The Khovanov homology of 3-strand pretzels, revisited

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ABSTRACT. We compute the reduced Khovanov homology of 3-stranded pretzel links. The coefficients are the integers with the "even" sign assignment. In particular, we show that the only homologically thin, non-quasi-alternating 3-stranded pretzels are $P(-p, p, r)$ with p an odd integer and r greater than or equal to p (these were shown to be homologically thin by Starkston and Qazaqzeh).

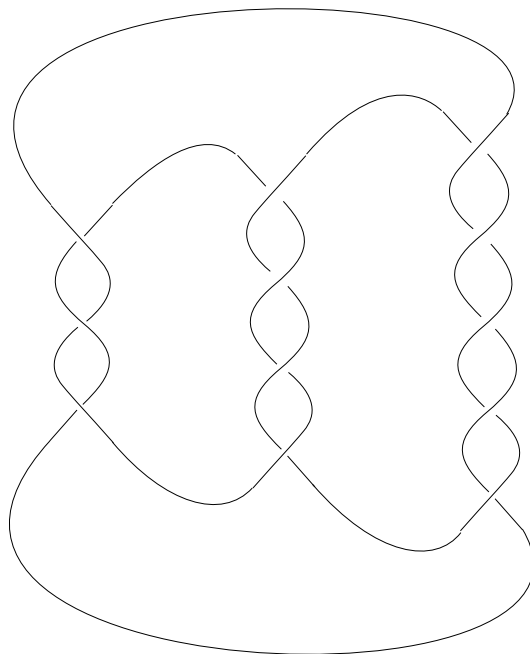## CONTENTS

1. Introduction

The purpose of this paper is to present a computation of the reduced Khovanov homology (introduced in [6]) of all 3-strand pretzel links. There have been several computations of Khovanov homology for partial families

FIGURE 1. The pretzel knot $P(-3, 4, 5)$.

of pretzel knots (see Starkston [10], Suzuki [11], Qazaqzeh [8]), and one by the author [7] computing the unreduced homology over $\mathbb{Q}$ for all 3-strand pretzels. Whereas [7] used the unoriented skein exact sequence in Khovanov homology, this paper will use a shorter and more conceptual argument relying on Bar-Natan's cobordism formulation of Khovanov homology for tangles (see [2]). We will determine the reduced homology over $\mathbb{Z}$, with the standard ("even") sign assignment.

One caveat is required: the reduced Khovanov homology of links with multiple components depends on which component has the basepoint. For 2- and 3-component 3-stranded pretzel links (those $P(p, q, r)$ where two or three of $\{p, q, r\}$ are even), we will only do the computation with one particular choice of basepoint. One could apply the same method with the other basepoint choices, but the details would be different enough that we decided not to write them up.

The usual diagram of the pretzel knot $P(-3, 4, 5)$ is shown in Figure 1. We will use the equivalent diagram in Figure 6. The general 3-strand preztel link is $P(p, q, r)$ where p, q, and r are arbitrary integers. Up to mirroring, though, we may assume that at most one of $\{p, q, r\}$ is negative. If none are negative, then the link is alternating and its Khovanov homology is determined by its signature and Jones polynomial, by a result of Lee from [12]. So we will restrict attention to $P(-p, q, r)$ with $\{p, q, r\}$ positive. We

may even assume $q \leq r$ for convenience, because of the symmetry of 3-stranded pretzels.

For some choices of p, q, and r, $P(-p, q, r)$ is quasi-alternating, and the same argument applies as with alternating links. Hence we may restrict attention to non-quasi-alternating pretzel links. Results of Champanerkar-Kofman in [4] and Greene in [5] imply that these are $P(-p, q, r)$ with p, q, r all positive, $p \leq q, r$ (or $p \leq q \leq r$), and $p \geq 2$. (Note that a simple isotopy of the standard diagram for $P(-1, q, r)$ yields an alternating diagram.)

The formula for the bigraded Khovanov homology of these links takes a bit of work to write down. Complicating matters is the issue of orientations; to get the right gradings, we must decide on the relative orientations of components of $P(-p, q, r)$ when it is a multi-component link. To avoid cluttering the introduction, we will state a simpler formula here. Recall the δ-grading on Khovanov homology; we will define it as $\delta = q/2 - h$, where q is the quantum grading and h is the homological grading. (In [9], Rasmussen defines it as $q - 2h$; the $1/2$ is a matter of preference.) With this collapse of the gradings, Khovanov homology becomes a singly-graded theory.

The link $P(-p, q, r)$ is a knot when at most one of $\{p, q, r\}$ are even. We need not consider the case of "only r even" separately from the case of "only q even," since these cases are interchanged by the symmetry of pretzel knots (here we're not requiring $q \leq r$). Below we state the δ-graded formula for knots; this has the advantage that we do not need to mention orientations at all.

Theorem 1.1. *Let p, q, r be as above, such that $P(-p, q, r)$ is a non-quasi-alternating knot (for links, see Theorem 2.6). Let H_δ be the reduced Khovanov homology of the knot $P(-p, q, r)$ in grading δ.*

- *If p, q, r are all odd, then $H_0 = \mathbb{Z}^{p^2-1}$ and $H_{-1} = \mathbb{Z}^{(q-p)(r-p)-1}$. All other H_δ are 0.*
- *If p is even, then $H_{\frac{q+r}{2}} = \mathbb{Z}^{p^2}$ and $H_{\frac{q+r}{2}-1} = \mathbb{Z}^{(q-p)(r-p)}$. All other H_δ are 0.*
- *If q is even, then $H_{\frac{-p+r}{2}} = \mathbb{Z}^{p^2-1}$ and $H_{\frac{-p+r}{2}-1} = \mathbb{Z}^{(q-p)(r-p)-1}$ in $\delta = \frac{-p+r}{2} - 1$. All other H_δ are 0.*

When p is odd and $p = q$ or $p = r$, the formula gives a -1 in the lower δ-grading. This should be interpreted as a 0, with a 1 added to the rank of the higher δ-grading.

In the course of proving this theorem (or rather Theorem 2.6 for links), we will see how the bigraded homology could be computed without any more real work. In fact, we will ignore gradings throughout most of the paper, and then deduce them at the end when needed.

1.1. Homological thinness. Starkston in [10] and Qazaqzeh in [8] were interested in the class of homologically thin, non-quasi-alternating pretzel knots. Starkston conjectured, and Qazaqzeh proved, that all $P(-p, p, r)$

pretzel knots with p odd and $r \geq p$ are homologically thin but not quasi-alternating. Our results here (e.g. Theorem 2.6) imply that these are the only homologically thin, non-quasi-alternating 3-column pretzel links. All other non-QA ones (including, e.g. $P(-p, p, r)$ with p even and $r \geq p$) are homologically thick. The same result could be deduced from the unreduced homology calculated in [7], but this paper's emphasis on the δ-grading makes it easier to see.

1.2. Acknowledgments. The author would like to thank Zoltán Szabó for many helpful discussions during the course of this work.

2. Khovanov homology computation.

We will use Bar-Natan's dotted cobordism formulation of Khovanov homology in this section; see Section 11.2 of [2]. In particular, if T is a tangle diagram, then its Khovanov chain complex is an object of the category $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{\cdot/l}))$.

2.1. Local preliminaries. The computation will use a lemma about the formal dotted-cobordism complex associated to a series of n half-twists. Effectively, the lemma is Proposition 25 of Khovanov's original paper [6], interpreted in the language of dotted cobordisms. We will give a proof here, to keep this paper as self-contained as possible.

First, though, we recall a fact about dotted cobordism pictures. Let D be a crossingless tangle diagram. If D contains a complete circle c, then D is isomorphic in the category $\mathrm{Mat}(\mathrm{Cob}^3_{\cdot/l})$ to $D' \oplus D'$, where D' is D with c removed. This "delooping" isomorphism is written down by Bar-Natan in [3]; the proof consists of a diagram, reproduced here for convenience (with a few modifications) in Figure 2.

Whenever we apply a delooping isomorphism to remove a circle, one of the two resulting summands will have a dot in the lower-right corner and one will not. The dot indicates the element whose q-grading is shifted by -1 rather than $+1$. The maps F and G in Figure 2 are inverses of each other, proving the delooping isomorphism.

