Representation theory of the cyclotomic Cherednik algebra via the Dunkl-Opdam subalgebra

Ben Webster

ABSTRACT. We give an alternate presentation of the cyclotomic rational Cherednik algebra, which has the useful feature of compatibility with the Dunkl-Opdam subalgebra. This presentation has a diagrammatic flavor, and it provides a simple explanation of several surprising facts about this algebra. It allows direct proof of the connection of category $\mathcal{O}$ to weighted KLR algebras, allows us to classify the simple Dunkl-Opdam modules over the Cherednik algebra and provides an algebraic construction of the KZ functor. Furthermore, one of prime motivations for considering this approach is to provide a better framework for connecting Cherednik algebras to Coulomb branches of 3-d gauge theories.

CONTENTS

1. Introduction

In this paper we consider the rational Cherednik algebra H in the cyclotomic case, i.e. that of the complex reflection group $G(\ell, 1, n)$. This is an algebra with a quite rich and interesting representation theory; this paper is dedicated to the proposition that this representation theory can be understood more clearly by choosing a different presentation. In particular, we can

classify the simple **Dunkl-Opdam modules** over the Cherednik algebra in this case. This is the analogue for the Cherednik algebra of Gelfand-Tsetlin modules over $U(\mathfrak{gl}_n)$; realizing both these algebras as Coulomb branches makes this analogy manifest. In fact, the approach we apply here can be generalized to any rational Galois order, as we will show in forthcoming work [33].

In Section 2, we describe the presentation needed for our results, and prove that it gives the Cherednik algebra. This presentation may not look obviously simpler than the familiar one introduced by Etingof and Ginzburg [10, (1.15)], but it does have a graphical calculus which allows it to be described in terms of small local relations (much like the KLR algebras [14, 24]). Furthermore, it has another dramatic advantage: it contains a manifest polynomial subalgebra defined by Dunkl and Opdam [9, Def. 3.7]. This subalgebra commutes with the Euler element (unlike the usual polynomial subalgebras, where all generators have weight ± 1 in the Euler grading). While exploited profitably in earlier papers of Dunkl and Griffeth [8, 13], there is much more this subalgebra can tell us about the representation theory of these algebras.

In Section 3, we turn to using this presentation to study the representation theory of the Cherednik algebra, using weight spaces for the Dunkl-Opdam polynomial subalgebra. This allows a new interpretation of previous work of the author relating category $\mathcal{O}$ of Cherednik algebras to weighted KLR algebras [31], a key step in proving Rouquier's conjecture on the decomposition numbers of category $\mathcal{O}$ for Cherednik algebras (this result was proved by other methods in [25, 18]). That work depended on a very indirect method using uniqueness of highest weight covers, whereas using this new presentation, it can be proven directly. Similarly, the Knizhnik-Zamolodchikov functor of [11], which had only been constructed analytically before, can be realized as a sum of weight spaces for the Dunkl-Opdam polynomial subalgebra (in particular, we can define the KZ functor over an arbitrary characteristic 0 field, not just $\mathbb{C}$). These results are only valid in characteristic 0, but this technique is also promising for studying the Cherednik algebra and coherent sheaves on Hilbert schemes in characteristic p.

In Section 4, we discuss the original motivation for this presentation: to exhibit an isomorphism between the spherical Cherednik algebra and the Coulomb branch of a certain 3-d gauge theory. While this paper was in preparation, this isomorphism was proven independently by Kodera-Nakajima [15]. This isomorphism looks quite strange in the usual presentation of the Cherednik algebra, and quite natural in the alternate one given here. It would be quite interesting to find a geometric description of the Cherednik algebra like the BFN construction of the Coulomb branch [20, 5], in terms of convolution in homology.

2. An alternate presentation

Let $\Bbbk$ be a field of characteristic coprime to ℓ and ζ be a primitive ℓth root of unity in $\Bbbk$. Let $\mathbb{K} = \Bbbk[\hbar]$. For most purposes, we can take $\Bbbk = \mathbb{C}$ and $\zeta = e^{2\pi i/\ell}$.

Let Γ be the group of $n \times n$ monomial matrices with entries given by ℓth roots of unity; this group is a wreath product of S_n with $\mathbb{Z}/\ell\mathbb{Z}$. It's generated by the permutation matrices (identified with S_n) and the matrices $t_j = \mathrm{diag}(1, \ldots, \zeta, \ldots, 1)$ with $(t_j)_{jj} = \zeta$ and all other diagonal entries 1.

Fix parameters $k, h_1, \ldots, h_{\ell-1}$, with the convention that $h_0 = h_\ell = 0$ and let

$$p(u) = \sum_{s=1}^{\ell-1} \sum_{r=1}^{\ell-1} \zeta^{-rs} h_r u^s. \tag{2.1}$$

We can equivalently fix the values

$$s_m = p(\zeta^m) + m\hbar \text{ for } m = 0, \ldots, \ell-1. \tag{2.2}$$

We'll consider the cyclotomic rational Cherednik algebra H for Γ, generated over $\mathbb{K}[\Gamma]$ by two alphabets of commuting variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. The former transform in the defining representation of Γ and the latter in its dual. That is:

$$t_i x_j = \zeta^{\delta_{ij}} x_j t_i \qquad t_i y_j = \zeta^{-\delta_{ij}} y_j t_i \tag{2.3}$$

The final relation is

$$[x, y] = \hbar\langle x, y\rangle - \sum_{s \in S} c_s \langle x, \alpha_s\rangle \langle \alpha_s^\vee, y\rangle \cdot s$$

We will use slightly different conventions here, following the conventions of [12, §2.1.3], so these relations take the form:

$$[x_i, y_i] = \hbar + k \sum_{j \neq i} \sum_{p=0}^{\ell-1} t_i^p t_j^{-p}(ij) + \sum_{s=1}^{\ell-1} \sum_{r=1}^{\ell} \zeta^{-rs}(h_r - h_{r-1}) t_i^s \tag{2.4}$$

$$= \hbar + k \sum_{j \neq i} \sum_{p=0}^{\ell-1} t_i^p t_j^{-p}(ij) + p(t_i) - p(\zeta^{-1} t_i)$$

$$[x_i, y_j] = -k \sum_{p=0}^{\ell-1} \zeta^p t_i^p t_j^{-p}(ij) \qquad\qquad (i \neq j) \tag{2.5}$$

Recall that this algebra contains the modified Dunkl-Opdam operators

$$u_i = y_i x_i + k \sum_{j > i} \sum_{p=0}^{\ell-1} t_i^p t_j^{-p}(ij) + p(t_i) \tag{2.6}$$

$$= x_i y_i - k \sum_{j < i} \sum_{p=0}^{\ell-1} t_i^p t_j^{-p}(ij) + p(\zeta^{-1} t_i) - \hbar \tag{2.7}$$

These differ from those defined in [13, (2.17)] by $z_i = u_i - p(\zeta^{-1}t_i) - \hbar$ and the reindexing of $1, \ldots, n$ by $i \mapsto n - i + 1$. Note that since t_i and z_i generate a commutative subalgebra, these elements u_i commute with each other and with t_i (and generate the same subalgebra). Accounting for reindexing, the equation [13, (3.5)] implies that if we let $r_j = (j, j + 1)$:

$$u_{r_j \cdot i} r_j - r_j u_i = \begin{cases} k\ell \pi_{j,j+1} & i = j \\ -k\ell \pi_{j,j+1} & i = j + 1 \\ 0 & j \notin \{i, i - 1\} \end{cases} \tag{2.8}$$

where $\pi_{j,m} = \frac{1}{\ell} \left(\sum_{p=0}^{\ell-1} t_j^p t_m^{-p} \right)$ is the projection to the invariants of $t_j t_m^{-1}$. Let $\mathcal{DO}_n$ denote the algebra generated by $\Bbbk\Gamma$ and u_i modulo the relations (2.8).

Consider the free $\mathbb{K}$ algebra $\tilde{A}$ generated by the group algebra $\mathbb{K}\Gamma$ and the symbols σ, τ, and u_i for $i = 1, \ldots, n$. We define $u_i, t_i \in \tilde{A}$ for any $i \in \mathbb{Z}$ by the rule $u_i = u_{i-n} + \hbar$, and $t_i = t_{i-n}\zeta^{-1}$.

Definition 2.1. *We let A be the quotient of this algebra by the relations* (2.8) *and:*

$$
\begin{align}
u_i t_j &= t_j u_i & i, j &\in \mathbb{Z} \tag{2.9a} \\
\sigma r_{j-1} &= r_j \sigma & j &= 2, \ldots, n - 1 \tag{2.9b} \\
\tau r_j &= r_{j-1} \tau & j &= 2, \ldots, n - 1 \tag{2.9c} \\
\sigma^2 r_n &= r_1 \sigma^2 & & \tag{2.9d} \\
\tau^2 r_1 &= r_n \tau^2 & & \tag{2.9e} \\
\sigma\tau &= u_1 - p(\zeta^{-1}t_1) + \hbar & & \tag{2.9f} \\
\tau\sigma &= u_n - p(t_n) & & \tag{2.9g} \\
u_i u_j &= u_j u_i & i, j &\in \mathbb{Z} \tag{2.9h} \\
u_i \sigma &= \sigma u_{i-1} & i &\in \mathbb{Z} \tag{2.9i} \\
u_i \tau &= \tau u_{i+1} & i &\in \mathbb{Z} \tag{2.9j} \\
t_i \sigma &= \sigma t_{i-1} & i &\in \mathbb{Z} \tag{2.9k} \\
t_i \tau &= \tau t_{i+1} & i &\in \mathbb{Z} \tag{2.9l}
\end{align}
$$

$$\tau(1,2)\sigma = \sigma(n-1,n)\tau + k\left(\sum_{p=0}^{\ell-1} \zeta^p t_n^p t_1^{-p} \right) \tag{2.9m}$$

Remark 2.2. Note that these relations are closely related to those for the degenerate DAHA given in [4, Def. 2.1], and should be regarded as a higher level version of this presentation.

We can represent these elements graphically as string diagrams on a cylinder with a seam. We'll draw these on the page with the cylinder cut along

the seam. The generators are:

t_m $\qquad\qquad$ u_m $\qquad\qquad$ $(m, m+1)$

σ $\qquad\qquad\qquad$ τ
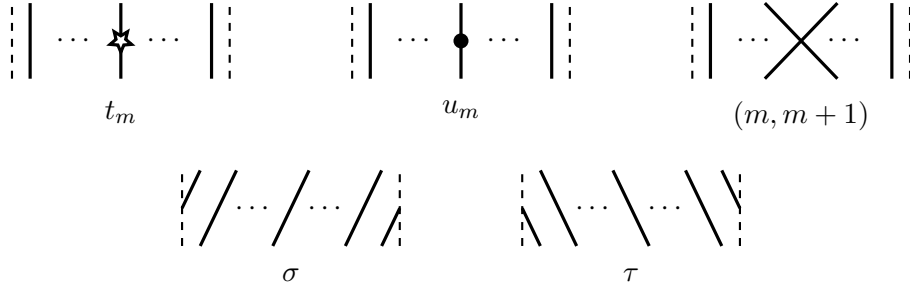
The relations of Γ and (2.8–2.9m) are determined by simple local rules such as:
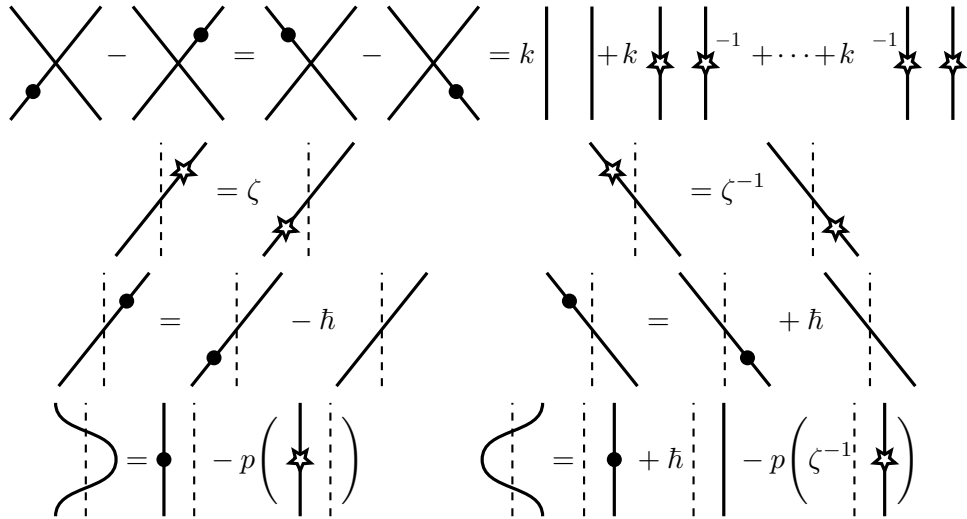
Consider the permutations $\chi_i = (i, i-1\ldots,1)$ and $\upsilon_i = (i, i+1, \ldots, n)$.

