New York Journal of Mathematics

New York J. Math. **25** (2019) 45–70.

The slope conjectures for 3-string Montesinos knots

Xudong Leng, Zhiqing Yang and Ximin Liu

Abstract. The (strong) slope conjecture relates the degree of the colored Jones polynomial of a knot to certain essential surfaces in the knot complement. We verify the slope conjecture and the strong slope conjecture for 3-string Montesinos knots satisfying certain conditions.

Contents

1. Introduction

The colored Jones polynomial is a generalization of the Jones polynomial, a celebrated knot invariant originally discovered by V. Jones via von

Received April 18, 2018.

2010 *Mathematics Subject Classification.* 57N10, 57M25.

Key words and phrases. slope conjecture; colored Jones polynomial; quadratic integer programming; boundary slope; incompressible surface.

Leng is the corresponding author. Yang is supported by the NFSC (No. 11271058). Liu is supported by the NFSC (No. 11431009).

ISSN 1076-9803/2019

Neumann algebras [13]. To find an intrinsic interpretation of the Jones polynomial, E. Witten introduced the Chern-Simons quantum field theory [23] which led to invariants of 3-manifolds as well as the colored Jones polynomial. Then N. Reshetikhin and V. Turaev constructed a mathematically rigorous mechanism by quantum groups [20] to produce these invariants. Also, these invariants can be obtained by skein theory, or the Tiemperley-Lieb algebra [18, 16].

Compared with the Jones polynomial, the colored Jones polynomial reveals much stronger connections between quantum algebra and 3-dimensional topology, for example, the Volume Conjecture, which relates the asymptotic behavior of the colored Jones polynomial of a knot to the hyperbolic volume of its complement. Another connection proposed by S. Garoufalidis [5], called the Slope Conjecture, predicts that the growth of maximal degree of the colored Jones polynomial of a knot determines some boundary slopes of the knot complement (see Conjecture 2.2(a)). As far as the authors know, the Slope Conjecture has been proved for knots with up to 10 crossings [5], adequate knots [3], 2-fusion knots [8], and some pretzel knots [15]. In [19], K. Motegi and T. Takata verify the conjecture for graph knots and prove that it is closed under taking connected sums. In [14], E. Kalfagianni and A. T. Tran prove the conjecture is closed under taking the (p, q)-cable with certain conditions on the colored Jones polynomial, and they formulate the Strong Slope Conjecture (see Conjecture 2.2(b)).

In this article we verify the Slope Conjecture and the Strong Slope Conjecture for 3-string Montesinos knots $M([r_0, \cdots, r_m], [s_0, \cdots, s_p], [t_0, \cdots, t_q])$ (see Figure 1) with $m, p, q \geq 1$ and certain conditions attached (see the conditions C(1), C(2) and C(3) in Section 2), where

$$[r_0, \cdots, r_m] = \frac{1}{r_0 - \frac{1}{r_1 - \frac{1}{\cdots - \frac{1}{r_m}}}},$$

and $[s_0, \cdots, s_p]$ and $[t_0, \cdots, t_q]$ are defined similarly. Note that our conventions for Montesinos knots coincide with those of [11].

This article is inspired by the work of C. R. S. Lee and R. van der Veen [15]. The goal of these articles is to provide evidences and data to the (Strong) Slope Conjecture for Montesinos knots. The reason to choose the family of Montesinos knots is that as a generalization of 2-bridge knots, it is large and representative, and meanwhile well parameterized. Moreover, Lee and van der Veen's method [15] to deal with the colored Jones polynomial and its degree and Hatcher and Oertel's algorithm [9] to determine the incompressible surfaces of Montesinos knots pave the way for the proof. The strategy of the proof is straight-forward: we first find out the maximal degree of the colored Jones polynomial and then choose the essential surface which matches the degree by the boundary slope and the Euler characteristic provided by the Hatcher-Oertel algorithm.

As we will see, for 3-string Montesinos knots $M([r_0,\cdots,r_m],[s_0,\cdots,s_p],[t_0,\cdots,t_q])$, the increasing of m, p, q does not cause too much complexity. Like the cases in [15], r_0, s_0 and s_0, particularly the discriminant Δ (see Theorem 2.4 and its proof in Section 3) still dominate the maximal degree of the colored Jones polynomial (Theorem 2.4) as well as the selection of the essential surface (Theorem 2.5). More specifically, when $\Delta < 0$, the degree of colored Jones polynomial is matched by a typical type I essential surface; when $\Delta \geq 0$, it is matched by a type II essential surface, but this type II surface generally (when at least one of m,p and q is greater than 1) is not a Seifert surface while it is in [15].

2. Preliminaries and main results

2.1. The Slope Conjectures. Let K denote a knot in S^3 and $N(K)$ denote its tubular neighbourhood. A surface S properly embedded in the knot exterior $E(K) = S^3 - N(K)$ is called *essential* if it is incompressible, ∂-incompressible, and non ∂-parallel. A fraction $\frac{p}{q} \in \mathbb{Q}\bigcup\{\infty\}$ is a *boundary slope* of K if $pm + ql$ represents the homology class of ∂S in the torus $\partial N(K)$, where m and l are the canonical meridian and longitude basis of $H_1(\partial N(K))$. The *number of sheets* of S, denoted by $\sharp S$, is the minimal number of intersections of ∂S and the meridional circle of $\partial N(K)$.

For the colored Jones polynomial, we follow the convention of [15] and denote the unnormalized *n-colored Jones polynomial* by $J_K(n;v)$. See Section 3 for details. Its value on the trivial knot is defined to be $[n] = \frac{v^{2n}-v^{-2n}}{v^2-v^{-2}}$, where $v = A^{-1}$, and A is the variable of the Kauffman bracket. The maximal degree of $J_K(n)$ is denoted by $d_+J_K(n)$.

A significant result made by S. Garoufalidis and T. Q. T. Le shows that the colored Jones polynomial is q-holonomic [7]. Furthermore, the degree of the colored Jones polynomial is a quadratic quasi-polynomial [6], which can be stated as follow.

Theorem 2.1. [6] *For any knot K, there exist an integer $p_K \in \mathbb{N}$ and quadratic polynomials $Q_{K,1}\ldots Q_{K,p_K} \in \mathbb{Q}[x]$ such that $d_+J_K(n) = Q_{K,j}(n)$ if $n = j$ (mod p_K) for sufficiently large n.*

Then the Slope Conjecture and the Strong Slope Conjecture can be formulated as follows:

Conjecture 2.2. *In the context of the above theorem, set $Q_{K,j}(x) = a_jx^2 + 2b_jx + c_j$, then for each j there exists an essential surface $S_j \subset S^3 - K$, such that:*

a.(Slope Conjecture [5]) a_j is a boundary slope of S_j,

b.(Strong Slope Conjecture [14]) there exists some b_i $(1 \leq i \leq p_K)$, s. t. $b_i = \frac{\chi(S_j)}{\sharp S_j}$, where $\chi(S_j)$ is the Euler characteristic of S_j.

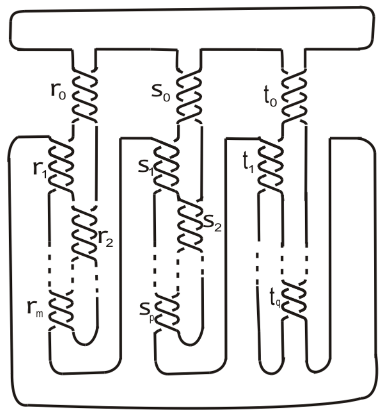

FIGURE 1. The Montesinos Knot $M([r_0, \cdots, r_m], [s_0, \cdots, s_p], [t_0, \cdots, t_q])$

2.2. Main results. A Montesinos knot is a knot formed by putting rational tangles together in a circle (see Figure 1). We denote the Montesinos knot obtained from rational tangles R_1, R_2, $\ldots R_N$ by $M(R_1, R_2, ..., R_N)$. For properties about Montesinos knots, the reader can refer to [1]. It is known that all Montesinos knots are semi-adequate [17], and the Slope Conjecture has been verified for adequate knots [3]. So we focus on a family of A-adequate and non-B adequate knots $M([r_0, \cdots, r_m], [s_0, \cdots, s_p], [t_0, \cdots, t_q])$ with $m, p, q \geq 1$ and conditions on $\{r_i\}$, $\{s_j\}$ and $\{t_k\}$ as follows.

C(1): For M to be knots rather than links, according to [9](Pg.456), let r_m, s_p and t_q be odd integers and all the rest be even integers such that $[r_0, \cdots, r_m]$,$[s_0, \cdots, s_p]$ and $[t_0, \cdots, t_q]$ are all of the type $\frac{odd}{odd}$.

C(2): For M to be A-adequate, let s_0, t_0 be positive integers and all the rest be negative integers.

C(3): To avoid the situation when $[r_0, \cdots, r_m] = [-2, -2, \ldots, -2, -1] = -1$, we need to add the restriction that when $r_m = -1$, $\exists r_i \leq -4$ $(0 \leq i \leq m-1)$.

Note that each of **C(1)** and **C(2)** is sufficient but unnecessary.

Equivalently, the three conditions above can be stated as follows.

- r_m, s_p and t_q are odd integers and all the rest of $\{r_i\}$, $\{s_j\}$ and $\{t_k\}$ be even integers;
- $r_i \leq -2$ for $0 \leq i \leq m-1, r_m \leq -1$ and when $r_m = -1$, $\exists r_i$ $(0 \leq i \leq m-1)$, s.t. $r_i \leq -4$;
- $s_0 \geq 2$, $s_j \leq -2$ for $1 \leq j \leq p-1$, $s_p \leq -1$;
- $t_0 \geq 2$, $t_k \leq -2$ for $1 \leq k \leq q-1$, $t_q \leq -1$.

It can be easily proved that any Odd/Odd-type fraction admits a unique $[even, \cdots, even, odd]$-expansion up to the equivalence of $[r_0, \cdots, r_m]$ and $[r_0, \cdots, r_m - 1, -1]$. Combining this with the theorem of classification of Montesinos knots (Theorem 8.2 of [4]) we can determine whether a given triple of rational numbers satisfies the conditions above.

Our main theorem is stated as follow.

Theorem 2.3. *The Slope Conjecture and the Strong Slope Conjecture (Conjecture 2.2(a) and (b)) are true for the Montesinos knots* $M([r_0,\cdots,r_m],[s_0,\cdots,s_p],[t_0,\cdots,t_q])$ *with* $m,p,q\geq 1$ *and* $\{r_i\}$, $\{s_j\}$ *and* $\{t_k\}$ *satisfying conditions C(1), C(2) and C(3).*

This theorem will be proved directly from the following two theorems. The first is about the degree of the colored Jones polynomial and the second is about the essential surface.