The lemma we need is the following:

Lemma 2.1. *The (formal) Khovanov chain complex of the positive n-half-twisted strand on the left side of Figure 3 is homotopy equivalent to the dotted-cobordism complex on the right side of Figure 3.*

Remark 2.2. We have ignored gradings in Lemma 2.1 because it's not necessary for our purposes to keep track of them now. We will be able to deduce them later given our knowledge of the boundary map. However, due to signs, it is still important to order of the crossings. We order the crossings in the n-half-twisted strand from bottom to top, and on the right side of the diagram, we show how the first few generators are labelled.
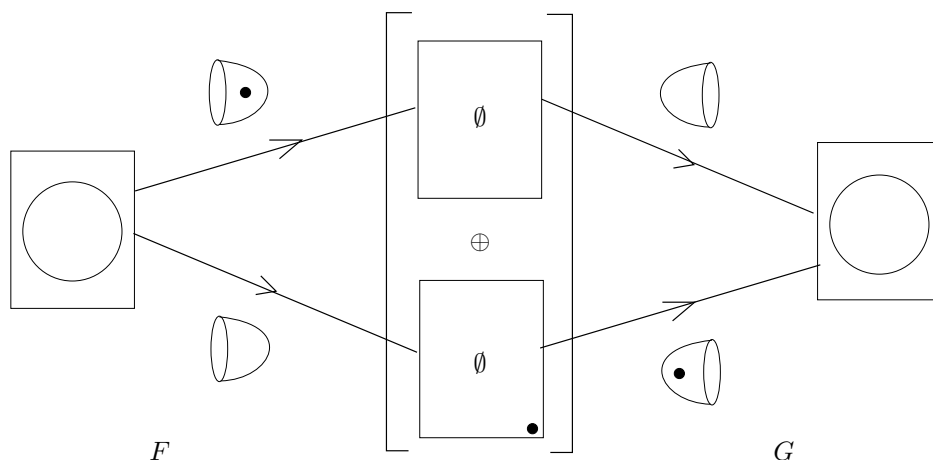
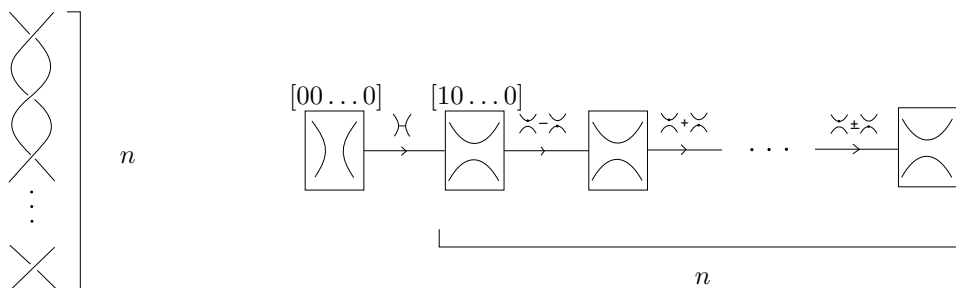FIGURE 2. Bar-Natan's delooping isomorphism, taken from [3].

FIGURE 3. Simplification of Kh(positive n-twisted tangle). There are n boxes (and $n-1$ maps) in the section labeled on the right. The final sign is $-$ for n even and $+$ for n odd.

Proof of Lemma 2.1. We will induct on n; the case $n = 1$ is true by the definition of the formal complex for a single crossing. Assume the lemma is true for $n - 1$; then we can use the induction hypothesis to replace the formal complex for the n-twisted strand by the one shown at the top of Figure 4. Delooping to get rid of the complete circles, we get the complex at the bottom of Figure 4.

Now we simplify using Gaussian elimination, as described in this context by Bar-Natan in [3]. Whenever we see an invertible matrix coefficient in the differential, we remove the two corresponding generators, and add in some "zig-zag" terms. Suppose the invertible coefficient is a, from generator g_1 to generator g_2 (write $a : g_1 \to g_2$ for convenience). Whenever we have $b : h \to g_2$ and $c : g_1 \to k$, we must add $-c \circ a^{-1} \circ b$ to the coefficient from h to k. Then the resulting complex with g_1 and g_2 removed is homotopy equivalent to the original one.

FIGURE 4. Inductive step of Lemma 2.1.
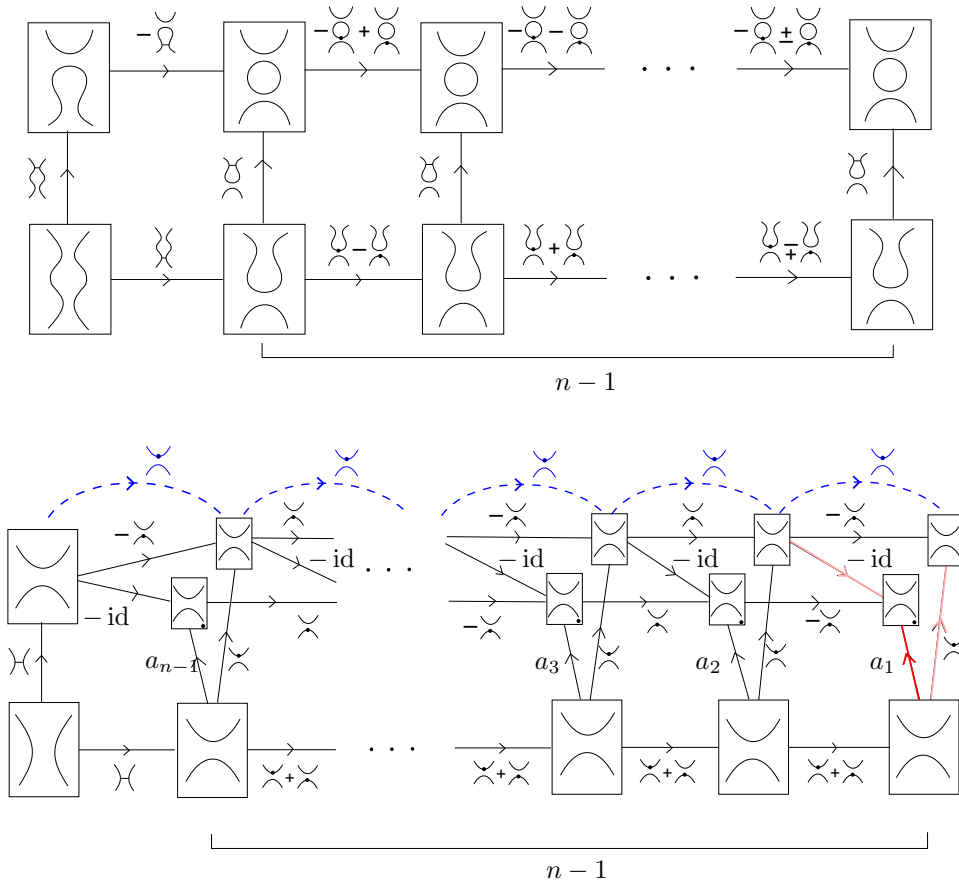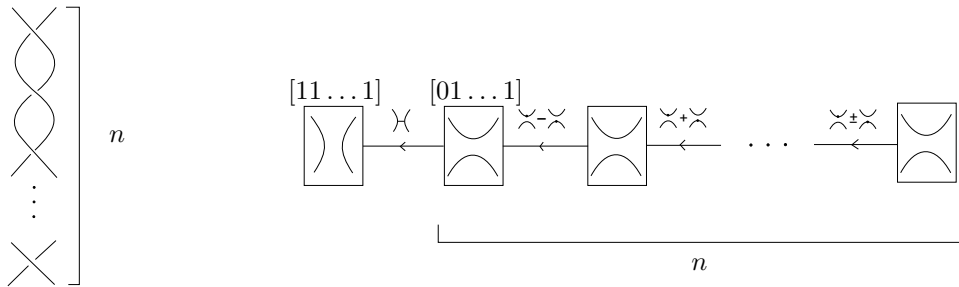

FIGURE 5. The negative analogue of Figure 3.

The edges labelled $a_1, \ldots, a_{n-1}$ all represent identity cobordisms with positive sign; we eliminate them in order, starting with a_1. Each elimination has one negative sign from the edges and another from the elimination formula, so the dotted maps in Figure 4 get positive signs. The result is the complex we want.   □
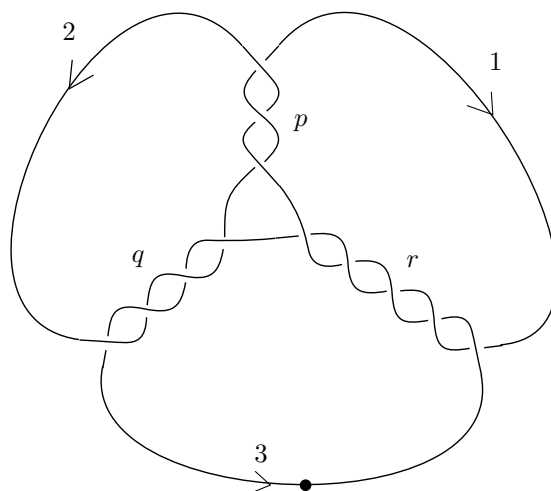
FIGURE 6. The pretzel knot $P(-3, 4, 5)$ (or $(-p, q, r)$ in general).