Theorem 2.3. *The algebras A and H are isomorphic via maps identifying the copies of $\mathbb{K}[\Gamma]$ and sending*

$$x_i \mapsto \chi_i \sigma \upsilon_i^{-1} \qquad y_i \mapsto \upsilon_i \tau \chi_i^{-1} \qquad u_i \mapsto u_i.$$

The elements $\chi_i \sigma \upsilon_i^{-1}$ and $\upsilon_i \tau \chi_i^{-1}$ have natural graphical representations:
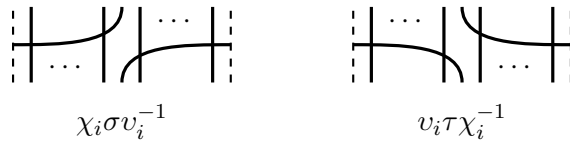
$\chi_i \sigma \upsilon_i^{-1}$ $\qquad\qquad\qquad$ $\upsilon_i \tau \chi_i^{-1}$

Proof. First we need to check the compatibility of this map with the action of Γ. Note that the images of x_i and y_i commute with transpositions except $(i, i \pm 1)$, and

$$(i, i \pm 1)\chi_i \sigma \upsilon_i^{-1}(i, i \pm 1) = \chi_{i\pm 1}\sigma \upsilon_{i\pm 1}^{-1}, \ \ (i, i \pm 1)\upsilon_i \tau \chi_i^{-1}(i, i \pm 1) = \upsilon_{i\pm 1}\tau \chi_{i\pm 1}^{-1}$$

This establishes equivariance for $S_n \subset \Gamma$. Furthermore,

$$\chi_i \sigma v_i^{-1} t_i = \chi_i \sigma t_n v_i^{-1} \qquad\qquad v_i \tau \chi_i^{-1} t_i = v_i \tau t_1 \chi_i^{-1}$$
$$= \chi_i t_{n+1} \sigma v_i^{-1} \qquad\qquad\qquad = v_i t_0 \tau \chi_i^{-1}$$
$$= t_{n+i} \chi_i \sigma v_i^{-1} \qquad\qquad\qquad = t_{i-n} v_i \tau \chi_i^{-1}$$
$$= \zeta^{-1} t_i \chi_i \sigma v_i^{-1} \qquad\qquad\qquad = \zeta t_i v_i \tau \chi_i^{-1}.$$

Similar calculations show that these elements commute with the other t_j's. Thus, these elements have the correct commutation relations with Γ and we need only check that they have the correct commutator with each other.

First, let us check that the images of x_i and x_j commute; we can assume that $j > i$. Thus, we have that:

$$\chi_j \sigma v_j^{-1} \chi_i \sigma v_i^{-1} = \chi_j \sigma \chi_i v_j^{-1} \sigma v_i^{-1}$$
$$= \chi_j (i+1, i, \ldots, 2) \sigma^2 (j-1, j, \ldots, n-1) v_i^{-1}$$
$$= \chi_i (j+1, i, \ldots, 2) \sigma^2 (i-1, j, \ldots, n-1) v_j^{-1}$$
$$= \chi_i \sigma v_i^{-1} \chi_j \sigma v_j^{-1}.$$

This proof is perhaps easier to imagine using a picture:
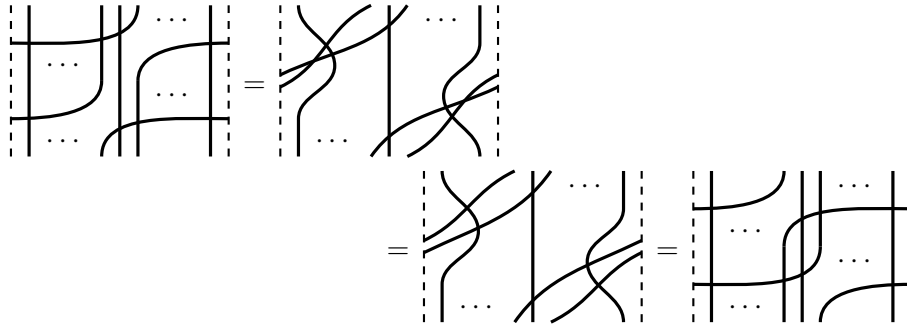
The key relation is (2.9d) which we use in the middle equality.

Next, we consider the commutation relation between x_i and y_i, given in (2.4). This we will prove in a few steps:

$$[\chi_i \sigma v_i^{-1}, v_i \tau \chi_i^{-1}] = \chi_i \sigma \tau \chi_i^{-1} - v_i \tau \sigma v_i^{-1}$$
$$= \chi_i (u_1 - p(\zeta^{-1} t_1) + \hbar) \chi_i^{-1} - v_i (u_n - p(t_n)) v_i^{-1} \quad (2.10)$$
$$= \hbar + p(t_i) - p(\zeta^{-1} t_i) + \chi_i u_1 \chi_i^{-1} - v_i u_n v_i^{-1}.$$

Note that

$$p(t_i) - p(\zeta^{-1} t_i)$$
$$= \sum_{s=1}^{\ell-1} \sum_{r=0}^{\ell} \zeta^{-rs} h_r t_i^s - \sum_{s=1}^{\ell-1} \sum_{r=0}^{\ell} \zeta^{-(r+1)s} h_r t_i^s = \sum \sum \zeta^{-rs} (h_r - h_{r-1}) t_i^s.$$
$$(2.11)$$

Similarly,

$$\chi_i u_1 \chi_i^{-1} = u_i + k \sum_{j=1}^{i-1} \sum_{p=0}^{\ell-1} t_j^p t_i^{-p}(j,i); \tag{2.12}$$

$$v_i u_n v_i^{-1} = u_i - k \sum_{j=i+1}^{n} \sum_{p=0}^{\ell-1} t_j^p t_i^{-p}(j,i). \tag{2.13}$$

Thus, combining (2.10–2.13), we can confirm (2.4) as follows:

$$[\chi_i \sigma v_i^{-1}, v_i \tau \chi_i^{-1}] = \hbar + \sum_{s=1}^{\ell-1} \sum_{r=0}^{\ell} \zeta^{-rs}(h_r - h_{r-1})t_i^s + k \sum_{i \neq j} \sum_{p=0}^{\ell-1} t_j^p t_i^{-p}(j,i). \tag{2.14}$$

Similarly, if $i \neq j$, then

$$[\chi_i \sigma v_i^{-1}, v_j \tau \chi_j^{-1}] = \chi_i v_j (\sigma(1,2)\tau - \tau(n-1,n)\sigma) v_i^{-1} \chi_j^{-1} = -k \sum_{p=0}^{\ell-1} \zeta^p t_i^p t_j^{-p}(i,j). \tag{2.15}$$

This confirms (2.5).

Thus, we have verified the existence of a map $\mathsf{H} \to A$. Note that

$$u_1 = x_1 y_1 + p(\zeta^{-1} t_1) - \hbar \mapsto \sigma\tau + p(\zeta^{-1} t_1) - \hbar = u_1.$$

By the relations (2.8) and (2.8), this implies $u_i \mapsto u_i$ for all i.

The inverse is defined by

$$\sigma \mapsto (1,\dots,i)x_i(i,\dots,n) \qquad \tau \mapsto (n,\dots,i)y_i(i,\dots,1) \qquad u_i \mapsto u_i \quad (2.16)$$

so this map is an isomorphism. $\qquad\square$

Lemma 2.4. *Under this isomorphism, the deformed Euler element* eu *of the Cherednik algebra matches* $u_1 + \cdots + u_n + n/2$.

Proof. This follows immediately from the fact that $u_i \mapsto u_i$ and the formula for the deformed Euler element given in [12, §2.3.5]. $\qquad\square$

Thus, considering the simultaneous eigenspaces of these operators gives a finer decomposition of the Euler eigenspaces, which we will study in the following section.

Remark 2.5. The map of commutator with eu is semi-simple on H, with all eigenvalues in $\mathbb{Z}$. Thus, this conjugation induces a $\mathbb{Z}$-grading on H, which is easy to describe in the presentation given above: the elements σ and τ have degrees 1 and -1, respectively, and all other generators have degree 0; we leave the verification of this based on the relations (2.8–2.9m) to the reader. Thus, in terms of diagrams, this grading measures the total winding number around the cylinder.

Lemma 2.6. *The elements* u_i, t_i *for* $i = 1,\dots,n$ *generate a subring* U *of* A *isomorphic to* $\mathbb{K}[u_1,\dots,u_n,t_1,\dots,t_n]/(t_1^\ell - 1,\dots,t_n^\ell - 1)$.

Proof. Obviously, the t_i generate a copy of the the group ring on $(\mathbb{Z}/\ell\mathbb{Z})^n$. The elements u_i commute by (2.9h). Furthermore, their images in the associated graded $\mathrm{gr}\,\mathsf{H} \cong \mathbb{K}[\Gamma] \otimes \mathbb{K}[\mathbf{x}, \mathbf{y}]$ are given by $x_1 y_1, \ldots, x_n y_n$. Since these are algebraically independent over the group algebra, the u_i are as well, and so they generate a copy of the polynomial ring. $\square$

The subring U has another special property:

Lemma 2.7. *The subalgebra $\mathsf{U} \subset \mathsf{H}$ is Harish-Chandra in the sense of [7, §1.3], that is, for any $a \in \mathsf{H}$, the bimodule $\mathsf{U}a\mathsf{U}$ is finitely generated as a left module or right module.*

Proof. This is easily seen from the fact that for each fixed element of the affine Weyl group, the diagrams tracing out affine permutations with all possible decorations by dots and stars form a bimodule over U which is finitely generated as a left or right module. Of course, every $a \in \mathsf{H}$ lies in one of these submodules. This completes the proof, since U is Noetherian. $\square$

Note that this presentation allows us to give a "strange" polynomial representation of the Cherednik algebra on the ring $\mathscr{U}$ of polynomials over $\mathbb{K}$ in the alphabets of variables $\mathbf{U} = \{U_1, \ldots, U_n\}$ and $\mathbf{T} = \{T_1, \ldots, T_n\}$ modulo the relations $T_i^\ell = 1$. As before, we define U_i, T_i for all $i \in \mathbb{Z}$, by the formula $U_i = U_{i-n} - \hbar, T_i = \zeta T_{i-n}$. To distinguish between the polynomial representation we wish to define and the action of Γ on polynomials induced by its linear action, we use f^σ to denote the image of f under the latter action of $\sigma \in \Gamma$. The desired representation sends

$$u_i \cdot f(\mathbf{U}; \mathbf{T}) = U_i f(\mathbf{U}; \mathbf{T}) \tag{2.17}$$

$$t_i \cdot f(\mathbf{U}; \mathbf{T}) = T_i f(\mathbf{U}; \mathbf{T}) \tag{2.18}$$

$$r_i \cdot f(\mathbf{U}; \mathbf{T}) = f^{r_i} + k\ell \frac{f^{(i,i+1)} - f}{U_{i+1} - U_i} \pi_{i,i+1} \tag{2.19}$$

$$\sigma \cdot f(\mathbf{U}; \mathbf{T}) = (u_1 - p(\zeta^{-1}t_1) + \hbar)\cdot \tag{2.20}$$

$$f(U_2, U_3, \ldots, U_n, U_{n+1}; T_2, T_3, \ldots, T_n, T_{n+1}) \tag{2.21}$$

$$\tau \cdot f(\mathbf{U}; \mathbf{T}) = f(U_0, U_1, \ldots, U_{n-2}, U_{n-1}; T_0, T_1, \ldots, T_{n-2}, T_{n-1}) \tag{2.22}$$

where, as before, $\pi_{i,i+1}$ is the $\mathbb{k}[\mathbf{U}]$-linear map that sends $T_1^{z_1} \cdots T_n^{z_n} \mapsto \delta_{z_i, z_{i+1}} T_1^{z_1} \cdots T_n^{z_n}$. This is an extension to the whole Cherednik algebra of the action by difference operators introduced by Kodera-Nakajima in [15, Thm. 1.5].