Theorem 2.4. *Let* $K=M([r_0,\cdots,r_m],[s_0,\cdots,s_p],[t_0,\cdots,t_q])$ *with* $m,p,q\geq 1$ *satisfying conditions C(1), C(2) and C(3), set* $A=-\frac{r_0+s_0+2}{2}, B=-(r_0+2), C=-\frac{r_0+t_0+2}{2}, \Delta=4AC-B^2$.
(1) If $\Delta<0$*, then* $p_K=\frac{s_0+t_0}{2}$*, and*

$$\begin{aligned}
&d_+J_K(n)=Q_{K,l}(n)\\
=&[\frac{2t_0^2}{s_0+t_0}-2(r_0+t_0+2)+6-2(m+p+q)-2(\sum_{even}^{m}r_i+\sum_{even}^{p}s_j+\sum_{even}^{q}t_k)]n^2\\
&+2[r_0+2(m+p+q)+\sum_{i=1}^{m}r_i+\sum_{j=1}^{p}s_j+\sum_{k=1}^{q}t_k]n\\
&-\frac{s_0+t_0}{2}\alpha_l^2-(s+t)\alpha_l-2(m+p+q)-2-2(\sum_{odd}^{m}r_i+\sum_{odd}^{p}s_j+\sum_{odd}^{q}t_k).
\end{aligned}$$

where α_l *is defined as follows. Let* $0\leq l<\frac{s_0+t_0}{2}$ *such that* $n=l \bmod \frac{s_0+t_0}{2}$ *, and set* v_l *to be the odd number nearest to* $\frac{2t_0}{s_0+t_0}l$*, then we set* $\alpha_l=-\frac{2t_0}{s_0+t_0}l+v_l-1$*. Note* $p_K=\frac{s_0+t_0}{2}$ *is a period of* $d_+J_K(n)$ *but may not be the least one. And* $\sum_{even}^{m}$ $(\sum_{odd}^{m})$ *means the summation is over all positive even (odd) numbers not greater than* m*, and* $\sum_{even}^{p}$ $(\sum_{odd}^{p})$ *and* $\sum_{even}^{q}$ $(\sum_{odd}^{q})$ *are defined similarly.*
(2) If $\Delta\geq 0$*, then* $p_K=1$ *and*

$$\begin{aligned}
d_+J_K(n)=&[6-2(m+p+q)-2(\sum_{even}^{m}r_i+\sum_{even}^{p}s_j+\sum_{even}^{q}t_k)]n^2\\
&+2[2(m+p+q)-4+(\sum_{i=1}^{m}r_i+\sum_{j=1}^{p}s_j+\sum_{k=1}^{q}t_k)]n\\
&+2-2(m+p+q)-2(\sum_{odd}^{m}r_i+\sum_{odd}^{p}s_j+\sum_{odd}^{q}t_k).
\end{aligned}$$

Theorem 2.5. *Under the same assumptions as Theorem 2.4,*
(1) When $\Delta<0$*, there exists an essential surface* S_1 *with boundary slope*

$$bs(S_1) = \frac{2t_0^2}{s_0+t_0} - 2(r_0+t_0+2) + 6 - 2(m+p+q) - 2(\sum_{even}^{m} r_i + \sum_{even}^{p} s_j + \sum_{even}^{q} t_k),$$

and

$$\frac{\chi_{(S_1)}}{\sharp S_1} = r_0 + 2(m+p+q) + \sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k.$$

(2) When $\Delta \geq 0$*, there exists an essential surface* S_2 *with boundary slope*

$$bs(S_2) = 6 - 2(m+p+q) - 2(\sum_{even}^{m} r_i + \sum_{even}^{p} s_j + \sum_{even}^{q} t_k),$$

and

$$\frac{\chi_{(S_2)}}{\sharp S_2} = 2(m+p+q) - 4 + (\sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k).$$

Note that in Theorem 2.4 the coefficient of the linear term of $d_+J_K(n)$ is always negative. This actually verifies another conjecture from [14] for this family of Montesinos knots, which can be stated as follow.

Conjecture 2.6. *(* [14, Conjecture 5.1]*) In the context of Theorem 2.1 and Conjecture 2.2, for any nontrivial knot in* S^3*, we have* $b_j \leq 0$.

Theorem 2.7. *Conjecture 2.6 is true for the Montesinos knots* $M([r_0,\cdots,r_m], [s_0,\cdots,s_p],[t_0,\cdots,t_q])$ $m,p,q \geq 1$ *satisfying the condition C(1), C(2) and C(3).*

3. Colored Jones polynomial and its degree

3.1. The colored Jones polynomial via KTGs. To compute the colored Jones polynomial of Montesinos knots, Lee and van der Veen apply the notion of *knotted trivalent graphs* (KTG) in [15] (see also [22, 21]). It is a natural generalization of knots and links and makes the skein theory [18] more convenient for Montesinos knots.

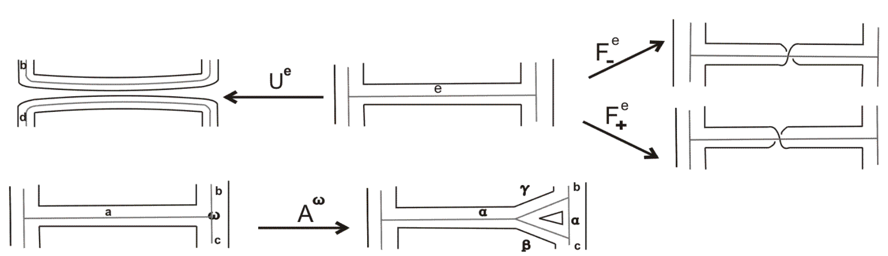

FIGURE 2. Operations on KTG: framing change F and unzip U applied to an edge e, triangle move A^ω applied to a vertex ω.

Definition 3.1. [15]

(1) A *framed graph* is a one dimensional simplicial complex γ together with an embedding $\gamma \to \Sigma$ of γ into a surface with boundary Σ as a spine.

(2) A *knotted trivalent graph* (KTG) is a trivalent framed graph embedded as a surface into $\mathbb{R}^3$, considered up to isotopy.

(3)Let Γ be a knotted trivalent graph, $E(\Gamma)$ be the set of its edges, and V be the set of its vertices. An *admissible coloring* of Γ is a map $\sigma : E(\Gamma) \to \mathbb{N}$ such that $\forall v \in V$ the following two conditions are satisfied:

- $a_v + b_v + c_v$ is even
- $a_v + b_v \geq c_v$, $b_v + c_v \geq a_v$ $c_v + a_v \geq b_v$ (triangle inequalities)

where a_v, b_v, c_v are the colors of the edges touching the vertex v.

The advantage of KTGs over knots or links is that they support powerful operations. In this article we will need the following three types of operations, the *framing change* $F^e_\pm$, the *unzip* U^e, and the *triangle move* A^ω, as illustrated in Figure 2.

The important thing is that these three types of operations are sufficient to produce any KTG from the θ graph (see the far left in Figure 3).

Theorem 3.2. [22, 21] *Any KTG can be generated from the θ graph by repeatedly applying the operations $F_\pm$, U and A defined above.*

By the above theorem, one can define the colored Jones polynomial of any KTG once he or she fixes the value of any colored θ graph and describes how it varies under the the above operations.

Definition 3.3. [15] The colored Jones polynomial of a KTG Γ with admissible coloring σ, denoted by $\langle \Gamma, \sigma \rangle$, is defined by the equations as follows.

$$\langle \theta; a, b, c \rangle = O^{\frac{a+b+c}{2}} \begin{bmatrix} & \frac{a+b+c}{2} & \\ \frac{-a+b+c}{2} & \frac{a-b+c}{2} & \frac{a+b-c}{2} \end{bmatrix},$$

$$\langle F^e_\pm(\Gamma), \sigma \rangle = f(\sigma(e))^{\pm 1} \langle \Gamma, \sigma \rangle,$$

$$\langle U^e(\Gamma), \sigma \rangle = \langle \Gamma, \sigma \rangle \sum_{\sigma(e)} \frac{O^{\sigma(e)}}{\langle \theta; \sigma(e), \sigma(b), \sigma(d) \rangle},$$

$$\langle A^\omega(\Gamma), \sigma \rangle = \langle \Gamma, \sigma \rangle \Delta(a, b, c, \alpha, \beta, \gamma).$$

Particularly, a knot K is a 0-frame KTG without vertices, and the colored Jones polynomial of K is defined to be $J_K(n+1) = (-1)^n \langle K, n \rangle$, where n is the color of the single edge of K, and $(-1)^n$ is to normalize the unknot as $J_O(n) = [n]$.

In above formulas, the quantum integer

$$[k] = \frac{v^{2k} - v^{-2k}}{v^2 - v^{-2}}, \text{ and } [k]! = [k][k-1] \dots [1].$$

The symmetric multinomial coefficient is defined as:

$$\begin{bmatrix} a_1 + a_2 + \dots a_r \\ a_1, a_2, \dots, a_r \end{bmatrix} = \frac{[a_1 + a_2 + ... + a_r]!}{[a_1]! \dots [a_r]!}.$$

The value of the k-colored unknot is defined as:

$$O^k = (-1)^k [k+1] = \langle O, k \rangle.$$

The the framing change f is defined as:

$$f(a) = (\sqrt{-1})^{-a} v^{-\frac{1}{2}a(a+2)}.$$

The summation in the equation of unzip is over all possible colorings of the edge e (derived from the triangle inequalities of the triple $(\sigma(e), \sigma(b), \sigma(d))$) that has been unzipped. Δ is the quotient of the $6j$-symbol and the θ, and

$$\Delta(a,b,c,\alpha,\beta,\gamma) = \Sigma \frac{(-1)^z}{(-1)^{\frac{a+b+c}{2}}} \begin{bmatrix} z+1 \\ \frac{a+b+c}{2}+1 \end{bmatrix} \begin{bmatrix} \frac{-a+b+c}{2} \\ z - \frac{a+\beta+\gamma}{2} \end{bmatrix} \begin{bmatrix} \frac{a-b+c}{2} \\ z - \frac{\alpha+b+\gamma}{2} \end{bmatrix} \begin{bmatrix} \frac{a+b-c}{2} \\ z - \frac{\alpha+\beta+c}{2} \end{bmatrix}.$$

The range of the summation in above formula is indicated by the binomials. Note that this Δ is not the one in Theorem 2.4.

The above definition agrees with the integer normalization in [2], where F. Costantino shows that $\langle \Gamma, \sigma \rangle$ is a Laurent polynomial in v independent of the choice of operations to produce the KTG.

3.2. The colored Jones polynomial of *K*. As illustrated in Figure 3, we obtain the colored Jones polynomial of the knot $K = M([r_0, \cdots, r_m], [s_0, \cdots, s_p], [t_0, \cdots, t_q])$ as follows. Starting from a θ graph , we first apply two A moves, then $(m+p+q)$ A moves on the three vertices of the lower triangle of the second graph, then one F move on each of the edges labelled by a_i, b_j and c_k, then unzip these twisted edges. The edges without labelling are actually colored by n. Note that an unzip applied to a twisted edge produces two twisted bands, each of which has the same twist number of the unzipped edge. Finally, to get the 0-frame colored Jones polynomial we need to cancel the framing produced by the operations and the writhe of the knot, which are denoted by $F(K)$ and $writhe(K)$ respectively and computed as follows.

$$F(K) = \sum_{i=0}^{m} r_i + \sum_{j=0}^{p} s_j + \sum_{k=0}^{q} t_k,$$

$$writhe(K) = \sum_{i=0}^{m} (-1)^{i+1} r_i + \sum_{j=0}^{p} (-1)^{j+1} s_j + \sum_{k=0}^{q} (-1)^{k+1} t_k.$$

so the result should be multiplied by

$$f(n)^{-2F(K)-2writhe(K)} = f(n)^{-4(\sum_{odd}^{m} r_i + \sum_{odd}^{p} s_j + \sum_{odd}^{q} t_k)},$$

where $\sum_{odd}^{m}$, $\sum_{odd}^{p}$ and $\sum_{odd}^{q}$ are defined in Theorem 2.4.