Taking the mirror image of this lemma gives us:

Corollary 2.3. *The (formal) Khovanov chain complex of the negative n-half-twisted strand on the left side of Figure 5 is homotopy equivalent to the dotted-cobordism complex on the right side of Figure 5.*

Strictly speaking, the global sign on any edge could turn out to be the opposite of the one depicted in Figure 5. But, up to isomorphism of complexes, we may assume the signs are as shown. Note that if we labelled the crossings in the n-half-twisted strand "up to down" rather than "down to up," we would get some complex that looks identical except for global signs on the edges (the relative signs on each edge are required by $d^2 = 0$). So by the same logic, the result is independent of this choice up to isomorphism.

2.2. Diagrams and orientations. Consider the pretzel link $P(-p, q, r)$, with a diagram D and basepoint drawn as in Figure 6. Order the crossings so that those in the p-strand come first, then those in the q-strand, then those in the r-strand. Within each strand, the crossings should be ordered from one end to the other; as noted above, it doesn't matter which end is which.

Since the Khovanov homology of links depends on relative orientations between the components, we need to be able to specify these orientations. We will do this here, although it will not be needed until we fix absolute gradings at the end.

There are three arrows in Figure 6, labelled 1, 2, and 3 (the directions are chosen somewhat arbitrarily to agree with the example in that diagram). We will say a pretzel link has orientation $+ + +$ if its orientations agree with the three arrows, or $- + +$ if they disagree at position 1 but agree at 2 and 3, etc. Since Khovanov homology is invariant under overall change

Pattern	n_+	n_-
$++$	r	$p+q$
$+-$	$p+q+r$	0
$-+$	p	$q+r$
$--$	q	$p+r$

TABLE 1. Values for n_+, the number of positive crossings in the diagram, and n_-, the number of negative crossings, given the orientation pattern.

of orientation, we may fix once and for all a $+$ in the third column, i.e. our links will always be oriented in agreement with arrow 3. Then we may simply write $++$, $-+$, etc. for the orientation of the link at positions 1 and 2.

When $P(-p, q, r)$ is a knot, the orientations at positions 1 and 2 are fixed. If q is even (and p and r are odd), like in Figure 6, the orientation is $++$. If only p is even, the orientation is $+-$. If only r is even, the orientation is $--$. Finally, if p, q, and r are all odd, the orientation is $-+$.

When $P(-p, q, r)$ is a two-component link, two out of the four orientation patterns are possible (depending on the parity of p, q, and r). When it is a three-component link, all four patterns are possible.

If we know the orientation pattern of the link, we can deduce the number of positive and negative crossings; these will arise in the grading formulas. We summarize them in Table 1 for convenience.

2.3. Computation of the complex. We now analyze the formal dotted-cobordism Khovanov complex of D. The tangle complex of each of the three twisted strands of D may be simplified using Lemma 2.1. The formal complex of D is then homotopy equivalent to the cube shown in Figure 7, with $(p+1)(q+1)(r+1)$ generators. Our goal will be to simplify this cube even further, using delooping and elimination, until we understand the differential completely.

While the resolution diagrams in Figure 7 and subsequent figures omit the basepoint for convenience, there should always be a basepoint on each diagram as indicated in Figure 6. This basepoint causes many cube differentials to be zero; those marked zero in Figure 7 are zero because they have a dot on the basepoint component.

2.3.1. Columns in the cube. We start by thinking of the cube as made up of vertical columns. Consider the columns which are not on either of the two "back walls" of the cube as drawn in Figure 7. We will consider these columns one at a time, starting with the one closest to the viewer in Figure 7.

The left side of Figure 8 shows any of these columns. According to the discussion above, we can choose (once and for all) to make the signs on
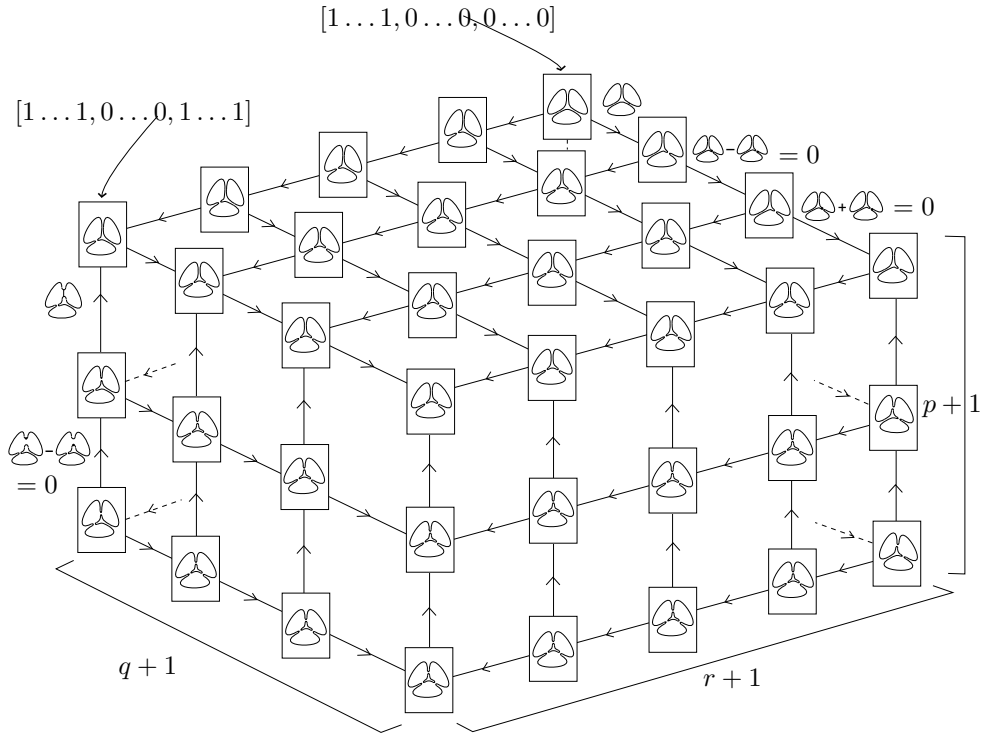
FIGURE 7. Chain complex homotopy equivalent to $Kh(P(-p, q, r))$, where $(p, q, r) = (-2, 3, 4)$ in the picture. All of the arrows should have labels; only some are shown for convenience. We have also shown the crossings-label of two generators to indicate the general pattern. The generator $[0 \ldots 0, 0 \ldots 0, 0 \ldots 0]$ is precisely the corner of the cube obscured to the viewer.

the p-columns as depicted. (This corresponds to choosing different global signs on various edges in Corollary 2.3.) We then apply delooping to get the column on the right of Figure 8.

2.3.2. Paths. We want to see what happens when we simplify each (non-back-wall) column by applying Gaussian elimination to the maps marked $-\operatorname{id}$ in Figure 8. We will get a complex like the one in Figure 9. It has two "walls" and a "floor"; note that we have applied delooping to the top vertices on each wall. In this diagram, when a circle is delooped, we depict it as a dashed circle in each resulting generator. There is a dot on one of the two dashed circles.