This representation is generated by the constant function 1, subject to the left ideal of relations generated by

$$(i, i+1) \cdot 1 = \tau \cdot 1 = 1 \qquad \sigma \cdot 1 = (u_1 - p(\zeta^{-1}t_1) + \hbar)$$

Note that if we transport structure from this representation to the Cherednik algebra H then the formulae for the action of x_i and y_i will be quite complicated.

Note also that the invariants of Γ acting on the ring $\mathscr{U}$ are simply the S_n-invariant functions in the variables U_i (for the usual action or equivalently, the dAHA action). Thus, the spherical Cherednik algebra $e\mathsf{H}e$ acts naturally on these symmetric polynomials.

In the discussion above, we can think of the parameters as formal variables, in which case, we'll obtain an action on $\mathscr{U}^\Gamma \otimes \Pi$, where

$$\Pi = \Bbbk[s_0, \ldots, s_{\ell-1}, k]^{S_\ell}$$

and s_i are as defined in (2.2).

3. Weighted KLR algebras

This presentation gives a concrete equivalence between a category of representations of the Cherednik algebra, and representations of a weighted KLR algebra, originally proven in [31]. In this section, we set $\hbar = -1$ for simplicity[1], and assume that we have numerical parameters $k, s_i \in \Bbbk$.

Definition 3.1. *Let $\mathsf{H}\text{-mod}_u$ be the category of H-modules on which the polynomial ring U acts locally finitely, with finite dimensional generalized weight spaces. We call modules in this category* **Dunkl-Opdam modules.** *In the terminology of [7], these are the "Harish-Chandra" modules for this subalgebra.*

By Lemma 2.4, any module where the Euler element eu acts with finite dimensional generalized weight spaces lies in this category. In particular, any module in the GGOR category $\mathcal{O}$ is a Dunkl-Opdam module.

Of course, for each pair $\mathbf{a} \in \Bbbk^n$ and $\mathbf{z} \in \mu_\ell(\Bbbk)^n$, we have an exact generalized weight space functor

$$W_{\mathbf{a},\mathbf{z}}(M) = \{m \in M \mid (u_i - a_i)^N m = (t_i - z_i)^N m = 0 \text{ for } N \gg 0\}.$$

Consider the additive quotient group $\Bbbk/\mathbb{Z}$; for an element $a \in \Bbbk$, we let $\bar{a}$ denote its coset in this quotient. We have a natural homomorphism $\gamma \colon \mu_\ell \to \Bbbk/\mathbb{Z}$ sending $\zeta^m \mapsto \frac{m}{\ell} \pmod{\mathbb{Z}}$. Let $\Sigma \colon \Bbbk \times \mu_\ell(\Bbbk) \to \Bbbk/\mathbb{Z}$ be the homomorphism $\Sigma(a, z) = \frac{\bar{a}}{\ell} + \gamma(z)$. Note that this is well-defined since the characteristic of $\Bbbk$ is coprime to ℓ.

Consider the length 0 element

$$\nu \cdot (\mathbf{a}, \mathbf{z}) = ((a_0 = a_n + 1, a_1, \ldots, a_{n-1}), (z_0 = \zeta z_n, \ldots, z_{n-1}))$$
$$\nu^{-1} \cdot (\mathbf{a}, \mathbf{z}) = ((a_2, a_3, \ldots, a_{n+1} = a_1 - 1), (z_2, \ldots, z_{n+1} = \zeta^{-1} z_1)).$$

The relations (2.9i–2.9l) show that:

Lemma 3.2. *The elements σ and τ induce natural transformations:*

$$\sigma \colon W_{(\mathbf{a},\mathbf{z})} \to W_{\nu \cdot (\mathbf{a},\mathbf{z})} \qquad \tau \colon W_{(\mathbf{a},\mathbf{z})} \to W_{\nu^{-1} \cdot (\mathbf{a},\mathbf{z})}.$$

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[1]The reader may doubt the simplicity of this choice, but due to some other notational choices, it really is for the best.

Lemma 3.3. *Let $v \in W_{\mathbf{a},\mathbf{z}}(M)$ be a weight vector that generates M. If for some $\mathbf{a}', \mathbf{z}'$ we have $W_{\mathbf{a}',\mathbf{z}'}(M) \neq 0$ then after some permutation $\rho \in S_n$, we have that $\Sigma(a_i, z_i) = \Sigma(a'_{\rho \cdot i}, z'_{\rho \cdot i})$.*

Proof. This is readily confirmed from the relations (2.9i–2.9l). The (2.8) shows that the action of Γ can only simultaneously permute $\mathbf{a}$ and $\mathbf{z}$, and Lemma 3.2 shows that Σ and τ act by simultaneous cyclic permutation of $\Sigma(a_i, z_i)$. □

Corollary 3.4. *If M is an indecomposable H-module, and we have $(\mathbf{a}, \mathbf{z})$ and $(\mathbf{a}', \mathbf{z}')$ such that $W_{\mathbf{a},\mathbf{z}}(M) \neq 0$ and $W_{\mathbf{a}',\mathbf{z}'}(M) \neq 0$ then the multisets $\{\Sigma(a_i, z_i)\}$ and $\{\Sigma(a'_i, z'_i)\}$ are equal.*

In particular, we can naturally organize the structure of modules over H by fixing which elements of $\Bbbk/\mathbb{Z}$ can appear as $\Sigma(a_i, z_i)$. Fix a subset D of $\Bbbk/\mathbb{Z}$, and let $\tilde{D} = \Sigma^{-1}(D)$.

Definition 3.5. *Let $\mathsf{H}\text{-mod}_D$ be the subcategory of $\mathsf{H}\text{-mod}_u$ killed by the functors $W_{\mathbf{a},\mathbf{z}}$ where $(a_i, z_i) \notin \tilde{D}$ for some i.*

We'll see that the structure of this category depends in a subtle way on the set D; we'll need a fair amount of combinatorics below to capture this structure. The most important aspect of it a quiver structure on D that we'll define below. We give D the structure of a quiver by adding an arrow $m \to m+k$ whenever both lie in D. Thus if $\Bbbk$ is a field of characteristic 0, if $k = a/e \in \mathbb{Q}$, then $\Bbbk/\mathbb{Z}$ is an infinite union of e-cycles, whereas if $k \in \Bbbk \setminus \mathbb{Q}$ then $\Bbbk/\mathbb{Z}$ is a union of infinite linear quivers.

3.1. Characteristic 0. Assume that $\Bbbk$ is a field of characteristic 0, and thus contains a canonically embedded copy of $\mathbb{Q}$. Accordingly, $\Bbbk$ is a $\mathbb{Q}$-vector space, and using the axiom of choice, we can choose a $\mathbb{Q}$-linear map $\Upsilon \colon \Bbbk \to \mathbb{R}$ which sends $1 \mapsto 1$. Note that making this choice, we have a divergence between two important cases: if $k \in \mathbb{Q}$, then we must have $\Upsilon(k) = k$; on the other hand, if $k \notin \mathbb{Q}$, then $\Upsilon(k)$ can be chosen freely. For example, in the latter case, we could without loss of generality assume that $\Upsilon(k) = 0$. Note that while our precise description of the attached weighted KLR algebra will depend on the choice of Υ, this choice is purely auxilliary, and changing it will result in two algebras which are isomorphic by [32, 2.15].

In this case, if $k = a/e \in \mathbb{Q}$, then $\Bbbk/\mathbb{Z}$ is an infinite union of e-cycles, whereas if $k \in \Bbbk \setminus \mathbb{Q}$ then $\Bbbk/\mathbb{Z}$ is a union of infinite linear quivers. Let $\mathsf{s}_i = \overline{s_i/\ell} \in \Bbbk/\mathbb{Z}$.

The category $\mathsf{H}\text{-mod}_D$ from Definition 3.5 has a natural description in terms of weighted KLR algebras.

Definition 3.6. *Let D_∞ be the quiver D with an additional vertex ∞ added, and an arrow $\infty \to \mathsf{s}_i$ added for each i such that $\mathsf{s}_i \in D$. This is what we*

often call a **Crawley-Boevey quiver***, after the observation by Crawley-Boevey that the points in Nakajima's quiver varieties can be seen as representations of the doubling of this quiver, with a 1-dimensional vector space at* ∞.

Choose a real number ϵ such that $0 < \epsilon \ll 1$. Consider the weighting of this quiver where each edge in D is weighted by $\Upsilon(k)$ and the new edge for s_i by $\Upsilon(\frac{p(\zeta^i)}{\ell}) - i\epsilon$. Note that this means that two new edges connected to the same vertex can never have the same weighting, since if $\Upsilon(\frac{p(\zeta^i)}{\ell}) = \Upsilon(\frac{p(\zeta^j)}{\ell})$, then $\Upsilon(\mathsf{s}_i - \mathsf{s}_j) = \frac{i-j}{\ell} \notin \mathbb{Z}$.

Example 3.7. For example, if $\ell = 2, k = 2/3$, and $s_0 = 0, s_1 = 1/3$ then we have that $\mathbb{k}/\mathbb{Z}$ breaks into 3-cycles

$$\bar{a} \to \bar{a} + \frac{2}{3} \to \bar{a} + \frac{4}{3} \to \bar{a} + 2 = \bar{a}.$$

If $D = \{\bar{0}, \overline{1/3}, \overline{2/3}\}$, then the Crawley-Boevey quiver is given by this 3-cycle with edges from $\bar{0}$ and $\overline{1/3}$ to ∞. On the other hand, if D is disjoint from $\{\bar{0}, \overline{1/3}, \overline{2/3}\}$, then the Crawley-Boevey quiver adds no edges.

On the other hand, if $k = \sqrt{2}$ (assuming this root exists in $\mathbb{k}$), then $\mathbb{k}/\mathbb{Z}$ will decompose into infinite chains $\cdots \to \bar{a} - \sqrt{2} \to \bar{a} \to \bar{a} + \sqrt{2} \to \cdots$. Note that k being an irrational algebraic number has no bearing on the structure of the category; the only thing which is significant is its order as an element of the group $\mathbb{k}/\mathbb{Z}$. Since $\Upsilon(k) = 0$, this graph has trivial weighting.