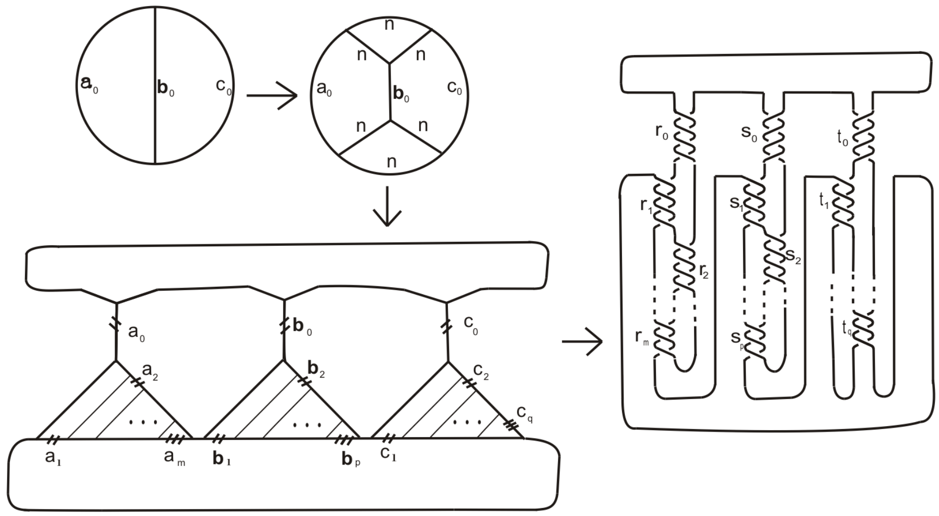

FIGURE 3. The operations to produce the knot K from a θ graph.

Lemma 3.4. *The colored Jones polynomial of the Montesinos knot $K = M([r_0, \cdots, r_m], [s_0, \cdots, s_p], [t_0, \cdots, t_q])$ with $m, p, q \geq 1$ and $\{r_i\}$, $\{s_j\}$ and $\{t_k\}$ satisfying conditions C(1), C(2) and C(3) is*

$$J_K(n+1) =$$

$$(-1)^n f(n)^{-4(\sum_{odd}^m r_i + \sum_{odd}^p s_j + \sum_{odd}^q t_k)} \sum_{(a_i,b_j,c_k)\in D_n} \langle\theta;a_0,b_0,c_0\rangle \Delta^2(a_0,b_0,c_0,n,n,n)$$

$$\prod_{i=0}^{m-1} \Delta(a_i,n,n,a_{i+1},n,n) \prod_{j=0}^{p-1} \Delta(b_j,n,n,b_{j+1},n,n) \prod_{k=0}^{q-1} \Delta(c_k,n,n,c_{k+1},n,n)$$

$$\prod_{i=0}^{m} f^{r_i}(a_i) \prod_{j=0}^{p} f^{s_j}(b_j) \prod_{k=0}^{q} f^{t_k}(c_k) \prod_{i=0}^{m} O^{a_i} \langle\theta;a_i,n,n\rangle^{-1} \prod_{j=0}^{p} O^{b_j} \langle\theta;b_j,n,n\rangle^{-1}$$

$$\prod_{k=0}^{q} O^{c_k} \langle\theta;c_k,n,n\rangle^{-1},$$

where the domain D_n is defined such that a_i,b_j,c_k are all even with $0 \leq a_i, b_j, c_k \leq 2n$, and a_0,b_0,c_0 satisfy the triangle inequality.

3.3. The degree of the colored Jones polynomial. To find out the maximal degree of the colored Jones polynomial of K, we need to analyze the the factors of the summands. The following lemma is from [15].

Lemma 3.5. [15]

$$d_+ f(a) = -\frac{a(a+2)}{2},$$

$$d_+O^a = 2a,$$

$$d_+\langle\theta; a, b, c\rangle = a(1-a) + b(1-b) + c(1-c) + \frac{(a+b+c)^2}{2},$$

$$d_+\Delta(a,b,c,\alpha,\beta,\gamma) = g(m+1, \frac{a+b+c}{2}+1) + g(\frac{-a+b+c}{2}, m - \frac{a+\beta+\gamma}{2}) + g(\frac{a-b+c}{2}, m - \frac{\alpha+b+\gamma}{2}) + g(\frac{a+b-c}{2}, m - \frac{\alpha+\beta+c}{2}),$$

where $g(n,k) = 2k(n-k)$ *and* $2m = a+b+c+\alpha+\beta+\gamma - max(a+\alpha, b+\beta, c+\gamma)$.

Now we can apply Lemma 3.4 and Lemma 3.5 to prove Theorem 2.4.

Proof of Theorem 2.4. Note that the maximal degree of $J_K(n+1)$ satisfies the inequality below.

$$d_+J_K(n+1) \le max_{(a_i,b_j,c_k,)\in D_n}\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q),$$

where

$$\begin{aligned}
&\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q)\\
=&-4(\sum_{odd}^{m} r_i + \sum_{odd}^{p} s_j + \sum_{odd}^{q} t_k)d_+f(n) + d_+\langle\theta; a_0, b_0, c_0\rangle + 2d_+\Delta(a_0,b_0,c_0,n,n,n)\\
&+\sum_{i=0}^{m-1} d_+\Delta(a_i,n,n,a_{i+1},n,n) + \sum_{j=0}^{p-1} d_+\Delta(b_j,n,n,b_{j+1},n,n)\\
&+\sum_{k=0}^{q-1} d_+\Delta(c_k,n,n,c_{k+1},n,n) + \sum_{i=0}^{m} r_i d_+f(a_i) + \sum_{j=0}^{p} s_j d_+f(b_j) + \sum_{k=0}^{q} t_k d_+f(c_k)\\
&+\sum_{i=0}^{m} d_+O^{a_i} + \sum_{j=0}^{p} d_+O^{b_j} + \sum_{k=0}^{q} d_+O^{c_k} - \sum_{i=0}^{m} d_+\langle\theta; a_i, n, n\rangle - \sum_{j=0}^{p} d_+\langle\theta; b_j, n, n\rangle\\
&-\sum_{k=0}^{q} d_+\langle\theta; c_k, n, n\rangle.
\end{aligned}$$

$\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q)$ is the highest degree of each term of the summation in Lemma 3.4. Equality holds when Φ has only one maximum or when it has multiple maxima and the coefficients of the maximal degree terms do not cancel out.

Generally, finding $max_{(a_i,b_j,c_k,)\in D_n}\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q)$ is a problem of quadratic integer programming, which is quite an involved topic [8]. In this case however, it can be solved by analyzing the partial derivatives of the real Φ.

For clarity, we divide the proof into two parts. One is the analysis for the contribution of the parameters a_i, b_j and c_k ($1 \le i \le m$, $1 \le j \le p$ and $1 \le k \le q$) to the maxima of Φ. The other is the analysis for the dependence of the maxima of Φ on a_0, b_0 and c_0.

(I) Contribution of a_i, b_j and c_k ($1 \le i \le m$, $1 \le j \le p$ and $1 \le k \le q$). From Lemma 3.5, we have

$$d_+\Delta(a,b,c,n,n,n) = \begin{cases} -\frac{1}{2}a^2 - a + (a+b+c+2)n - bc & \text{if } a \ge b,c; \\ -\frac{1}{2}b^2 - b + (a+b+c+2)n - ac & \text{if } b \ge a,c; \\ -\frac{1}{2}c^2 - c + (a+b+c+2)n - ab & \text{if } c \ge a,b, \end{cases}$$

and

$$d_+\Delta(a,n,n,b,n,n) = \begin{cases} -\frac{a^2}{2} - a - b^2 - ab + 2an + 4bn + 2n - 2n^2 & \text{if } a+b \ge 2n; \\ -\frac{1}{2}b^2 + b + 2nb & \text{if } a+b \le 2n. \end{cases}.$$

When $m \ge 2$, for $1 \le i \le m-1$ we have,

$$\begin{aligned} &\partial_{a_i}\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q) \\ =&a_i - r_i(a_i+1) + 1 - 2n + \partial_{a_i}d_+\Delta(a_{i-1},n,n,a_i,n,n) + \partial_{a_i}d_+\Delta(a_i,n,n,a_{i+1},n,n) \\ =&\begin{cases} -(r_i+2)a_i - r_i - a_{i-1} - a_{i+1} + 4n & \text{if } a_{i-1}+a_i \ge 2n \text{ and } a_i+a_{i+1} \ge 2n; \\ -(r_i+1)a_i - a_{i-1} - r_i + 2n + 1 & \text{if } a_{i-1}+a_i \ge 2n \text{ and } a_i+a_{i+1} \le 2n; \\ -(r_i+1)a_i - r_i + 2n - a_{i+1} + 1 & \text{if } a_{i-1}+a_i \le 2n \text{ and } a_i+a_{i+1} \ge 2n; \\ -r_i(a_i+1) + 2 & \text{if } a_{i-1}+a_i \le 2n \text{ and } a_i+a_{i+1} \le 2n. \end{cases} \end{aligned}$$

For $i = m$ ($m \ge 2$), we have

$$\begin{aligned} &\partial_{a_m}\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q) \\ =&a_m - r_m(a_m+1) + 1 - 2n + \partial_{a_m}d_+\Delta(a_{m-1},n,n,a_m,n,n) \\ =&\begin{cases} -(r_m+1)a_m - a_{m-1} - r_m + 2n + 1 & \text{if } a_{m-1}+a_m \ge 2n; \\ -r_m(a_m+1) + 2 & \text{if } a_{m-1}+a_m \le 2n. \end{cases} \end{aligned}$$

Since $r_i \le -2$ ($1 \le i \le m-1$) and $r_m \le -1$, from the two equations above it is easy to verify that we have $\partial_{a_i}\Phi > 0$ in all cases.

When $m = 1$, by a similar calculation we have $\partial_{a_1}\Phi > 0$. So we can conclude that $\partial_{a_i}\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q) > 0$ with $m \ge 1$ and $1 \le i \le m$. Similarly, we have $\partial_{b_j}\Phi > 0$, $\partial_{c_k}\Phi > 0$ with $p,q \ge 1$ and $1 \le j \le p$, $1 \le k \le q$. So $\Phi(a_0,\cdots,a_m,b_0,\cdots,b_p,c_0,\cdots,c_q)$ achieves its maxima only when $a_i = 2n$, $b_j = 2n$ and $c_k = 2n$, where $m,p,q \ge 1$, $1 \le i \le m$, $1 \le j \le p$ and $1 \le k \le q$.