The arrows in Figure 9 all represent components of the differential. For visual clarity, some grid lines are also shown in Figure 9, but they do not represent components of the differential; only the arrows with a direction contribute to the differential.
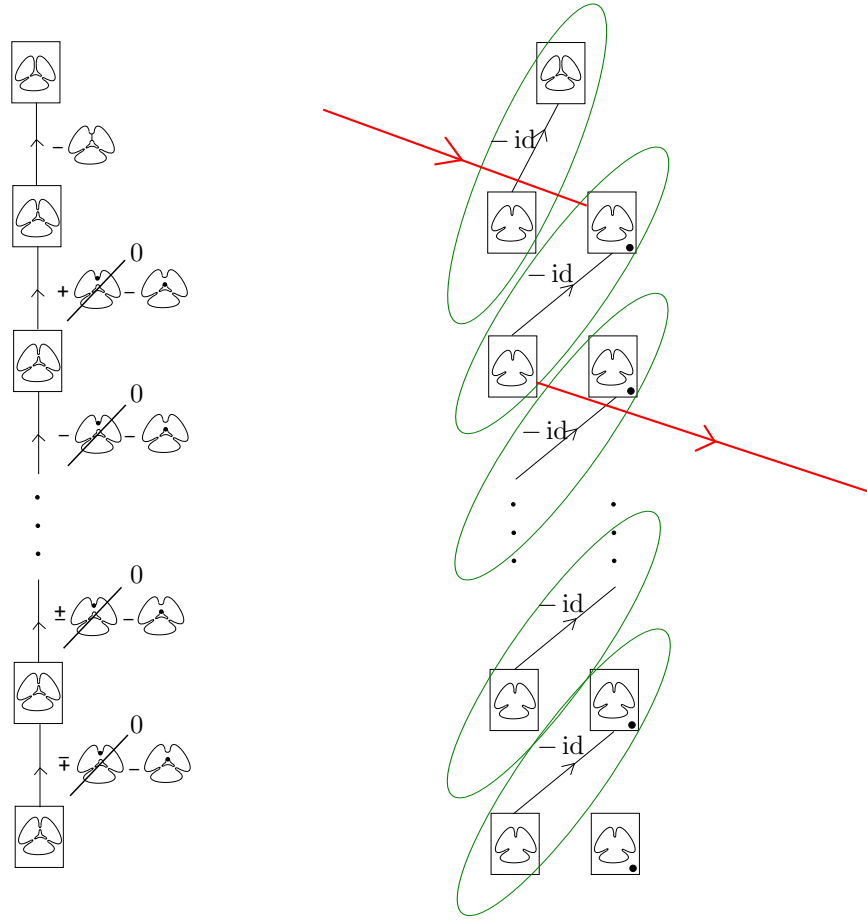
FIGURE 8. A vertical column of the cube, not on the back walls. The red arrows on the right are meant to suggest the zig-zag maps which result from cancelling circled pairs of generators.

There are more differentials like the red ones, based at each wall generator except for the column where the walls intersect. These arrows all point at floor generators, and they are due to zig-zag maps arising from the eliminations we did in the columns.

Such a component, arising from repeated eliminations, looks like a path of arrows. The colored paths on the left of Figure 10 are examples. The paths must start and end in the positive x- or y-direction, and must alternate such "horizontal" steps (always positive) with vertical steps down (via the eliminated identity components in the interior columns). Paths starting on the left wall must start in the y-direction, and paths starting on the right wall must start in the x-direction.
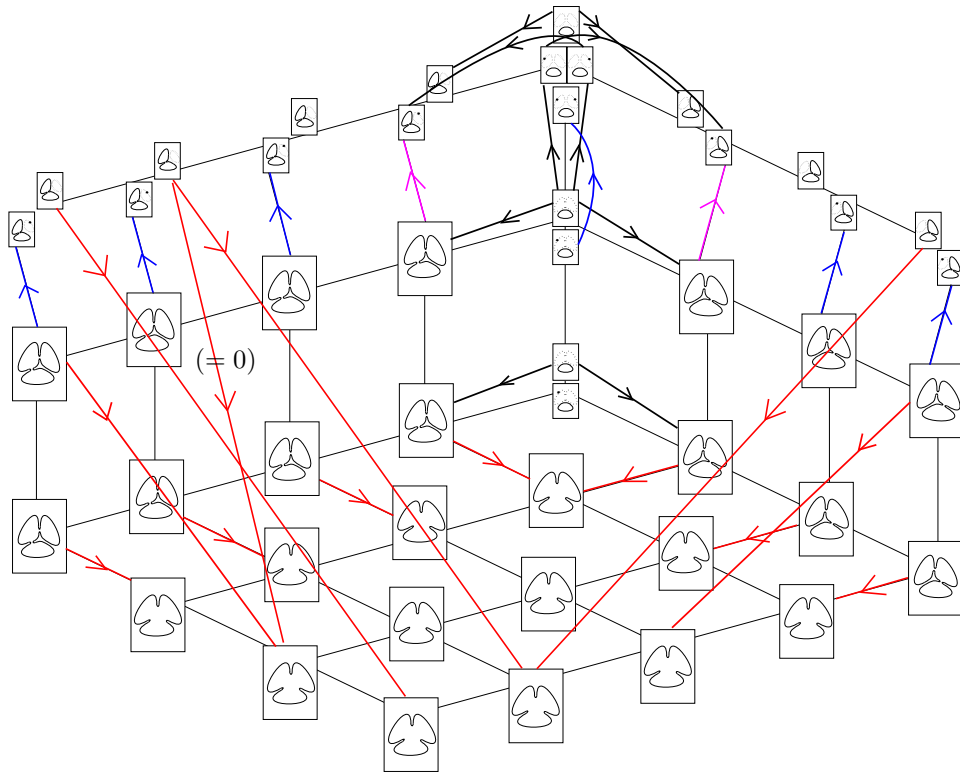
FIGURE 9. The cube after doing eliminations of all non-back-wall columns. All arrows represent components of the differential; there are more differentials like the red arrows that are not shown.

Each horizontal step in the path is $\pm$ id, except for steps outward from the very tops of the walls. These top steps are join maps before delooping; after, the "top" component is $\pm$ id and the "bottom" one is $\pm$(dotted id). However, none of the dotted-id maps can contribute to nonzero maps remaining after elimination, because they (eventually) put a dot on the basepoint. Hence the only nonzero red arrows come from paths with each step $\pm$ id.

Let w be a (remaining) wall generator and f be a floor generator. Then dw contains n copies of f where n is the signed count of paths from w to f.

2.3.3. Signs on paths. To actually compute n, we need to pin down the signs on the paths. It is easiest to do this with an example. In Figure 10, there is one path connecting w to f and three paths connecting w' to f'. These paths are marked with various colors. There is also a pattern drawn in orange at the right of Figure 10. To compute the sign of a path, project it down to the floor of the cube and walk along it backwards (starting from the floor generator). Start with a $+$ sign. On the first step (and all odd-numbered steps), the sign flips if the path traverses a black edge. On the
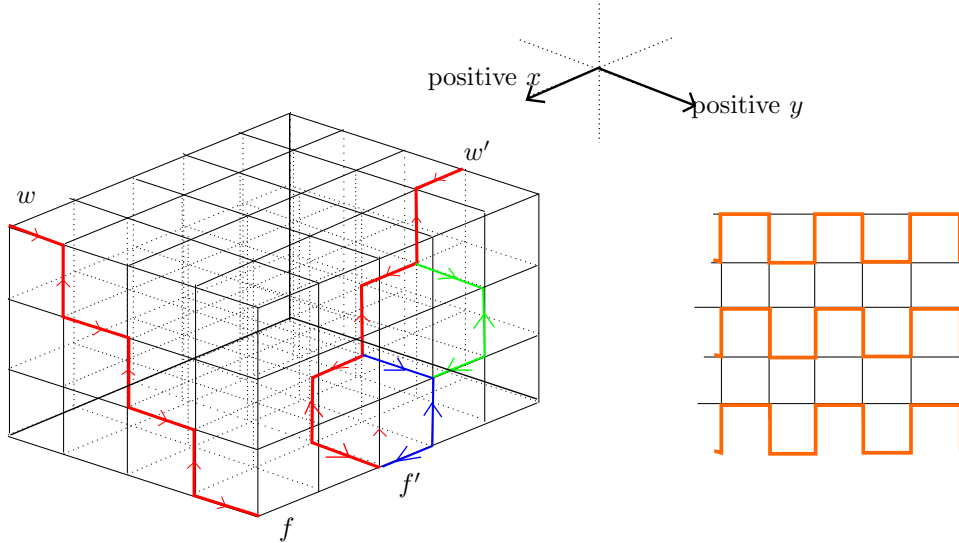
FIGURE 10. On the left: paths which contribute to the differential. In this diagram, $(p, q, r) = (3, 4, 5)$. On the right: pattern of positive signs (orange) and negative signs (black) on the bottom layer of the cube.

second step (and all even-numbered steps), the sign flips if the path traverses an orange edge. The resulting sign is the sign of the path. (Equivalently, we draw the opposite of the orange pattern every other level, and then always flip on black edges.)

For instance, the path from f back up to w goes B, O, B, O (where B denotes a black edge and O denotes an orange edge), so its sign is $+$ after four flips. The red path from f' back up to w' goes B, B, O, B, so it gets $-$ after one flip. The blue path (O B O B) gets $+$ (zero flips), and the green path (O B B B) gets $-$ (one flip).

To see why these signs are correct, note that the orange pattern is just the standard pattern of positive signs in a double complex. In the full triple complex we are considering, the vertical arrows are all negative and the pattern of positive horizontal signs switches every vertical layer. But every time we eliminate a vertical arrow in a column simplification, the negative sign on the arrow cancels the negative sign from the Gaussian elimination. Thus the sign on the resulting arrow is just the product of the signs on its horizontal components, and this pattern continues to hold even after many cancellations. Hence the signs on the paths are as described.

