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Algebras, synchronous games, and chromatic numbers of graphs

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Abstract. We associate to each synchronous game a *-algebra whose representations determine whether the game has a perfect deterministic strategy, perfect quantum strategy or one of several other types of strategies studied in the theory of non-local games. Applying these results to the graph coloring game allows us to develop a correspondence between various chromatic numbers of a graph and the question of whether ideals in a free algebra are proper; this latter question can then be approached via non-commutative Gröbner basis methods. Furthermore, we introduce several new chromatic numbers guided by the algebra. One of these new chromatic numbers, χ_{alg}, is called the algebraic chromatic number, and one of our main results is the algebraic 4-colorability theorem: all graphs G satisfy $\chi_{alg}(G) \leq 4$.

Contents

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1. Introduction

In recent years, the theory of non-local games has received considerable attention, especially due to its connections with theoretical computer science, quantum information theory, and Connes' embedding conjecture. In such a game, two players, Alice and Bob, play cooperatively against a referee. They can agree on a strategy ahead of time, but cannot communicate once the game begins. The criterion by which they win or lose is public information, and is based on responses they give to questions posed by the referee. In this paper, we focus on *synchronous games*, those that satisfy certain symmetry properties for Alice and Bob. A major problem is to compute the maximum probability with which Alice and Bob can win a given game. This of course depends on the type of strategy they employ, the two primary kinds of strategies being *deterministic* and *quantum*; there are many kinds of quantum strategies (q, qa, qc), but all rely on Alice and Bob sharing a so-called entangled state prior to the start of the game which they use to determine responses to the questions posed by the referee, see Section 2 for precise definitions.

Interestingly many graph invariants can be rephrased in terms of synchronous games. For example, given a graph G and positive integer c, there is the so-called c-coloring game (G, c); this game has a perfect deterministic strategy if and only if $\chi(G) \leq c$, and it has a perfect q-strategy if and only if the quantum chromatic number $\chi_q(G) \leq c$. As another example, given two graphs G and H, there is the graph homomorphism game which has a perfect deterministic strategy if and only if there is a graph homomorphism $G \to H$, and has a q-strategy if and only if there is a so-called quantum homomorphism $G \stackrel{q}{\to} H$.

The first aim of our paper is to associate to every synchronous game $\mathcal{G}$ a $*$-algebra $\mathcal{A}(\mathcal{G})$. This algebra $\mathcal{A}(\mathcal{G})$ has the property that its representation theory completely determines whether $\mathcal{G}$ has a perfect deterministic or quantum strategy. Specifically, we prove the following in Theorem 3.2.

Theorem 1.1. *Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game.*

(1) *$\mathcal{G}$ has a perfect deterministic strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathbb{C}$.*
(2) *$\mathcal{G}$ has a perfect q-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(\mathcal{G})$ to the bounded operators $B(\mathcal{H})$ on a finite dimensional Hilbert space $\mathcal{H} \neq 0$.*
(3) *$\mathcal{G}$ has a perfect qc-strategy if and only if there exists a unital C*-algebra $\mathcal{C}$ with a faithful trace and a unital *-homomorphism $\pi : \mathcal{A}(\mathcal{G}) \to \mathcal{C}$.*

Given the fundamental role $\mathcal{A}(\mathcal{G})$ plays in determining the types of strategies $\mathcal{G}$ has, our next aim is to investigate the algebra $\mathcal{A}(\mathcal{G})$ itself. We do so in the specific case of the c-coloring game, introducing a new invariant that we now describe. Fixing a graph G, by considering the c-coloring game as c varies, we obtain a system of $*$-algebras $\mathcal{A}(G, c)$.

Definition 1.2. *The algebraic chromatic number $\chi_{alg}(G)$ of G is the smallest c for which $\mathcal{A}(G, c) \neq 0$.*

It is immediate from this definition that χ_{alg} provides a universal lower bound on all chromatic numbers (χ_q, χ_{qa}, χ_{qc}, χ) of G. Indeed, if any of these latter chromatic numbers equals c, then Theorem 1.1 tells us that $\mathcal{A}(G, c)$ has a non-trivial $*$-representation, hence $\mathcal{A}(G, c) \neq 0$ and so $\chi_{alg}(G) \geq c$. We prove several basic properties of χ_{alg} in Section 4.

The main theorem is the following somewhat surprising result. This appears as Theorem 6.1 in the paper.

Theorem 1.3 (Algebraic 4-colorability theorem)**.** *For every graph G, we have $\chi_{alg}(G) \leq 4$.*

Our proof of this theorem is computer-assisted: in Section 5, we reduce the problem from one concerning $*$-algebras to one concerning $\mathbb{Q}$-algebras defined by generators and relations; in Section 6 we then apply a Gröbner basis approach to prove that the identity is not in the ideal generated by the relations, thereby proving $\mathcal{A}(G, 4) \neq 0$, yielding Theorem 1.3. Despite considerable effort on our part, as well as the part of several others, we are unable to find a simple direct proof that these algebras are non-trivial that does not rely on the Gröbner basis approach, which in turn requires extensive calculations. One reason that other efforts have failed is that although our results show that the $*$-algebras $\mathcal{A}(G, 4)$ are non-trivial, many of them nonetheless have no non-trivial $*$-representations as algebras of operators on a Hilbert space. For example, $\mathcal{A}(K_5, 4)$ is a free $*$-algebra defined by 20 generators and roughly 60 relations. This algebra contains 4 non-zero self-adjoint idempotents that sum to -1, and consequently there can be no non-zero $*$-homomorphism of this algebra into the algebra of operators on a Hilbert space. Nonetheless, via our computational approach, we can prove $\mathcal{A}(K_5, 4) \neq 0$. These properties of $\mathcal{A}(K_5, 4)$ are explained more fully in Remark 6.2.

Finally, we consider variants on our definition of χ_{alg}, introducing two further chromatic numbers: the *hereditary chromatic number* $\chi_{hered}(G)$ built out of systems of hereditary ideals (see Definition 3.6), and the *locally commuting chromatic number* $\chi_{lc}(G)$ which arises by imposing extra commuting conditions in $\mathcal{A}(G,c)$ for adjacent vertices (see Definition 7.1). Currently we know of no graph G where χ_{hered} is not equal to the common value of χ_q, χ_{qa}, and χ_{qc}. We find this particularly exciting since it indicates that the Gröbner basis problem for hereditary ideals could potentially yield an algorithm for computing quantum chromatic numbers or help in determining if χ_q, χ_{qa}, and χ_{qc} coincide.

We study χ_{lc} in considerable detail in Sections 7 and 8. In these sections we also introduce the corresponding *locally commuting clique number* ω_{lc}. We prove in Theorem 7.4 that $\omega = \omega_{lc}$ and in Theorem 8.18 we prove χ_{lc} is not equal to χ, χ_q, or χ_{vect}.

2. Synchronous games and strategies

We lay out some definitions and a few basic properties of games and strategies. We will primarily be concerned with the c-coloring game and the graph homomorphism game.

2.1. Definitions of games and strategies. By a **two-person finite input-output game** we mean a tuple $\mathcal{G} = (I_A, I_B, O_A, O_B, \lambda)$ where I_A, I_B, O_A, O_B are finite sets and

$$\lambda : I_A \times I_B \times O_A \times O_B \to \{0,1\}$$

is a function that represents the rules of the game, sometimes called the predicate. The sets I_A and I_B represent the inputs that Alice and Bob can receive, and the sets O_A and O_B, represent the outputs that Alice and Bob can produce, respectively. A referee selects a pair $(v,w) \in I_A \times I_B$, gives Alice v and Bob w, and they then produce outputs (answers), $a \in O_A$ and $b \in O_B$, respectively. They win the game if $\lambda(v,w,a,b) = 1$ and lose otherwise. Alice and Bob are allowed to know the sets and the function λ and cooperate before the game to produce a strategy for providing outputs, but while producing outputs, Alice and Bob only know their own inputs and are not allowed to know the other person's input. Each time that they are given an input and produce an output is referred to as a **round** of the game.

We call such a game **synchronous** provided that: (i) Alice and Bob have the same input sets and the same output sets, which we denote by I and O, respectively, and (ii) λ satisfies:

$$\forall v \in I, \;\; \lambda(v,v,a,b) = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases},$$

that is, whenever Alice and Bob receive the same inputs then they must produce the same outputs. To simplify notation we write a synchronous game as $\mathcal{G} = (I, O, \lambda)$.

A *graph* G is specified by a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$, satisfying $(v,v) \notin E(G)$ and $(v,w) \in E(G) \implies (w,v) \in E(G)$. The **c-coloring game** for G has inputs $I_A = I_B = V(G)$ and outputs $O_A = O_B = \{1, ..., c\}$ where the outputs are thought of as different colors. They win provided that whenever Alice and Bob receive adjacent vertices, i.e., $(v,w) \in E$, their outputs are different colors and when they receive the same vertex they output the same color. Thus, $(v,w) \in E(G) \implies \lambda(v,w,a,a) = 0, \forall a,\ \lambda(v,v,a,b) = 0, \forall v \in V(G), \forall a \neq b$ and the rule function is equal to 1 for all other tuples. It is easy to see that this is a synchronous game.

Given two graphs G and H, a *graph homomorphism from G to H* is a function $f : V(G) \to V(H)$ with the property that $(v,w) \in E(G) \implies (f(v), f(w)) \in E(H)$. The **graph homomorphism game** from G to H has inputs $I_A = I_B = V(G)$ and outputs $O_A = O_B = V(H)$. They win provided that whenever Alice and Bob receive inputs that are an edge in G, their outputs are an edge in H and that whenever Alice and Bob receive the same vertex in G they produce the same vertex in H. This is also a synchronous game.

A **deterministic strategy** for a game is a pair of functions, $h : I_A \to O_A$ and $k : I_B \to O_B$ such that if Alice and Bob receive inputs (v,w) then they produce outputs $(h(v), k(w))$. A deterministic strategy wins every round of the game if and only if

$$\forall (v,w) \in I_A \times I_B, \lambda(v,w,h(v),k(w)) = 1.$$

Such a strategy is called a **perfect deterministic strategy**.

It is not hard to see that for a synchronous game, any perfect deterministic strategy must satisfy, $h = k$. In particular, a perfect deterministic strategy for the c-coloring game for G is a function $h : V(G) \to \{1, ..., c\}$ such that $(v,w) \in E(G) \implies h(v) \neq h(w)$. Thus, a perfect deterministic strategy is precisely a c-coloring of G. Similarly, a perfect deterministic strategy for the graph homomorphism game is precisely a graph homomorphism.

Finally, it is not difficult to see that if K_c denotes the complete graph on c vertices then a graph homomorphism exists from G to K_c if and only if G has a c-coloring. This is because any time $(v,w) \in E(G)$ then a graph homomorphism must send them to distinct vertices in K_c. Indeed, the rule function for the c-coloring game is exactly the same as the rule function for the graph homomorphism game from G to K_c.

A **random strategy** for a game G is a conditional probability density $p(a,b|v,w)$, which represents the probability that, given inputs $(v,w) \in I_A \times I_B$, Alice and Bob produce outputs $(a,b) \in O_A \times O_B$. Thus, $p(a,b|v,w) \geq 0$ and for each (v,w),

$$\sum_{a \in O_A, b \in O_B} p(a,b|v,w) = 1.$$

In this paper we identify random strategies with their conditional probability density, so that a random strategy will simply be a conditional probability density $p(a,b|v,w)$.

A random strategy is called **perfect** if

$$\lambda(v,w,a,b)=0 \implies p(a,b|v,w)=0,\ \forall (v,w,a,b)\in I_A\times I_B\times O_A\times O_B.$$

Thus, for each round, a perfect strategy gives a winning output with probability 1.

We next discuss **local** random strategies, which are also sometimes called **classical**, meaning not quantum. They are obtained as follows: Alice and Bob share a probability space (Ω, P), for each input $v\in I_A$, Alice has a random variable, $f_v:\Omega\to O_A$ and for each input $w\in I_B$, Bob has a random variable, $g_w:\Omega\to O_B$ such that for each round of the game, Alice and Bob will evaluate their random variables at a point $\omega\in\Omega$ via a formula that has been agreed upon in advance. This yields conditional probabilities,

$$p(a,b|v,w)=P(\{\omega\in\Omega \mid f_v(\omega)=a, g_w(\omega)=b\}).$$

The set of all conditional probability densities $p(a,b|v,w)$ that can be obtained in this fashion is denoted $C_{loc}(n_1,n_2,k_1,k_2)$, where $n_1=|I_A|$ and $n_2=|I_B|$ are the cardinalities of Alice and Bob's input sets, respectively, and $k_1=|O_A|, k_2=|O_B|$ are the respective cardinalities of Alice and Bob's output sets.

A density $p(a,b|v,w)$ will be a perfect strategy for a game $\mathcal{G}$ if and only if

$$\forall (v,w)\in I_A\times I_B,\ P(\{\omega\in\Omega \mid \lambda(v,w,f_v(\omega),g_w(\omega))=0\})=0,$$

or equivalently,

$$\forall (v,w)\in I_A\times I_B,\ P(\{\omega\in\Omega \mid \lambda(v,w,f_v(\omega),g_w(\omega))=1\})=1.$$

If we have a perfect local strategy and set

$$\Omega_1=\cap_{v\in I_A, w\in I_B}\{\omega\in\Omega \mid \lambda(v,w,f_v(\omega),g_w(\omega))=1\},$$

then $P(\Omega_1)=1$ since I_A and I_B are finite sets; in particular, Ω_1 is nonempty. If we choose any $\omega\in\Omega_1$ and set $h(v)=f_v(\omega)$ and $k(w)=g_w(\omega)$, then it is easily checked that this is a perfect deterministic strategy.