(II) Dependence on a_0, b_0 and c_0. Denote $\Phi(a_0,2n,\cdots,2n,b_0,2n,\cdots,2n,c_0,2n,\cdots,2n)$ by $T(a_0,b_0,c_0)$. Then we have

$$\begin{aligned} &\partial_{a_0}T(a_0,b_0,c_0) \\ =&b_0 + c_0 + 2 - 2n - (r_0+1)a_0 + 2\partial_{a_0}d_+\Delta(a_0,b_0,c_0,n,n,n) + \partial_{a_0}d_+\Delta(a_0,n,n,2n,n,n) \\ =&\begin{cases} -(r_0+2)(a_0+1) - a_0 + b_0 + c_0 + 1 & \text{if } a_0 \ge b_0,c_0; \\ -(r_0+1)(a_0+1) + b_0 - c_0 + 2 & \text{if } b_0 \ge a_0,c_0; \\ -(r_0+1)(a_0+1) + c_0 - b_0 + 2 & \text{if } c_0 \ge a_0,b_0. \end{cases} \end{aligned}$$

Since $r_0 \le -2$, we always have $\partial_{a_0} T(a_0, b_0, c_0) > 0$. See Figure 4. The domain of the real function $T(a_0, b_0, c_0)$ is the hexahedron $AB'CD'C'$ defined by the set

$$H_n = \{(a_0, b_0, c_0) \mid a_0+b_0 \le c_0, a_0+c_0 \le b_0, b_0+c_0 \le a_0, 0 \le a_0, b_0, c_0 \le 2n.\}$$

Note that for any (b_0', c_0') (b_0' and c_0' are even integers and $0 \le b_0', c_0' \le 2n$), there exists an even integer a_0' ($0 \le a_0' \le 2n$) such that (a_0', b_0', c_0') is in the triangle region $AB'C$ or $B'CC'$. So $T(a_0, b_0, c_0)$ must achieve its maxima in the triangle region $AB'C$ or $B'CC'$. Note that in the tetrahedron $AB'CC'$ we have

$$\partial_{b_0} T(a_0, b_0, c_0) = a_0 - b_0 - c_0 + 1 - s_0 - s_0 b_0 < 0.$$

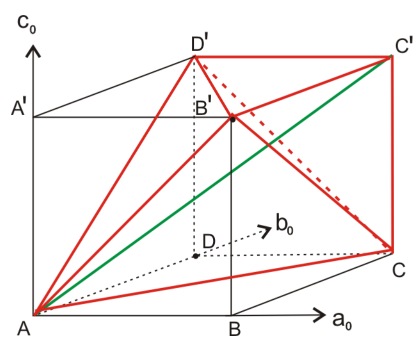

FIGURE 4. The feasible region of real $T(a_0, b_0, c_0)$ is the hexahedron $AB'CD'C'$ in the cube $ABCD-A'B'C'D'$ with edge length $2n$.

So any maximum of $T(a_0, b_0, c_0)$ must occur in the triangle region $AB'C$ with $a_0 = b_0+c_0$. Now we focus on the following 2-variable function $R(b_0, c_0)$ restricted in the triangle domain $T_n = \{(b_0, c_0) \mid b_0, c_0 \ge 0, b_0 + c_0 \le 2n\}$.

$$\begin{aligned} &R(b_0, c_0) = T(b_0 + c_0, b_0, c_0) \\ &= -\frac{r_0+s_0+2}{2} b_0^2 - (r_0+2) b_0 c_0 - \frac{r_0+t_0+2}{2} c_0^2 - (r_0+s_0) b_0 - (r_0+t_0) c_0 \\ &\quad + [6 - 2(m+p+q)] n^2 + 4n - 2n(n+1)\Big(\sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k\Big) \\ &\quad + 2n(n+2)\Big(\sum_{odd}^{m} r_i + \sum_{odd}^{p} s_j + \sum_{odd}^{q} t_k\Big). \end{aligned} \tag{3.1}$$

To analyse $R(b_0, c_0)$, we set $A = -\frac{r_0+s_0+2}{2}$, $B = -(r_0+2)$, $C = -\frac{r_0+t_0+2}{2}$ and $\Delta = 4AC - B^2$.

Although Theorem 2.4 (also Theorem 2.5) is divided into two cases by the range of Δ, it is also natural to consider the range of A and C in its proof. First we note that, if $A=-\frac{r_0+s_0+2}{2}\geq 0$, then

$$\Delta = 4AC-B^2=(s_0+t_0)(r_0+2)+s_0t_0\leq -s_0(s_0+t_0)+s_0t_0=-s_0^2<0.$$

Similarly, when $C\geq 0$ we have $\Delta<0$.

Then we can divided the cases into

(1) Either A or $C\geq 0$ ($\Rightarrow \Delta<0$),
(2a) Both $A<0$ and $C<0$, and $\Delta<0$,
(2b) Both $A<0$ and $C<0$, $\Delta>0$ and $r_0<-2$,
(2c,d) Both $A<0$ and $C<0$, and $\Delta=0$,
(2e) $r_0=-2$ ($\Rightarrow A<0$ and $C<0$, and $\Delta>0$).

So the case $\Delta<0$ consists of the cases (1) and (2a) while the case $\Delta\geq 0$ consists of the other cases above. The case when $r_0=-2$ is discussed separately because of some technical reason presented in Part (2) of this proof.

The argument of the case (1) is similar to that of the corresponding case in the paper of Lee-van der Veen [15].

(1) When $A\geq 0$ or $C\geq 0$, we have

$$\partial_{b_0}R=-(r_0+s_0+2)b_0-(r_0+2)c_0-(r_0+s_0)>0$$

or

$$\partial_{c_0}R=-(r_0+t_0+2)b_0-(r_0+2)c_0-(r_0+t_0)>0$$

respectively.

Then the maxima must be on the line $b_0+c_0=2n$. Set $Q(b)=R(b_0,2n-b_0)$, we have

$$\begin{aligned}Q(b_0)&=R(b_0,2n-b_0)\\&=-\frac{s_0+t_0}{2}b_0^2+[2t_0n-s_0+t_0]b_0-2(r_0+t_0+2)n^2-2(r_0+t_0)n\\&+[6-2(m+p+q)]n^2+4n-2n(n+1)(\sum_{i=1}^{m}r_i+\sum_{j=1}^{p}s_j+\sum_{k=1}^{q}t_k)\\&+2n(n+2)(\sum_{odd}^{m}r_i+\sum_{odd}^{p}s_j+\sum_{odd}^{q}t_k).\end{aligned}$$

$Q(b_0)$ is a quadratic function in b_0 with negative leading coefficient, and its real maximum is at $\hat{b_0}=\frac{2t_0n-s_0+t_0}{s_0+t_0}$, $\hat{b_0}\in(0,2n)$ for n sufficiently large. Since we define the colored Jones polynomial in Definition 3.3 by $J_K(n+1)=(-1)^n\langle K,n\rangle$, by now what we have computed are all about the $(n+1)$-th colored Jones polynomial rather than the n-th. So we have to make the switch $N=n+1$ to get the formula of $d_+J_K(N)$. Set $N=h(\frac{s_0+t_0}{2})+l$, where $0\leq l<\frac{s_0+t_0}{2}$, then $\hat{b_0}=t_0h-1+\frac{2t_0l}{s_0+t_0}$. Let $\bar{b_0}$ be the even number nearest to $\hat{b_0}$, note t_0h-1 is odd, then we have $\bar{b_0}=t_0h-1+v_l$, where v_l is the odd number nearest to $\frac{2t_0}{s_0+t_0}l$ (v_l can be considered as the source of the periodicity in the case (1) of Theorem 2.4), so $\bar{b_0} = \frac{2t_0}{s_0+t_0}N - \frac{2t_0}{s_0+t_0}l + v_l - 1$. Set $\bar{b_0} = \frac{2t_0}{s_0+t_0}N + \alpha_l$, where $\alpha_l = -\frac{2t_0}{s_0+t_0}l + v_l - 1$. Then we have

$$
\begin{aligned}
&max_{(a_i,b_j,c_k)\in D_n}\Phi(a_i,b_j,c_k) = Q(\bar{b_0})\\
=&[\frac{2t_0^2}{s_0+t_0} - 2(r_0+t_0+2) + 6 - 2(m+p+q) - 2(\sum_{even}^{m} r_i + \sum_{even}^{p} s_j + \sum_{even}^{q} t_k)]N^2\\
&+ 2[r_0 + 2(m+p+q) + \sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k]N\\
&- \frac{s_0+t_0}{2}\alpha_l^2 - (s+t)\alpha_l - 2(m+p+q) - 2 - 2(\sum_{odd}^{m} r_i + \sum_{odd}^{p} s_j + \sum_{odd}^{q} t_k).
\end{aligned}
$$

When $\hat{b_0}$ is not odd, the maximum is unique. Otherwise, Φ has exactly 2 maxima, we need to consider the possibility that the coefficients of the 2 maximal-degree terms may cancel out. From Lemma 3.4 and Definition 3.3, and the fact that we take $a_i = b_j = c_k = 2n$ when $i,j,k \geq 1$, it is easy to see that for the leading coefficient of each term of the summation, without counting the factors independent of a_0, b_0, c_0, the f's contribute $(-1)^{\frac{1}{2}(a_0r_0+b_0s_0+c_0t_0)}$, the Δ's contribute $(-1)^{-\frac{1}{2}(a_0+b_0+c_0)}$, the O's and the θ's contribute none, and altogether it is

$$C = (-1)^{\frac{1}{2}[(r_0-1)a_0+(s_0-1)b_0+(t_0-1)c_0]}.$$

Furthermore, since any maximum of $Q(b_0)$ must occur on $a_0 = b_0 + c_0 = 2n$, we have

$$
\begin{aligned}
\tilde{C}(b_0) &= \frac{1}{2}[(r_0-1)a_0 + (s_0-1)b_0 + (t_0-1)c_0]\\
&= \frac{1}{2}[2n(r_0-1) + (s_0-1)b_0 + (t_0-1)(2n-b_0)].
\end{aligned}
$$

If there are two maxima $Q(b_0)$ and $Q(b_0+2)$, we must have

$$\tilde{C}(b_0) - \tilde{C}(b_0+2) = t_0 - s_0.$$

Since s_0 and t_0 are even, $t_0 - s_0$ must be even and the coefficients of the two maximal terms will not cancel out. So we have $d_+J_K(N) = Q(\bar{b_0})$.

(2) When $A < 0$ and $C < 0$, for any fixed $\tilde{c_0}$, by Equation 3.1, $R(b_0, \tilde{c_0})$ is a quadratic function in b_0 with negative leading coefficient, whose axis of symmetry (in the plane $c_0 = \tilde{c_0}$ of the b_0c_0R-coordinates) intersects the line $\partial_{b_0}R = 0$ and is perpendicular to the b_0c_0-plane.

