part.

The last two results show that it is possible to get Brownian type parts for concave operators which does not satisfy the condition (1.2). The investigation in this direction will be continued in a future paper and will concern even non-expansive operators.

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