Remark 2.4. Note for future purposes that when two paths "differ by a cube," as the red/blue paths or the blue/green paths are related, then they have the opposite sign. To see why this is true, color everything with an orange/black pattern which flips every level, so sign flips are always black edges (as mentioned above). Consider (e.g.) the cube spanned by the red

and blue paths, and the squares on the top and bottom of this cube. These squares each have a "source" vertex with two outgoing edges and a "sink" vertex with two incoming edges. Since the pattern is such that every square has an odd number of orange/black edges, the outgoing edges match color precisely when the incoming edges don't. Furthermore, the outgoing edges match color on the top iff the outgoing edges match on the bottom, and the same is true for incoming edges.

Now, the difference between the red and blue paths has two outgoing edges, two incoming edges, and two vertical edges which are positive. If the outgoing edges match, the incoming edges don't, so the total sign difference is -1, and similarly if the incoming edges match. Thus any paths differing like the red and blue paths must differ in sign.

2.3.4. Cancelling more differentials.

Figure 11 shows the complex of Figure 9 from the top and rotated a bit clockwise. For now we will assume $p + 2 \leq q$, r; this is the generic case. Later we can look back and see what happens when $q = p$ or $q = p + 1$ (without loss of generality we may assume $q \leq r$).

No paths are long enough to hit the floor generators circled with a 1 in Figure 11, so these generators survive to homology. There are $(q - p - 1)(r - p - 1)$ of them. We want to determine which other floor generators survive to homology.

First, look at the blue differentials in Figure 9. No other arrows have the same tip as these, so they may all be cancelled without picking up extra zig-zag maps. The purple arrows are almost like the blue ones, except for the presence of black arrows with the same tip as the purple ones. If we wanted, we could still cancel these and pick up some extra maps, but instead we will leave them be for now; later we will be able to cancel them without extra maps.

After cancelling the blue arrows, nearly all of the second-to-top-row wall generators are gone. Now consider the uncircled dots in the leftmost column of Figure 11. Each gets matched up with a wall generator in the leftmost wall column, via arrows like the three red arrows on the far left of Figure 9. Each arrow counts only one path, so it is automatically $\pm\,\mathrm{id}$. The leftmost dot lying in the circle "2" would get matched with the missing generator from the second-to-top row. Cancelling arrows, we see that all the uncircled dots die but the circled dot survives to homology. No extra maps are generated because no other arrows share a tail with any of the cancelled arrows.

Having dealt with the leftmost column, we move inward and do the same thing with the next one. All the dots marked 2 (resp. 3) survive to homology, and uncircled dots in their columns (resp. rows) do not.

With the dots marked 5 and 6, all we can say right now is that they are potentially homology generators. To see why, look at the dot marked f_1. This dot is the first to get hit by two wall generators w_1 (from the north wall)
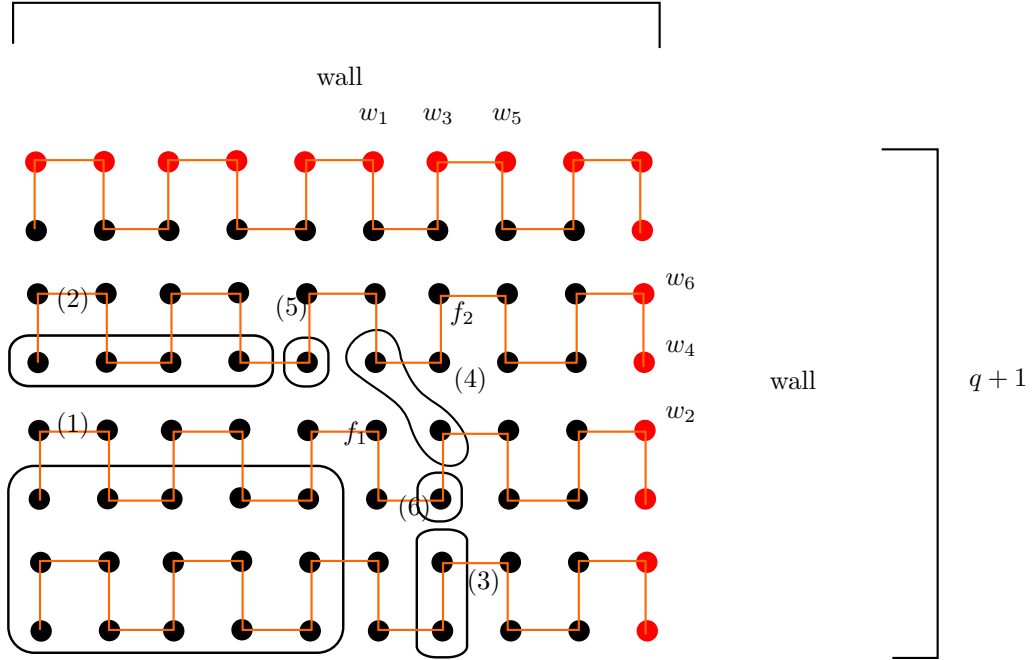
FIGURE 11. Top-down view of the remaining generators, for $(p, q, r) = (3, 7, 9)$. The pattern of positive signs on the bottom layer is also shown (in orange) for convenience.

and w_2 (from the east wall), i.e. it is the first place where our cancellation process runs into complications.

To determine what happens here, we want to identify the maps from w_1 to (5) and from w_2 to (6). The parity of p becomes important here:

Proposition 2.5. *If p is even, the maps $w_1 \to$ (5) and $w_2 \to$ (6) are zero. If p is odd, these maps are $\pm \mathrm{id}$.*

Proof. Without loss of generality, consider $w_1 \to$ (5). There are p paths contributing to this matrix coefficient, and they are arranged in a sequence like the red/blue/green paths in Figure 10. As noted in Remark 2.4, since each path in the sequence differs from the next by a cube, the signs alternate along the sequence. Hence the total sum is 0 if p is even and ± 1 if p is odd. □

If p is even, we can choose (arbitrarily) to cancel the arrow from w_1 to f_1, and we pick up no extra maps from w_2 to (5) or (6). Hence both (5) and (6) survive to homology. If p is odd, we can still cancel the same arrow; we get an additional map from w_2 to (5), but this just means w_2 dies in homology and a rank 1 summand of $\langle (5), (6) \rangle$ survives.

The picture looks similar if we move one step up and to the right. If p is even, then the wall generator w_3 hits only one of the dots in circle 4, and

w_4 hits the other one. The maps have coefficient ± 1, so neither dot in (4) survives to homology.

If p is odd, we have to be careful with signs. Using the sign algorithm discussed above, the path from w_3 to the bottom dot in (4) gets $p + 1$ plus signs, for a net $+$. The path from w_4 to the left dot in (4) gets $p + 1$ minus signs, for a net $+$ as well. The paths from w_3 to the left dot in (4) have signs $-, +, \ldots, -$, so the net sign is $-$. Finally, the paths from w_4 to the bottom dot in (4) also have signs $-, +, \ldots, -$ for a net $-$. To make it easier for the reader to follow these calculations, the sign pattern is overlaid in Figure 11 for convenience (only the orange edges are shown, to avoid cluttering the diagram). The picture looks qualitatively the same for all odd p, so checking what happens when $p = 3$ yields the general pattern.

Now that we know these signs, the homology is easy; the wall generators contribute a $\mathbb{Z}$ summand, and the floor generators contribute $\frac{\mathbb{Z}^2}{(1,-1)} \cong \mathbb{Z}$.

Finally, consider the dot marked f_2. If p is odd, this dot gets hit by both w_5 and w_6, so it does not survive to homology. However, if p is even, the maps from w_5 and w_6 to f_2 are zero, so f_2 survives to homology.

The rest of the floor generators do not contribute (generators at the bottoms of the walls are excluded from being floor generators). Indeed, the process of cancelling these generators using red arrows from the walls proceeds without needing to consider non-straight-line paths.

At this point, we can look back at the purple arrows on the walls in Figure 9; they no longer share tails with any other nonzero arrows, so we can cancel them just like we did the blue arrows.

2.3.5. Remaining cases. Counting everything up, we have found $(q - p)(r - p)$ surviving floor generators if p is even and $(q - p)(r - p) - 1$ if p is odd. We obtained this formula assuming $p + 2 \leq q, r$, but at this point it is not too difficult to look back and see what happens in the remaining cases.

First suppose $q = p + 1$. In Figure 11, the missing "features" are the dots circled 1, 6, and 3. If p is even, the dot circled 5 survives to homology, and there are $r - p$ surviving floor generators in total. This number agrees with our existing formula, so we do not need to modify it.