If $r_0 < -2$ (the case when $r_0 = -2$ will be analysed at the end of this proof), we consider the real value of $R(b_0, c_0)$ on the line $\partial_{b_0}R = 0$:

$$
\begin{cases}
R(b_0,c_0) = -\frac{r_0+s_0+2}{2}b_0^2 - (r_0+2)b_0c_0 - \frac{r_0+t_0+2}{2}c_0^2 - (r_0+s_0)b_0 - (r_0+t_0)c_0 + \text{const}\\
\partial_{b_0}R = -(r_0+s_0+2)b_0 - (r_0+2)c_0 - (r_0+s_0) = 0.
\end{cases}
\tag{3.2}
$$

Then we have

$$R\mid_{\partial_{b_0}R=0} = R(b_0, -\frac{r_0+s_0+2}{r_0+2}b_0 - \frac{r_0+s_0}{r_0+2})$$
$$= [-\frac{r_0+s_0+2}{2(r_0+2)^2}\Delta]b_0^2 + \frac{r_0+s_0+2}{(r_0+2)^2}[(r_0+2)(r_0+t_0)-(r_0+t_0+2)(r_0+s_0)]b_0 + const. \tag{3.3}$$

If we imagine $R(b_0, c_0)$ as the surface of a mountain, then $R\mid_{\partial_{b_0}R=0}$ is just the ridge of it.

If $\Delta \neq 0$, $R\mid_{\partial_{b_0}R=0}$ is a quadratic function in b whose axis of symmetry is perpendicular to the b_0c_0-plane at the point P with coordinates

$$(b_0\mid_P, c_0\mid_P) = (\frac{(r_0+2)(r_0+t_0)-(r_0+t_0+2)(r_0+s_0)}{\Delta}, \frac{(r_0+2)(r_0+s_0)-(r_0+s_0+2)(r_0+t_0)}{\Delta}).$$

(P is actually the intersection of $\partial_{b_0}R = 0$ and $\partial_{c_0}R = 0$).

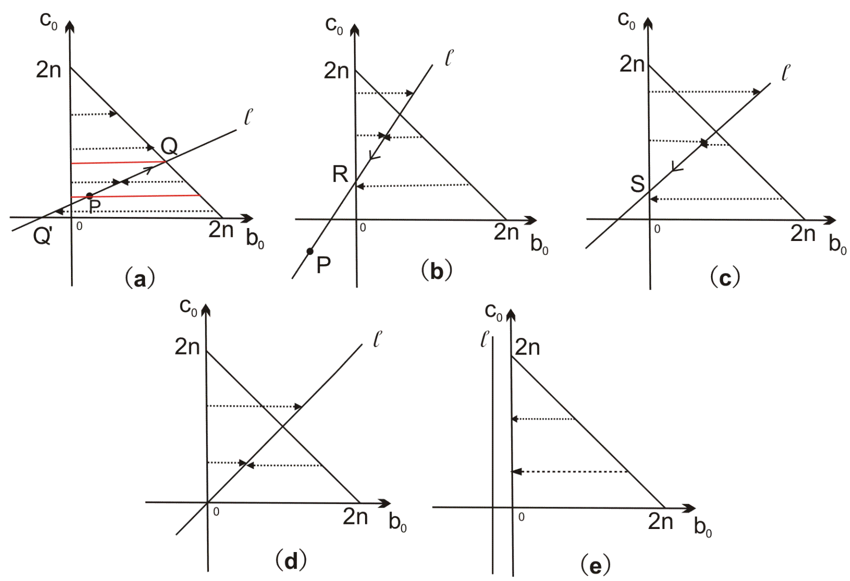

FIGURE 5. $R(b_0, c_0)$ is restricted to the triangle domain T_n: $(0,0)$-$(0,2n)$-$(2n,0)$, and the arrows indicate its increasing direction; ℓ denotes the line $\partial_{b_0}R = 0$.

(a) If A and $C < 0, \Delta < 0$, by Equation 3.3, $R\mid_{\partial_{b_0}R=0}$ is a quadratic function in b_0 with positive leading coefficient. And we have $b_0\mid_P \geq 0$, $c_0\mid_P \geq 0$. See Figure 5(a). Let ℓ denotes the line $\partial_{b_0}R = 0$. The dotted arrows indicate the increasing directions of $R(b_0, c_0)$. On each horizontal line in b_0c_c-plane, the monotonicity of $R(b_0, c_0)$ is separated by ℓ, and on ℓ the monotonicity is separated by the point P. Let $c_0\mid_Q$ be the c_0-coordinate of the point Q. Then we divide the triangle domain T_n into 3 parts (with two red horizontal segment as dividing lines in the figure):

- the part T_n^1 with $c_0 \geq c_0\mid_Q$,

- the part T_n^2 with $c_0\,|_P\leq c_0 \leq c_0\,|_Q$,
- the part T_n^3 with $c_0 \leq c_0\,|_P$.

Let $R\,|_{T_n^i}$ denote the function $R(b_0, c_0)$ restricted on T_n^i $(i = 1, 2, 3)$. By observing the dotted arrows (increasing directions), we find that $R\,|_{T_n^1}$ achieves its maxima (or maximum) on the segment $[Q,(0, 2n)]$, $R\,|_{T_n^2}$ achieves its maximum on the point Q and the maxima (or maximum) of $R\,|_{T_n^3}$ is less than $R(Q')$, here $R(Q')$ is value of $R(b_0, c_0)$ on the point Q'. For sufficiently large n, we must have $R(Q) > R(Q')$, so any maximum of $R(b_0, c_0)$ must be on the segment $[Q,(0, 2n)]$ in the line $b_0 + c_0 = 2n$, then the argument will be the same as that of case (1).

(b) If $A, C < 0$, and $\Delta > 0$, $R\,|_{\partial_{b_0}R=0}$ is a quadratic function in b_0 with negative leading coefficient. And we have $b_0\,|_P\leq 0$, $c_0\,|_P\leq 0$. See Figure 5(b). By a similar analysis with (a), we can conclude that any maximum must occur on OR. Since

$$R(0, c_0) = -\frac{r_0 + t_0 + 2}{2}c_0^2 - (r_0 + t_0)c_0 + const$$

and $C = -\frac{r_0+t_0+2}{2} < 0$, it is easy to verify that $R(0, c_0)$ decreases in $[0, +\infty)$. So the maximum is unique and must occur at $O = (0, 0)$, and we have

$$\begin{aligned} d_+J_K(n + 1) &= R(0, 0)\\ &= [6 - 2(m + p + q)]n^2 + 4n - 2n(n + 1)(\sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k)\\ &+ 2n(n + 2)(\sum_{odd}^{m} r_i + \sum_{odd}^{p} s_j + \sum_{odd}^{q} t_k). \end{aligned}$$

Let $N = n + 1$, we have

$$\begin{aligned} d_+J_K(N) =&[6 - 2(m + p + q) - 2(\sum_{even}^{m} r_i + \sum_{even}^{p} s_j + \sum_{even}^{q} t_k)]N^2\\ &+ 2[2(m + p + q) - 4 + (\sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k)]N\\ &+ 2 - 2(m + p + q) - 2(\sum_{odd}^{m} r_i + \sum_{odd}^{p} s_j + \sum_{odd}^{q} t_k). \end{aligned}$$

(c) If $A, C < 0$,$\Delta = 0$, and $(r_0 + s_0)^2 + (r_0 + t_0)^2 \neq 0$, $R\,|_{\partial_{b_0}R=0}$ is a decreasing linear function in b_0. See Figure 5(c). Any maximum must occur on OS. Since $R(0, c_0)$ decreases in $[0, +\infty)$, the maximum is unique and must be on $O = (0, 0)$. So in this case we still have

$$d_+J_K(N) = R(0, 0).$$

(d) If $A, C < 0$, $\Delta = 0$, and $(r_0+s_0)^2+(r_0+t_0)^2 = 0$, then we immediately have $r_0 = -4, s_0 = 4, t_0 = 4$, $R(b_0, c_0) = -(b_0 - c_0)^2 + const$, the maxima are $R(0,0) = R(2,2) = \ldots = R(k,k)$, where $k = n$ when n is even, $k = n-1$ when n is odd. See Figure 5(d). By a similar argument with the end of (1), we can conclude that there are no cancellations between the the highest-degree coefficients, so

$$d_+J_K(N) = R(0,0).$$

(e) If $r_0 = -2$ (then we must have $A < 0$ and $C < 0$ and $\Delta > 0$, see also Remark 4.4), Equation 3.2 is converted to

$$\begin{cases} R(b_0, c_0) = -\frac{1}{2}s_0 b_0^2 - \frac{1}{2}t_0 c_0^2 - (s_0-2)b_0 - (t_0-2)c_0 + const \\ \partial_{b_0} R = -s_0 b_0 - (s_0 - 2) = 0. \end{cases}$$

See Figure 5(e). Since $\partial_{b_0} R(b_0, c_0) = 0 \ \Rightarrow\ b_0 = \frac{2-s_0}{s_0}$ and $b_0 \leq 0$ when $s_0 \geq 2$, $R(b_0, c_0)$ must achieve any of its maximum in c_0-axis. Since $R(0, c_0)$ decreases in $[0, +\infty)$, the unique maximum must be on $O = (0,0)$, and we have

$$d_+J_K(N) = R(0,0).$$

□

4. Essential surface

4.1. Hatcher and Oertel's edgepath system. The edgepath system of Hatcher-Oertel is actually based on the work of [10]. Like many other topics in geometric topology, the main ideal of the algorithm is to treat the object of study combinatorially. In this mechanism, properly embedded surfaces in a Montesinos knot complement with non-empty and non-meridional boundary are formed by saddles, and the edgepath system describes how these saddles are combined. For details please refer to [9, 12]. Briefly speaking, the *candidate surfaces*, which are the surfaces listed out to include all the essential surfaces (with non-empty and non-meridional boundary), are associated to certain *edgepath systems* in a 1-dimensional diagram $\mathcal{D}$ in the uv-plane (see Figure 6(b)). The vertices of $\mathcal{D}$ correspond to projective curve systems $[a, b, c]$ on the 4-punctured sphere carried by the train track in Figure 6(a) via $u = \frac{b}{a+b}$, $v = \frac{c}{a+b}$.

Specifically, the vertices of $\mathcal{D}$ are:

(1) the vertices corresponding to the arcs with slope $\frac{p}{q}$ denoted by $\langle \frac{p}{q} \rangle$, with the projective curve systems $[1, q-1, p]$, and the uv-coordinates $(\frac{q-1}{q}, \frac{p}{q})$,

(2) the vertices corresponding to the circles with slope $\frac{p}{q}$ denoted by $\langle \frac{p}{q} \rangle^\circ$, with the projective curve systems $[0, p, q]$ and the uv-coordinates $(1, \frac{p}{q})$,

(3) the vertices corresponding to the arcs with slope ∞ denoted by $\langle \infty \rangle$, with the uv-coordinates $(-1, 0)$.