The extra edges in the Crawley-Boevey quiver still attached to $\bar{0}$ and $\overline{1/3}$, but these are now on different components.

3.2. Weighted KLR algebras. Consider the **reduced weighted KLR algebra** R_D attached to the quiver D_∞ with its chosen weighting as defined in [31, §4.1] (see also [32, §3.1]). Choose $\epsilon \in \mathbb{R}$ to be smaller than $|\Upsilon(a_i - a_j)|/n$ for any pair i and j with $\Upsilon(a_i - a_j) \neq 0$.

Definition 3.8. *We let a* **weighted KLR diagram** *be a collection of curves in* $\mathbb{R} \times [0,1]$ *with each curve mapping diffeomorphically to* $[0,1]$ *via the projection to the y-axis. Each curve is allowed to carry any number of dots, and has a label that lies in* D. *We draw:*

- *a dashed line* $\Upsilon(k)$ *units to the right of each strand, which we call a* **ghost***,*
- *red lines at* $x = \Upsilon(\frac{p(\zeta^i)}{\ell}) - i\epsilon$ *labeled with the fundamental weight for* $\mathsf{s}_i \in D$.

We now require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no dots lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and dots on crossings.

The intersection of such a diagram with $y = 0$ or $y = 1$ gives a **loading**, that is, a labeling of a finite subset of $\mathbb{R}$ with vertices of the quiver D. For every pair of n-tuples $\mathbf{a}$ and $\mathbf{z}$ with $\Sigma(a_i, z_i) \in D$ for all i, we can define a loading $e(\mathbf{a}, \mathbf{z})$ as follows: we label the real number $\Upsilon(\frac{a_i}{\ell}) + i\epsilon$ with the element $\Sigma(a_i, z_i) \in D$.

Definition 3.9. *Consider the algebra R_D spanned by weighted KLR diagrams whose top and bottom both give loadings of the form $e(\mathbf{a}, \mathbf{z})$ with $\Sigma(a_i, z_i) \in D$ modulo the local relations*

$$\text{(diagram)} = \text{(diagram)} \quad \text{for } i \neq j \tag{3.1a}$$

$$\text{(diagram)} = \text{(diagram)} + \text{(diagram)} \qquad \text{(diagram)} = \text{(diagram)} + \text{(diagram)} \tag{3.1b}$$

$$\text{(diagram)} = 0 \quad \text{and} \quad \text{(diagram)} = \text{(diagram)} \tag{3.1c}$$

$$\text{(diagram)} = \text{(diagram)} \quad \text{for } i + k \neq j \tag{3.1d}$$

$$\text{(diagram)} = \text{(diagram)} \quad \text{for } i + k \neq j \tag{3.1e}$$

$$\text{(diagram)} = \text{(diagram)} - \text{(diagram)} + h\,\text{(diagram)} \tag{3.1f}$$

$$\text{(3.1g)}$$

$$\text{(3.1h)}$$

$$\text{(3.1i)}$$

$$\text{(3.1j)}$$

$$\text{(3.1k)}$$

$$\text{(3.1l)}$$

$$\text{(3.1m)}$$

For the relations (3.1m), we also include their mirror images.
    This algebra is graded with

$$\deg \underset{i \quad j}{\times} = -2\delta_{i,j} \qquad \deg \underset{i \quad j}{\times} = \delta_{j,i-k} \qquad \deg \underset{i \quad j}{\times} = \delta_{j,i+k}$$

$$\deg \; \Big| \hspace{-0.3em}\bullet_{i} = 2 \qquad \deg \; \overset{\times}{\underset{i \quad j}{}} = \delta_{j,i} \qquad \deg \; \overset{\times}{\underset{i \quad j}{}} = \delta_{j,i}$$

and we'll also consider the completion $\widehat{R}_D$ of this algebra with respect to its grading.

We let $e(\mathbf{a}, \mathbf{z})$ denote the idempotent in R_D or the completion $\widehat{R}_D$ given by a diagram of vertical lines whose x-values are determined by the corresponding loading.

It will often be technically more convenient for us to think of R_D or $\widehat{R}_D$ as a category whose objects are loadings and whose morphisms are elements of R_D matching the source loading at the bottom and target loading at the top; this is the standard trick for considering a ring with set of idempotents summing to the identity as a category, discussed in [16, §3.1].

Remark 3.10. As we've defined it, the algebra R_D is infinite rank as a module over $\mathbb{K}[y_1, \ldots, y_n]$, since we consider the x-values of the strands at the top and bottom of the diagram as fixed. However, if two loadings are related by an isotopy (i.e. the straight line diagram relating them has no crossings), they are equivalent objects in the category R_D. This is **equivalence** of loadings, as discussed in [32, Def. 2.9]. As in [32, Def. 2.13], we usually take "weighted KLR algebra" to mean the algebra Morita equivalent to R_D where we keep only one loading from each equivalence class.

3.3. The isomorphism. We'll now compare this KLR algebra with the category of Dunkl-Opdam modules using the approach of Drozd-Futorny-Ovsienko [7]. They introduce a category $\mathcal{H}$ whose objects are pairs $(\mathbf{a}, \mathbf{z}) \in \mathbb{k}^n \times \mu_\ell(\mathbb{k})^n$, considered as maximal ideals $\mathfrak{m}_{(\mathbf{a}, \mathbf{z})} \subset \mathsf{U}$. The morphisms in this category are given by:

$$\mathrm{Hom}_{\mathcal{H}}((\mathbf{a}, \mathbf{z}), (\mathbf{a}', \mathbf{z}')) = \varinjlim \mathsf{H}/(\mathfrak{m}_{(\mathbf{a}', \mathbf{z}')}^N \mathsf{H} + \mathsf{H} \mathfrak{m}_{(\mathbf{a}, \mathbf{z})}^N)$$

with the obvious composition by multiplication. As an inverse limit, this Hom-space has a natural induced topology.

Theorem 3.11 ([7, Th. 17]). *The category of Dunkl-Opdam modules is equivalent to the representations of the category $\mathcal{H}$ which are continuous in the discrete topology, via a functor sending the module M to the representation $(\mathbf{a}, \mathbf{z}) \mapsto W_{\mathbf{a}, \mathbf{z}}(M)$.*

Note that this category has a "polynomial representation" induced by the representation of H on $\mathcal{U}$; this sends

$$(\mathbf{a}, \mathbf{z}) \mapsto \varinjlim \mathcal{U}/\mathfrak{m}_{(\mathbf{a}, \mathbf{z})}^N \cong \mathbb{K}[[U_1 - a_1, \ldots, U_n - a_n]].$$

This module does not have the discrete topology, and thus does not have a corresponding Dunkl-Opdam module. Since the action of H on $\mathcal{U}$ is faithful, the same is true of the action of $\mathcal{H}$ on the completions.

Note that the extended affine Weyl group $S_n \ltimes \mathbb{Z}^n$ acts on $\tilde{D}^n$ by permutations and translations sending

$$(\mathbf{a}, \mathbf{z}) \mapsto ((a_1 + m_1, \ldots, a_n + m_n), (\zeta^{m_1} z_1, \ldots, \zeta^{m_n} z_n)).$$

Two pairs lie in the same orbit if and only if their images in D agree up to permutation of the entries. For purposes of understanding this action, it's useful to extend $\mathbf{a}$ and $\mathbf{z}$ to arbitrary integers via $a_i = a_{i-n} - 1$, and $z_i = z_{i-n} \zeta^{-1}$.

For two pairs $(\mathbf{a}, \mathbf{z})$ and $(\mathbf{a}', \mathbf{z}') = w \cdot (\mathbf{a}, \mathbf{z})$ with w in the extended affine Weyl group, we let $\xi(\mathbf{a}, \mathbf{z}, w)$ be the straight-line diagram connecting these loadings.

It's worth noting how these diagrams look for various values of $\mathbf{a}, \mathbf{z}$ and w. If $w = r_m$, then this straight line diagram $\xi(\mathbf{a}, \mathbf{z}, r_m)$ moves the strand corresponding to (a_m, z_m) to the right by ϵ and that for (a_{m+1}, z_{m+1}) to the left. This will result in a diagram which is the same up to isotopy, unless:

(1) If $\Upsilon(a_m) = \Upsilon(a_{m+1})$, then the resulting strands will cross.
(2) If $\Upsilon(a_m - k\ell) = \Upsilon(a_{m+1})$ then the the mth strand crosses the ghost of the $m + 1$st strand moving rightward.
(3) If $\Upsilon(a_m + k\ell) = \Upsilon(a_{m+1})$ then the $m + 1$st strand crosses the ghost of the mth strand moving leftward.

The diagram $\xi(\mathbf{a}, \mathbf{z}, \nu)$ moves each strand ϵ steps to the left, except that corresponding to a_n, which moves $1 - (n-1)\epsilon$ steps to the right; not that this ensures that this strand does not cross any strands with the same label, nor the ghost of any with adjacent labels. Similarly, $\xi(\mathbf{a}, \mathbf{z}, \nu^{-1})$ pushes all strands ϵ units to the right, except that for a_1, which moves $1 - (n-1)\epsilon$ units to the left. Unlike diagrams coming from elements of S_n, these can create red and black crossings.

Let

$$\theta_m = (u_m - u_{m+1}) r_m - k\ell \pi_{m,m+1}. \tag{3.2}$$

Lemma 3.12. *There is a fully faithful functor*

$$\Xi \colon \widehat{R}_D \to \mathcal{H}. \tag{3.3}$$

such that Ξ sends the loading $e(\mathbf{a}, \mathbf{z})$ to the object $(\mathbf{a}, \mathbf{z})$. On morphisms, the dot $y_m e(\mathbf{a}, \mathbf{z})$ on the strand corresponding to (a_m, z_m) is sent to

$$\Xi(y_m e(\mathbf{a}, \mathbf{z})) = (u_m - a_m) e(\mathbf{a}, \mathbf{z}), \tag{3.4}$$

and we have that $\Xi(\xi(\mathbf{a}, \mathbf{z}, r_m)) = e(\mathbf{a}, \mathbf{z}) r_m$ if $z_m \neq z_{m+1}$ and if $z_m = z_{m+1}$, then

$$\Xi(\xi(\mathbf{a}, \mathbf{z}, r_m)) = \begin{cases} e(\mathbf{a}, \mathbf{z}) \frac{1}{u_m - u_{m+1} - k\ell} \theta_m & a_m - k\ell \neq a_{m+1} \neq a_m \\ e(\mathbf{a}, \mathbf{z}) \theta_m & a_m - k\ell = a_{m+1} \neq a_m \\ e(\mathbf{a}, \mathbf{z}) \frac{1}{u_{m+1} - u_m + k\ell} (r_m - 1) & a_m - k\ell \neq a_{m+1} = a_m \\ e(\mathbf{a}, \mathbf{z})(1 - r_m) & a_m - k\ell = a_{m+1} = a_m. \end{cases} \tag{3.5}$$

Furthermore,

$$\Xi(\xi(\mathbf{a},\mathbf{z},\nu)) = \begin{cases} \sigma & a_n = p(z_n) \\ \sigma \frac{1}{u_n - p(z_n)} & a_n \neq p(z_n) \end{cases} \qquad \Xi(\xi(\mathbf{a},\mathbf{z},\nu^{-1})) = \tau.$$

$$(3.6)$$

In the formulas above, we have used that if f is a n-variable polynomial such that $f(a_1,\ldots,a_n) \neq 0$, then $f(u_1,\ldots,u_n)e(\mathbf{a},\mathbf{z})$ can be inverted, using the geometric series.