Thus, a perfect classical random strategy exists if and only if a perfect deterministic strategy exists. An advantage to using a perfect classical random strategy over a perfect deterministic strategy, is that it is difficult for an observer to construct a deterministic strategy even after observing the outputs of many rounds.

The idea behind **nonlocal games** is to allow other, larger sets of conditional probabilities, namely, those that can be obtained by allowing Alice and Bob to run quantum experiments to obtain their outputs. Currently, there are several competing mathematical models that try to describe the sets of probability densities that can be obtained in this fashion.

We begin with the most frequently used model. We keep the values of n_1, n_2, k_1, k_2 as above. Recall that a *projection valued measure* (*PVM*) is a set $\{P_i\}_{i=1}^k$ of orthogonal projections on some Hilbert space $\mathcal{H}$ with $\sum_{i=1}^k P_i = I$.

An (n_1, n_2, k_1, k_2)-tuple, $\big(p(a,b|v,w)\big)$, $(v,w) \in I_A \times I_B$, $(a,b) \in O_A \times O_B$, is called a *quantum correlation* if there exist two finite dimensional Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$, PVMs $\{P_{v,a}\}_{a=1}^{k_1}$, $v \in I_A$ on $\mathcal{H}_A$ and $\{Q_{w,b}\}_{b=1}^{k_2}$, $w \in I_B$ on $\mathcal{H}_B$, together with a unit vector $h \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$p(a,b|v,w) = \langle (P_{v,a} \otimes Q_{w,b})h, h\rangle.$$

Such a tuple $\big(p(a,b|v,w)\big)$ is called a **quantum correlation** and the set of all such quantum correlations is denoted by $C_q(n_1, n_2, k_1, k_2)$.

Note that $C_q(n_1, n_2, k_1, k_2)$ can be thought of as a subset of $[0,1]^{n_1 n_2 k_1 k_2}$. Its closure in $[0,1]^{n_1 n_2 k_1 k_2}$ is denoted by $C_{qa}(n_1, n_2, k_1, k_2)$, and a tuple $\big(p(a,b|v,w)\big)$ in $C_{qa}(n_1, n_2, k_1, k_2)$ is called a **quantum approximate correlation**.

The next model we discuss is similar to $C_q(n_1, n_2, k_1, k_2)$ but it only considers one Hilbert space $\mathcal{H}$ as opposed to the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Consider a unit vector $h \in \mathcal{H}$, and PVM's $\{P_{v,a}\}_{a=1}^{k_1}$, $v \in I_A$ and $\{Q_{w,b}\}_{b=1}^{k_2}$, $b \in I_B$ on $\mathcal{H}$ such that $P_{v,a}Q_{w,b} = Q_{w,b}P_{v,a}, \forall v, w, a, b$. Then let

$$p(a,b|v,w) = \langle P_{v,a}Q_{w,b}h, h\rangle.$$

A density $\big(p(a,b|v,w)\big)$ arising in this manner is called a **quantum commuting correlation**; the set of such tuples is denoted by $C_{qc}(n_1, n_2, k_1, k_2)$.

Note that in this last model, if we set $h_{v,a} = P_{v,a}h$ and $k_{w,a} = Q_{w,b}h$ then we have two sets of vectors such that $p(a,b|v,w) = \langle h_{v,a}, k_{w,b}\rangle$. This idea is used to define the quantum vector correlations. A tuple $\big(p(a,b|v,w)\big)$ is called a **quantum vector correlation**, if there exists a Hilbert space $\mathcal{H}$ and vectors $\{h_{v,a}\}, v \in I_A, a \in O_A$ and $\{k_{w,b}\}, w \in I_B, b \in O_B$ with $p(a,b|v,w) = \langle h_{v,a}, k_{w,b}\rangle$ such that

- $\langle h_{v,a}, h_{v,c}\rangle = 0, a \neq c$,
- $\langle k_{w,b}, k_{w,c}\rangle = 0, b \neq c$,
- $\sum_{a=1}^{k_1} h_{v,a} = \sum_{b=1}^{k_2} k_{w,b}, \forall v, w$ and this common value is a unit vector,
- $\langle h_{v,a}, k_{w,b}\rangle \geq 0, \forall v, w, a, b$.

The set of all quantum vector correlations is denoted by $C_{vect}(n_1, n_2, k_1, k_2)$. This larger set is often useful for purposes of semidefinite programming.

Finally, the largest natural set of densities are the **non-signalling correlations**. A tuple $\big(p(a,b|v,w)\big)$ is called non-signalling provided:

- $p(a,b|v,w) \geq 0$, $\forall v, w, a, b$,
- $\sum_{a,b} p(a,b|v,w) = 1$, $\forall v, w$,
- $\sum_b p(a,b|v,w) = \sum_b p(a,b|v,w')$, $\forall w, w'$ and this common value is denoted by $p_A(a|v)$,

- $\sum_a p(a,b|v,w) = \sum_a p(a,b|v',w)$, $\forall v, v'$ and this common value is denoted by $p_B(b|w)$.

The set of all such correlations is denoted by $C_{nsb}(n_1,n_2,k_1,k_2)$.

For $t \in \{loc, q, qa, qc, vect, nsb\}$, when the values n_1,n_2,k_1,k_2 are understood, we simply use C_t to denote the corresponding set of conditional probabilities. It is known that

$$C_{loc} \subsetneq C_q \subsetneq C_{qa} \subseteq C_{qc} \subsetneq C_{vect} \subsetneq C_{nsb}.$$

Proofs of these containments can be found in [7], [11], and [27]. The strict containments are intended to indicate that for some values of n_1,n_2,k_1,k_2 these containments are strict. The precise values of n_1,n_2,k_1,k_2 for which these containments are strict is still a subject of ongoing research. The fact that $C_q \subsetneq C_{qa}$ is the most difficult and was only recently shown by [30] for large values of n_1,n_2 and by [6] for smaller values, i.e., $n_1,n_2 \geq 5$. The question of whether or not $C_{qa} = C_{qc}$ for any number of experiments and any number of outputs is known to be equivalent to Connes' embedding conjecture due to results of [25].

We say that $p(a,b|v,w)$ is a **perfect t-strategy** for a game provided that it is a perfect strategy that belonging to the set C_t.

Given a graph G we set $\chi_t(G)$ equal to the least c for which there exists a perfect t-strategy for the c-coloring game for G. The above inclusions imply that

$$\chi(G) = \chi_{loc}(G) \geq \chi_q(G) \geq \chi_{qa}(G) \geq \chi_{qc}(G) \geq \chi_{vect}(G) \geq \chi_{nsb}(G).$$

Currently, it is unknown if there are any graphs that separate $\chi_q(G), \chi_{qa}(G)$ and $\chi_{qc}(G)$ or whether these three parameters are always equal. Examples of graphs are known for which $\chi(G) > \chi_q(G)$, for which $\chi_{qc}(G) > \chi_{vect}(G)$ and for which $\chi_{vect}(G) > \chi_{nsb}(G)$. For details, see [5], [27] and [26]. Other versions of quantum chromatic type graph parameters appear in [1] and a comparison of those parameters with χ_{qa} and χ_{qc} can be found in [1, Section 1].

Similarly, we say that there is a *t-homomorphism* from G to H if and only if there exists a perfect t-strategy for the graph homomorphism game from G to H. It is unknown if q-homomorphisms, qa-homomorphisms and qc-homomorphisms are distinct or coincide.

Finally, we close this section by showing that it is enough to consider so-called symmetric games. Note that in a synchronous game there is no requirement that $\lambda(v,w,a,b) = 0 \implies \lambda(w,v,b,a) = 0$. That is, the rule function does not need to be **symmetric** in this sense. The following shows that it is enough to consider synchronous games with this additional symmetry.

Given $\mathcal{G} = (I, O, \lambda)$ a synchronous game, we define $\lambda_s : I \times I \times O \times O \to \{0,1\}$ by setting $\lambda_s(v,w,a,b) = \lambda(v,w,a,b)\lambda(w,v,b,a)$ and set $\mathcal{G}_s = (I, O, \lambda_s)$. Then it is easily seen that $\mathcal{G}_s$ is a synchronous game with the property that $\lambda_s(v,w,a,b) = 0 \iff \lambda_s(w,v,b,a) = 0$.

2.2. A few properties of strategies. In the remainder of this section, we prove the following slight extension of [27].

Proposition 2.1. *Suppose $\mathcal{G} = (I, O, \lambda)$ is a synchronous game and that $p(a, b|v, w) = \langle h_{v,a}, k_{w,b}\rangle$ is a perfect vect-strategy for $\mathcal{G}$, where the vectors $h_{v,a}$ and $k_{w,b}$ are as in the definition of a vector correlation (see* [24, 6.15]*). Then $h_{v,a} = k_{v,a}, \forall v \in I, a \in O$.*

Proof. By definition, for each $v \in I$ the vectors $\{h_{v,a} : a \in O\}$ are mutually orthogonal and $\{k_{v,a} : a \in O\}$ are mutually orthogonal. So,

$$1 = \sum_{a,b\in O} p(a, b|v, v) = \sum_{a\in O} p(a, a|v, v) = \sum_{a\in O} \langle h_{v,a}, k_{v,a}\rangle \leq \sum_{a\in O} \|h_{v,a}\|\|k_{v,a}\| \leq \big(\sum_{a\in O} \|h_{v,a}\|^2\big)^{1/2}\big(\sum_{a\in O} \|k_{v,a}\|^2\big)^{1/2} = 1.$$

Thus, the inequalities are equalities, which forces $h_{v,a} = k_{v,a}$ for all $v \in I$ and all $a \in O$. □

Corollary 2.2. *Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game and let $t \in \{loc, q, qa, qc, vect\}$. If $p(a, b|v, w)$ is a perfect t-strategy for $\mathcal{G}$, then we have $p(a, b|v, w) = p(b, a|w, v)$ for all $v, w \in I$ and all $a, b \in O$.*

Proof. If $p(a, b|v, w)$ is a perfect t-strategy, then it is a perfect vect-strategy and hence there exist vectors as in the definition such that,

$$p(a, b|v, w) = \langle h_{v,a}, h_{w,b}\rangle = \langle h_{w,b}, h_{v,a}\rangle = p(b, a|w, v),$$

where the middle equality follows since the inner products are assumed to be non-negative. □

This corollary readily yields the following result.

Proposition 2.3. *Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game and let $t \in \{loc, q, qa, qc, vect\}$. Then $p(a, b|v, w)$ is a perfect t-strategy for $\mathcal{G}$ if and only if $p(a, b|v, w)$ is a perfect t-strategy for $\mathcal{G}_s$.*

3. The *-algebra of a synchronous game

We begin by constructing a *-algebra, defined by generators and relations, that is affiliated with a synchronous game. The existence or non-existence of various types of perfect quantum strategies for the game then corresponds to the existence or non-existence of various types of representations of this algebra. This leads us to examine various ideals in the algebra.

3.1. Relations, generators, and the basic *-algebra. Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game and assume that the cardinality of I is $|I| = n$ while the cardinality of O is $|O| = m$. We will often identify I with $\{0, ..., n - 1\}$ and O with $\{0, ..., m - 1\}$. We let $\mathbb{F}(n, m)$ denote the free product of n copies of the cyclic group of order m and let $\mathbb{C}[\mathbb{F}(n, m)]$ denote the complex *-algebra of the group. We regard the group algebra as both a *-algebra, where for each group element g we have $g^* = g^{-1}$, and as an (incomplete) inner product space, with the group elements forming an orthonormal set and the inner product is given by

$$\langle f, h \rangle = \tau(fh^*),$$

where τ is the trace functional.

For each $v \in I$ we have a unitary generator $u_v \in \mathbb{C}[\mathbb{F}(n,m)]$ such that $u_v^m = 1$. If we set $\omega = e^{2\pi i/m}$ then the eigenvalues of each u_v is the set $\{\omega^a : 0 \le a \le m-1\}$. The orthogonal projection onto the eigenspace corresponding to ω^a is given by

$$e_{v,a} = \frac{1}{m} \sum_{k=0}^{m-1} \left(\omega^{-a} u_v\right)^k, \tag{3.1}$$

and these satisfy

$$1 = \sum_{a=0}^{m-1} e_{v,a} \text{ and } u_v = \sum_{a=0}^{m-1} \omega^a e_{v,a}.$$

The set $\{e_{v,a} : v \in I, 0 \le a \le m-1\}$ is another set of generators for $\mathbb{C}[\mathbb{F}(n,m)]$.

We let $\mathcal{I}(\mathcal{G})$ denote the 2-sided *-closed ideal in $\mathbb{C}[\mathbb{F}(n,m)]$ generated by the set

$$\{e_{v,a}e_{w,b} \mid \lambda(v,w,a,b) = 0\}$$

and refer to it as **the ideal of the game** $\mathcal{G}$. We define the ***-algebra of** $\mathcal{G}$ to be the quotient

$$\mathcal{A}(\mathcal{G}) = \mathbb{C}[\mathbb{F}(n,m)]/\mathcal{I}(\mathcal{G}).$$

A familiar case occurs when we are given two graphs G and H and $\mathcal{G}$ is the graph homomorphism game from G to H. Then $\mathcal{A}(\mathcal{G}) = \mathcal{A}(G,H)$, where the algebra on the right hand side is the algebra introduced in [24], so we shall continue that notation in this instance. Recall that $\mathcal{A}(G, K_c)$ is then the algebra of the c-coloring game for G.

Definition 3.1. *We say that a game has a perfect algebraic strategy if $\mathcal{A}(\mathcal{G})$ is nontrivial. Given graphs G and H, we write $G \xrightarrow{alg} H$ if $\mathcal{A}(G,H)$ is nontrivial. We define the algebraic chromatic number of G to be*

$$\chi_{alg}(G) = \min\{c \mid \mathcal{A}(G, K_c) \textit{ is nontrivial}\}$$

The following is a slight generalization of [24, Theorem 4.7].