The edges of $\mathcal{D}$ are:

(1) the *non-horizontal* edges connecting the vertex $\frac{p}{q}$ to the vertex $\frac{r}{s}$ with $\mid ps - qr \mid = 1$, denoted by $\langle \frac{r}{s} \rangle$—$\langle \frac{p}{q} \rangle$,

(2) the *horizontal* edges connecting $\langle \frac{p}{q} \rangle^\circ$ to $\langle \frac{p}{q} \rangle$, denoted by $\langle \frac{p}{q} \rangle$—$\langle \frac{p}{q} \rangle^\circ$,

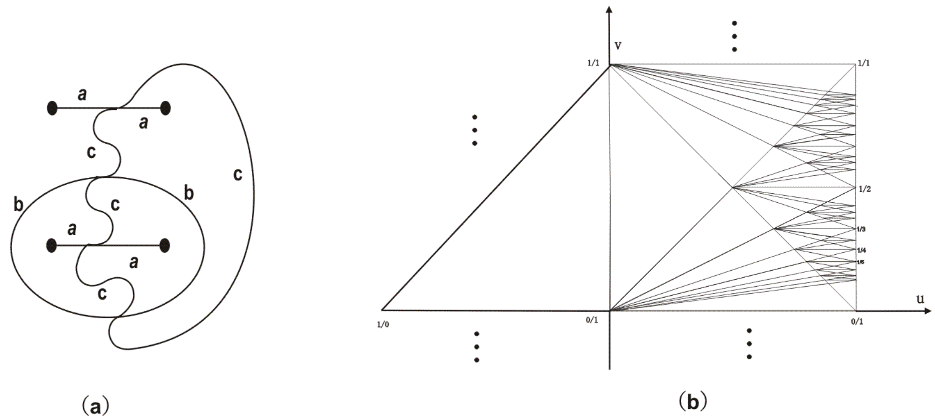

FIGURE 6. (a)The train track in a 4-punctured sphere. (b)The diagram $\mathcal{D}$ in the uv-plane.

(3) the *vertical* edges connecting $\langle z\rangle$ to $\langle z\pm 1\rangle$, denoted by $\langle z\pm 1\rangle$—$\langle z\rangle$, here $z\in\mathbb{Z}$,

(4) the *infinity* edges connecting $\langle z\rangle$ to $\langle\infty\rangle$ denoted by $\langle\infty\rangle$—$\langle z\rangle$,

(5) the *constant* edges which are points on the horizontal edge $\langle\frac{p}{q}\rangle$—$\langle\frac{p}{q}\rangle^\circ$ with the form $\frac{k}{m}\langle\frac{p}{q}\rangle+\frac{m-k}{m}\langle\frac{p}{q}\rangle^\circ$,

(6) the *partial* edges which are parts of non-horizontal edges $\langle\frac{r}{s}\rangle$—$\langle\frac{p}{q}\rangle$ with the form $\frac{k}{m}\langle\frac{r}{s}\rangle+\frac{m-k}{m}\langle\frac{p}{q}\rangle$—$\langle\frac{p}{q}\rangle$.

An *edgepath* denoted by γ in $\mathcal{D}$ is a piecewise linear path starting and ending at rational points of $\mathcal{D}$. An *edgepath system* denoted by $\Gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ is an n-tuple of edgepaths. An edgepath system of a candidate surface satisfies the following properties.

(E1) The starting point of γ_i is on the horizontal edge $\langle\frac{p_i}{q_i}\rangle$—$\langle\frac{p_i}{q_i}\rangle^\circ$, and if it is not the vertex $\langle\frac{p_i}{q_i}\rangle$, γ_i is constant.

(E2) γ_i is minimal, that is, it never stops or retraces itself, nor does it ever go along two sides of the same triangle of $\mathcal{D}$ in succession.

(E3) The ending points of γ_i's are rational points of $\mathcal{D}$ with their u-coordinates equal and v-coordinates adding up to zero.

(E4) γ_i proceeds monotonically from right to left, "monotonically" in the weak sense that motion along vertical edges is permitted.

A candidate edgepath system and the corresponding candidate surfaces are called *type I*, *type II* or *type III*, if the u-coordinate of the ending points of the candidate edgepath system is positive, zero or negative respectively.

4.2. Criteria for incompressibility. In [9], a series of candidate surfaces are associated to each candidate edgepath system, then every essential surface in knot complement with non-empty boundary of finite slope is isotopic to one of the candidate surfaces. However, not all of these candidate surfaces are essential, to further detect the incompressibility, Hatcher and Oertel developed the notion of r-value in [9].

Definition 4.1. [9] The r-value of an edge $\langle\frac{t}{s}\rangle$—$\langle\frac{p}{q}\rangle$ is defined to be $q-s$, where $0<s<q$. Particularly, when the edge is vertical its r-value is 0. The r-value of an edgepath is defined to be the r-value of its final edge.

By Corollary 2.4 through Proposition 2.10 of [9], a series of criteria for incompressibility are established. Here we just extract the part useful for us from Corollary 2.4, Proposition 2.7 and Proposition 2.8(a) of [9].

Theorem 4.2. [9] *For a 3-string Montesinos knot, any candidate surface associated to the edgepath system $\{\gamma_i\}$ is incompressible if it satisfies one of the conditions below:*
(1) If $\{\gamma_i\}$ contains no vertical edges, the cycle of r-values of $\{\gamma_i\}$ is not the type $(1,1,r_3)$ or $(1,2,r_3)$;
(2) If the cycle of r-values is the type of $(1,2,r_3)$, the final edges of each edgepath must be all increasing or decreasing;
(3) If $\{\gamma_i\}$ contains some vertical edges, the cycle of r-values is not the type $(0,2,1)$ or $(0,2,0)$.

4.3. Formulas for boundary slope and Euler characteristic. The boundary slope of an essential surface S is computed by $\tau(S)-\tau(S_0)$, where $\tau(S)$ is the *total number of twist* (or twist for short) of S, and S_0 is a Seifert surface in the list of candidate surfaces. For a candidate surface S associated to a candidate edgepath system Γ, we have [12]

$$\tau(S)=\sum_{\gamma_i\in\Gamma_{non-const}}\sum_{e_{i,j}\in\gamma_i}-2\sigma(e_{i,j})|e_{i,j}|. \tag{4.1}$$

In above formula, $|e|$ is the *length* of an edge e, which is defined to be 0, 1, or $\frac{k}{m}$ for a constant edge, a complete edge or a partial edge $\frac{k}{m}\langle\frac{r}{s}\rangle+\frac{m-k}{m}\langle\frac{p}{q}\rangle$—$\langle\frac{p}{q}\rangle$, respectively. And $\sigma(e)$ is the *sign* of a non-constant edge e, which is defined to be $+1$ or -1 according to whether the edge is increasing or decreasing (from right to left in uv-plane) respectively for a non-∞ edge; for an ∞ edge the sign is defined to be 0.

For the Euler characteristic of a candidate surface S, we use the formulas (3.4) and (3.5) of [12] to compute the ratio $\frac{-\chi(S)}{\sharp S}$.

Lemma 4.3. [12] *Let S be a candidate surface associated to a candidate edgepath system $\Gamma=(\gamma_1,\cdots,\gamma_N)$.*
(1) If S is type I, denote the u-coordinate of its ending points by u_0, then

$$\frac{-\chi(S)}{\sharp S}=\sum_{\gamma_i\in\Gamma_{non-const}}|\gamma_i|+N_{const}-N+(N-2-\sum_{\gamma_i\in\Gamma_{const}}\frac{1}{q_i})\frac{1}{1-u_0}.$$

Here $\Gamma_{non-const}$ (Γ_{const}) denotes the set consist of non-constant (constant) edges, $|\gamma_i|$ denotes the length of the edgepath γ_i, N_{const} denotes the number of constant edges in Γ.

(2) If S is type II, then

$$\frac{-\chi(S)}{\sharp S} = \sum_{i=1}^{N}(\mid \gamma_{i,>0} \mid)+ \mid \Gamma(+0) \mid -2.$$

Here $\gamma_{i,>0}$ denotes the part of γ with positive u-coordinate, and $\Gamma(+0)$ denotes the sum of the v-coordinates of γ_i's when they first reach the v-axis.

4.4. Proof of Theorem 2.5. Now we are ready to prove Theorem 2.5.

Proof. First we note that for $[r_0,\cdots,r_m]$ satisfying the condition C(2), in the diagram $\mathcal{D}$, $\langle[r_0,\ldots,r_m]\rangle$ is connected to $\langle[r_0,\ldots,r_{m-1}]\rangle$ by an increasing edge and is connected to $\langle[r_0,\ldots,r_m+1]\rangle$ by a decreasing edge from right to left. This is easy to proof by induction and similar facts exist for $[s_0,\cdots,s_p]$ and $[t_0,\cdots,t_q]$.

By the method of [9] (Pg.461), since the condition C(1) in Section 2 implies that the three tangles of the knot $M([r_0,\cdots,r_m],[s_0,\cdots,s_p],[t_0,\cdots,t_q])$ are all of the form $\frac{odd}{odd}$, we directly find the edgepath system of a Seifert surface S_0 as follows. (For simplicity, we just use $[r_0,\ldots,r_m]$ to denote the vertices instead of $\langle[r_0,\ldots,r_m]\rangle$. The edgepaths go from right to left in the same row and the far left vertex of a row is connected to the far right vetex of the row next below. The arrow $\leftharpoonup$ / $\leftharpoondown$ indicates the edge is increasing/ decreasing from right to left.)

$$\begin{aligned}
\delta_1 :\ & \leftharpoonup [r_0,r_1,\ldots,r_{m-1},-1] \leftharpoondown [r_0,r_1,\ldots,r_{m-1},-2] \leftharpoondown \cdots \leftharpoondown [r_0,r_1,\ldots,r_{m-1},r_m]\\
& \leftharpoonup [r_0,r_1,\ldots,r_{m-3},-1] \leftharpoondown [r_0,r_1,\ldots,r_{m-3},-2] \leftharpoondown \cdots \leftharpoondown [r_0,r_1,\ldots,r_{m-3},r_{m-2}]\\
& \cdots\cdots\\
& \leftharpoonup [r_0,r_1,r_2,-1] \leftharpoondown [r_0,r_1,r_2,-2] \cdots \leftharpoondown [r_0,r_1,r_2,r_3]\\
& \langle 0\rangle \leftharpoonup [r_0,-1] \leftharpoondown [r_0,-2],\ldots, \leftharpoondown [r_0,r_1].\\
\delta_2 :\ & \leftharpoonup [s_0,s_1,\ldots,s_{p-1},-1] \leftharpoondown [s_0,s_1,\ldots,s_{p-1},-2] \leftharpoondown \cdots \leftharpoondown [s_0,s_1,\ldots,s_{p-1},s_p]\\
& \leftharpoonup [s_0,s_1,\ldots,s_{p-3},-1] \leftharpoondown [s_0,s_1,\ldots,s_{p-3},-2] \leftharpoondown \cdots \leftharpoondown [s_0,s_1,\ldots,s_{p-3},s_{p-2}]\\
& \cdots\cdots\\
& \leftharpoonup [s_0,s_1,s_2,-1] \leftharpoondown [s_0,s_1,s_2,-2] \cdots \leftharpoondown [s_0,s_1,s_2,s_3]\\
& \langle 0\rangle \leftharpoondown [s_0,-1] \leftharpoondown [s_0,-2],\ldots, \leftharpoondown [s_0,s_1].\\
\delta_3 :\ & \leftharpoonup [t_0,t_1,\ldots,t_{q-1},t_q]\\
& \leftharpoonup [t_0,t_1,\ldots,t_{q-3},-1] \leftharpoondown [t_0,t_1,\ldots,t_{q-3},-2] \leftharpoondown \cdots \leftharpoondown [t_0,t_1,\ldots,t_{q-3},t_{q-2}]\\
& \cdots\cdots\\
& \leftharpoonup [t_0,t_1,t_2,-1] \leftharpoondown [t_0,t_1,t_2,-2] \cdots \leftharpoondown [t_0,t_1,t_2,t_3]\\
& \langle 0\rangle \leftharpoondown [t_0,-1] \leftharpoondown [t_0,-2],\ldots, \leftharpoondown [t_0,t_1].
\end{aligned}$$

Note that the edgepaths of the Seifert surface should avoid the vertices with even denominators, in this case we let m and p be odd and q be even, but the parity of them won't affect our expressions of further results.