Proof. First, note that the space $e(\mathbf{a}',\mathbf{z}') \cdot R_D \cdot e(\mathbf{a},\mathbf{z})$ has a basis over polynomials in the dots which is in bijection with the elements of the extended affine Weyl group sending $(\mathbf{a},\mathbf{z})$ to $(\mathbf{a}',\mathbf{z}')$. Writing a reduced expression of this element, times a power of the length 0 rotation shows how to write this basis vector (modulo those corresponding to shorter elements of the Weyl group) as a product of straight-line diagrams. More precisely, we see that $\widehat{R}_D$ is generated over the dots by the diagrams $\xi(\mathbf{a},\mathbf{z},r_m)$ and $\xi(\mathbf{a},\mathbf{z},\nu^{\pm})$.

We will thus define Ξ by describing the images of these elements. The algebra R_D has a polynomial representation of the weighted KLR algebra introduced in [32, Prop. 2.7]; in the categorical framework, we can think of this as a functor that sends each loading to the polynomial ring $\mathbb{K}[Y_1,\ldots,Y_n]$. After completion, we obtain an action of $\widehat{R}_D$ that sends each loading to $\mathbb{K}[[Y_1,\ldots,Y_n]]$. We'll compare polynomial representations by using the isomorphism of this ring to $\mathbb{K}[[U_1 - a_1,\ldots,U_n - a_n]]$ which sends $Y_i \mapsto U_i - a_i$.

Note first that this is compatible with (3.4).

In order to calculate the images of $\xi(\mathbf{a},\mathbf{z},r_m)$ and $\xi(\mathbf{a},\mathbf{z},\nu^{\pm})$, note that the action of θ_m and $r_m - 1$ in the polynomial representation can be described as:

$$\theta_m \cdot f = (u_m - u_{m+1} - k\ell\pi_{m,m+1})f^{r_m}$$

$$(r_m - 1) \cdot f = \frac{u_m - u_{m+1} - k\ell\pi_{m,m+1}}{u_m - u_{m+1}}(f^{r_m} - f)$$

The formulas of (3.5) show that:

$$\Xi(\xi(\mathbf{a},\mathbf{z},r_m)) \cdot fe(\mathbf{a}',\mathbf{z}')$$

$$= \begin{cases} f^{r_m}e(\mathbf{a},\mathbf{z}) & a_m - k\ell \neq a_{m+1} \neq a_m \\ (u_m - u_{m+1} - k\ell)f^{r_m}e(\mathbf{a},\mathbf{z}) & a_m - k\ell = a_{m+1} \neq a_m \\ \frac{f^{r_m} - f}{u_{m+1} - u_m}e(\mathbf{a},\mathbf{z}) & a_m - k\ell \neq a_{m+1} = a_m \\ (f - f^{r_m})e(\mathbf{a},\mathbf{z}) & a_m - k\ell = a_{m+1} = a_m. \end{cases}$$

The four cases in (3.5) correspond to:

(1) There are only crossings in the diagram ξ that act trivially on the polynomial representation.
(2) There is a ghost crossing in ξ corresponding to an arrow

$$\Sigma(a_{m+1}, z_{m+1}) \to \Sigma(a_m, z_m)$$

in D, where the strand moves left to right.

(3) There is a crossing of strands with the same label $\Sigma(a_m, z_m) = \Sigma(a_{m+1}, z_{m+1})$, but no ghost crossing.

(4) There is both a strand and a ghost crossing, corresponding to a loop at $\Sigma(a_m, z_m) = \Sigma(a_{m+1}, z_{m+1})$.

Thus, these match the formulae of [32, Prop. 2.7].

In the case of $\xi(\mathbf{a}, \mathbf{z}, \nu^{\pm})$, this same correspondence is easily confirmed. The straight line diagram $\xi(\mathbf{a}, \mathbf{z}, \nu)$:

- only has a ghost crossing with an adjacent label if $\Upsilon(a_n) > \Upsilon(a_m - k\ell) > \Upsilon(a_n) + 1$ for some m, which is impossible if a_n and a_m lie in the same component of D (since then they would differ by a multiple of $k\ell$), and

- only has a red/black crossing if $\Upsilon(a_n) \leq \Upsilon(p(z_n)) < \Upsilon(a_n) + 1$, but this red/black crossing only has an interesting action if $a_n = p(z_n)$. Note that in this case, if $z_n = \zeta^m$, we have that the label on the corresponding strand is $\Sigma(a_n, z_n) = \mathsf{s}_m$, so this gives the node labeling the corresponding red line.

Thus, by the formulae of [32, Prop. 2.7], we have that $\xi(\mathbf{a}, \mathbf{z}, \nu^{\pm})$ acts by the identity unless $p(z_n) = a_n$, in which case it acts by the identity times a dot on the strand corresponding to (a_n, z_n). This matches the action of the elements on the RHS of (3.6) under the action (2.20).

A similar analysis shows that under the representation of [32, Prop. 2.7], the diagram $\xi(\mathbf{a}, \mathbf{z}, \nu^{-1})$ always acts by the identity. This matches with (2.22), completing the proof.

This shows that we have a functor $R_D \to \mathcal{H}$, which we wish to show is fully faithful after completion. First note that Ξ intertwines the grading topology with that on $\mathcal{H}$ on the subalgebras $\mathbb{K}[y_1, \ldots, y_n]$. Since $e(\mathbf{a}', \mathbf{z}') \cdot R_D \cdot e(\mathbf{a}, \mathbf{z})$ is finitely generated as a right module over $\mathbb{K}[y_1, \ldots, y_n]$, the grading topology on this space is the same as that induced by any finite set of generators over $\mathbb{K}[y_1, \ldots, y_n]$; similarly, each Hom space in $\mathcal{H}$ is finitely generated as a right module over the suitable completion of U, and thus has a similar description of its topology. This shows that Ξ induces a continuous functor $R_D \to \mathcal{H}$, which thus extends to the completion.

This functor must be injective on R_D, since the polynomial representation remains faithful after completion by [29, Lem. 2.5]. On the other hand, we can easily show that generating morphisms of the category $\mathcal{H}$ lie in the image by inverting the formulas (3.4–3.6). $\qquad \square$

Let $\widehat{R}_D$-mod$_{\mathrm{fd}}$ be the category of modules over the algebra $\widehat{R}_D$ such that $e(\mathbf{a}, \mathbf{z})M$ is finite dimensional for all $(\mathbf{a}, \mathbf{z})$; if, as in Remark 3.10, we replace $\widehat{R}_D$ by the Morita equivalent algebra where we take one loading from each equivalence class, these are genuinely finite-dimensional modules. Note that these are precisely the finite-dimensional representations of R_D on which

the dots act nilpotently. Combining Theorem 3.11 and Lemma 3.12, we find that:

Theorem 3.13. *The functor* $\mathsf{W}\colon \mathsf{H}\text{-mod}_D \to \widehat{R}_D\text{-mod}_{\mathrm{fd}}$ *sending* $M \mapsto \oplus_{\mathbf{a},\mathbf{z}}W_{\mathbf{a},\mathbf{z}}(M)$ *is an equivalence.*

3.4. Category $\mathcal{O}$. For a fixed choice of parameters $k, s_i \in \Bbbk$, we let

$$D = \{\mathsf{s}_i + mk \mid m \in [-n, n]\} \subset \Bbbk/\mathbb{Z}.$$

This is a union of finite linear quivers if $k \notin \mathbb{Q}$ or n is small; it is a union of e-cycles if $k = a/e$ in reduced form, and $n > e/2$. Taking the limit as $n \to \infty$, we just obtain the set $\{\mathsf{s}_i + mk \mid m \in \mathbb{Z}\}$, which is a union of infinity linear quivers (A_∞) or of e-cycles.

The category $\mathsf{H}\text{-mod}_D$ has a natural subcategory $\mathcal{O}^+$ consisting of finitely generated modules on which x_i acts nilpotently, considered by [11]; we can equally well consider $\mathcal{O}^-$, where y_i acts nilpotently, which is the Ringel dual of $\mathcal{O}^+$ by [11, 4.11]. In [31, Th. A], this category is related to a quotient of the weighted KLR algebra: the steadied quotient. We'll only be interested in a special case of this notion (which in general depends on a choice of stability condition).

Definition 3.14. *We'll say that a loading is* **unsteady** *(for the positive stability condition) if there exists a real number* $\delta \geq \Upsilon(\frac{p(\zeta^i)}{\ell})$ *such that a non-empty set of points in the loading have x-value* $> \delta + |\Upsilon(k)|$, *and all others have x-value* $\leq \delta$.

There is also a negative stability condition where all signs above are reversed: we have $\delta \leq \Upsilon(\frac{p(\zeta^i)}{\ell})$, *a non-empty set of points have x-value* $< \delta - |\Upsilon(k)|$, *and all others have x-value* $\geq \delta$.

The quotient of R_D by the two-sided ideal generated by the idempotents $e(\mathbf{a},\mathbf{z})$ which correspond to unsteady loadings (for one stability condition) is called the **steadied quotient**; we denote these by $R_D(\pm)$ for the positive/negative stability condition.

Note that these algebras have a number of desirable properties: they are cellular and highest weight (since new edges connected to the same vertex in D always have different weightings) by [31, Th. B].

Theorem 3.15. *The functor* W *induces an equivalence* $\mathcal{O}^\pm \cong R_D(\pm)\text{-mod}$.

Proof. Since the proof is the same in both cases, we consider the case of $\mathcal{O}^-$. The pair $(\mathbf{a},\mathbf{z})$ corresponds to an unsteady loading if and only if there exists $I \subset [1, n]$ and a real number $\delta \leq \Upsilon(-s_i)$ for all i such that $\Upsilon(a_i) < \delta - |\Upsilon(k)|$ if $i \in I$ and $\Upsilon(a_i) \geq \delta$ if $i \notin I$. Note that permuting an element of I past one in $[1, n] \setminus I$ gives an isomorphism between the corresponding weight functors, so without loss of generality, we can assume that $I = [1, q]$. Similarly, we have an isomorphism of $W_{\mathbf{a},\mathbf{z}} \cong W_{\mathbf{a}_g,\mathbf{z}}$ where $\mathbf{a}_g = (a_1 - g\ell, \ldots, a_q - g\ell, a_{q+1}, \ldots, a_n)$, since the corresponding loadings are

connected by a crossingless diagram. If $N \in \mathcal{O}$, then N must be killed by $W_{\mathbf{a}_g, \mathbf{z}}$ for $g \gg 0$, since the Euler eigenvalues of N are bounded below. Thus, $\mathsf{W}(N)$ is killed by $e(\mathbf{a}, \mathbf{z})$ for any unsteady loading, and thus the action on it factors through the steadied quotient.

On the other hand, any pair $(\mathbf{a}, \mathbf{z})$ with $\sum \Upsilon(a_j)$ sufficiently negative must be unsteady, since if a strand is more than $n|\Upsilon(k)|$ left of a red line, it must be destabilizing. Thus, if the action on M factors through the steadied quotient, then $\mathsf{h}(M)$ has Euler eigenvalues which are bounded below. Since $\mathsf{h}(M)$ is finitely generated, and the action of eu is locally finite, this shows that $\mathsf{h}(M)$ lies in category $\mathcal{O}$. $\qquad\square$

Perhaps a few remarks are called for about the match of this result with [31, Thm. 4.7]. Theorem 3.15 is more general, since it does not assume that $\Bbbk = \mathbb{C}$. To recover [31, Thm. 4.7], we consider the case where $\Upsilon \colon \mathbb{C} \to \mathbb{R}$ is given by taking real part.