Theorem 3.2. *Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game.*

(1) *$\mathcal{G}$ has a perfect deterministic strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathbb{C}$.*
(2) *$\mathcal{G}$ has a perfect q-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(\mathcal{G})$ to $B(\mathcal{H})$ for some non-zero finite dimensional Hilbert space.*

(3) *$\mathcal{G}$ has a perfect qc-strategy if and only if there exists a unital C*-algebra $\mathcal{C}$ with a faithful trace and a unital *-homomorphism $\pi : \mathcal{A}(\mathcal{G}) \to \mathcal{C}$.*

Hence, if $\mathcal{G}$ has a perfect qc-strategy, then it has a perfect algebraic strategy and so $\chi_{qc}(G) \geq \chi_{alg}(G)$ for every graph G.

Proof. We start with the third statement. Since the game is synchronous, any perfect strategy $p(a,b|v,w)$ must also be synchronous. By [26, Theorem 5.5], any synchronous density is of the following form: $p(a,b|v,w) = \tau(E_{v,a}E_{w,b})$, where $\tau : \mathcal{C} \to \mathbb{C}$ is a tracial state for a unital C*-algebra $\mathcal{C}$ generated by projections $\{E_{v,a}\}$ satisfying $\sum_a E_{v,a} = I$ for all v.

If we take the GNS representation [12] of $\mathcal{C}$ induced by τ, then the image of $\mathcal{C}$ under this representation will be a quotient of $\mathcal{C}$ with all the same properties and the additional property that τ is a faithful trace on the quotient.

Now if, in addition, $p(a,b|v,w)$ belongs to the smaller family of perfect q-strategies, then by [26, Theorem 5.3] the C*-algebra $\mathcal{C}$ will be finite dimensional. Hence, the second statement follows.

Lastly, if $p(a,b|v,w)$ belongs to the smaller family of perfect loc-strategies, then the C*-algebra $\mathcal{C}$ will be abelian, and hence, the first statement follows. □

Remark 3.3. After this paper was written a characterisation of the existence of perfect qa-strategies was given in [14, Corollary 3.7]: $\mathcal{G}$ has a perfect qa-strategy if and only if there exists a unital *-homomorphism of $\mathcal{A}(\mathcal{G})$ into the von Neumann algebra $\mathcal{R}^\omega$.

3.2. Hereditary chromatic number χ_{hered}. The *-algebra $\mathbb{C}[\mathbb{F}(n,m)]$ also possesses an order defined as follows: let $\mathcal{P}$ be the cone generated by all elements of the form f^*f for $f \in \mathbb{C}[\mathbb{F}(n,m)]$. If $h,k \in \mathbb{C}[\mathbb{F}(n,m)]$ are self-adjoint elements, we write $h \leq k$ if $k - h \in \mathcal{P}$. Next, notice that $\mathcal{P}$ induces a cone on $\mathcal{A}(\mathcal{G})$, which we regard as the positive elements, by setting

$$\mathcal{A}(\mathcal{G})^+ = \{p + \mathcal{I}(\mathcal{G}) : p \in \mathcal{P}\}.$$

Given two self-adjoint elements $h,k \in \mathcal{A}(\mathcal{G})$, we again write $h \leq k$ if and only if $k - h \in \mathcal{A}(\mathcal{G})^+$. In the language of Ozawa [25] this makes $\mathcal{A}(\mathcal{G})$ into a **semi-pre-C*-algebra**.

A self-adjoint vector subspace $V \subseteq \mathbb{C}[\mathbb{F}(n,m)]$ is called **hereditary** provided that $0 \leq f \leq h$ and $h \in V$ implies that $f \in V$.

Problem 3.4. *Let $\mathcal{G}$ be a synchronous game. Find conditions on the game so that the 2-sided ideal $\mathcal{I}(\mathcal{G})$ is hereditary.*

Later we will see an example of a game such that $\mathcal{I}(\mathcal{G})$ is not hereditary. The following result shows why the hereditary condition is important.

Proposition 3.5. *Let $\mathcal{G}$ be a synchronous game and let $\mathcal{I}(\mathcal{G})$ be the ideal of the game. Then $\mathcal{I}(\mathcal{G})$ is a hereditary subspace of $\mathbb{C}[\mathbb{F}(n,m)]$ if and only if $\big(\mathcal{A}(\mathcal{G})^+\big) \cap \big(-\mathcal{A}(\mathcal{G})^+\big) = (0)$.*

Proof. Let $x = x^* \in \mathbb{C}[\mathbb{F}(n,m)]$. We begin by characterizing when the equivalence class $x+\mathcal{I}(\mathcal{G})$ is contained in $\mathcal{A}(\mathcal{G})^+ \cap \big(-\mathcal{A}(\mathcal{G})^+\big)$. By definition, this occurs if and only if there are elements $p = p^*, q = q^*$ in $\mathcal{I}(\mathcal{G})$ such that $x + p \geq 0$ and $-x + q \geq 0$. This is equivalent to $0 \leq x + p \leq p + q$.

Now suppose that $x = x^*$ and that the equivalence class $x + \mathcal{I}(\mathcal{G})$ is non-zero in $\mathcal{A}(\mathcal{G})$. If the class is contained in $\big(\mathcal{A}(\mathcal{G})^+\big) \cap \big(-\mathcal{A}(\mathcal{G})^+\big)$ then choosing p and q as in the previous paragraph, the element $x + p$ demonstrates that $\mathcal{I}(\mathcal{G})$ is not hereditary.

Conversely, if $\mathcal{I}(\mathcal{G})$ is not hereditary, then there exists $x = x^* \notin \mathcal{I}(\mathcal{G})$ and $q \in \mathcal{I}(\mathcal{G})$ such that $0 \leq x \leq q$. The inequality $0 \leq x$ implies that $x + \mathcal{I}(\mathcal{G}) \in \mathcal{A}(\mathcal{G})^+$, while $0 \leq q - x$ implies that $q - x + \mathcal{I}(\mathcal{G}) = -x + \mathcal{I}(\mathcal{G}) \in \mathcal{A}(\mathcal{G})^+$. Clearly, this element is non-zero. □

We let $\mathcal{I}^h(\mathcal{G})$ denote the smallest ideal that contains $\mathcal{I}(\mathcal{G})$ which is a hereditary subspace; we refer to this as the **hereditary closure** of $\mathcal{I}(\mathcal{G})$. We define the **hereditary *-algebra of the game** $\mathcal{G}$ to be the quotient

$$\mathcal{A}^h(\mathcal{G}) = \mathbb{C}[\mathbb{F}(n,m)]/\mathcal{I}^h(\mathcal{G}).$$

Note that $\mathcal{A}^h(\mathcal{G})$ is a quotient of $\mathcal{A}(\mathcal{G})$.

Definition 3.6. *We say that a game has a perfect hereditary strategy if $\mathcal{A}^h(\mathcal{G})$ is nontrivial. Given graphs G and H, we write $G \stackrel{hered}{\longrightarrow} H$ if $\mathcal{A}^h(G,H)$ is nontrivial. We define the hereditary chromatic number of G by*

$$\chi_{hered}(G) = \min\{c \mid \mathcal{A}^h(G, K_c) \text{ is nontrivial}\}.$$

We define the positive cone in $\mathcal{A}^h(\mathcal{G})$ by setting

$$\mathcal{A}^h(\mathcal{G})^+ = \{p + \mathcal{I}^h(\mathcal{G}) : p \in \mathcal{P}\},$$

so that $\mathcal{A}^h(\mathcal{G})$ is also a semi-pre-C^*-algebra. The following is immediate:

Proposition 3.7. *Let $\mathcal{G}$ be a synchronous game, then*

$$\big(\mathcal{A}^h(\mathcal{G})^+\big) \cap \big(-\mathcal{A}^h(\mathcal{G})^+\big) = (0).$$

Thus, the "positive" cone on $\mathcal{A}^h(\mathcal{G})$ is now a **proper** cone and $\mathcal{A}^h(\mathcal{G})$ is an ordered vector space.

Proposition 3.8. *If $K_n \stackrel{hered}{\longrightarrow} K_c$ then $n \leq c$. Consequently, $\chi_{hered}(K_n) = n$.*

Proof. Suppose, that $K_n \stackrel{hered}{\longrightarrow} K_c$. We let $E_{v,i} = e_{v,i} + \mathcal{I}^h(K_n, K_c) \in \mathcal{A}^h(K_n, K_c)$, where $\mathcal{A}^h$ is defined as in §3. Then we have that $\sum_{i=0}^{c-1} E_{v,i} = I$ for all v. Set $P_i = \sum_v E_{v,i}$. Since $E_{v,i}E_{w,i} = 0$, we have that $P_i = P_i^* = P_i^2$. Hence, $Q_i = I - P_i$ is also a "projection" in the sense that $Q_i = Q_i^* = Q_i^2$. Now $\sum_i P_i = cI - \sum_i Q_i$. But also,

$$\sum_i P_i = \sum_v \sum_i E_{v,i} = nI.$$

Hence, $\sum_i Q_i^2 = \sum_i Q_i = (c-n)I$. By definition, $\sum_i Q_i^2 \in \mathcal{A}^h(G,H)^+$. If $c < n$, then $(c-n)I \in -(\mathcal{A}^h(G,H)^+)$ and by Proposition 3.7, we have $I = 0$; that is, $1 \in \mathcal{I}^h(K_n, K_c)$ which contradicts our hypothesis that $K_n \overset{hered}{\longrightarrow} K_c$. Hence, $c \geq n$. This shows that $\chi_{hered}(K_n) \geq n$.

For the other inequality, note that if $K_n \overset{hered}{\longrightarrow} K_c$, then $\chi_{hered}(K_n) \leq c$. By the results of [24], if G is any graph with $c = \chi_{qc}(G)$, then there is a unital *-homomorphism from $\mathcal{A}(G, K_c)$ into a C*-algebra with a trace. The kernel of this homomorphism is a hereditary ideal and so must contain $\mathcal{I}^h(G, K_c)$. Hence, this latter ideal is proper and $c \geq \chi_{hered}(G)$. Thus, $\chi_{hered}(K_n) \leq \chi_{qc}(K_n) \leq \chi(K_n) = n$. Hence, $n = \chi_{hered}(K_n) \leq c$. ☐

The above proof also shows that:

Proposition 3.9. *If $K_n \overset{alg}{\to} K_c$ and $-I \notin \mathcal{A}(K_n, K_c)^+$, then $n \leq c$.*

3.3. The C^*-chromatic number χ_{C^*}. The next natural question is if $\mathcal{A}^h(\mathcal{G})$ is a **pre-C^*-algebra** in the sense of [25]. The answer is that this cone will need to satisfy one more hypothesis.

Definition 3.10. *Let $\mathcal{G}$ be a synchronous game and let $\mathcal{I}^c(\mathcal{G})$ denote the intersection of the kernels of all unital $*$-homomorphisms from $\mathbb{C}[\mathbb{F}(n,m)]$ into the bounded operators on a Hilbert space (possibly 0 dimensional) that vanish on $\mathcal{I}(\mathcal{G})$. Let $\mathcal{A}^c(\mathcal{G}) = \mathbb{C}[\mathbb{F}(n,m)]/\mathcal{I}^c(\mathcal{G})$.*

Proposition 3.11. *Let $\mathcal{G}$ be a synchronous game. Then $\mathcal{I}^h(\mathcal{G}) \subseteq \mathcal{I}^c(\mathcal{G})$ and*

$$\mathcal{I}^c(\mathcal{G}) = \{x \in \mathbb{C}[\mathbb{F}(n,m)] \,:\, x^*x + \mathcal{I}(\mathcal{G}) \leq \epsilon 1 + \mathcal{I}(\mathcal{G}),\, \forall \epsilon > 0,\, \epsilon \in \mathbb{R}\}.$$

*There exists a (non-zero) Hilbert space $\mathcal{H}$ and a unital *-homomorphism $\pi : \mathbb{C}[\mathbb{F}(n,m)] \to B(\mathcal{H})$ that vanishes on $\mathcal{I}(\mathcal{G})$ if and only if $\mathcal{A}^c(\mathcal{G}) \neq (0)$.*

Proof. The kernel of every *-homomorphism is a hereditary ideal and the intersection of hereditary ideals is a hereditary ideal, hence $\mathcal{I}^c(\mathcal{G})$ is a hereditary ideal containing $\mathcal{I}(\mathcal{G})$. So, $\mathcal{I}^h(\mathcal{G}) \subseteq \mathcal{I}^c(\mathcal{G})$.

We have that $x \in \mathcal{I}^c(\mathcal{G})$ if and only if $x + \mathcal{I}(\mathcal{G})$ is in the kernel of every *-homomorphism of $\mathcal{A}(\mathcal{G})$ into the bounded operators on a Hilbert space. In [25, Theorem 1] it is shown that this is equivalent to $x + \mathcal{I}(\mathcal{G})$ being in the "ideal of infinitesimal elements" of $\mathcal{A}(\mathcal{G})$, which is the ideal defined by the right-hand side of the above formula. The last result comes from the fact that the ideal of infinitesimal elements is exactly the intersection of the kernels of all such representations. ☐

Definition 3.12. *We say that a game $\mathcal{G}$ has a* **perfect C*-strategy** *provided that $\mathcal{A}^c(\mathcal{G})$ is nontrivial. We write $G \overset{C^*}{\longrightarrow} H$ provided that for given graphs G and H, the algebra $\mathcal{A}^c(G,H)$ is nontrivial. We define the C^*-chromatic number of G to be*

$$\chi_{C^*}(G) = \min\{c \mid \mathcal{A}^c(G, K_c) \text{ is nontrivial}\}.$$

It is easily seen that our definition of $G \xrightarrow{C^*} H$ is equivalent to the definition given in [24], since they only consider the case of non-trivial C*-algebras.