If p is odd, (5) does not survive to homology. There are $r - p - 1$ surviving floor generators if $r > p + 1$, and 1 if $r = p + 1$. Thus the only case where the formula needs modification is when $r = p + 1$; there the formula would give 0 surviving generators instead of the correct number 1.

Now suppose $q = p$. If p is odd, our formula predicts -1 floor generators, so clearly it needs modification. But each circled floor generator gets hit by a corresponding wall generator, by an arrow parallel to the one from w_1 to (5). These arrows are nonzero by the argument of Proposition 2.5, and they can be cancelled one at a time. Hence no floor generators survive to homology, and we just need to change the -1 to a 0.

If p is even, the formula requires more drastic modification. The quantity $(r-p)(q-p)$ is zero, but the generators (2), (5), and f_2 survive to homology, for a total of 1 if $r = q = p$ and $r - p$ if $r > q - p$.

2.4. Relative gradings. In fact, the floor generators we have found in homology are all in the lowest possible δ-grading of $Kh(P(-p, q, r))$, and they comprise the entire homology in this grading. We can see this by analyzing the relative gradings of various generators in the complex.

Any time we deloop, the two resulting summands differ by 2 in the q-grading; the summand without the dot is 2 steps higher. The summands have the same homological grading. Also, nonzero components of the differential must preserve the q-grading and increase the homological grading by one. These two facts will be all we need to determine the relative gradings of the generators. The second fact holds before and after any eliminations; we will be sloppy and not always identify which stage of the elimination process contains the nonzero component in the differential (it should be clear from context).

In terms of the grading $\delta = q/2 - h$, two summands from a delooping differ in δ-grading by 1, and the differential decreases δ-grading by 1. For simplicity, we will focus on the δ-grading here.

In the internal columns (right side of Figure 8), there are only two δ-gradings, say g and $g + 1$; the generators on the left have grading $g + 1$, and those on the right have grading g. This observation follows immediately from the two properties stated above; note that g is the same from column to column because of the differential components connecting the columns. All the floor generators (i.e. those surviving after cancellation of the identity maps in the column) have δ-grading g. In particular, this is true for the homology generators we have found so far.

Now look at generators on the walls (except those in the column where the walls intersect). All wall generators not on top of the walls have nonzero components of their differential on floor generators (because of the zig-zag maps). Hence all these generators must lie in δ-grading $g+1$. The top nodes on the walls contribute two generators each; the higher one's differential hits a floor generator, so the δ-gradings of these two generators are again g and $g + 1$. Note that all the ones in grading g do not survive to homology (they die when the blue and purple differentials in Figure 9 are cancelled).

Finally, in the column where the walls intersect, all nodes contribute two generators except the top node which contributes four. In the two-generator nodes, the δ-gradings are $g+2$ and $g+1$ because the black arrows are nonzero. In the four-generator node, the gradings are $g + 2$, $g + 1$, and g, and the generator in grading g gets cancelled with the blue arrows.

We have now isolated the homology in δ-grading g; except in the special cases, it is $\mathbb{Z}^{(q-p)(r-p)}$ if p is even and $\mathbb{Z}^{(q-p)(r-p)-1}$ if p is odd. It is also not hard to see what happens between gradings $g + 2$ and $g + 1$. Consider the grading-$g + 2$ generators in the two-generator nodes of Figure 9. Each

has two nonzero black arrows pointing to generators on the walls. As long as at least one of these wall generators still exists on each level after the cancellations we have done, the grading-$g + 2$ generators do not survive to homology. But all of the wall generators still exist except for one of the two on the very bottom level (depending on which one we chose to cancel). So we can cancel each grading-$g + 2$ generator with a wall generator using a black arrow.

All the remaining generators are in δ-grading $g + 1$, so we can conclude our complex has no more differentials. We just need to count generators to determine the rest of the homology; at no point has any torsion appeared, so the homology is free. The count of generators is easy if we only care about the δ-graded theory; for the bigradings, we refer the reader to the appendix.

To count the generators in grading $g + 1$, first consider the generic case $(p + 2 \leq q, r)$. Note that we started with $qr + q + r + 1$ generators in grading g (qr from the floor generators and $q + r + 1$ from the top layer). There were $(p + 1)(r + 1) + (p + 1)(q + 1) - p$ wall generators in grading $g + 1$ and $p + 1$ generators in grading $g + 2$. If p is even, $(q - p)(r - p)$ of the floor generators survive to homology, so $(qr + q + r + 1) - (q - p)(r - p)$ cancellations occurred between gradings $g + 1$ and g. None of the grading-$g + 2$ generators survive, so $p + 1$ cancellations occurred between gradings $g + 2$ and $g + 1$. Thus the remaining number of generators in grading $g + 1$ is

$$
\begin{aligned}
&(p + 1)(r + 1) + (p + 1)(q + 1) - p \\
&- ((qr + q + r + 1) - (q - p)(r - p)) - (p + 1) \\
&= p^2,
\end{aligned}
$$

after some simplifying. If, instead, p is odd, then one fewer floor generator survives to homology. Hence there is one more cancellation between gradings $g + 1$ and g, and only $p^2 - 1$ generators remain in grading $g + 1$.

In summary, in the generic case, if p is even the δ-graded homology is $\mathbb{Z}_{(g)}^{(q-p)(r-p)} \oplus \mathbb{Z}_{(g+1)}^{p^2}$. If p is odd, it is $\mathbb{Z}_{(g)}^{(q-p)(r-p)-1} \oplus \mathbb{Z}_{(g+1)}^{p^2-1}$.

For the special cases, if p is odd and $q = r = p + 1$, one extra floor generator survives to homology. Hence we also have an extra generator in grading $g + 1$, and the homology is $\mathbb{Z}_{(g)}^{(q-p)(r-p)} \oplus \mathbb{Z}_{(g+1)}^{p^2}$ (like in the generic case of p even), which simplifies to $\mathbb{Z}_{(g)} \oplus \mathbb{Z}_{(g+1)}^{p^2}$.

If p is odd and $q = p$, the answer is similar. Since the generic formula undershot the homology in grading g by 1 (by having -1 rather than 0), it also undershot the homology in grading $g + 1$ by 1. So the homology is $\mathbb{Z}_{(g+1)}^{p^2}$.

If p is even and $q = p$, we understated the homology in grading g by a larger amount: $r - p$ for $r > p$ or 1 for $r = p$. Hence, if $r > p$, the homology is $\mathbb{Z}_{(g)}^{r-p} \oplus \mathbb{Z}_{(g+1)}^{p^2+r-p}$. If $r = p = q$, it is $\mathbb{Z}_{(g)} \oplus \mathbb{Z}_{(g+1)}^{p^2+1}$

2.5. Absolute gradings. All we need to do to finish the δ-graded computation is to identify g. We will do this by computing the grading of the grading-g generator in the top (4-generator) node in Figure 9. No deloopings were performed in the making of this generator, so its q-grading is not affected by any of the delooping shifts. Its dot-degree is -2, and its pattern of crossing resolutions is $[1 \ldots 1, 0 \ldots 0, 0 \ldots 0]$ which has weight p. Hence its q-grading is $-2 + p + n_+ - 2n_-$. The numbers n_+ and n_- of positive and negative crossings can be computed from the orientation of the link and Table 1.

Similarly, the homological grading of this generator is $p - n_-$. Hence its δ-grading g is $q/2 - h = (-2 - p + n_+)/2$. We have completed the computation of the δ-graded Khovanov homology of $P(-p, q, r)$, which we state as a theorem (to be compared with Theorem 1.1):

Theorem 2.6. *Let p, q, and r be positive integers with $p \le q \le r$. Let H_δ be the reduced Khovanov homology of $P(-p, q, r)$ in grading δ (with basepoint chosen as in Figure 6).*

- *If p is even and $p + 1 \le q, r$, then $H_{\frac{-p+n_+}{2}} = \mathbb{Z}^{p^2}$ and $H_{\frac{-2-p+n_+}{2}} = \mathbb{Z}^{(q-p)(r-p)}$. All other H_δ are 0.*
- *If p is odd, $p + 1 \le q$, and $p + 2 \le r$, then $H_{\frac{-p+n_+}{2}} = \mathbb{Z}^{p^2-1}$ and $H_{\frac{-2-p+n_+}{2}} = \mathbb{Z}^{(q-p)(r-p)-1}$. All other H_δ are 0.*
- *If p is odd and $q = r = p+1$, then $H_{\frac{-p+n_+}{2}} = \mathbb{Z}^{p^2}$ and $H_{\frac{-2-p+n_+}{2}} = \mathbb{Z}$. All other H_δ are 0.*
- *If p is even, $q = p$, and $r > p$, then $H_{\frac{-p+n_+}{2}} = \mathbb{Z}^{p^2+r-p}$ and $H_{\frac{-2-p+n_+}{2}} = \mathbb{Z}^{r-p}$. All other H_δ are 0.*
- *If p is even and $p = q = r$, then $H_{\frac{-p+n_+}{2}} = \mathbb{Z}^{p^2+1}$ and $H_{\frac{-2-p+n_+}{2}} = \mathbb{Z}$. All other H_δ are 0.*
- *If p is odd and $q = p$, then $H_{\frac{-p+n_+}{2}} = \mathbb{Z}^{p^2}$. All other H_δ are 0.*

The values of n_+ and n_- depend on the orientation of the link $P(-p, q, r)$ and can be computed from Table 1.