This theorem allows us to recover in an interesting way the classification of modules in category $\mathcal{O}$. The best known version of this classification is due to Ginzburg, Guay, Opdam and Rouquier:

Theorem 3.16 ([11, Prop. 2.11]). *For every simple module S in category $\mathcal{O}^-$, the subspace U of elements with minimal weight under eu is an irreducible module over $G(\ell, 1, n)$ and this describes a bijection between simples in $\mathcal{O}$ and over $G(\ell, 1, n)$.*

Of course, simple modules over $G(\ell, 1, n)$ are indexed by ℓ-multipartitions with n total boxes, and the corresponding module over $G(\ell, 1, n)$ has a basis indexed by standard tableaux on the corresponding Young diagram. Since there are several notions of standard tableau on a multi-partition, let us clarify that we just mean a filling with $[1, n]$ which increases in rows and columns.

This construction is carried out in the style of Vershik and Okounkov [22] in work of Pushkarev [23] and Ogievetsky and Poulain d'Andecy [21]. These papers show that, in particular, the subalgebra generated by t_i (denoted j_i in [21]) and the Jucys-Murphy elements (denoted $\tilde{j}_i$ in *loc. cit.*) has simple spectrum, with elements in the spectrum in canonical bijection with standard tableaux as discussed above.

In [21, Prop. 11], they define a representation V_ξ of $\Bbbk \Gamma$ with a basis v_{S} for tableaux S of shape ξ, if the entry c is in the ith row and jth column of the mth component, then t_c acts by the scalar ζ^m, and $(c, c+1)$ acts by switching c and $c+1$ if these are in different components, and by the Young normal form if they are in the same component. These are a complete list of the irreps.

The most important tool in this construction is the algebra they denote $\mathfrak{A}_{\ell, n}$ in [21, §3]. This is simply our algebra $\mathcal{DO}_n$ under an isomorphism

$$\tilde{x}_m \mapsto \frac{1}{k\ell} u_m \qquad x_m \mapsto t_m \qquad \bar{s}_i \mapsto (i, i+1).$$

In *loc. cit.*, the algebra $\Bbbk\Gamma$ is written as a quotient of $\mathcal{DO}_n$ by setting $u_1 = 0$, but this is not the correct map to use for the elements of minimal eu-weight in a module.

Of course, $\mathcal{DO}_n$ acts on the subspace U, and does so via a quotient map to $\Bbbk\Gamma$, but not this most obvious one. Since τ acts by 0 on U, the product $\sigma\tau = u_1 - p(\zeta^{-1}t_1) - 1$ does as well. Thus, we have unique surjective homomorphism $\eta\colon \mathcal{DO}_n \to \Bbbk\Gamma$ splitting the usual inclusion and killing the 2-sided ideal generated by $u_1 - p(\zeta^{-1}t_1) - 1$. In particular, if we have a weight $(\mathbf{a}, \mathbf{z})$ that appears in V_ξ with $z_1 = \zeta^m$ for $m \in [0, \ell - 1]$, then $a_1 = p(\zeta^{m-1}) + 1$.

Making small changes in arguments of [21, §4], we can see that the weights of V_ξ correspond to the tableaux S of shape ξ as follows:

Lemma 3.17. *If the entry c is in the ith row and jth column of the mth component, then u_c and t_c act in the vector v_{S} by the scalars $a_c = p(\zeta^{m-1}) + 1 + k\ell(j - i)$ and $z_c = \zeta^m$.*

All of these weights will give isomorphic idempotents $e(\mathbf{a}, \mathbf{z})$ in R_D, which match the loading $\mathbf{i}_\xi$ introduced in [31, Def. 2.11]; of course, we can see directly from the cellular structure of [31, Th. B] that these must be the lowest weights, showing the compatibility with the GGOR perspective.

Finally, we turn to considering the KZ functor of $\mathcal{O}^\pm$. This functor has a categorical interpretation: it is represented by the sum of all self-dual projectives, with multiplicities given by the dimensions of simple modules over Hecke algebras at roots of unity. The functors $W_{\mathbf{a}, \mathbf{z}}$ are also represented by projectives and thus it is natural to try to express the KZ functor in terms of them.

Choose a fixed lift $\varphi\colon D \to \Bbbk$, where $\Sigma(\varphi(d), 1) = d$. Choose an integer

$$N \gg \max_{\substack{i \in [1, \ell] \\ d \in D}} (|\Upsilon(p(\zeta^i))|, |\Upsilon(k)|, |\Upsilon(\varphi(d))|).$$

For each n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in D^n$, let

$$\mathbf{a}_{\mathbf{d}}^\pm = (\varphi(d_1) \mp N, \varphi(d_2) \mp 2N, \ldots, \varphi(d_n) \mp nN) \qquad \mathbf{1} = (1, \ldots, 1).$$

Theorem 3.18. *The functor* KZ *on* $\mathcal{O}^\pm$ *is isomorphic to the sum* $\displaystyle\bigoplus_{\mathbf{d} \in D^n} W_{\mathbf{a}_{\mathbf{d}}^\pm, \mathbf{1}}$.

Proof. As before, the argument is identical for the two different signs, and so we consider $\mathcal{O}^-$. We need only show that there is an isomorphism between the representing projectives. For $\mathbf{d} \in D^N$, we can define a loading which places a dot with label d_m at $x = mN$. Let $e_{s,n} \in R_D(-)$ be the sum of the idempotents for these loadings. From the isomorphisms of Theorems 3.13 and 3.15, we know that $\oplus_{\mathbf{d} \in D^n} W_{\mathbf{a}_{\mathbf{d}}^-, \mathbf{1}}$ corresponds to the projective over $R_D(-)$ given by $R_D(-)e_{s,N}$. The isomorphism [31, Thm. 4.5] sends this to the idempotent $e_{D_{s,N}}$ in the notation of [31, Sec. 2.5], which [31, Thm. 3.9] shows corresponds to the KZ functor. $\qquad\square$

The endomorphisms of the functor $\oplus_{\mathbf{d} \in D^n} W_{\mathbf{a}_{\mathbf{d}}^{\pm}, \mathbf{1}}$ are isomorphic to the cyclotomic KLR algebra with n strands corresponding to the highest weight $\sum_{i=1}^{\ell} \omega_{s_i}$. Previous work of Brundan and Kleshchev [6] has constructed an isomorphism of these to the cyclotomic Hecke algebras which naturally act by monodromy on KZ.

3.5. The classification of Dunkl-Opdam modules.

The equivalence of Theorem 3.13 allows us to classify all simple Dunkl-Opdam modules over H, not just those in category $\mathcal{O}^{\pm}$.

For a general Dunkl-Opdam module, of course, there is no maximal or minimal weight under eu. Instead, we must look for some other patterns within the weights.

A **charged segment** is a g-tuple (for some $g \leq n$) of elements $\mathbf{q} = (q_1, \ldots, q_g)$ of $\Bbbk/\mathbb{Z}$, which satisfy $q_{i+1} - q_i = k$. We'll use **lifted segment** to mean a similar g-tuple $\mathbf{a}$ in $\Bbbk$ satisfying $a_{i+1} - a_i = k\ell$ Choose a large negative integer $P \ll 0$, and let $\Lambda(q_1, \ldots, q_g)$ for a charged segment be the unique lifted segment $(a_1, \ldots, a_g)$ of elements of $\Bbbk$ such that $\Sigma(a_i, z_i) = q_i$, $a_{i+1} - a_i = k\ell$, $z_{i+1} = z_i$, and $\Upsilon(a_1)$ is minimized subject to $P \leq \Upsilon(a_i)$; this means that $P \leq \Upsilon(a_1) < P + 1$ if $\Upsilon(k) \geq 0$ and $P \leq \Upsilon(a_g) < P + 1$ if $\Upsilon(k) \leq 0$. A charged multisegment is an m-tuple of charged segments. The size of a multisegment is the sum of the lengths of the segments.

As usual, we can associate to any lifted segment $\mathbf{a}$ and $z \in \mu_\ell(\Bbbk)$, a 1-dimensional representation of the algebra $\mathcal{DO}_g$ by letting S_g act trivially, the elements t_i act by the scalar z and u_i act by the scalar a_i; to a charged segment $\mathbf{q}$, we associate the 1-dimensional representation for the distinguished lift $\Lambda(q_1, \ldots, q_g)$.

Note that by the usual theory of modules over degenerate affine Hecke algebras, based on work of Zelevinsky [35] and refined further by Suzuki [27], we can associate a simple $\mathcal{DO}_g$ module $L(\mathbf{Q})$ to any multisegment $\mathbf{Q}$ of size g by inducing up the tensor product of the 1-dimensional modules attached to segments ordered, and taking the unique simple quotient. Note that we have to be careful about the order of lifted segments; if two lifted segments with the same z of the form $(a, a + k\ell, \ldots)$ and $(a - hk\ell, a - (h-1)k\ell, \ldots)$ with $h \in \mathbb{Z}_{>0}$ appear, they must be in this order in the induction.

Let

$$\mathcal{DO}_{g, n-g} = \mathcal{DO}_g \otimes \mathcal{DO}_{n-g} \subset \mathcal{DO}_n$$

be the subalgebra generated by t_i, u_i for all $i \in [1, n]$ and the Young subgroup $S_g \times S_{n-g}$. Given a multisegment $\mathbf{Q}$ of size g and an ℓ-multipartition ξ of size $n - g$, we have a $\mathcal{DO}_{g, n-g}$ module $L(\mathbf{Q}) \otimes V_\xi$ by taking outer tensor of these modules, where V_ξ has the $\mathcal{DO}_{n-g}$ module structure via the homomorphism η discussed in the previous section.

We can construct a module over H_n by considering

$$\mathcal{M}(\mathbf{Q}, \xi) = \mathsf{H}_n \otimes_{\mathcal{DO}_{g, n-g}} (L(\mathbf{Q}) \otimes V_\xi).$$

Note that that this definition depends on the choice of P. We assume from now on that $P < \Upsilon(p(\zeta^m)) - 2n|\Upsilon(k\ell)|$.

Lemma 3.19. *Every simple Dunkl-Opdam module S is a quotient of $\mathcal{M}(\mathbf{Q}, \xi)$ for some $\mathbf{Q}, \xi$.*

Proof. For simplicity, we'll assume throughout the proof that $\Upsilon(k) \geq 0$.

By assumption, we have that $W_{\mathbf{a},\mathbf{z}}(S) \neq 0$ for some $(\mathbf{a}, \mathbf{z})$. We claim that we can choose $(\mathbf{a}, \mathbf{z})$ so that $\Upsilon(a_i) > P$ for all i. We'll prove this by induction on the sum Π of the quantity $P - \Upsilon(a_i) + 1$ over the indices i such that $\Upsilon(a_i) \leq P$. Obviously, this is 0 if and only if $\Upsilon(a_i) > P$ for all i.

Consider the equivalence relation on the indices $[1, n]$ obtained by transitive closure of the relation that $i \sim j$ if we have that $a_i = a_j \pm k\ell$ and $z_i = z_j$. Note that we have $|\Upsilon(a_i) - \Upsilon(a_j)| \leq n|\Upsilon(k\ell)|$ for any $i \sim j$. We will use several times the fact that

(∗) if two consecutive indices satisfy $i \not\sim i + 1$, then $\theta_m \colon W_{(\mathbf{a},\mathbf{z})}(S) \to W_{s_i \cdot (\mathbf{a},\mathbf{z})}(S)$ is an isomorphism, so we can reorder these without changing whether the weight space is non-zero.