The following is immediate.

Proposition 3.13. *If G is a graph, then $\chi_{qc}(G) \geq \chi_{C^*}(G) \geq \chi_{hered}(G) \geq \chi_{alg}(G)$.*

This motivates the following question.

Problem 3.14. *Let $\mathcal{G}$ be a synchronous game. Is $\mathcal{I}^c(\mathcal{G}) = \mathcal{I}^h(\mathcal{G})$?*

Problem 3.15. *If $\mathcal{I}^h(\mathcal{G}) \neq \mathbb{C}[\mathbb{F}(n,m)]$, then does there exist a non-zero Hilbert space $\mathcal{H}$ and a unital *-homomorphism, $\pi : \mathbb{C}[\mathbb{F}(n,m)] \to B(\mathcal{H})$ such that $\pi(\mathcal{I}^h(\mathcal{G})) = (0)$, that is, if $\mathcal{I}^h(\mathcal{G}) \neq \mathbb{C}[\mathbb{F}(n,m)]$, then is $\mathcal{I}^c(\mathcal{G}) \neq \mathbb{C}[\mathbb{F}(n,m)]$?*

3.4. Determining if an ideal is hereditary. Here we mention some literature on determining if an ideal $\mathcal{I}$ is hereditary and the issue of computing its "hereditary closure." In the real algebraic geometry literature, a hereditary ideal is called a real ideal. For a finitely generated left ideal $\mathcal{I}$ in $\mathbb{R}(\mathbb{F}(k))$ the papers [3, 2] present a theory and a numerical algorithm to test (up to numerical error) if $\mathcal{I}$ is hereditary. The algorithm also computes the "hereditary radical" of $\mathcal{I}$. The computer algorithm relies on numerical optimization (semidefinite programming) and hence it is not exact but approximate.

For two sided ideals [4] and [15] contain some theory. Also the first author and Klep developed and crudely implemented a hereditary ideal algorithm under NCAlgebra. However, it is too memory consuming to be effective, so we leave this topic for future work.

A moral one can draw from this literature is that computing hereditary closures is not broadly effective at this moment.

3.5. Clique numbers. The clique number of a graph $\omega(G)$ is defined as the size of the largest complete subgraph of G. It is not hard to see that G contains a complete subgraph of size c if and only if there is a graph homomorphism from K_c to G. Hence, there is a parallel theory of quantum clique numbers that we shall not pursue here, other than to remark that for each of the cases $t \in \{loc, q, qa, qc, C^*, hered, alg\}$ we define the **t-clique number** of G by

$$\omega_t(G) = \max\{c \mid K_c \xrightarrow{t} G\},$$

so that

$$\omega(G) = \omega_{loc}(G) \leq \omega_q(G) \leq \omega_{qa}(G) \\ \leq \omega_{qc}(G) \leq \omega_{C^*}(G) \leq \omega_{hered}(G) \leq \omega_{alg}(G).$$

Lovasz [17] introduced his theta function $\vartheta(G)$ of a graph. The famous Lovasz sandwich theorem [9] says that for every graph G, if $\overline{G}$ denotes its graph complement, then $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$. In [24, Proposition 4.2] they showed the following improvement of the Lovasz sandwich theorem:

$$\omega_{C^*}(G) \leq \vartheta(\overline{G}) \leq \chi_{C^*}(G).$$

We shall show later that $\chi_{alg}(K_5) = 4$ while $\vartheta(\overline{K_5}) = 5$. Hence the sandwich inequality fails for the algebraic version.

This motivates the following problem:

Problem 3.16. *Is $\omega_{hered}(G) \leq \vartheta(\overline{G}) \leq \chi_{hered}(G)$ for all graphs?*

4. The case of 1, 2 and 3 colors

It is a classic result that deciding if $\chi(G) \leq 3$ is an NP-complete problem. In [10] it was shown that deciding if $\chi_q(G) \leq 3$ is NP-hard, and, in particular, there is no known algorithm for deciding if this latter inequality is true. For these reasons it is interesting to see what can be said about the new inequalities, $\chi_{C^*}(G) \leq 3$, $\chi_{hered}(G) \leq 3$, and $\chi_{alg}(G) \leq 3$. Addressing the first two inequalities would require one to compute $\mathcal{I}^c(G, K_3)$ and $\mathcal{I}^h(G, K_3)$, and unfortunately these ideals contain elements not just determined by simple algebraic relations. However, studying $\mathcal{A}(G, K_2)$ and $\mathcal{A}(G, K_3)$ is rewarding, as we shall now see. Throughout the section, we use the notation $E_{v,i} = e_{v,i} + \mathcal{I}(K_n, K_c)$.

Proposition 4.1. *Let G be a graph. Then $\chi_{alg}(G) = 1$ if and only if G is an empty graph. Hence, $\chi_{alg}(G) = 1 \iff \chi(G) = 1$.*

Proof. For each vertex we only have one idempotent $E_{v,1}$ and since these sum to the identity, necessarily $E_{v,1} = I$. But if there is an edge (v, w) then $I = I \cdot I = E_{v,1}E_{w,1} = 0$. □

Proposition 4.2. *If G is a graph, then $\chi_{alg}(G) = 2 \iff \chi(G) = 2$.*

Proof. If $\chi(G) = 2$, then $1 \leq \chi_{alg}(G) \leq \chi(G) = 2$. Proposition 4.1 shows that $\chi_{alg}(G) \neq 1$, so $\chi_{alg}(G) = 2$.

Now suppose $\chi_{alg}(G) = 2$. Again by Proposition 4.1, we know $\chi(G) > 1$. Fix a vertex v and set $P_0 = E_{v,0}$ and $P_1 = E_{v,1}$. Note $P_0 + P_1 = I$. Let $(v, w) \in E(G)$, then $P_0E_{w,0} = 0$ and $P_1E_{w,1} = 0$. Hence, $E_{w,0} = (P_0 + P_1)E_{w,0} = P_1E_{w,0}$ and similarly, $E_{w,0} = E_{w,0}P_1$. Also, $P_1 = P_1(E_{w,0} + E_{w,1}) = P_1E_{w,0} = E_{w,0}$.

Thus, whenever $(v, w) \in E(G)$, then $E_{v,i} = E_{w,i+1}$, i.e., there are two projections and they flip. By using a path from v to an arbitrary w in the connected component of v, we see that $\{E_{w,0}, E_{w,1}\} = \{P_0, P_1\}$.

Now we wish to 2-color the connected component of v. Define the color of any such vertex w to be 0 if $E_{w,0} = P_0$ and 1 if $E_{w,0} = P_1$. This yields a 2-coloring. Applying this argument on each connected component of G, we have shown $\chi(G) \leq 2$. Since we already know $\chi(G) > 1$, we have shown the result. □

Proposition 4.3. *If* $(v,w) \in E(G)$ *then* $E_{v,i}E_{w,j} = E_{w,j}E_{v,i} \in \mathcal{A}(G, K_3)$ *for all* i, j. *In particular, if* G *is complete, then* $\mathcal{A}(G, K_3)$ *is abelian.*

Proof. For $0 = E_{v,0}E_{v,1} = E_{v,0}(E_{w,0} + E_{w,1} + E_{w,2})E_{v,1} = E_{v,0}E_{w,2}E_{v,1}$. Similarly, $E_{v,i}E_{w,j}E_{v,k} = 0$ whenever $\{i, j, k\} = \{0, 1, 2\}$.

Now $E_{w,0} = (E_{v,0} + E_{v,1} + E_{v,2})E_{w,0}(E_{v,0} + E_{v,1} + E_{v,2}) = E_{v,1}E_{w,0}E_{v,1} + E_{v,2}E_{w,0}E_{v,2}$. Similarly, $E_{w,j} = E_{v,j+1}E_{w,j}E_{v,j+1} + E_{v,j+2}E_{w,j}E_{v,j+2}$.

Hence, for $i \neq j$, $E_{v,i}E_{w,j} = E_{v,i}E_{w,j}E_{v,i} = E_{w,j}E_{v,i}$, while when $i = j$, $E_{v,i}E_{w,i} = 0 = E_{w,i}E_{v,i}$. □

In the case that the algebra $\mathcal{A}(G, K_3)$ can be represented as operators on a Hilbert space, the above result was already shown in [10, Lemma 2].

Theorem 4.4. $\chi_{alg}(K_j) = j$ *for* $1 \leq j \leq 4$.

Proof. We know $\chi_{alg}(K_1) = 1$ and $\chi_{alg}(K_2) = 2$ by Propositions 4.1 and 4.2, respectively. Now if $\chi_{alg}(K_3) = 2$ then by Proposition 4.2, we see $\chi(K_3) = 2$, which is a contradiction. Hence, $3 \leq \chi_{alg}(K_3) \leq \chi(K_3) = 3$.

Finally, if $\chi_{alg}(K_4) = 3$, then by Proposition 4.3, we have that $\mathcal{A}(K_4, K_3)$ is a non-zero abelian complex *-algebra. But every unital, abelian ring contains a proper maximal ideal M, and forming the quotient we obtain a field $\mathbb{F}$. The map $\lambda 1 \to \lambda 1 + M$ embeds $\mathbb{C}$ as a subfield. Now we use the fact that $\mathcal{A}(K_4, K_3)$ is generated by projections and that the image of each projection in $\mathbb{F}$ is either 0 or 1 in order to see that the range of the quotient map is just $\mathbb{C}$. Thus, $\mathbb{F} = \mathbb{C}$ and we have a unital homomorphism $\pi : \mathcal{A}(K_4, K_3) \to \mathbb{C}$ and again using the fact that the image of each projection is 0 or 1 and that the projections commute, we see that π is a *-homomorphism. Hence, by [24, Theorem 4.12], we have $\chi(K_4) \leq 3$, a contradiction.

Thus, $3 < \chi_{alg}(K_4) \leq \chi(K_4) = 4$ and the result follows. □

Corollary 4.5. $I \in \mathcal{I}(K_4, K_3)$.

5. *-Algebra versus free algebra

The original motivation for the construction of the algebra of a game comes from projective quantum measurement systems which are always given by orthogonal projections on a Hilbert space, i.e., operators satisfying $E = E^2 = E^*$. This is why we have defined the algebra of a game to be a *-algebra. But a natural question is whether or not one really needs a *-algebra or is there simply a free algebra with relations that suffices? In this section we show that as long as one introduces the correct relations then the assumption that the algebra be a *-algebra is not necessary.

To this end let $\mathcal{F}(nm) := \mathbb{C}\langle x_{v,a} \mid 0 \leq v \leq n-1, 0 \leq a \leq m-1\rangle$ be the free unital complex algebra on nm generators and let $\mathcal{B}(n,m) = \mathcal{F}(nm)/\mathcal{I}(n,m)$ where $\mathcal{I}(n,m)$ is the two-sided ideal generated by

$$x_{v,a}^2 - x_{v,a}, \forall v, a; \quad 1 - \sum_{a=0}^{m-1} x_{v,a}, \forall v; \quad x_{v,a}x_{v,b}, \forall v, \forall a \neq b.$$

We let $p_{v,a}$ denote the coset of $x_{v,a}$ in the quotient so that

$$p_{v,a}^2 = p_{v,a}, \forall v, a; \quad 1 = \sum_{a=0}^{m-1} p_{v,a}, \forall v; \quad p_{v,a}p_{v,b} = 0, \forall v, \forall a \neq b.$$

Proposition 5.1. *There is an isomorphism* $\pi : \mathcal{B}(n,m) \to \mathbb{C}[\mathbb{F}(n,m)]$ *with* $\pi(p_{v,a}) = e_{v,a}, \forall v, a$, *where* $e_{v,a}$ *are defined as in the previous section.*

Proof. Let $\rho : \mathcal{F}(nm) \to \mathbb{C}[\mathbb{F}(n,m)]$ be the unital algebra homomorphism with $\rho(x_{v,a}) = e_{v,a}$. Then ρ vanishes on $\mathcal{I}(n,m)$ and so induces a quotient homomorphism $\pi : \mathcal{B}(n,m) \to \mathbb{C}[\mathbb{F}(n,m)]$. It remains to show that π is one-to-one.

To this end set $\omega = e^{2\pi i/m}$ and let $y_v = \sum_{a=0}^{m-1} \omega^a p_{v,a}$. It is readily checked that $y_v^m = \sum_{a=0}^{m-1} \omega^{am} p_{v,a} = 1$. Since $p_{v,a} = \frac{1}{m} \sum_{k=0}^{m-1} (\omega^{-a} y_v)^k$ we have that $\{y_v : 0 \leq v \leq n-1\}$ generates $\mathcal{B}(n,m)$.

Now by the universal property of $\mathbb{C}[\mathbb{F}(n,m)]$ there is a homomorphism $\gamma : \mathbb{C}[\mathbb{F}(n,m)] \to \mathcal{B}(n,m)$ with $\gamma(u_v) = y_v$ and hence, this is the inverse of π. ☐

Corollary 5.2. *Let* $\mathcal{G} = (I, O, \lambda)$ *be a symmetric synchronous game with* $|I| = n$ *and* $|O| = m$. *Then* $\mathcal{A}(\mathcal{G})$ *is isomorphic to the quotient of* $\mathcal{F}(nm)$ *by the 2-sided ideal generated by*

$$x_{v,a}^2 - x_{v,a}, \forall v, a; \quad 1 - \sum_{a=0}^{m-1} x_{v,a}, \forall v$$

and

$$x_{v,a}x_{w,b}, \forall v, w, a, b \text{ such that } \lambda(v,w,a,b) = 0.$$

Proof. Recall that $\mathcal{A}(\mathcal{G}) = \mathbb{C}[\mathbb{F}(n,m)]/\mathcal{I}(\mathcal{G})$, where $\mathcal{I}(\mathcal{G})$ is the smallest *-closed ideal containing the set $S = \{e_{v,a}e_{w,b} \mid \lambda(v,w,a,b) = 0\}$. In our case, $\mathcal{G}$ is symmetric and so the set S is already closed under the * operation. As a result, $\mathcal{I}(\mathcal{G})$ is the ideal generated by the set S.