The cycle of r-value of the above edgepath system is $(-r_0-2, s_0, t_0)$. Note that r_0, s_0 and t_0 are even, and $r_0 \leq -2$, $s_0, t_0 \geq 2$. Any candidate surface associated to the above edgepath system is essential by Theorem 4.2(1) when $r_0 \leq -4$ or by Theorem 4.2(3) when $r_0 = -2$.

Remark 4.4. If $r_0 = -2$, then $[r_0, -1] = -1$ and there is a vertical edge $\langle 0\rangle \leftharpoondown \langle -1\rangle$ in the above edgepath δ_1 and in the edgepath β_1 presented in the second part of this proof. Note that when $r_0 = -2$, we must have $A = -\frac{1}{2}s_0 < 0$, $C = -\frac{1}{2}t_0 < 0$ and $\Delta = \frac{1}{4}s_0t_0 > 0$. So this situation can only happen in Case (2) of the Theorem 2.5 and won't affect the expressions of our results.

By Formular 4.1, the twist of S_0 is

$$\begin{aligned}\tau(S_0) &= 2[(-r_m-1)+(-1)+(-r_{m-2}-1)+(-1)+\cdots+(-r_1-1)-1]\\ &\quad+2[(-s_p-1)+(-1)+(-s_{p-2}-1)+(-1)+\cdots+(-s_1-1)+1]\\ &\quad+2[(-1)+(-t_{q-1}-1)+(-1)+\cdots+(-t_1-1)+1]\\ &= 2-2(m+p+q)-2(\sum_{odd}^{m} r_i+\sum_{odd}^{p} s_j+\sum_{odd}^{q} t_k).\end{aligned}$$

(1) When $\Delta < 0$, we claim that there exists a type I candidate edgepath system having ending points with u-coordinate u_0 on the left of the vertices $\langle\frac{1}{r_0+1}\rangle$, $\langle\frac{1}{s_0+1}\rangle$ and $\langle\frac{1}{t_0+1}\rangle$, and $u_0 = \frac{s_0t_0}{s_0t_0+s_0+t_0}$. In fact, u_0 is just the solution of the equation $v_1(u)+v_2(u)+v_3(u) = 0$, where the linear functions $v = v_1(u)$, $v = v_2(u)$ and $v = v_3(u)$ are determined by the lines through the edges $\langle -1\rangle$–$\langle -\frac{1}{2}\rangle$–, ...,–$\langle\frac{1}{r_0+1}\rangle$, $\langle 0\rangle$—$\langle\frac{1}{s_0+1}\rangle$ and$\langle 0\rangle$—$\langle\frac{1}{t_0+1}\rangle$ respectively. Denote by $u_{[r_0+1]}$, $u_{[s_0+1]}$ and $u_{[t_0+1]}$ the u-coordinates of $\langle\frac{1}{r_0+1}\rangle$, $\langle\frac{1}{s_0+1}\rangle$ and $\langle\frac{1}{t_0+1}\rangle$ respectively. With direct calculations we have

$$u_0 - u_{[r_0+1]} = \frac{-\Delta}{(r_0+1)(s_0t_0+s_0+t_0)} < 0,$$

$$u_0 - u_{[s_0+1]} = \frac{-s_0^2}{(s_0+1)(s_0t_0+s_0+t_0)} < 0,$$

$$u_0 - u_{[t_0+1]} = \frac{-t_0^2}{(t_0+1)(s_0t_0+s_0+t_0)} < 0,$$

so u_o must be on the left of $u_{[r_0+1]}$, $u_{[s_0+1]}$ and $u_{[t_0+1]}$. Suppose the edgepath of the $[r_0,r_1,\ldots,r_{m-1},r_m]$-tangle ends on the edge $\langle\frac{1}{r_0+k+1}\rangle$—$\langle\frac{1}{r_0+k}\rangle$, where $0 \leq k \leq -r_0-2$, then we obtain a type I candidate edgepath system below, with its ending points having u-coordinates u_0.

$$\begin{aligned}\gamma_1: &\leftharpoondown [r_0, r_1, \ldots, r_{m-1}, -1] \leftharpoondown [r_0, r_1, \ldots, r_{m-1}, -2] \leftharpoondown \cdots \leftharpoondown [r_0, r_1, \ldots, r_{m-1}, r_m] \\ &\leftharpoondown [r_0, r_1, \ldots, r_{m-2}, -1] \leftharpoondown [r_0, r_1, \ldots, r_{m-2}, -2] \leftharpoondown \cdots \leftharpoondown [r_0, r_1, \ldots, r_{m-2}, r_{m-1}+2] \\ &\cdots\cdots \\ &\leftharpoondown [r_0, -1] \leftharpoondown [r_0, -2], \ldots, \leftharpoondown [r_0, r_1+2] \\ &(\frac{-s_0 t_0}{s_0+t_0} - 1 - r_0 - k)[r_0+k+1] + (\frac{s_0 t_0}{s_0+t_0} + 2 + r_0 + k)[r_0+k] \leftharpoondown [r_0+k] \cdots \leftharpoondown [r_0+2].\end{aligned}$$

$$\begin{aligned}\gamma_2: &\leftharpoondown [s_0, s_1, \ldots, s_{p-1}, -1] \leftharpoondown [s_0, s_1, \ldots, s_{p-1}, -2] \leftharpoondown \cdots \leftharpoondown [s_0, s_1, \ldots, s_{p-1}, s_p] \\ &\leftharpoondown [s_0, s_1, \ldots, s_{p-2}, -1] \leftharpoondown [s_0, s_1, \ldots, s_{p-2}, -2] \leftharpoondown \cdots \leftharpoondown [s_0, s_1, \ldots, s_{p-2}, s_{p-1}+2] \\ &\cdots\cdots \\ &\frac{s_0}{s_0+t_0}\langle 0\rangle + \frac{t_0}{s_0+t_0}[s_0, -1] \leftharpoondown [s_0, -1] \leftharpoondown [s_0, -2], \ldots, \leftharpoondown [s_0, s_1+2].\end{aligned}$$

$$\begin{aligned}\gamma_3: &\leftharpoondown [t_0, t_1, \ldots, t_{q-1}, -1] \leftharpoondown [t_0, t_1, \ldots, t_{q-1}, -2] \leftharpoondown \cdots \leftharpoondown [t_0, t_1, \ldots, t_{q-1}, t_q] \\ &\leftharpoondown [t_0, t_1, \ldots, t_{q-2}, -1] \leftharpoondown [t_0, t_1, \ldots, t_{q-2}, -2] \leftharpoondown \cdots \leftharpoondown [t_0, t_1, \ldots, s_{q-2}, t_{q-1}+2] \\ &\cdots\cdots \\ &\frac{t_0}{s_0+t_0}\langle 0\rangle + \frac{s_0}{s_0+t_0}[t_0, -1] \leftharpoondown [t_0, -1] \leftharpoondown [t_0, -2], \ldots, \leftharpoondown [t_0, t_1+2].\end{aligned}$$

In the above edgepath system, the length of the partial edges above are calculated via u_0 by [12, Formula (3.1)].

The cycle of r-value of the above edgepath system is $(1, s_0, t_0)$. Any candidate surface associated to this edgepath system is essential by Theorem 4.2(1) or (2).

By Formula (4.1), the twist of an essential surface S_1 associated to the above edgepath system is

$$\begin{aligned}\tau(S_1) &= \sum_{\gamma_i \in \Gamma_{non-const}} \sum_{e_{i,j} \in \gamma_i} -2\sigma(e_{i,j})|e_{i,j}| \\ &= 2[(-r_m - 1) + (-r_{m-1} - 2) + \cdots + (-r_1 - 2)] + 2(k-1) + 2(\frac{-s_0 t_0}{s_0+t_0} - 1 - r_0 - k) \\ &+ 2[(-s_p - 1) + (-s_{p-1} - 2) + \cdots + (-s_1 - 2)] + 2\frac{s_0}{s_0+t_0} \\ &+ 2[(-t_q - 1) + (-t_{q-1} - 2) + \cdots + (-t_1 - 2)] + 2\frac{t_0}{s_0+t_0} \\ &= \frac{2t_0^2}{s_0+t_0} - 2(r_0 + t_0 + 2) + 8 - 4(m+p+q) - 2(\sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k).\end{aligned}$$

So the boundary slope of S_1 is

$$\begin{aligned}bs(S_1) &= \tau(S_1) - \tau(S_0) \\ &= \frac{2t_0^2}{s_0+t_0} - 2(r_0 + t_0 + 2) + 6 - 2(m+p+q) - 2(\sum_{even} r_i + \sum_{even} s_j + \sum_{even} t_k).\end{aligned}$$

By Lemma 4.3 (1), we have

$$
\begin{aligned}
&-\frac{\chi(S_1)}{\sharp S_1} = \sum_{\gamma_i\in\Gamma_{non-const}} |\gamma_i| + N_{const} - N + (N-2-\sum_{\gamma_i\in\Gamma_{const}}\frac{1}{q_i})\frac{1}{1-u_0}\\
&= (-r_m-1)+(-r_{m-1}-2)+\cdots+(-r_1-2)+(k-1)+(\frac{-s_0t_0}{s_0+t_0}-r_0-k-1)\\
&+(-s_p-1)+(-s_{p-1}-2)+\cdots+(-s_1-2)+\frac{s_0}{s_0+t_0}\\
&+(-t_q-1)+(-t_{q-1}-2)+\cdots+(-t_1-2)+\frac{t_0}{s_0+t_0}-3+\frac{s_0t_0+s_0+t_0}{s_0+t_0}\\
&= -r_0-2(m+p+q)-(\sum_{i=1}^{m} r_i+\sum_{j=1}^{p} s_j+\sum_{k=1}^{q} t_k).
\end{aligned}
$$

So far we have proved the case (1) of Theorem 2.5.