Let i be the index that minimizes $\Upsilon(a_i)$. If for any i, we have that $\Upsilon(a_i) \leq P$, then we have $\Upsilon(a_j) < \Upsilon(p(\zeta^m))$ for all $j \sim i$ and all m. If we let j be the largest index such that $j \sim i$, then by (∗) we can assume that $j = n$ without loss of generality. In this case, have that $\Upsilon(a_n) < \Upsilon(p(\zeta^m))$ for all m, so σ induces an isomorphism $W_{\mathbf{a},\mathbf{z}}(S) \cong W_{\nu \cdot (\mathbf{a},\mathbf{z})}(S)$. The weight $\nu \cdot (\mathbf{a}, \mathbf{z})$ has strictly fewer indices in the equivalence class of i, so we can reduce to the case where $i = n$.

In this case, $(\mathbf{a}', \mathbf{z}') = \nu \cdot (\mathbf{a}, \mathbf{z})$ has almost all indices the same, but $a_1' = a_n + 1$, so either Π has dropped by exactly 1, or we have strictly fewer indices $\Upsilon(a_i) \leq P$, in which case Π drops by at least 1.

Thus, after performing this operation finitely many times, we must have Π drop to 0. Thus, we can assume that $\Upsilon(a_i) > P$ for all i.

Now assume that $(\mathbf{a}, \mathbf{z})$ minimizes $\sum \Upsilon(a_i)$ amongst weights satisfying this condition; that is, we minimize the eigenvalue of eu on this weight space. Consider the intertwiner $\tau \colon W_{\mathbf{a},\mathbf{z}}(S) \to W_{\nu^{-1} \cdot (\mathbf{a},\mathbf{z})}(S)$. Since the latter weight space has lower Euler eigenvalue, either we must have $\Upsilon(a_1) - 1 \leq P$, or this map is 0; the latter can only happen if $a_1 = p(\zeta^{m-1}) + 1, z_1 = \zeta^m$ for some m, since $\sigma\tau = u_1 - p(\zeta^{-1}t_1) - 1$ must act by 0. That is, we must have exactly one of the options:

(1) $a_1 = p(\zeta^{m-1}) + 1, z_1 = \zeta^m$
(2) $P < \Upsilon(a_1) - 1 \leq P + 1$

Using (∗) again, we see the same is true of any index i such that i is not equivalent to any lower index.

Thus, as before, we can decompose the indices $[1, n]$ according the equivalence relation $\sim$, and the lowest index in every equivalence class satisfies exactly one of (1) or (2). This in turn breaks the indices into two classes which we call types (1') and (2'): either they are greater than or less than

$P + n|\Upsilon(k\ell)|$. All elements of an equivalence class containing an element satisfying (1) will necessarily be of type (1'), and those containing an element satisfying (2) will necessarily be of type (2'). The fact $(*)$ shows that we can assume that $[1, g]$ consists of indices of type (1') and $[g+1, n]$ of type (2').

Now, we consider the module over $\mathcal{DO}_n$ generated by $W_{\mathbf{a},\mathbf{z}}$, and consider any simple K $\mathcal{DO}_n$-submodule of this space; WLOG, we can assume this has non-trivial intersection with $W_{\mathbf{a},\mathbf{z}}$. Let K' be the subspace in K given by the sum of all weight spaces such that $[1, g]$ consists of indices of type (1') and $[g+1, n]$ of type (2'); by assumption, this is a non-trivial module over $\mathcal{DO}_{g,n-g}$. We have an obvious map $\mathcal{DO}_n \otimes_{\mathcal{DO}_{g,n-g}} K' \to K$, and applying $(*)$ shows that this is an isomorphism. In particular K' must be a simple $\mathcal{DO}_{g,n-g}$-module, and thus $K' \cong L \otimes V$ for L a simple $\mathcal{DO}_g$-module and V a $\mathcal{DO}_{n-g}$-module.

First, we claim that $L = L(\mathbf{Q})$ for some $\mathbf{Q}$. The module L corresponds to some lifted multisegment; let a be first entry in one of these lifted segments which maximizes $\Upsilon(a)$. By assumption $\Upsilon(a) > P$. We can assume that $a + hk\ell$ does not appear as the first entry in one of these lifted segments for all $h \in \mathbb{Z}_{>0}$. Thus, the subspace K' contains a weight with $a_1 = a$; applying τ maps to a weight space with lower Euler eigenvalue, and is an isomorphism since the index a is of type (1'). This is only possible if $\Upsilon(a) \leq P + 1$, so the same is true of the initial element of each segment. This shows that L has the form $L(\mathbf{Q})$.

Now, assume $g < n$. Using $(*)$ again, we can also write $K = \mathcal{DO}_n \otimes_{\mathcal{DO}_{n-g,g}} (V \otimes L)$; the fact that τ acts trivially on any vector in $V \otimes L$ in this embedding shows that V is killed by $u_1 - p(\zeta^{-1}t_1) - 1$, and thus must be of the form V_ξ with ξ having $n - g$ boxes.

Thus, the inclusion of $\mathcal{DO}_{g,n-g}$-modules $L(\mathbf{Q}) \otimes V_\xi \to S$ induces the desired surjection. $\qquad\square$

Let c_ξ be the eigenvalue of $\mathsf{eu} \in \mathcal{DO}_{n-g}$ acting on V_ξ.

Definition 3.20. *Let* $\Delta(\mathbf{Q}, \xi)$ *be the quotient of* $\mathcal{M}(\mathbf{Q}, \xi)$ *by the image of any map from* $\mathcal{M}(\mathbf{Q}', \xi')$ *with* $\mathbf{Q}'$ *of greater size than* $\mathbf{Q}$ *or* $c_{\xi'} < c_\xi$.

Remark 3.21. If $\mathbf{Q} = \emptyset$, then we can easily check that these are the Verma modules in category $\mathcal{O}$. We should take pains here to emphasize that in general, these are *not* the standard modules of a quasi-hereditary structure on Dunkl-Opdam modules; consideration of the special case $n = 1$ shows there is no such structure. However, these are the proper standards of a standardly stratified structure one can easily derive from the approach of [30, §5.4].

In particular, if we just subtract ℓ from P and all elements of $\mathbf{Q}$, then the module $\Delta(\mathbf{Q}, \xi)$ will be unchanged.

Theorem 3.22. *For fixed $P \ll 0$, every simple Dunkl-Opdam module S is the unique simple quotient of $\Delta(\mathbf{Q}, \xi)$ for a unique $\mathbf{Q}$ and ξ.*

Proof. Consider a simple Dunkl-Opdam module S. By Lemma 3.19, we have that S is a quotient of some $M(\mathbf{Q}, \xi)$, and we can choose $(\mathbf{Q}, \xi)$ with ξ having a minimal number of boxes, and c_ξ minimal amongst the possible ξ with the minimal number of boxes.

In this case, S is a quotient of $M(\mathbf{Q}, \xi)$ but not of any of the $M(\mathbf{Q}', \xi')$ whose images we kill to get $\Delta(\mathbf{Q}, \xi)$. Thus, the map to $M(\mathbf{Q}, \xi) \to S$ must factor through $\Delta = \Delta(\mathbf{Q}, \xi)$.

Now we must show that Δ is unique, and has a unique simple quotient. Let $(\mathbf{a}, \mathbf{z})$ be a weight space in $J = L(\mathbf{Q}) \otimes V_\xi$. Then for any weight $(\mathbf{a}', \mathbf{z}')$ satisfying $\Upsilon(a_i') > P$, if we let $\{w_1, \ldots, w_k\}$ be the finite set of elements of $\widehat{W}$ such that $w_p \cdot (\mathbf{a}, \mathbf{z}) = (\mathbf{a}', \mathbf{z}')$, then $W_{\mathbf{a}', \mathbf{z}'}(\Delta)$ is spanned by $d_k v$ for d_k a sequence of intertwining operators tracing out w_k (or equivalently, Ξ applied to the weighted KLR diagram $\xi(\mathbf{a}, \mathbf{z}, w_k)$) and $v \in W_{\mathbf{a}', \mathbf{z}'}(\Delta) \cap J$. Note that all intermediate steps of these interwining operators pass through $(\mathbf{a}'', \mathbf{z}'')$ with

$$P < \min_i(\Upsilon(a_i'), \Upsilon(a_i)) \leq \Upsilon(a_k'') \leq \max_i(\Upsilon(a_i'), \Upsilon(a_i)).$$

Now, assume that $\sum a_i = \sum a_i'$, that is, that these have the same Euler eigenvalue. If w_k is not in S_n, then we can arrange this sequence of intertwiners so that a τ appears before a σ using the relations (2.9b,2.9c,2.9m). Thus, this sequence factors through a weight space with lower Euler eigenvalue that still satisfies $\Upsilon(a_i'') > P$ for all i. By assumption, this weight space is zero.

That is, we must have

$$W_{\mathbf{a}', \mathbf{z}'}(\Delta) \subset \mathcal{DO}_n \cdot J = J.$$

This shows that J is uniquely characterized as the sum of the weight spaces in Δ which minimize eu among those with $\Upsilon(a_i) > P$. Since this space is a simple $\mathcal{DO}_n$-module, any submodule N of Δ with $N \cap J \neq 0$ must have $J \subset N$ and so $N = \Delta$. That is, N is proper if and only if $N \cap J = 0$; as usual, this implies that the sum of all proper submodules is proper and Δ has a unique simple quotient.

On the other hand, this also show that $(\mathbf{Q}, \xi)$ can be reconstructed from this simple quotient by considering the $\mathcal{DO}_n$ action on the sum of the weight spaces in Δ which minimize eu among those with $\Upsilon(a_i) > P$. $\square$

It's worth noting the similarity of this result to that for the trigonometric Cherednik algebra (also known as degenerate double affine Hecke algebra) by Suzuki [28, Cor. 8.3]; if we replace the equations (2.9f,2.9g) by $\sigma\tau = \tau\sigma = 1$, then one can check that we get a slight variation on the usual presentation of the trigonometric Cherednik algebra, and our result reduces to Suzuki's.

3.6. Positive characteristic. Lemma 3.12 fails as stated if $\Bbbk$ is a field of characteristic p; its very statement uses the existence of $\mathbb{Q}$-linear maps $\Bbbk \to \mathbb{R}$. However, the functor W and the general strategy of computing its endomorphisms remain valid. The result is quite interesting because of its relationship to the coherent sheaves on the degree n Hilbert scheme of $\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z})$. More precisely, consider the case where $\Bbbk = \mathbb{F}_p$ for $p \nmid \ell$, and D is the (finite) set of all pairs possible in this field; let $\mathsf{Coh}_{\mathrm{pun}}(\mathsf{Hilb}^n(\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z})))$ be the category of coherent sheaves on the Hilbert scheme supported on a formal neighborhood of the punctual Hilbert scheme. In the case of $\ell = 1$, this is a well-established result of Bezrukavnikov, Finkelberg and Ginzburg:

Proposition 3.23 ([1, Thms. 1.3.2 & 1.4.1]). *For $p \gg 0$ and k generic, we have that*
$$D^b(\mathsf{H}\text{-mod}_D) \cong D^b(\mathsf{Coh}_{\mathrm{pun}}(\mathsf{Hilb}^n(\mathbb{C}^2))).$$

This result is extended to $\ell > 1$ in [2].

We'll discuss the computation of $\mathrm{End}(\mathsf{W})$ in a more general context in future work [34], where we can give more detailed context; the combinatorial description of this endomorphism algebra is a cylindrical version of the KLR algebra which has not yet been introduced in the literature. This modified KLR algebra is actually a more useful object for algebraic geometers than the Cherednik algebra, since even in characteristic 0, it appears as the endomorphisms of a tilting bundle on the Hilbert scheme, and thus can describe all coherent sheaves, not just those set-theoretically supported on the punctual Hilbert scheme.