In Proposition 5.1, we showed that the quotient map $\rho : \mathcal{F}(nm) \to \mathbb{C}[\mathbb{F}(n,m)]$ given by $\rho(x_{v,a}) = e_{v,a}$, induces an isomorphism $\mathcal{B}(n,m) := \mathcal{F}(nm)/\mathcal{I}(n,m) \simeq \mathbb{C}[\mathbb{F}(n,m)]$. Thus, $\mathcal{A}(\mathcal{G})$ is the quotient of $\mathcal{F}(nm)$ by $\mathcal{I}(n,m)$ and the 2-sided ideal generated by $\rho^{-1}(S)$; this is precisely the 2-sided ideal given in the statement of the corollary. ☐

Corollary 5.3. *A symmetric synchronous game* $\mathcal{G} = (I, O, \lambda)$ *has a perfect algebraic strategy if and only if 1 is not in the 2-sided ideal of* $\mathcal{F}(nm)$ *generated by*

$$x_{v,a}^2 - x_{v,a}, \forall v, a; \quad 1 - \sum_{a=0}^{m-1} x_{v,a}, \forall v,$$

and

$$x_{v,a}x_{w,b}, \forall v, w, a, b \text{ such that } \lambda(v,w,a,b) = 0.$$

Before moving on to discuss base change in §5.1, we remark that when $m = 3$ the condition $x_{v,a}x_{v,b} = 0, \forall a \neq b$ is a consequence of the other hypotheses and is not needed in the definition of the ideal $\mathcal{I}(n, m)$. To see this, suppose a unital algebra contains 3 idempotents, p_1, p_2, p_3 with $p_1 + p_2 + p_3 = 1$. Then $p_1 + p_2 = 1 - p_3$ is idempotent, and squaring yields $p_1p_2 + p_2p_1 = 0$. Thus, $0 = p_1(p_1p_2 + p_2p_1) = p_1p_2 + p_1p_2p_1$ and $0 = (p_1p_2 + p_2p_1)p_1 = p_1p_2p_1 + p_2p_1$, from which it follows that $p_1p_2 = p_2p_1$ and so $2p_1p_2 = 0$. Similarly, we see $p_ip_j = 0$ for $i \neq j$.

For $m \geq 4$ however, it is necessary to include the relation $x_{v,a}x_{v,b}$ in the ideal in order to guarantee that the quotient $\mathcal{B}(n, m)$ is isomorphic to $\mathbb{C}[\mathbb{F}(n, m)]$. This is because in a complex algebra it is possible to have 4 idempotents that sum to the identity but whose products are not 0. We thank Heydar Radjavi for pointing this out to us.

If a set of self-adjoint projections on a Hilbert space, $P_1, ..., P_m$ sum to the identity then it is easily checked that they project onto orthogonal subspaces and so $P_iP_j = 0, \forall i \neq j$. Thus, in any C*-algebra when self-adjoint idempotents sum to the identity, their pairwise products are 0. But the situation is not so clear for self-adjoint idempotents in a *-algebra and we have not been able to resolve this question. So we ask:

Problem 5.4. *Let $\mathcal{A}$ be a unital *-algebra and let $p_1, ..., p_m$ satisfy $p_i = p_i^2 = p_i^*$ and $p_1 + \cdots + p_m = 1$. Then does it follow that $p_ip_j = 0, \forall i \neq j$?*

5.1. Change of field. The following is important for Gröbner basis calculations. Let $1 \in \mathbb{K} \subseteq \mathbb{C}$ be any subfield. We set $\mathcal{F}_{\mathbb{K}}(nm)$ equal to the free $\mathbb{K}$-algebra on nm generators $x_{v,i}$, so that $\mathcal{F}_{\mathbb{C}}(nm) = \mathcal{F}(nm)$. Given any symmetric synchronous game $\mathcal{G}$ with n inputs and m outputs, we let $\mathcal{I}_{\mathbb{K}}(\mathcal{G}) \subseteq \mathcal{F}_{\mathbb{K}}(nm)$ be the 2-sided ideal generated by

$$x_{v,a}^2 - x_{v,a}, \forall v, a; \quad 1 - \sum_{a=0}^{m-1} x_{v,a}, \forall v,$$

and

$$x_{v,a}x_{w,b}, \forall v, w, a, b \text{ such that } \lambda(v, w, a, b) = 0.$$

We let $\mathcal{A}_{\mathbb{K}}(\mathcal{G}) = \mathcal{F}_{\mathbb{K}}(nm)/\mathcal{I}_{\mathbb{K}}(\mathcal{G})$. By Corollary 5.3, $\mathcal{G}$ has a perfect algebraic strategy if and only if $1 \notin \mathcal{I}_{\mathbb{C}}(\mathcal{G})$, or equivalently $\mathcal{A}_{\mathbb{C}}(\mathcal{G}) \neq 0$.

We show that this computation is independent of the field $\mathbb{K}$.

Proposition 5.5. *If $\mathbb{K}$ is a field containing $\mathbb{Q}$, then*

$$\mathcal{A}_{\mathbb{K}}(\mathcal{G}) = \mathcal{A}_{\mathbb{Q}}(\mathcal{G}) \otimes_{\mathbb{Q}} \mathbb{K}.$$

Furthermore, $\mathcal{A}_{\mathbb{K}}(\mathcal{G}) = 0$ if and only if $\mathcal{A}_{\mathbb{Q}}(\mathcal{G}) = 0$.

Proof. By definition, we have a short exact sequence

$$0 \to \mathcal{I}_{\mathbb{Q}} \to \mathcal{F}_{\mathbb{Q}} \to \mathcal{A}_{\mathbb{Q}} \to 0$$

of $\mathbb{Q}$-vector spaces. Since $\mathbb{K}$ is flat over $\mathbb{Q}$, we obtain a short exact sequence

$$0 \to \mathcal{I}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K} \to \mathcal{F}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K} \to \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K} \to 0.$$

Since the generators are independent of the field, one checks that $\mathcal{F}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K} = \mathcal{F}_{\mathbb{K}}$ and that the image of $\mathcal{I}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K} \to \mathcal{F}_{\mathbb{K}}$ is equal to $\mathcal{I}_{\mathbb{K}}$. Since this latter map is injective, we see $\mathcal{I}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K} = \mathcal{I}_{\mathbb{K}}$. Hence, the above short exact sequence shows $\mathcal{A}_{\mathbb{K}}(\mathcal{G}) = \mathcal{A}_{\mathbb{Q}}(\mathcal{G}) \otimes_{\mathbb{Q}} \mathbb{K}$.

Lastly, $\mathbb{K}/\mathbb{Q}$ is faithfully flat. So, $\mathcal{A}_{\mathbb{K}}(\mathcal{G}) = 0$ if and only if $\mathcal{A}_{\mathbb{Q}}(\mathcal{G}) = 0$. □

6. Algebraic 4-colorability theorem

This section gives a machine-assisted proof which analyzes 4 algebraic colors. We prove:

Theorem 6.1. *For any graph G, we have $\chi_{alg}(G) \leq 4$.*

Remark 6.2. As mentioned in the Introduction, our proof of Theorem 6.1 is computer-assisted, using a Gröbner basis approach to show $\mathcal{A}(G, 4) \neq 0$. One may be tempted to find a simple direct proof of this fact without the use of computational techniques; we have spent considerable effort to find such a proof, but an essential difficulty in such attempts is that many of the $*$-algebras $\mathcal{A}(G, 4)$ have no non-trivial $*$-representations as algebras of operators on a Hilbert space. For example, consider the $*$-algebra $\mathcal{A}(K_5, 4)$. If we let $e_{v,i} = x_{v,i} + \mathcal{I}(K_5, K_4), 1 \leq v \leq 5, 1 \leq i \leq 4$ denote the canonical self-adjoint idempotent generators of this algebra, then we have that $e_{v,i}e_{v,j} = 0, \forall i \neq j$ and that $e_{v,i}e_{w,i} = 0, \forall v \neq w$. Arguing as in the proof of Proposition 3.8, if we set $p_i = \sum_v e_{v,i}, 1 \leq i \leq 4$, then $p_i = p_i^* = p_i^2, 1 \leq i \leq 4$. Hence, $q_i = 1 - p_i$ is also a set of self-adjoint idempotents. Finally,

$$\sum_{i=1}^{4} q_i = 4 \cdot 1 - \sum_{i=1}^{4} p_i = 4 \cdot 1 - \sum_{v=1}^{5} \sum_{i=1}^{4} e_{v,i} = 4 \cdot 1 - 5 \cdot 1 = -1,$$

so that the algebra $\mathcal{A}(K_5, 4)$ contains a set of 4 self-adjoint idempotents that sum to -1. From this it follows that any $*$-homomorphism from $\mathcal{A}(K_5, 4)$ into the algebra of operators on a Hilbert space must send each q_i to 0, and hence map -1 to 0 as well.

This proves the algebra $\mathcal{A}(K_5, 4)$ has no non-trivial $*$-representation on a Hilbert space. So to prove that the algebra is non-trivial, we need a proof that avoids representations.

Theorem 6.1 is equivalent to the statement that for any graph G, we have $1 \notin \mathcal{I}(G, K_4)$. A natural, albeit computational, way to prove such a statement is through the use of (noncommutative) Gröbner bases. For a brief effective exposition to noncommutative Gröbner basis algorithms, see Chapter 12.3 [8] or [21, 28, 16].

For those readers already familiar with (commutative) Gröbner bases, we explain the key differences with the noncommutative setting. Let $\mathcal{I} = (p_1, \dots p_k)$ be a two-sided ideal, and prescribe a monomial order. A noncommutative Gröbner basis $\mathcal{B}$ of $\mathcal{I}$ is a set of generators such that the leading term of any element of $\mathcal{I}$ is in the monomial ideal generated by the leading terms of $\mathcal{B}$. A noncommutative Gröbner basis is produced in the same way as in the commutative case. Let m_j be the leading term of p_j and notice that any two m_j, m_k have as many as 4 possible least common multiples, each of which produces syzygyies. One repeatedly produces syzygyies and reduces to obtain a Gröbner basis in the same way as the commutative setting. However, unlike the commutative case, a Gröbner basis can be infinite. Very fortunately the Gröbner bases that arise in our coloring computations below are finite. The key property we use is that p is in $\mathcal{I}$ if and only if the reduction of p by a Gröbner basis for $\mathcal{I}$ yields 0.

Recall that $\mathcal{I}(G, K_4)$ is generated by the following relations:

$$x_{v,i}x_{v,j} \ \ \forall i \neq j; \ \ 1 - \sum_{i=0}^{3} x_{v,i} \ \ \forall v; \ \ x_{v,i}x_{w,i} \ \ \forall (v,w) \in E(G), \forall i.$$

To prove Theorem 6.1 we will make use of the following theorem:

Theorem 6.3. *For any $n \geq 3$ a Gröbner basis for $\mathcal{I}(K_n, K_4)$ under the graded lexographic ordering with*

$$x_{0,0} < x_{0,1} < x_{0,2} < x_{0,3} < x_{1,0} < x_{1,1} < \ldots < x_{n-1,3} \tag{6.1}$$

consists of relations of the following forms:

(1)
$$x_{v,i}x_{v,j}$$
with $i, j \leq 2$ $i \neq j$

(2)
$$x_{v,i}^2 - x_{v,i}$$
with $i \leq 2$

(3)
$$x_{v,3} + x_{v,2} + x_{v,1} + x_{v,0} - 1$$

(4)
$$x_{v,i}x_{w,i}$$
with $v \neq w$, $i \leq 2$

(5)
$$x_{v,2}x_{w,1} + x_{v,2}x_{w,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0} + x_{v,0}x_{w,2} + x_{v,0}x_{w,1} \\ -x_{v,2} - x_{v,1} - x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} + 1$$
with $v \neq w$

(6)
$$x_{v,2}x_{w,0}x_{v,1} - x_{v,1}x_{w,2}x_{v,0} - x_{v,1}x_{w,0}x_{v,2} - x_{v,0}x_{w,2}x_{v,0} \\ -x_{v,0}x_{w,1}x_{v,2} - x_{v,0}x_{w,1}x_{v,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0} \\ +x_{v,0}x_{w,2} + x_{v,0}x_{w,1} + x_{w,2}x_{v,0} + x_{w,1}x_{v,2} + x_{w,1}x_{v,0} \\ +x_{w,0}x_{v,2} - x_{v,2} - x_{v,1} - x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} + 1$$
with $v \neq w$

(7)

$$x_{v,2}x_{w,0}x_{u,1} - x_{v,1}x_{w,2}x_{u,0} - x_{v,1}x_{w,0}x_{u,2} - x_{v,0}x_{w,2}x_{u,0} - x_{v,0}x_{w,1}x_{u,2}$$
$$-x_{v,0}x_{w,1}x_{u,0} + x_{v,2}x_{u,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0}$$
$$+2x_{v,1}x_{u,2} + 2x_{v,1}x_{u,0} + x_{v,0}x_{w,2} + x_{v,0}x_{w,1} + 2x_{v,0}x_{u,2}$$
$$+x_{v,0}x_{u,1} + x_{w,2}x_{u,0} + x_{w,1}x_{u,2} + x_{w,1}x_{u,0} + x_{w,0}x_{u,2}$$
$$-x_{v,2} - 2x_{v,1} - 2x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} - 2x_{u,2} - x_{u,1} - 2x_{u,0} + 2$$

with $v \neq w \neq u \neq v$

Specifically $\mathcal{I}(K_n, K_4)$ *has a Gröbner basis that does not contain 1 and thus* $1 \notin \mathcal{I}(K_n, K_4)$.