(2) When $\Delta \geq 0$, we choose the following candidate edgepath system.

$$
\begin{aligned}
\beta_1 :& \leftharpoondown [r_0,r_1,\ldots,r_{m-1},-1] \leftharpoondown [r_0,r_1,\ldots,r_{m-1},-2] \leftharpoondown \cdots \leftharpoondown [r_0,r_1,\ldots,r_{m-1},r_m]\\
&\leftharpoondown [r_0,r_1,\ldots,r_{m-2},-1] \leftharpoondown [r_0,r_1,\ldots,r_{m-2},-2] \leftharpoondown \cdots \leftharpoondown [r_0,r_1,\ldots,r_{m-2},r_{m-1}+2]\\
&\ldots\ldots\\
&\langle 0\rangle \leftharpoondown [r_0,-1] \leftharpoondown [r_0,-2],\ldots,\leftharpoondown [r_0,r_1+2].
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 :& \leftharpoondown [s_0,s_1,\ldots,s_{p-1},-1] \leftharpoondown [s_0,s_1,\ldots,s_{p-1},-2] \leftharpoondown \cdots \leftharpoondown [s_0,s_1,\ldots,s_{p-1},s_p]\\
&\leftharpoondown [s_0,s_1,\ldots,s_{p-2},-1] \leftharpoondown [s_0,s_1,\ldots,s_{p-2},-2] \leftharpoondown \cdots \leftharpoondown [s_0,s_1,\ldots,s_{p-2},s_{p-1}+2]\\
&\ldots\ldots\\
&\langle 0\rangle \leftharpoondown [s_0,-1] \leftharpoondown [s_0,-2],\ldots,\leftharpoondown [s_0,s_1+2].
\end{aligned}
$$

$$
\begin{aligned}
\beta_3 :& \leftharpoondown [t_0,t_1,\ldots,t_{q-1},-1] \leftharpoondown [t_0,t_1,\ldots,t_{q-1},-2] \leftharpoondown \cdots \leftharpoondown [t_0,t_1,\ldots,t_{q-1},t_q]\\
&\leftharpoondown [t_0,t_1,\ldots,t_{q-2},-1] \leftharpoondown [t_0,t_1,\ldots,t_{q-2},-2] \leftharpoondown \cdots \leftharpoondown [t_0,t_1,\ldots,s_{q-2},t_{q-1}+2]\\
&\ldots\ldots\\
&\langle 0\rangle \leftharpoondown [t_0,-1] \leftharpoondown [t_0,-2],\ldots,\leftharpoondown [t_0,t_1+2].
\end{aligned}
$$

The cycle of r-value of the above edgepath system is $(-r_0-2, s_0, t_0)$. Any candidate surface associated to this edgepath system is essential by Theorem 4.2(1) or (3).

The twist of an essential surface S_2 associated to the above edgepath system is

$$
\begin{aligned}
\tau(S_2) =& 2[(-r_m-1)+(-r_{m-1}-2)+\cdots+(-r_1-2)-1]\\
&+2[(-s_p-1)+(-s_{p-1}-2)+\cdots+(-s_1-2)+1]\\
&+2[(-t_q-1)+(-t_{q-1}-2)+\cdots+(-t_1-2)+1]\\
&= 8-4(m+p+q)-2(\sum_{i=1}^{m} r_i+\sum_{j=1}^{p} s_j+\sum_{k=1}^{q} t_k).
\end{aligned}
$$

The boundary slope of S_2 is

$$bs(S_2) = \tau(S_2) - \tau(S_0) = 6 - 2(m+p+q) - 2(\sum_{even}^{m} r_i + \sum_{even}^{p} s_j + \sum_{even}^{q} t_k).$$

By Lemma 4.3 (2), we have

$$\begin{aligned} -\frac{\chi(S_2)}{\sharp S_2} &= (-r_m - 1) + (-r_{m-1} - 2) + \cdots + (-r_1 - 2) + 1 \\ &+ (-s_p - 1) + (-s_{p-1} - 2) + \cdots + (-s_1 - 2) + 1 \\ &+ (-t_q - 1) + (-t_{q-1} - 2) + \cdots + (-t_1 - 2) + 1 - 2 \\ &= 4 - 2(m+p+q) - (\sum_{i=1}^{m} r_i + \sum_{j=1}^{p} s_j + \sum_{k=1}^{q} t_k). \end{aligned}$$

□

Acknowledgments

The authors would like to thank Ying Zhou for her Matlab program to check some cases of Theorem 2.4. Nathan Dunfield's Python program (available at http://dunfield.info/montesinos) is applied to check Theorem 2.5 for some cases. The first author is deeply indebted to Miaowang Li for her constant encouragement and numerous helpful suggestions.

References

[1] Burde, Gerhard; Zieschang, Heiner. Knots. Second edition. De Gruyter Studies in Mathematics, 5. *Walter de Gruyter & Co., Berlin*, 2003. xii+559 pp. ISBN: 3-11-017005-1. MR1959408, Zbl 1009.57003. 48

[2] Costantino, Francesco. Integrality of Kauffman brackets of trivalent graphs. *Quantum Topol.* **5** (2014), no. 2, 143–184. MR3229040, Zbl 1298.57009, arXiv:0908.0542, doi: 10.4171/QT/49. 52

[3] Futer, David; Kalfagianni, Efstratia; Purcell, Jessica S. Slopes and colored Jones polynomials of adequate knots. *Proc. Amer. Math. Soc.* **139** (2011), no. 5, 1889–1896. MR2763776, Zbl 1232.57007, arXiv:1002.0256, doi: 10.1090/S0002-9939-2010-10617-2. 46, 48

[4] Futer, David; Kalfagianni, Efstratia; Purcell, Jessica S. Guts of surfaces and the colored Jones polynomial. Lecture Notes in Mathematics, 2069. *Springer, Heidelberg*, 2013. x+170 pp. ISBN: 978-3-642-33301-9; 978-3-642-33301-6. MR3024600, Zbl 1270.57002, arXiv:1108.3370, doi: 10.1007/978-3-642-33302-6. 48

[5] Garoufalidis, Stavros. The Jones slopes of a knot. *Quantum Topol.* **2** (2011), no. 1, 43–69. MR2763086, Zbl 1228.57004, arXiv:0911.3627, doi: 10.4171/QT/13. 46, 47

[6] Garoufalidis, Stavros. The degree of a q-holonomic sequence is a quadratic quasi-polynomial. *Electron. J. Comb.* **18** (2011), no. 2, Paper 4, 23 pp. MR2795781, Zbl 1298.05042, arXiv:1005.4580. 47

[7] Garoufalidis, Stavros, Lê, Thang T. Q. The colored Jones function is q-holonomic. *Geom. Topol.* **9** (2005), 1253–1293. MR2174266, Zbl 1078.57012, arXiv:math/0309214, doi: 10.2140/gt.2005.9.1253. 47

[8] Garoufalidis, Stavros; van der Veen, Roland. Quadratic integer programming and the slope conjecture. *New York J. Math.* **22** (2016), 907–932. MR3548130, Zbl 1361.57009, arXiv:1405.5088. 46, 54

[9] Hatcher, Allen; Oertel Ulrich. Boundary slopes for Montesinos knots. *Topology* **28** (1989), no. 4, 453–480. MR1030987, Zbl 0686.57006, doi: 10.1016/0040-9383(89)90005-0. 46, 48, 61, 62, 63, 64

[10] Hatcher, Allen; Thurston, William. Incompressible surfaces in 2-bridge knot complements. *Invent. Math.* **79** (1985), no. 2, 225–246. MR0778125, Zbl 0602.57002, doi: 10.1007/BF01388971. 61

[11] Hirasawa, Mikami; Murasugi, Kunio. Genera and fibredness of Montesinos knots. *Pac. J. Math.* **225** (2006), no. 1, 53–83. MR2233725, Zbl 1124.57002, doi: 10.2140/pjm.2006.225.53. 46

[12] Ichihara, Kazuhiro; Mizushima, Shigeru. Bounds on numerical boundary slopes for Montesinos knots. *Hiroshima Math. J.* **37** (2007), no. 2, 211–252. MR2345368, Zbl 1138.57004, arXiv:math/0503190, doi: 10.32917/hmj/1187916319. 61, 63, 66

[13] Jones, Vaughan F. R. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)* **12** (1985) no. 1, 103–111. MR0766964, Zbl 0564.57006, doi: 10.1090/S0273-0979-1985-15304-2. 46

[14] Kalfagianni, Efstratia; Tran, Anh T. Knot cabling and the degree of the colored Jones polynomial. *New York J. Math.* **21** (2015), 905–941. MR3425627, Zbl 1331.57022, arXiv:1501.01574. 46, 47, 50

[15] Lee, Christine Ruey Shan; van der Veen, Roland. Slopes for pretzel knots. *New York J. Math.* **22** (2016), 1339–1364. MR3576292, Zbl 1379.57017, arXiv:1602.04546. 46, 47, 50, 51, 53, 57

[16] Lickorish, W. B. Raymond. An introduction to knot theory. Graduate Texts in Mathematics, 175. *Springer-Verlag, New York*, 1997. x+201 pp. ISBN: 0-387-98254-X. MR1472978, Zbl 0886.57001, doi: 10.1007/978-1-4612-0691-0. 46

[17] Lickorish, W. B. Raymond.; Thistlethwaite Morwen B. Some links with non-trivial polynomials and their crossing-numbers. *Comment. Math. Helv.* **63** (1988), no. 4, 527–539. MR0966948, Zbl 0686.57002, doi: 10.1007/BF02566777. 48

[18] Masbaum, Gregor; Vogel, Pierre. 3-valent graphs and the Kauffman bracket. *Pacific J. Math.* **164** (1994), no. 2, 361–381. MR1272656, Zbl 0838.57007, doi: 10.2140/pjm.1994.164.361. 46, 50

[19] Motegi, Kimihiko; Takata, Toshie. The slope conjecture for graph knots. *Math. Proc. Camb. Philos. Soc.* **162** (2017), no. 3, 383–392. MR3628196, Zbl 1375.57017, arXiv:1501.01105, doi: 10.1017/S0305004116000566. 46

[20] Reshetikhin, Nikolai; Turaev, Vladimir G. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103** (1991), no. 3, 547–597. MR1091619, Zbl 0725.57007, doi: 10.1007/BF01239527. 46

[21] Thurston, Dylan P. The algebra of knotted trivalent graphs and Turaev's shadow world. *Invariants of knots and 3-manifolds* (Kyoto, 2001), 337–362, Geom. Topol. Monogr., 4. *Geom. Topol. Publ., Coventry*, 2002. MR2048108, Zbl 1163.57301, arXiv:math/0311458, doi: 10.2140/gtm.2002.4.337. 50, 51

[22] van der Veen, Roland. The volume conjecture for augmented knotted trivalent graphs. *Algebr. Geom. Topol.* **9** (2009), no. 2, 691–722. MR2496886, Zbl 1166.57006, arXiv:0805.0094, doi: 10.2140/agt.2009.9.691. 50, 51

[23] Witten, Edward. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.* **121** (1989), no. 3, 351–399. MR0990772, Zbl 0667.57005, doi: 10.1007/BF01217730. 46

(Xudong Leng) School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
xudleng@163.com

(Zhiqing Yang) School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
yangzhq@dlut.edu.cn

(Ximin Liu) School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
ximinliu@dlut.edu.cn

This paper is available via http://nyjm.albany.edu/j/2019/25-2.html.