3. Appendix.

Here we wrap up some loose ends, computing the bigraded homology and making some remarks on the unreduced version.

3.1. Bigradings. We already have a reference point for the absolute bigradings, computed in Section 2.5. We only need to compute the relative bigradings. Since we know the δ-gradings, we may focus on the homological grading and use it to compute the q-grading later. This grading is actually much simpler to compute; every "forward" step in the cube (vertically upwards or horizontally out) increases homological grading by 1, regardless

of whether there are nonzero differentials connecting the generators. So we can just look through the homology generators we have found, note where they are in the cube, and deduce their homological gradings.

For the lower δ-grading $(-2 - p + n_+)/2$, all the homology generators are on the floor of the cube in Figure 9. Hence the homological gradings correspond to the diagonals of slope -1 in the square of Figure 11. The dot marked f_2 has homological grading p more than the homological grading of the generators we considered before, which was $p - n_-$. Hence we can start at f_2, with h-grading $2p - n_-$, and count dots in subsequent lower diagonals to get the homological grading of generators in this δ-grading. This grading can be used along with δ to obtain the q-grading; in particular, the q-grading of f_2 is $2(\delta + h) = -2 - p + n_+ + 4p - 2n_-$, which simplifies to $-2 + 3p + n_+ - 2n_-$.

For convenience, we will write down the formulas more explicitly. Each of the circled groups of dots in Figure 11 potentially contributes a term to the Khovanov homology, which we will describe by its Poincaré polynomial P_{Kh} (over $\mathbb{Q}$, say) because it is free. We will write the formal variables in this Poincaré polynomial by Q and H, to avoid confusion with the q we already have. Let $\phi_{p,q,r}(x) = \sum_{n=0}^{(r-p)+(q-p)-4} c_n x^n$, where the sequence c_n is defined by the pattern

$$(1, 2, 3, \ldots, (q - p - 2),$$
$$(q - p - 1), (q - p - 1), \ldots, (q - p - 1),$$
$$(q - p - 2), \ldots, 2, 1)$$

and there are $r - q + 1$ instances of $q - p - 1$ on the middle line.

Proposition 3.1. *The dots in Figure 11 represent the following possible contributions to the polynomial $P_{Kh}(P(-p, q, r))$; whether these summands appear in the formula depends on whether the dots survive to homology. (Here, and in the rest of the paper, sums indexed from 0 to a negative number should be interpreted as empty.)*

- *The dots in circle (1) contribute*

$$a_1 := Q^{6+3p+n_+-2n_-} H^{2p-n_-+4} \phi_{p,q,r}(Q^2 H)$$

- *The dots in circle (2) contribute*

$$a_2 := Q^{4+3p+n_+-2n_-} H^{2p-n_-+3} \sum_{n=0}^{r-p-3} (Q^2 H)^n.$$

- *The dots in circle (3) contribute*

$$a_3 := Q^{4+3p+n_+-2n_-} H^{2p-n_-+3} \sum_{n=0}^{q-p-3} (Q^2 H)^n.$$

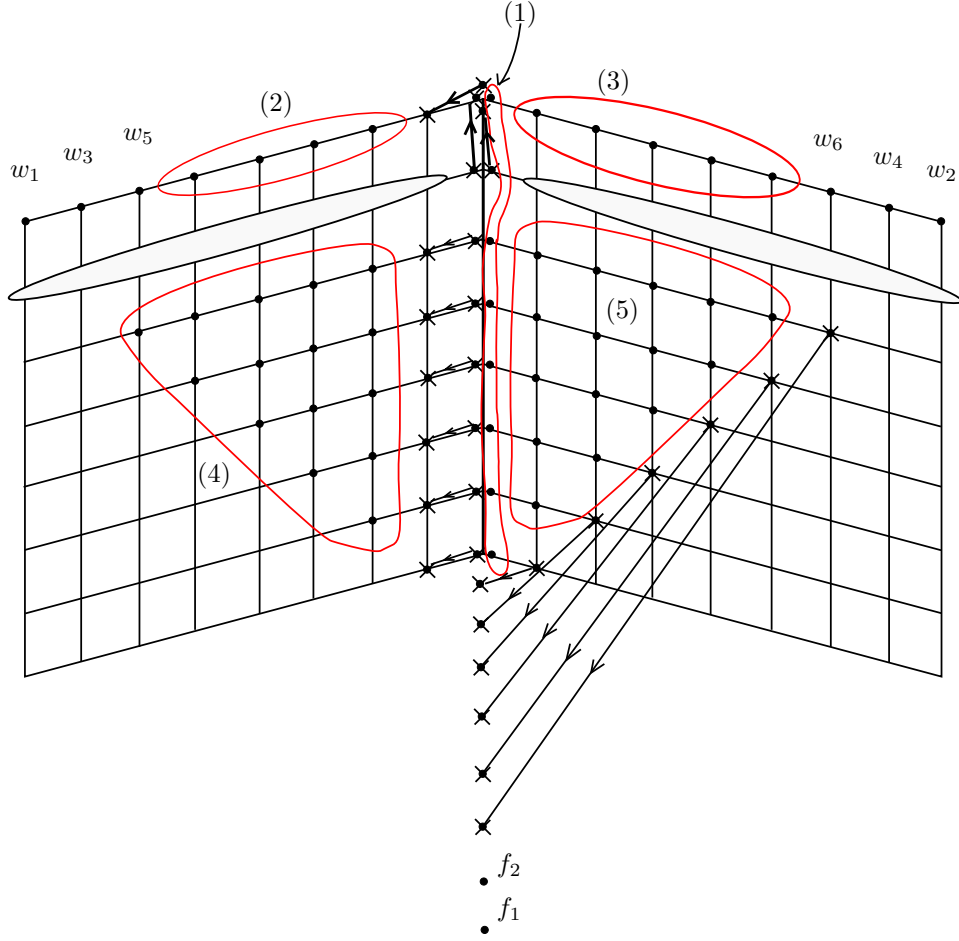- *The dots in circle (4) each contribute $Q^{3p+n_+-2n_-} H^{2p-n_-+1}$.*

FIGURE 12. Generators remaining in δ-grading $(-p + n_+)/2$.

- *The dots in circles (5) and (6) each contribute*

$$Q^{2+3p+n_+-2n_-} H^{2p-n_-+2}.$$

- *The dot f_2 contributes $Q^{-2+3p+n_+-2n_-} H^{2p-n_-}$.*

Before we go through and total up the contributions for each case of $P(-p, q, r)$, we will analyze the bigradings in the higher δ-grading $(-p + n_+)/2$. Figure 12 shows the relevant generators, after cancelling all arrows. Dots marked with an x have been cancelled, while unmarked dots contribute to homology. Their contributions to P_{Kh} are summarized below; before stating the formulas, though, we make another definition. Given a positive

integer k, let $\psi(k) = \sum_{n=0}^{2k} c_n x^n$, where c_n is defined by the pattern

$$(1, 1, 2, 2, \ldots, (k/2 - 1), (k/2 - 1),$$
$$k/2, k/2, k/2,$$
$$(k/2 - 1), (k/2 - 1), \ldots, 1, 1)$$

if k is even and

$$(1, 1, 2, 2, \ldots, (k-1)/2, (k-1)/2,$$
$$(k+1)/2,$$
$$(k-1)/2, (k-1)/2, \ldots, 1, 1)$$

if k is odd.