Remark 6.4. Each of the relations (1)–(7) correspond to a set of relations obtained by taking all choices of v, w, u in $V(K_n)$. However because of the monomial ordering chosen, the leading terms are always of the forms:

(1) $x_{v,i}x_{v,j}$ (2) $x_{v,i}^2$ (3) $x_{v,3}$ (4) $x_{v,i}x_{w,i}$
(5) $x_{v,2}x_{w,1}$ (6) $x_{v,2}x_{w,0}x_{v,1}$ (7) $x_{v,2}x_{w,0}x_{u,1}$

Additionally for every $1 \leq i \leq 7$, all the vertices of K_n which appear in the terms of relation (i) also appear in the leading term of (i).

Proof. Let the ideal generated by these relations be denoted by $\mathcal{J}$, we will first show that these relations form a Gröbner basis for $\mathcal{J}$, and then show that $\mathcal{J} = \mathcal{I}(K_n, K_4)$.

Before we begin our calculations pertaining to an algebra over $\mathbb{C}$ we note that all of the coefficients that appear will be in $\mathbb{Q}$. Section 5.1 bears on this.

To see that these relations form a Gröbner basis we must show that the syzygy between any two polynomials in this list is zero when reduced by the list. First by Remark 6.4 each of the relations has variables corresponding to at most three different vertices of K_n and reducing by a relation will not introduce variables corresponding to different vertices. Thus when calculating and reducing the syzygy between any two relations, variables corresponding to at most 6 vertices of K_n will be involved. Therefore we can verify that all syzygies reduce to zero by looking at the case $n = 6$ which we verify using NCAlgebra 5.0 and NCGB running under Mathematica (see the file QCGB-9-20-16.nb, available at: https://github.com/NCAlgebra/UserNotebooks). This proves that the relations (1) – (7) form a Gröbner basis.

We now show that $\mathcal{J} = \mathcal{I}(K_n, K_4)$. We will first show that all of the generators of $\mathcal{J}$ are contained in $\mathcal{I}(K_n, K_4)$. The elements of types (1), (3), and (4) are self-evidently in $\mathcal{I}(K_n, K_4)$ since they are elements of the generating set of $\mathcal{I}(K_n, K_4)$. For type (2) we note that under the relations generating $\mathcal{I}(K_n, K_4)$ that

$$x_{v,i}(1 - \sum_{j=0}^{3} x_{v,j}) = x_{v,i} - x_{v,i}^2 - \sum_{j \neq i} x_{v,i}x_{v,j} = x_{v,i} - x_{v,i}^2,$$

and thus elements of type (2) are in $\mathcal{I}(K_n, K_4)$. For type (5) we use the relations generating $\mathcal{I}(K_n, K_4)$ to get that

$$\begin{aligned} x_{v,3}x_{w,3} &= (1 - x_{v,2} - x_{v,1} - x_{v,0})(1 - x_{w,2} - x_{w,1} - x_{w,0}) \\ &= x_{v,2}x_{w,1} + x_{v,2}x_{w,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0} + x_{v,0}x_{w,2} \\ &+ x_{v,0}x_{w,1} - x_{v,2} - x_{v,1} - x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} + 1. \end{aligned}$$

Finally type (6) is obtained by reducing

$$(x_{v,2}x_{w,1} + x_{v,2}x_{w,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0} + x_{v,0}x_{w,2} + x_{v,0}x_{w,1}$$
$$-x_{v,2} - x_{v,1} - x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} + 1)x_{v,1} - x_{v,2}(x_{w,1}x_{v,1})$$

using the relations of types (1)–(5), and type (7) is obtained by reducing

$$(x_{v,2}x_{w,1} + x_{v,2}x_{w,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0} + x_{v,0}x_{w,2} + x_{v,0}x_{w,1}$$
$$-x_{v,2} - x_{v,1} - x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} + 1)x_{u,1} - x_{v,2}(x_{w,1}x_{u,1})$$

using the relations of types (1)–(5). These two reductions are verified with Mathematica in QCGB-9-20-16.nb. Thus all of the generating relations of $\mathcal{J}$ are in $\mathcal{I}(K_n, K_4)$ and we have that $\mathcal{J} \subset \mathcal{I}(K_n, K_4)$.

Next we will show that all the generators of $\mathcal{I}(K_n, K_4)$ are contained in $\mathcal{J}$. The only generating relations of I that are not immediately seen to be in $\mathcal{J}$ are

$$x_{v,3}x_{v,j}, x_{v,i}x_{v,3},$$

and

$$x_{v,3}x_{w,3}.$$

To see that $x_{v,i}x_{v,3}$ is in $\mathcal{J}$ we consider $x_{v,i}(x_{v,3} + x_{v,2} + x_{v,1} + x_{v,0} - 1)$. This is an element of $\mathcal{J}$ since $(x_{v,3} + x_{v,2} + x_{v,1} + x_{v,0} - 1)$ is in $\mathcal{J}$, and when multiplied out all terms except $x_{v,i}x_{v,3}$ are in J, and thus $x_{v,i}x_{v,3}$ is in $\mathcal{J}$, similarly $x_{v,3}x_{v,j}$ is in $\mathcal{J}$. Finally, we consider the equation

$$x_{v,3}x_{w,3} = (x_{v,3} + x_{v,2} + x_{v,1} + x_{v,0} - 1)(x_{w,3} + x_{w,2} + x_{w,1} + x_{w,0} - 1)$$
$$-(x_{v,2}x_{w,1} + x_{v,2}x_{w,0} + x_{v,1}x_{w,2} + x_{v,1}x_{w,0} + x_{v,0}x_{w,2} + x_{v,0}x_{w,1}$$
$$-x_{v,2} - x_{v,1} - x_{v,0} - x_{w,2} - x_{w,1} - x_{w,0} + 1) - x_{v,2}x_{w,2} - x_{v,1}x_{w,1} - x_{v,0}x_{w,0},$$

the right-hand side is a sum of relations in $\mathcal{J}$ and is thus in $\mathcal{J}$, and thus the left-hand side is also in $\mathcal{J}$, specifically $x_{v,3}x_{w,3}$ is in $\mathcal{J}$. Therefore all of the generating relations of $\mathcal{I}(K_n, K_4)$ are in $\mathcal{J}$, so that $\mathcal{I}(K_n, K_4) \subset \mathcal{J}$. Since we have shown inclusion both ways, we have that $\mathcal{I}(K_n, K_4) = \mathcal{J}$ and we are done. □

Lemma 6.5. *If G, H are graphs such that $V(H) = V(G)$ and $E(H) \supset E(G)$, then $\mathcal{I}(H, K_m) \supset \mathcal{I}(G, K_m)$ and thus $1 \notin \mathcal{I}(H, K_m) \implies 1 \notin \mathcal{I}(G, K_m)$.*

Proof. The relations generating $\mathcal{I}(H, K_m)$ contains the relations generating $\mathcal{I}(G, K_m)$ and thus the result follows. □

Proof of Theorem 6.1. Let G be a graph on n vertices. By Theorem 6.3, $1 \notin \mathcal{I}(K_n, K_4)$. Additionally $E(G) \subset E(K_n)$, and thus by Lemma 6.5, $1 \notin \mathcal{I}(G, K_4)$. Therefore $\chi_{alg}(G) \leq 4$. □

Problem 6.6. *What is the complexity of deciding if $\chi_{alg}(G) = 4$, i.e., of deciding if $1 \in \mathcal{I}(G, K_3)$?*

7. Locally commuting algebra

In the final two sections of the paper, we consider a variant of χ_{alg}. We show that by adding mild commutativity relations to the algebra $\mathcal{A}(G, c)$, we obtain a chromatic number χ_{lc} which exhibits behavior much more akin to χ, χ_q, χ_{qa}, and χ_{qc}. For example, in this section we prove $\chi_{lc}(K_n) = n$, and in the following section we show it behaves well with respect to products. We also obtain an *a priori* new type of clique number ω_{lc}, although we prove in this section that $\omega_{lc} = \omega$, see Theorem 7.4.

Since our algebras were initially motivated by quantum chromatic numbers, it is natural to look to quantum mechanics for further relations to impose. In the case of a graph, we can imagine each vertex as corresponding to a laboratory and think of two vertices as connected whenever those laboratories can conduct a joint experiment. In this case, all of the measurement operators for the two labs should commute, i.e., whenever (v, w) is an edge, then the commutator $[e_{v,i}, e_{w,j}] := e_{v,i}e_{w,j} - e_{w,j}e_{v,i} = 0$. Note that this commutation rule is exactly the rule that we were able to derive in the case of three colors in Proposition 4.3. This motivates the following definitions.

Definition 7.1. *Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game with $|I| = n$ and $|O| = m$. We say that $v, w \in I$ are* **adjacent** *and write $v \sim w$ provided that $v \neq w$ and there exists $a, b \in O$ such that $\lambda(v, w, a, b) = 0$. We define the* **locally commuting ideal** *of the game to be the 2-sided ideal $\mathcal{I}_{lc}(\mathcal{G})$ in $\mathbb{C}[\mathbb{F}(n, m)]$ generated by the set*

$$\{e_{v,a}e_{w,b} \mid \lambda(v, w, a, b) = 0\} \cup \{[e_{v,a}, e_{w,b}] \mid \ v \sim w, \forall a, b \in O\}.$$

We set $\mathcal{A}_{lc}(\mathcal{G}) = \mathbb{C}[\mathbb{F}(n, m)]/\mathcal{I}_{lc}(\mathcal{G})$ and call this the **locally commuting algebra of** *$\mathcal{G}$.*

In the case that G and H are graphs and $\mathcal{G}$ is the graph homomorphism game from G to H we set

$$\mathcal{I}_{lc}(G, H) = \mathcal{I}_{lc}(\mathcal{G})$$

and

$$\mathcal{A}_{lc}(G, H) = \mathcal{A}_{lc}(\mathcal{G}).$$

We write $G \overset{lc}{\to} H$ if $\mathcal{I}_{lc}(G, H) \neq \mathbb{C}[\mathbb{F}(n, m)]$ and set

$$\chi_{lc}(G) = \min\{c \mid G \overset{lc}{\to} K_c\}.$$

We similarly define

$$\omega_{lc}(G) = \max\{c \mid K_c \stackrel{lc}{\to} G\}.$$

Note that in the case of the graph homomorphism game from G to H we have that $I = V(G)$ and $v \sim w \iff (v,w) \in E(G)$. Thus, the relationship $\sim$ extends the concept of adjacency to the inputs of a general synchronous game.

Thus, $\mathcal{A}_{lc}(G, K_c)$ is the universal *-algebra generated by self-adjoint projections $\{E_{v,i} : v \in V(G), 1 \le i \le c\}$ satisfying

- $\sum_{i=1}^{c} E_{v,i} = I, \forall v,$
- $v \sim w \implies E_{v,i}E_{w,i} = 0, \forall i,$
- $v \sim w \implies [E_{v,i}, E_{w,j}] = 0, \forall i, j$

and $\chi_{lc}(G)$ is the least c for which such a non-trivial *-algebra exists.

Lemma 7.2. *If there exists a morphism $G \to H$, then $\mathcal{A}_{lc}(G,H) \neq 0$, i.e. $G \stackrel{lc}{\to} H$.*

Proof. Let $\phi \colon G \to H$ be a graph homomorphism. Consider the map $\mathcal{A}_{lc}(G,H) \to \mathbb{C}$ sending $e_{v,\phi(v)}$ to 1 and $e_{v,x}$ to 0 for $x \neq \phi(v)$. It is easy to see this is a well-defined $\mathbb{C}$-algebra map and hence surjective. As a result, $\mathcal{A}_{lc}(G,H) \neq 0$. □

Corollary 7.3. *We have $\chi_{lc}(G) \le \chi(G)$ and $\omega(G) \le \omega_{lc}(G)$.*

Proof. There is a graph homomorphism $G \to K_{\chi(G)}$ so by Lemma 7.2, we have $G \stackrel{lc}{\to} K_{\chi(G)}$ and hence $\chi_{lc}(G) \le \chi(G)$. The inequality for ω is shown in an analogous fashion. □

We are now ready to prove the main result of this section.

Theorem 7.4. *Let H be a graph and $n \ge 1$. Then*

1. *$\mathcal{A}_{lc}(K_n, H)$ is the abelianization of $\mathcal{A}(K_n, H)$,*
2. *$\mathcal{A}_{lc}(K_n, H) \neq 0$ if and only if H has an n-clique,*
3. *$\omega(H) = \omega_{lc}(H)$,*
4. *$\chi_{lc}(K_n) = n$.*

Proof. It is immediate from the definition that $\mathcal{A}_{lc}(K_n, H)$ is the quotient of $\mathcal{A}(K_n, H)$ where we impose all commutation relations among the generators, hence $\mathcal{A}_{lc}(K_n, H)$ is the abelianization of $\mathcal{A}(K_n, H)$.

Applying Theorem 3.2 (1) to the graph homomorphism game, we see $\mathcal{A}(K_n, H)$ has an abelian representation if and only if there is a graph homomorphism $K_n \to H$, i.e. if H has an n-clique. We conclude by noting that $\mathcal{A}(K_n, H)$ has an abelian representation if and only if its abelianization $\mathcal{A}_{lc}(K_n, H)$ does if and only if $\mathcal{A}_{lc}(K_n, H) \neq 0$.