This also fits into a more general context about Coulomb branches (as discussed in Section 4) in characteristic p, which we do not have the space to develop here.

4. Coulomb branches

The isomorphism of Theorem 2.3 makes it easy to see the relationship between the cyclotomic Cherednik algebra and quantum Coulomb branches. Consider the GL_n representation $V = \mathfrak{gl}_n \oplus (\mathbb{C}^n)^{\oplus \ell}$, and consider the BFN space
$$\mathbb{X} = \left\{ (g(t), v(t)) \in GL_n((t)) \times_{GL_n[[t]]} V[[t]] \mid g(t) \cdot v(t) \in V[[t]] \right\}$$
as discussed in [20, 5]. For an action of GL_N on any space, we will use the term **equivariant parameters** to mean the equivariant Chern classes of the trivial bundle with fiber $\mathbb{C}^N$. The BFN space has:

(1) an action of $\mathbb{C}^*$ by loop rotation with equivariant parameter $\ell\hbar$;
(2) an obvious action of $GL_n[[t]]$; we will identify the Chern classes of the tautological bundle for this action with the elementary symmetric polynomials $e_i(\mathbf{U})$, and thus the Chern roots with U_i;
(3) an action of GL_ℓ on the multiplicity space of $\mathbb{C}^n$; we will identify the Chern roots of the tautological bundle with $-s_i + \ell\hbar$;

(4) an action of $\mathbb{C}^*$ by scalar multiplication on $\mathfrak{gl}_n$ with equivariant parameter k.

All of these actions commute. We let G be the product of the first two, and H the product of the last two. Consider the $G \times H$-equivariant Borel-Moore homology $\mathfrak{A} = H_*^{G \times H}(\mathbb{X})$; this algebra is the quantum Coulomb branch of the gauge theory attached to V.

This algebra acts naturally on the $G \times H$-equivariant homology of $V[[t]]$, which is the same as that of a point, that is, a polynomial ring over $\Bbbk$ in the equivariant parameters $\hbar, e_i(\mathbf{U}), e_i(\mathbf{s}), k$.

Theorem 4.1. *There is an isomorphism of $e\mathsf{H}e$ with the quantum Coulomb branch $\mathfrak{A}$. This isomorphism is induced by the isomorphism $\mathscr{U}^\Gamma \otimes \Pi \cong H_{G \times H}^*(*)$ discussed above.*

In [3], the commutative Coulomb branch of the corresponding gauge theory is described as the cone $\mathrm{Sym}^n(\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z}))$; by the uniqueness of quantizations shown by Losev [17, 19], we must have that $\mathfrak{A}$ is isomorphic to $e\mathsf{H}e$, which is a well-known quantization of this variety. However, having a concrete understanding of this isomorphism is of course, more useful, and more revealing about the structure of both algebras. Since a proof of this result was recently given by Kodera-Nakajima [15], we will only sketch the isomorphism below. However, we believe it is of some independent interest, since this isomorphism is quite straightforward given the isomorphism of Theorem 2.3.

Let us prove a slightly stronger (but none the less easier) version of this theorem. The BFN space can be replaced by its Iwahori analogue. Let $I = \{g(t) \in GL_n[[t]] \mid g(0) \in B\}$ be the standard Iwahori corresponding to the standard Borel B of upper triangular invertible matrices. This analogue is defined by:

$$\mathfrak{I} = \{v(0) \in \mathfrak{b} \oplus (\mathbb{C}^n)^{\oplus \ell} \mid v(t) \in V[[t]]\}$$
$$\mathbb{X}' = \{(g(t), v(t)) \in GL_n((t)) \times_I \mathfrak{I} \mid g(t) \cdot v(t) \in \mathfrak{I}\},$$

and the quantum Coulomb branch can be replaced by its Iwahori version $\mathfrak{A}' = H_*^{I \times \mathbb{C}^*}(\mathbb{X}')$; see [4, §4] for a more detailed discussion of this variety. Similarly, we can replace $e\mathsf{H}e$ by $e'\mathsf{H}e'$ where

$$e' = \frac{1}{\ell^n} \sum_{\mathbf{i} \in (\mathbb{Z}/\ell\mathbb{Z})^n} t_1^{i_1} \cdots t_n^{i_n}$$

is just the idempotent symmetrizing for the action of $A = (\mathbb{Z}/\ell\mathbb{Z})^n$. More generally, for any character η of the group A, we have an idempotent

$$e_\eta = \frac{1}{\ell^n} \sum_{\mathbf{i} \in (\mathbb{Z}/\ell\mathbb{Z})^n} \eta(t_1^{-i_1} \cdots t_n^{-i_n}) t_1^{i_1} \cdots t_n^{i_n} \tag{4.1}$$

the idempotent of the group algebra $\mathbb{C}[A]$ projecting to this isotypic component. We let $E_\eta = e_\eta \cdot 1 \in \mathscr{U}$ and $E' = E_1$; this is effectively the same sum as

(4.1), but with the substitution $t_i \mapsto T_i$. Since $\mathscr{U} \cong \mathbb{C}[U_1, \ldots, U_n] \otimes_{\mathbb{C}} \mathbb{C}[A]$, we have that

$$\mathscr{U} = \bigoplus_{\eta} \mathbb{C}[U_1, \ldots, U_n] E_{\eta}.$$

Thus, both algebras $\mathfrak{A}'$ and $e'\mathsf{H}e'$ act naturally on $\mathscr{U}^A \cong H^*_{I \times \mathbb{C}^*}(*)$, identifying the variables U_i with the Euler classes of the tautological line bundles on the classifying space of I.

Lemma 4.2. *There is an isomorphism of $e'\mathsf{H}e'$ with the flag quantum Coulomb branch $\mathfrak{A}'$. This isomorphism is induced by the obvious isomorphism $e'\mathscr{U} = \mathscr{U}^A \cong H^*_{I \times \mathbb{C}^*}(*)$.*

This extension is also proven by Braverman-Etingof-Finkelberg [4, §4.2] with a similar proof.

Proof. In both cases, we have a copy of polynomial multiplication, given by the $e'u_i$ in $e'\mathsf{H}e'$ and the Chern classes of tautological bundles in $\mathfrak{A}'$. We also have copies of S_n which act as in dAHA. In $\mathfrak{A}'$, this is given by the pullback of the action of S_n on the Springer sheaf. Finally, the shift element $e'y_n^{\ell-1}\tau e'$ agrees with the shift correspondence

$$X_{\tau} = \{(V_{\bullet}, V'_{\bullet} \mid V_i = V'_{i+1}\}$$

and $e'x_1^{\ell-1}\sigma e'$ agrees with the correspondence

$$X_{\sigma} = \{(V_{\bullet}, V'_{\bullet} \mid V_i = V'_{i-1}\}$$

To see that these act the same way, we need only check their commutation with u_i, as in (2.9i–2.9j), and that they act correctly on the unit. The commutation is clear, since the shift correspondence simply reindexes the tautological line bundles.

The element $e'y_n^{\ell-1}\tau e'$ and $[X_{\tau}]$ both send E' to E'. We claim that the element $e'x_1^{\ell-1}\sigma e'$ sends E' to

$$(U_1 + \hbar - p(\zeta^{-1})) \cdots (U_1 + (\ell - 1)\hbar - p(\zeta))(U_1 + \ell\hbar - p(1))E'$$
$$= (U_1 - s_{\ell-1} + \ell\hbar) \cdots (U_1 - s_1 + \ell\hbar)E'. \quad (4.2)$$

In order to do this computation, we have to leave $\mathscr{U}^A$, and consider elements of $\mathscr{U}$ transforming over another character $\eta \colon A \to \mathbb{C}^*$. Consider the character $\eta_i(t_j) = \zeta^{\delta_{ij}}$; note that $e_{\eta\eta_i}x_i = x_i e_{\eta}$.

Recall that $x_1 = \sigma v_1^{-1}$. Note that

$$v_1^{-1}t_1 = t_n v_1^{-1} \qquad v_1^{-1}u_1 = u_n v_1^{-1} + a \qquad (4.3)$$

where a is a diagram given by permutations of length $< n - 1$. Thus combining (4.3) with (2.9i) and (2.9k), if we have a polynomial $f(u_1, t_1)$, then

$$x_1 \cdot f(U_1, T_1)e_{\eta} = f(U_1 + \hbar, \zeta^{-1}T_1)(U_1 + \hbar - p(\zeta^{-1}))E_{\eta_i\eta} + (1 - e_{\eta_i\eta}) \cdot a'(f)$$

for a correction term $a'(f)$.

Now, let us apply this to the proof of (4.2). First, note that $\sigma \cdot E' = (U_1 + \hbar - p(\zeta^{-1}))E_{\eta_1}$. Thus, we have that:

$$e'x_1^{\ell-1}\sigma \cdot E' = e'x_1^{\ell-1} \cdot (U_1 + \hbar - p(\zeta^{-1}))E_{\eta_i}$$
$$= e'x_1^{\ell-2} \cdot (U_1 + 2\hbar - p(\zeta^{-2}))(U_1 + \hbar - p(\zeta^{-1}))E_{\eta_i^2} +$$
$$e'x_1^{\ell-2}(1 - e_{\eta_i^2}) \cdot a'(u_1 + \hbar - p(\zeta^{-1}))$$

Since $e'x_1^{\ell-2}(1 - e_{\eta_i^2}) = 0$, this correction term vanishes. Applying this inductively, we find that

$$e'x_1^{\ell-1}\sigma E' = e'x_1^{\ell-2} \cdot (U_1 + 2\hbar - p(\zeta^{-2}))(U_1 + \hbar - p(\zeta^{-1}))E_{\eta_i^2}$$
$$= e'x_1^{\ell-3} \cdot (U_1 + 3\hbar - p(\zeta^{-3}))(U_1 + 2\hbar - p(\zeta^{-2}))(U_1 + \hbar - p(\zeta^{-1}))E_{\eta_i^3}$$
$$\vdots$$
$$= (U_1 + \hbar - p(\zeta^{-1})) \cdots (U_1 + (\ell - 1)\hbar - p(\zeta))(U_1 + \ell\hbar - p(1))E'$$

This shows equation (4.2).

On the other hand, $[X_\sigma] \cdot 1$ is the class of the subspace of flags such that $\rho \cdot v(t) \in V[[t]]$ where

$$\rho = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{-1} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The obstruction to this is the constant term of the first component of $v(t)$. This is a section of ℓ copies of the tautological bundle on the affine Grassmannian, which transform according to the standard representation of GL_ℓ, and trivially with respect to the loop $\mathbb{C}^*$. Thus, $[X_\sigma] \cdot 1$ is just the Euler class of this bundle, which agrees with (4.2) by the convention we have chosen for Chern roots. This completes the proof that we have a map $e'\mathsf{H}e' \to \mathfrak{A}'$.

We note that $\mathbb{X}'$ has a cell decomposition pulling back the Schubert decomposition, and this map hits the fundamental class of each cell. Using the shift elements constructed above, we see that the map from $e'\mathsf{H}e'$ hits the classes of Schubert cells for all simple reflections. Multiplying the classes of the simple reflections in the reduced decomposition of an element of the Weyl group hits the class of the corresponding Schubert cell, plus those of shorter length, by a standard argument (see, for example, [26, Lemma 3.13]). Thus, the map is surjective, and the proof is completed. $\square$

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(Ben Webster) DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO & PERIMETER INSTITUTE FOR MATHEMATICAL PHYSICS, WATERLOO, ON, CANADA
ben.webster@uwaterloo.ca

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