Proposition 3.2. *The dots in Figure 12 represent the following contributions to the polynomial $P_{Kh}(P(-p, q, r))$; for w_1 through w_6, whether their summands appear depends on the parity of p (with a few special cases).*

- *The dots in circle (1) contribute*

$$b_1 := Q^{-p+n_+ -2n_-} H^{-n_-} \sum_{n=0}^{p-2} (Q^2 H)^n + Q^{p+n_+ -2n_-} H^{p-n_-}.$$

- *The dots in circle (2) contribute*

$$b_2 := Q^{4+p+n_+ -2n_-} H^{2+p-n_-} \sum_{n=0}^{p-4} (Q^2 H)^n.$$

- *The dots in circle (3) contribute*

$$b_3 := Q^{2+p+n_+ -2n_-} H^{1+p-n_-} \sum_{n=0}^{p-3} (Q^2 H)^n.$$

- *The dots in circle (4) contribute*

$$b_4 := Q^{6-p+n_+ -2n_-} H^{3-n_-} \psi_{p-2}(Q^2 H).$$

- *The dots in circle (5) contribute*

$$b_5 := Q^{4-p+n_+ -2n_-} H^{2-n_-} \psi_{p-2}(Q^2 H).$$

- *The dots w_1 and w_2 each contribute $Q^{2+3p+n_+ -2n_-} H^{1+2p-n_-}$.*
- *The dots w_3 and w_4 each contribute $Q^{3p+n_+ -2n_-} H^{2p-n_-}$.*
- *The dots w_5 and w_6 each contribute $Q^{-2+3p+n_+ -2n_-} H^{-1+2p-n_-}$.*

Generically, when p is even, both w_5 and w_6 contribute to homology, while both w_3 and w_4 die. (If p is 2, however, w_5 coincides with an X in Figure 12, so it does not count.)A rank-one summand of $\{w_1, w_2\}$ survives to homology. When p is odd, both $\{w_5, w_6\}$ and $\{w_3, w_4\}$ contribute rank-1 summands, and both w_1 and w_2 die.

For the special cases: when p is odd and $q = r = p + 1$, then w_1, w_3, and w_5 contribute to homology. If p is even, $q = p$, and $r > p$, then w_3 and

all the dots in the line to its left in Figure 11 survive, while if $r = p = q$ then only w_4 survives. If p is odd and $q = p$, then (besides w_4 and w_6) the farthest-left dot on the top line of Figure 11 survives. (Note that we have stopped saying "rank-1 summand," for convenience, choosing instead to pick arbitrarily which generator survives in some cases.)

We are ready to state the formula for the bigraded reduced Khovanov homology of $P(-p, q, r)$. There will be several cases, and we will use the polynomials a_i and b_i from Proposition 3.1 and Proposition 3.2 respectively.

Theorem 3.3. *Let p, q, and r be positive integers with $p \leq q \leq r$. The reduced (even) Khovanov homology of $P(-p, q, r)$ in grading δ (with basepoint chosen as in Figure 6) is free over $\mathbb{Z}$. Let P_{Kh} be its Poincaré polynomial over $\mathbb{Q}$; then P_{Kh} is given by the following formulas when $p \geq 3$. (Sums from 0 to a negative number are taken to be empty, as before.)*

When $p = 2$, note that each case of "p even" in the formula has a term of the form $2Q^{-2+3p+n_+-2n_-}H^{-1+2p-n_-}$. The coefficient 2 here should be replaced with a 1 when $p = 2$, because the dot w_5 is missing from the count of generators.

- *If p is even and $p + 2 \leq q, r$, then*

$$P_{Kh} = \sum_{n=1}^{3} a_i + \sum_{n=1}^{5} b_i + 2Q^{2+3p+n_+-2n_-}H^{2p-n_-+2}$$
$$+ Q^{-2+3p+n_+-2n_-}H^{2p-n_-} + Q^{2+3p+n_+-2n_-}H^{1+2p-n_-}$$
$$+ 2Q^{-2+3p+n_+-2n_-}H^{-1+2p-n_-}.$$

- *If p is odd and $p + 2 \leq q, r$, then*

$$P_{Kh} = \sum_{n=1}^{3} a_i + \sum_{n=1}^{5} b_i + Q^{2+3p+n_+-2n_-}H^{2p-n_-+2} + Q^{3p+n_+-2n_-}H^{2p-n_-+1}$$
$$+ Q^{3p+n_+-2n_-}H^{2p-n_-} + Q^{-2+3p+n_+-2n_-}H^{-1+2p-n_-}.$$

- *If p is even, $q = p + 1$, and $r > q$, then*

$$P_{Kh} = a_2 + \sum_{n=1}^{5} b_i + Q^{2+3p+n_+-2n_-}H^{2p-n_-+2} + Q^{-2+3p+n_+-2n_-}H^{2p-n_-}$$
$$+ Q^{2+3p+n_+-2n_-}H^{1+2p-n_-} + 2Q^{-2+3p+n_+-2n_-}H^{-1+2p-n_-}.$$

- *If p is odd, $q = p + 1$, and $r > q$, then*

$$P_{Kh} = a_2 + \sum_{n=1}^{5} b_i + Q^{3p+n_+-2n_-}H^{2p-n_-+1}$$
$$+ Q^{3p+n_+-2n_-}H^{2p-n_-} + Q^{-2+3p+n_+-2n_-}H^{-1+2p-n_-}.$$

- *If p is even and $r = q = p + 1$, then*

$$P_{Kh} = \sum_{n=1}^{5} b_i + Q^{-2+3p+n_+-2n_-} H^{2p-n_-}$$

$$+ Q^{2+3p+n_+-2n_-} H^{1+2p-n_-} + 2Q^{-2+3p+n_+-2n_-} H^{-1+2p-n_-}.$$

- *If p is odd and $r = q = p + 1$, then*

$$P_{Kh} = \sum_{n=1}^{5} b_i + Q^{3p+n_+-2n_-} H^{2p-n_-+1}$$

$$+ Q^{3p+n_+-2n_-} H^{2p-n_-} + Q^{-2+3p+n_+-2n_-} H^{-1+2p+n_-}$$

$$+ Q^{2+3p+n_+-2n_-} H^{1+2p-n_-}.$$

- *If p is even, $q = p$, and $r > p + 1$, then*

$$P_{Kh} = a_2 + \sum_{n=1}^{5} b_i + Q^{-2+3p+n_+-2n_-} H^{2p-n_-} + Q^{2+3p+n_+-2n_-} H^{2p-n_-+2}$$

$$+ Q^{2+3p+n_+-2n_-} H^{1+2p-n_-} \sum_{n=0}^{r-p-1} (Q^2 H)^n$$

$$+ Q^{3p+n_+-2n_-} H^{2p-n_-}$$

$$+ 2Q^{-2+3p+n_+-2n_-} H^{-1+2p-n_-}.$$

- *If p is even, $q = p$, and $r = p + 1$, then*

$$P_{Kh} = \sum_{n=1}^{5} b_i + Q^{-2+3p+n_+-2n_-} H^{2p-n_-}$$

$$+ Q^{2+3p+n_+-2n_-} H^{1+2p-n_-} + Q^{3p+n_+-2n_-} H^{2p-n_-}$$

$$+ 2Q^{-2+3p+n_+-2n_-} H^{-1+2p-n_-}.$$

- *If p is even and $p = q = r$, then*

$$P_{Kh} = \sum_{n=1}^{5} b_i + Q^{-2+3p+n_+-2n_-} H^{2p-n_-}$$

$$+ 2Q^{3p+n_+-2n_-} H^{2p-n_-} + 2Q^{-2+3p+n_+-2n_-} H^{-1+2p-n_-}.$$

- *If p is odd and $q = p$, then*

$$P_{Kh} = \sum_{n=1}^{5} b_i$$

$$+ Q^{-2+3p+n_+-2n_-} H^{-1+2p-n_-} + Q^{3p+n_+-2n_-} H^{2p-n_-}$$

$$+ Q^{3p+n_+-2n_-+2(r-p)} H^{2p-n_-+(r-p)}.$$

The values of n_+ and n_- depend on the orientation of the link $P(-p, q, r)$ and can be computed from Table 1.

Proof. The proof consists of carefully looking at Figure 11 and Figure 12 and counting up the contributing generators. To save space, and since we have already indicated how to do this counting, we will omit a more detailed proof here. □

3.2. Unreduced homology. Although we focused on the reduced Khovanov homology in this paper, it would not be too difficult to use this computation to obtain the unreduced homology. The Lee spectral sequences on reduced and unreduced homology, together with the exact sequence relating two copies of the reduced homology with the unreduced homology, give lots of information about the unreduced homology given the reduced version. In the examples the author computed, this information was enough to determine the unreduced homology. As described in [7], the unreduced homology consists only of knight's-move pairs and exceptional pairs (see Bar-Natan [1]); while [7] works only over $\mathbb{Q}$, the integral unreduced homology ends up having copies of $\mathbb{Z}_2$ as expected in the knight's-move pairs and is free otherwise. We will forego a more rigorous discussion because it would lengthen the paper without necessarily adding more insight.

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