Statement (3) and follows immediately from (2): the quantity $\omega_{lc}(H)$ is the minimal n for which $\mathcal{A}_{lc}(K_n, H) \neq 0$, and we have shown this $n = \omega(H)$. Statement (4) is also immediate: $\chi_{lc}(K_n)$ is the minimal c for which

$\mathcal{A}_{lc}(K_n, K_c) \neq 0$; since non-vanishing of this algebra is equivalent to K_c containing an n-clique, the minimum such c is n itself. □

Problem 7.5. *We do not know if the Lovasz sandwich result holds in this context, i.e., if* $\omega_{lc}(G) \leq \vartheta(\overline{G}) \leq \chi_{lc}(G)$.

8. Some further properties of $\mathcal{A}_{lc}$ and χ_{lc}

Our main goal is to distinguish χ_{lc} from χ, χ_q, and χ_{vect}. This is done in Theorem 8.18 of Subsection 8.3, and will follow from analysis of how χ_{lc} behaves with respect to graph suspension and certain graph products. Subsection 8.1 is concerned with the study of graph suspension. As a corollary, we also obtain a more refined version of Theorem 7.4: the theorem only tells us when $\mathcal{A}_{lc}(K_n, H)$ is non-zero, but tells us nothing more about the structure of the algebra; in Corollary 8.10, we compute the algebra structure of $\mathcal{A}_{lc}(K_n, H)$. In Subsection 8.2, we turn to the behavior of χ_{lc} under graph products. We use our results to explicitly compute $\chi_{lc}(C_5 \boxtimes K_3)$, which is used in the proof of Theorem 8.18.

Throughout this section we shall write $\simeq$ to indicate that two algebras are isomorphic. We shall use $\mathbb{C}^n$ to denote the abelian algebra of complex-valued functions on n points. For ease of notation, we will frequently write $v \in G$ instead of $v \in V(G)$.

Recall $\mathcal{A}_{lc}(G, H)$ is the quotient of $\mathbb{C}\langle e_{vx} \mid v \in G, x \in H\rangle$ by the ideal generated by the following relations

(1) $\sum_{x \in H} e_{vx} = 1$,
(2) $e_{vx}^2 = e_{vx}$,
(3) $e_{vx}e_{vy} = 0$ for $x \neq y$,
(4) $e_{vx}e_{wy} = 0$ if $v \sim w$ and $x \not\sim y$, and
(5) $[e_{vx}, e_{wy}] = 0$ for $v \sim w$.

and recall that we write $G \stackrel{lc}{\to} H$ if $\mathcal{A}_{lc}(G, H) \neq 0$.

8.1. Behavior of χ_{lc} under suspension and a more refined version of Theorem 7.4.

Lemma 8.1. *If* $G \stackrel{lc}{\to} H$ *and* $H \stackrel{lc}{\to} K$, *then* $G \stackrel{lc}{\to} K$.

Proof. If $\mathcal{A}_{lc}(G, H)$ and $\mathcal{A}_{lc}(H, K)$ are non-zero, then we must prove that $\mathcal{A}_{lc}(G, K)$ is non-zero as well. To see this, consider the map

$$\mathbb{C}\langle e_{vr} \mid v \in G, r \in K\rangle \to \mathcal{A}_{lc}(G, H) \otimes \mathcal{A}_{lc}(H, K)$$

given by

$$e_{vr} \mapsto \sum_{x \in H} e_{vx} \otimes e_{xr}$$

and suppose that it vanishes on $\mathcal{I}_{lc}(G, K)$. Hence, there would be a well-defined map on the quotient,

$$\mathcal{A}_{lc}(G, K) \to \mathcal{A}_{lc}(G, H) \otimes \mathcal{A}_{lc}(H, K)$$

$$e_{vr} \mapsto \sum_{x\in H} e_{vx} \otimes e_{xr}.$$

If $1 = 0$ in $\mathcal{A}_{lc}(G,K)$, then the same would be true in $\mathcal{A}_{lc}(G,H)\otimes\mathcal{A}_{lc}(H,K)$, since this map sends units to units.

Thus it remains to show that the above map vanishes on $\mathcal{I}_{lc}(G,K)$. In order to do this, it is sufficient to check that each generating relation is sent to zero. This is easily checked, for example,

$$\sum_{r\in K}\sum_{x\in H} e_{vx}\otimes e_{xr} = \sum_{x\in H} e_{vx}\otimes \sum_{r\in K} e_{xr} = \sum_{x\in H} e_{vx}\otimes 1 = 1.$$

Checking the other relations is left to the reader. □

Corollary 8.2. *If $G \stackrel{lc}{\to} H$, then $\chi_{lc}(G) \le \chi_{lc}(H)$.*

Proof. Let $c = \chi_{lc}(H)$. Then we have $H \stackrel{lc}{\to} K_c$ and hence $G \stackrel{lc}{\to} K_c$. Thus, $\chi_{lc}(G) \le c = \chi_{lc}(H)$. □

We also have the following consequence of the proof of Lemma 8.1.

Theorem 8.3. *The assignment*

$$(\textit{Graphs}) \times (\textit{Graphs}) \longrightarrow (\mathbb{C}\textit{-algebras})$$

$$(G,H) \longmapsto \mathcal{A}_{lc}(G,H)$$

is a functor, which is covariant in the first factor and contravariant in the second.

Proof. If $\phi : G \to G'$ is a morphism, then we have a map $\mathcal{A}_{lc}(G,H) \to \mathcal{A}_{lc}(G',H)$ given by $e_{v,x} \mapsto e_{\phi(v),x}$. On the other hand, if $\phi : H \to K$ is a morphism, then we have $H \stackrel{lc}{\to} K$ and so from the proof of Lemma 8.1, we have

$$\mathcal{A}_{lc}(G,K) \to \mathcal{A}_{lc}(G,H) \otimes \mathcal{A}_{lc}(H,K).$$

Since ϕ is a morphism of graphs, we have a map $\mathcal{A}_{lc}(H,K) \to \mathbb{C}$ as in the proof of Lemma 7.2. Composing with the above, we have $\mathcal{A}_{lc}(G,K) \to \mathcal{A}_{lc}(G,H)$. Explicitly, this map is given by sending $e_{vx} \in \mathcal{A}_{lc}(G,K)$ to $\sum_{\phi(r)=x} e_{vr}$. □

We now show how the functor $\mathcal{A}_{lc}$ interacts with various natural graph operations. To begin, recall that if G is a graph, its suspension ΣG is defined by adding a new vertex v and an edge from v to each of the vertices of G.

Given an algebra $\mathcal{A}$ we shall let $\mathcal{A}^c$ denote the algebra of c-tuples with entries from $\mathcal{A}$, i.e., the tensor product $\mathcal{A}\otimes\mathbb{C}^c \simeq \mathcal{A}^{\oplus c}$ where $\mathbb{C}^c$ can be identfied with the algebra of $\mathbb{C}$-valued functions on c points.

Proposition 8.4. *Let G and H be any graphs. For $y \in H$ we let N_y denote the neighborhood of y, i.e., the induced subgraph of H with vertices adjacent to y; notice $y \notin N_y$. Then we have an algebra isomorphism*

$$\mathcal{A}_{lc}(\Sigma G, H) \simeq \bigoplus_{y \in H} \mathcal{A}_{lc}(G, N_y).$$

In particular, if H is vertex transitive and y is any vertex of H with neighborhood N, then

$$\mathcal{A}_{lc}(\Sigma G, H) \simeq \mathcal{A}_{lc}(G, N)^{|H|}.$$

Proof. Let u be the new vertex added to ΣG, i.e., $u \in \Sigma G \setminus G$. Since u is adjacent to every vertex of G, we see e_{ux} commutes with e_{vy} for all $v \in G$ and $x, y \in H$. Furthermore, the defining relations of $\mathcal{A}_{lc}$ tell us $e_{ux}e_{uy} = \delta_{x,y}e_{ux}$ where δ denotes the Kronecker delta function. So, considering the ring $\mathcal{A}_{lc}(G, H)[e_{ux}]$ of polynomials in e_{ux} with coefficients in $\mathcal{A}_{lc}(G, H)$, we have an isomorphism

$$\mathcal{A}_{lc}(\Sigma G, H) \simeq \mathcal{A}_{lc}(G, H)[e_{ux}]/(\sum_x e_{ux} - 1, e_{ux}e_{uy} = \delta_{x,y}e_{ux}).$$

In other words, the e_{ux} for $x \in H$ are commuting orthogonal idempotents, which shows

$$\mathcal{A}_{lc}(\Sigma G, H) \simeq \bigoplus_{y \in H} \mathcal{A}_{lc}(G, H)e_{uy} \simeq \bigoplus_{y \in H} \mathcal{A}_{lc}(G, H)/(e_{vx} \colon x \not\sim y),$$

where the last equality comes from the fact that $e_{vx}e_{uy} = 0$ for $x \not\sim y$.

Now note that e_{vx} remains non-zero in the quotient $\mathcal{A}_{lc}(G, H)/(e_{vx} \colon x \not\sim y)$ if and only if $x \sim y$. Thus,

$$\mathcal{A}_{lc}(G, H)/(e_{vx} \colon x \not\sim y) \simeq \mathcal{A}_{lc}(G, N_y),$$

which establishes the first assertion of the proposition. The second assertion easily follows from the first since all neighborhoods are isomorphic. □

Corollary 8.5. *For all non-empty graphs G, we have $\mathcal{A}_{lc}(\Sigma G, K_1) = 0$. If $c \geq 2$, then*

$$\mathcal{A}_{lc}(\Sigma G, K_c) \simeq \mathcal{A}_{lc}(G, K_{c-1})^c.$$

Proof. This is an immediate consequence of Proposition 8.4 using that K_c is vertex transitive. □

Corollary 8.6. *For all graphs G, we have $\chi_{lc}(\Sigma G) = \chi_{lc}(G) + 1$.*

Proof. By the above isomorphism, the least c such that $\mathcal{A}_{lc}(\Sigma G, K_{c+1}) \neq (0)$ is equal to the least c such that $\mathcal{A}_{lc}(G, K_c) \neq (0)$. □

Remark 8.7. In [19] an example of a graph G is given for which $\chi_q(\Sigma G) = \chi_q(G)$. Hence, either $\chi_{lc}(\Sigma G) \neq \chi_q(\Sigma G)$ or $\chi_{lc}(G) \neq \chi_q(G)$.

Corollary 8.8. *If $c \geq n$, then*

$$\mathcal{A}_{lc}(K_n, K_c) \simeq \mathbb{C}^{c(c-1)\dots(c-n+1)}.$$

If $c < n$, then $\mathcal{A}_{lc}(K_n, K_c) = 0$.

Proof. One easily checks that $\mathcal{A}_{lc}(K_1, G) \simeq \mathbb{C}^{|G|}$ for any graph G. In particular, our desired statement holds for $n = 1$. The proof then follows from induction on n by applying Corollary 8.5 and using that $K_n = \Sigma K_{n-1}$. □

Remark 8.9. In Theorem 7.4, we proved $\chi_{lc}(K_n) = n$. Corollary 8.8 gives another proof of this result which tells us the specific structure of $\mathcal{A}_{lc}(K_n, K_c)$ whereas the theorem merely tells us it is non-zero.

Using Proposition 8.4, we can easily understand iterated suspensions.

Corollary 8.10. *If H is a graph, then*

$$\mathcal{A}_{lc}(K_n, H) \simeq \bigoplus_{S \subseteq H} \mathbb{C}^{n!} \simeq \mathbb{C}^{Nn!},$$

where S ranges over the n-cliques of H, and N denotes the number of n-cliques.

Remark 8.11 ($\omega_{lc} = \omega$)**.** Corollary 8.10 gives another proof that $\omega_{lc} = \omega$. The corollary gives more information than our previous proof in Theorem 7.4. The theorem tells us $\mathcal{A}_{lc}(K_n, H) \neq 0$ if and only if $n \geq \omega(H)$, but the corollary tells us the structure of algebra $\mathcal{A}_{lc}(K_n, H)$.

Proof of Corollary 8.10. We leave the $n = 1$ case to the reader. Iteratively applying Proposition 8.4, we see

$$\mathcal{A}_{lc}(K_n, H) \simeq \bigoplus_{x \in H} \mathcal{A}_{lc}(K_{n-1}, N_x) \simeq \cdots \simeq \bigoplus_{(x_{n-1},\dots,x_2,x_1)} \mathcal{A}_{lc}(K_1, N_{x_{n-1}} \dots N_{x_2} N_{x_1})$$

where the index of the direct sum runs over all sequences $(x_{n-1}, \dots, x_2, x_1)$ with $x_{i+1} \in N_{x_i} N_{x_{i-1}} \dots N_{x_1}$; recall that $N_{x_i} N_{x_{i-1}} \dots N_{x_1}$ is the set of all $z \in N_{x_{i-1}} \dots N_{x_1}$ that are adjacent to x_i.

We show by induction that the $x_1, \dots, x_i$ form an i-clique and $N_{\{x_1,\dots,x_i\}}$ equals $N_{x_i} N_{x_{i-1}} \dots N_{x_1}$; recall the definition given in the statement of the corollary that $N_S = \{z \in H \mid z \sim x \ \forall x \in S\}$ for any clique S. For $i = 1$ this is just the definition. For $i > 1$, observe that by construction $x_i \in N_{x_{i-1}} \dots N_{x_1} = N_{\{x_1,\dots,x_{i-1}\}}$ and since $x_1, \dots, x_{i-1}$ forms an $(i-1)$-clique, we see $x_1, \dots, x_i$ forms an i-clique. Next, $N_{x_i} N_{\{x_1,\dots,x_{i-1}\}}$ is the set of $z \in N_{\{x_1,\dots,x_{i-1}\}}$ that are adjacent to x_i, which is the definition of $N_{\{x_1,\dots,x_i\}}$.

This shows

$$\mathcal{A}_{lc}(K_n, H) \simeq \bigoplus_{(x_{n-1},\dots,x_2,x_1)} \mathcal{A}_{lc}(K_1, N_{\{x_1,\dots,x_{n-1}\}}),$$

and by the $n = 1$ case, each summand is isomorphic to $|N_{\{x_1,\ldots,x_{n-1}\}}|$ copies of $\mathbb{C}$. Now notice that $N_{\{x_1,\ldots,x_{n-1}\}}$ is independent of the order of the sequence, and hence this term arises $(n-1)!$ times. This shows $\mathcal{A}_{lc}(K_n, H) \simeq \bigoplus_{T\subseteq H} \mathbb{C}^{|N_T|(n-1)!}$ as T ranges over the $(n-1)$-cliques in H and $N_T = \{z \in H \mid z \sim x \ \forall x \in T\}$. This yields the desired statement since $|N_T|$ is equal to the number of n-cliques containing T. □

8.2. Two explicit examples, and the behavior of χ_{lc} under graph products. We end the paper by considering how χ_{lc} interacts with several graph products. We then use our results to calculate χ_{lc} in two examples. Consider the graph products $G \,\square\, H$ and $G \boxtimes H$; they both have vertex set $V(G) \times V(H)$. In the former, $(v, x) \sim (w, y)$ if and only if $v = w$ and $x \sim y$, or $v \sim w$ and $x = y$. In the latter, $(v, x) \sim (w, y)$ if and only if $v \sim w$ and $x \sim y$, or $v = w$ and $x \sim y$, or $v \sim w$ and $x = y$. The products $\square$ and $\boxtimes$ are referred to as the Cartesian and strong products, respectively.

For any pair of graphs we have $\chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\}$. We show that the same is true for χ_{lc}.

Theorem 8.12 (χ_{lc} of Cartesian product)**.** *For any graphs G and H, we have*

$$\chi_{lc}(G \,\square\, H) = \max\{\chi_{lc}(G), \chi_{lc}(H)\}.$$

Proof. We have at least $|H|$ maps $G \to G \,\square\, H$, so Lemma 7.2 and Corollary 8.2 show $\chi_{lc}(G) \leq \chi_{lc}(G \,\square\, H)$. Similarly for H, so $\max\{\chi_{lc}(G), \chi_{lc}(H)\} \leq \chi_{lc}(G \,\square\, H)$. To prove the result, it now suffices to show we have a map

$$\mathcal{A}_{lc}(G \,\square\, H, K_c) \to \mathcal{A}_{lc}(G, K_c) \otimes \mathcal{A}_{lc}(H, K_c).$$

Indeed, if $\mathcal{A}_{lc}(G, K_c)$ and $\mathcal{A}_{lc}(H, K_c)$ are non-zero, then so is $\mathcal{A}_{lc}(G \,\square\, H, K_c)$ since the above map would send 0 to 0 and 1 to 1, and if $0 = 1$ in $\mathcal{A}_{lc}(G \,\square\, H, K_c)$, then $0 = 1$ in $\mathcal{A}_{lc}(G, K_c) \otimes \mathcal{A}_{lc}(H, K_c)$, which is not the case. Taking $c = \max\{\chi_{lc}(G), \chi_{lc}(H)\}$, this would then show $\chi_{lc}(G \,\square\, H, K_c) \geq c$.

We now construct the above map. We define it by:

$$e_{(x,y),k} \mapsto \sum_{i \in \mathbb{Z}/c} e_{x,i} \otimes e_{y,k-i}$$

and show it is well-defined. First suppose that $(x, y) \sim (x', z)$ and $k \not\sim \ell$. Then $k = \ell$ and without loss of generality $x = x'$ and $y \sim z$. Then

$$e_{(x,y),k} e_{(x,z),\ell} \mapsto \sum_{i,j} e_{x,i} e_{x,j} \otimes e_{y,k-i} e_{z,\ell-j} = 0$$

since $e_{x,i} e_{x,j} = 0$ if $i \neq j$, and if $i = j$, then $k - i = \ell - j$ and so $e_{y,k-i} e_{z,\ell-j} = 0$.

Next, if $y \sim z$, then the images of $e_{(x,y),k} e_{(x,z),\ell}$ and $e_{(x,z),\ell} e_{(x,y),k}$ are equal since

$$e_{(x,y),k} e_{(x,z),\ell} \mapsto \sum_{i,j} e_{x,i} e_{x,j} \otimes e_{y,k-i} e_{z,\ell-j}$$

and $e_{y,k-i}e_{z,\ell-j} = e_{z,\ell-j}e_{y,k-i}$ as $y \sim z$, and $e_{x,i}e_{x,j} = \delta_{ij}e_{x,i} = e_{x,j}e_{x,i}$.

We next see that $\sum_k e_{(x,y),k}$ maps to

$$\sum_i \sum_k e_{x,i} \otimes e_{y,k-i} = \sum_i \sum_k e_{x,i} \otimes e_{y,k} = \sum_i e_{x,i} \otimes \sum_k e_{y,k} = 1 \otimes 1.$$

If $k \neq \ell$, then

$$e_{(x,y),k}e_{(x,y),\ell} \mapsto \sum_{i,j} e_{x,i}e_{x,j} \otimes e_{y,k-i}e_{y,\ell-j} = 0$$

since $e_{x,i}e_{x,j} = 0$ if $i \neq j$, and if $i = j$, then $k-i \neq \ell-j$ and so $e_{y,k-i}e_{y,\ell-j} = 0$.

Lastly,

$$e^2_{(x,y),k} \mapsto \sum_{i,j} e_{x,i}e_{x,j} \otimes e_{y,k-i}e_{y,k-j} = \sum_i e^2_{x,i} \otimes e^2_{y,k-i}$$

since $e_{x,i}e_{x,j} = 0$ if $i \neq j$. Thus, $e^2_{(x,y),k}$ and $e_{(x,y),k}$ have the same image. This completes the proof that the map is well-defined. ☐

Lemma 8.13. *If $G \overset{\text{lc}}{\to} K$ and $H \overset{\text{lc}}{\to} K'$, then $G \cdot H \overset{\text{lc}}{\to} K \cdot K'$ for any $\cdot \in \{\times, \square, \boxtimes\}$.*

Proof. It suffices to construct a map

$$\mathcal{A}_{lc}(G \cdot H, K \cdot K') \to \mathcal{A}_{lc}(G, K) \otimes \mathcal{A}_{lc}(H, K').$$

We define it by

$$e_{(x,y),(k,k')} \mapsto e_{x,k} \otimes e_{y,k'}.$$

One readily checks that this map is well-defined. For example, in the case of the Cartesian product $\square$ we show that if $(x,y) \sim (z,w)$ and $(k,k') \not\sim (\ell,\ell')$, then $e_{(x,y),(k,k')}e_{(z,w),(\ell,\ell')}$ maps to 0. Without loss of generality, we can assume that $x = z$ and $y \sim w$. Then the image is $e_{x,k}e_{x,\ell} \otimes e_{y,k'}e_{w,\ell'}$, which is automatically 0 if $k \neq \ell$. So, we may assume $k = \ell$, in which case $k' \not\sim \ell'$ since $(k,k') \not\sim (\ell,\ell')$. But then $e_{y,k'}e_{w,\ell'} = 0$. ☐

Corollary 8.14. *We have $\chi_{lc}(G \boxtimes H) \leq \chi_{lc}(G)\chi_{lc}(H)$.*

Proof. This follows immediately from Lemma 8.13 after observing that $K_n \boxtimes K_m = K_{nm}$. ☐

In [5], the authors showed that $8 = \chi(C_5 \boxtimes K_3) = \chi_q(C_5 \boxtimes K_3) > \chi_{vect}(C_5 \boxtimes K_3) = 7$. Thus, separating χ_q from χ_{vect}. Later, [26] showed that $\chi_{qc}(C_5 \boxtimes K_3) = 8$, separating the potentially smaller χ_{qc} from χ_{vect}. We show below that $\chi_{lc}(C_5 \boxtimes K_3) = 8$ as well.

Before considering $C_5 \boxtimes K_3$ we begin with a simpler example.

Example 8.15 ($C_5 \boxtimes K_2$)**.** Let $G = C_5 \boxtimes K_2$. It is easy to see that $\omega(G) = 4$ and $\chi(G) = 5$, so a priori χ_{lc} could be 4 or 5. We show

$$\chi_{lc}(C_5 \boxtimes K_2) = 5.$$

We need to show that $\mathcal{A}_{lc}(G, K_4) = 0$. The graph G is made up of 2 pentagons stacked on top of each other. Let one of the pentagons have vertices x, y, z, w, s labeled clockwise and let the other pentagon have vertices x', y', z', w', s' with x and x' having the same neighbors. For ease of notation, we denote $e_{v,i}$ by v_i. It is not difficult to see that

$$1 = \sum_{\sigma \in S_4} (s_{\sigma(3)} s'_{\sigma(4)} + s'_{\sigma(3)} s_{\sigma(4)}) x_{\sigma(1)} x'_{\sigma(2)} y_{\sigma(3)} y'_{\sigma(4)} (z_{\sigma(1)} z'_{\sigma(2)} + z'_{\sigma(1)} z_{\sigma(2)}).$$

Multiplying on both the left and right by $w_1 w'_2$, we see that the only non-zero terms in the sum must have $\sigma(3), \sigma(4) \in \{3, 4\}$ or else $w_1 w'_2 s_{\sigma(3)} s'_{\sigma(4)} + s'_{\sigma(3)} s_{\sigma(4)}) = 0$. However, this forces $\sigma(1), \sigma(2) \in \{1, 2\}$ and so $(z_{\sigma(1)} z'_{\sigma(2)} + z'_{\sigma(1)} z_{\sigma(2)}) w_1 w'_2 = 0$. We therefore see

$$w_1 w'_2 = w_1 w'_2 (s_3 s'_4 + s'_3 s_4)(x_1 x'_2 + x'_1 x_2)(y_3 y'_4 + y'_3 y_4)(z_1 z'_2 + z'_1 z_2) w_1 w'_2 = 0.$$

Similarly, we find $w_i w'_j = 0$ for all i, j. As a result,

$$1 = \sum_{i,j} w_i w'_j = 0$$

and so $\mathcal{A}_{lc}(G, K_4) = 0$.

Example 8.16 ($C_5 \boxtimes K_3$)**.** Let $G = C_5 \boxtimes K_3$. We see $\omega = 6$ and $\chi = 8$, so a priori χ_{lc} could be 6, 7, or 8. We show

$$\chi_{lc}(C_5 \boxtimes K_3) = 8 = \chi(C_5 \boxtimes K_3).$$

We must show $\mathcal{A}_{lc}(G, K_7) = 0$. We follow the same notational conventions as in Example 8.15. Let x, y, z, w, s be the vertices of C_5 labeled clockwise and denote the next two copies of C_5 by $x', \ldots, s'$ resp. $x'', \ldots, s''$ where x, x', x'' have the same neighbors in G. We also let $v_i = e_{v,i}$.

As in the previous example,

$$w_1 w'_2 w''_3 = w_1 w'_2 w''_3 SXYZ w_1 w'_2 w''_3,$$

where $S = \sum_{i,j,k} s_i s'_j s''_k$ and analogously for X, Y, Z. We show that every term occurring in the sum on the righthand side of the above equation is 0. The indices i, j, k occurring in the sum S must all lie in $\{4, 5, 6, 7\}$ otherwise the term vanishes (since it is multiplied by $w_1 w'_2 w''_3$). Our goal is to show that all terms in the sum in the righthand side vanish, so we can fix a summand in S and assume i, j, k equal $4, 5, 6$ respectively. Then the indices in X must be 3 of $\{1, 2, 3, 7\}$. We also see that the indices in Z must be 3 of $\{4, 5, 6, 7\}$. Fix a summands $x_a x'_b x''_c$ and $z_p z'_q z''_r$ of X and Z, respectively. Then $\{1, 2, \ldots, 7\} \backslash \{a, b, c, p, q, r\}$ has size at most 2. Therefore, every summand t of Y satisfies $XtZ = 0$. So, $w_1 w'_2 w''_3 = 0$, and analogously we see $w_i w'_j w''_k = 0$ for all i, j, k. So,

$$1 = \sum_{i,j} w_i w'_j w''_k = 0$$

showing that $\mathcal{A}_{lc}(G, K_7) = 0$. As a result, $\chi_{lc}(G) = 8$.

8.3. Distinguishing χ_{lc} from χ, χ_q, and χ_{vect}.

Problem 8.17. *Since the definition of χ_{lc} is not obviously related to representations on Hilbert spaces, it is unclear how to relate it to χ_t for $t \in \{loc, q, qa, qc, vect\}$. Where does χ_{lc} fit within this hierarchy?*

As a partial answer to Problem 8.17, we have:

Theorem 8.18. *$\chi_{lc} \neq \chi$, $\chi_{lc} \neq \chi_q$, and $\chi_{lc} \neq \chi_{vect}$.*

Proof. By Remark 8.7 we know $\chi_{lc} \neq \chi_q$. In Example 8.16, we established $\chi_{lc}(C_5 \boxtimes K_3) = 8$ and in [5], the authors showed that $\chi_{vect}(C_5 \boxtimes K_3) = 7$, so $\chi_{lc} \neq \chi_{vect}$.

Finally, in [19], the authors construct a graph G with $\chi_q(G) = 3$ and $\chi(G) > 3$. Since $\chi_q(G) = 3$, by Theorem 3.2 there is a non-trivial finite-dimensional representation $\pi\colon \mathcal{A}(G, K_3) \to B(\mathcal{H})$, and by Proposition 4.3, π factors through the quotient map $\mathcal{A}(G, K_3) \to \mathcal{A}_{lc}(G, K_3)$. This yields a non-trivial finite-dimensional representation of $\mathcal{A}_{lc}(G, K_3)$, hence $\chi_{lc}(G) \leq 3 < \chi(G)$. □

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