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Corrigendum to "Effective separability of finitely generated nilpotent groups", *New York J. Math.* 24 (2018), 83–145.

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ABSTRACT. In previous work [4], the author claimed a characterization for $\mathrm{F}_G(n)$ and lower asymptotic bounds for $\mathrm{Conj}_G(n)$ when G is a finitely generated nilpotent group. However, a counterexample to the characterization of $\mathrm{F}_G(n)$ for finitely generated nilpotent groups was communicated to us by Khalid Bou-Rabee which also had consequences to the lower asymptotic bound provided for $\mathrm{Conj}_G(n)$. The purpose of this note to explain what is incorrect in [4] along with the counterexample provided to us. We will also explain what remains correct in [4] and how we obtain weaker lower bounds for $\mathrm{F}_N(n)$ and $\mathrm{Conj}_N(n)$ which are found in the author's thesis and a forthcoming preprint.

CONTENTS

1. Introduction

The following is found in [4]. The numbering and any unexplained terminology is also taken from [4].

Theorem 1.1. *Let N be an infinite, finitely generated nilpotent group. Then there exists a $\psi_{\mathrm{RF}}(N) \in \mathbb{N}$ such that $\mathrm{F}_G(n) \approx (\log(n))^{\psi_{\mathrm{RF}}(N)}$. Additionally, one may compute $\psi_{\mathrm{RF}}(N)$ given a basis for $\gamma_c(N/T(N))$ where c is the step length of $N/T(N)$.*

Theorem 1.8(ii). *Let N be an infinite, finitely generated nilpotent group. Suppose that N is not virtually abelian. There exists a $\psi_{Lower}(N) \in \mathbb{N}$ such that $n^{\psi_{Lower}(N)} \preceq \mathrm{Conj}_N(n)$. Additionally, one can compute $\psi_{Lower}(N)$ given a basis for $\gamma_c(N/T(N))$ where c is the step length of $N/T(N)$.*

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Khalid Bou-Rabee provided an example of a torsion free, finitely generated nilpotent group G where Theorem 1.1 predicts $\mathrm{F}_G(n) \approx (\log(n))^5$, but where it can be shown that $\mathrm{F}_G(n) \preceq (\log(n))^4$. Thus, the asymptotic lower bound produced for $\mathrm{F}_N(n)$ in [4, Theorem 1.1] is incorrect. Upon inspection of the original article, it turns out that [4, Proposition 4.10] is false which we provide counterexamples for. Since the proof of Theorem 1.8(ii) relied on this proposition, its proof is incomplete as well.

2. A counterexample to [4, Proposition 4.10] and [4, Theorem 1.1]

The following group was communicated to us by Khalid Bou-Rabee:

$$G = \{x, y, w, z, u, v \mid [x, y] = [w, z] = 1, [x, w] = [y, z] = u,$$
$$[x, z] = v, [y, w] = v^{-1}, u \text{ and } v \text{ are central}\}.$$

The following proposition was one of the main tools from [4].

Proposition 4.10. *Let N be a torsion free, finitely generated nilpotent group with a cyclic series $\{H_i\}_{i=1}^{h(N)}$ and a compatible generating subset $\{x_i\}_{i=1}^{h(N)}$. Let $\varphi : N \to Q$ be a surjective group morphism to a finite p-group where $p > B(N, H_i, \{x_i\})$. Suppose that $\varphi([x_{\vec{a}}]) \neq 1$ for all $[x_{\vec{a}}] \in W(N, H_i, \{x_i\}) \cap Z(N)$. Also, suppose that $\varphi(x_i) \neq 1$ for $x_i \in Z(N)$ and $\varphi(x_i) \neq \varphi(x_j)$ for all $x_i, x_j \in Z(N)$ where $i \neq j$. Then $\varphi(x_t) \neq 1$ for $1 \leq t \leq h(N)$ and $\varphi(x_i) \neq \varphi(x_j)$ for $1 \leq i < j \leq h(N)$. Finally, $|Q| \geq p^{h(N)}$.*

The following proposition produces infinitely many primes p in such a way that there exists a surjective group morphism $\psi_p : G \to Q_p$ to a finite p-group Q_p satisfying the hypotheses of Proposition 4.10 and where $|Q_p| = p^4$, and since Proposition 4.10 predicts $|Q_p| = p^5$, we have a collection of counterexamples for Proposition 4.10. Before starting, we introduce some notation. Let

$$E = \{p \in \mathbb{P} \mid 4 \text{ divides } p - 1\}.$$

For $p \in E$, we let $\{a_p, b_p\}$ be the two distinct solutions to the equation $T^2 + 1 \equiv 0 \bmod p$. Finally, we let A_p and B_p be the normal closures of the subgroups $\langle x^{a_p} y \rangle$ and $\langle x^{b_p} y \rangle$ in G, respectively.

Proposition 2.1. *If $p \in E$, then $\pi_p(A_p) \cap Z(G/G^p) \cong \mathbb{F}_p$ and $\pi_p(B_p) \cap Z(G/G^{p^k}) \cong \mathbb{F}_p$. Moreover, $|G/G^p \cdot A_p| = |G/G^p \cdot B_p| = p^4$ and $Z(G/G^p \cdot A_p) \cong Z(G/G^p \cdot B_p) \cong \mathbb{F}_p$. We also have that $\pi_p(A_p) \cap \pi_p(B_p) \cong \{1\}$ and $\langle \pi_p(A_p), \pi_p(B_p) \rangle \cong Z(G/G^p)$. Finally, $\pi_{G^p \cdot A_p}(u), \pi_{G^p \cdot A_p}(v), \pi_{G^p \cdot B_p}(u), \pi_{G^p \cdot B_p}(v) \neq 1$. Additionally, $\pi_{G^p \cdot A_p}(u) \neq \pi_{G^p \cdot A_p}(v)$ and $\pi_{G^p \cdot B_p}(u) \neq \pi_{G^p \cdot B_p}(v)$.*

Proof. For the first statement, it is sufficient to prove that $|G/G^p \cdot A_p| = p^4$ and that $Z(G/G^p) \cap \pi(A_p) \cong \mathbb{F}_p$. By direct calculation, we have that $A_p \cap Z(G) \cong \langle u^{a_p} v^{-1}, u v^{a_p} \rangle$. Since $(u^{a_p} v^{-1})^{-a_p} = u^{-(a_p)^2} v^{a_p} = u v^{a_p} \bmod G^p$, we have $\pi_{p^k}(A_p) \cap Z(G/G^p) \cong \langle \pi_p(u v^{a_p}) \rangle \cong \mathbb{F}_p$. Since $G/G^p \cdot A_p$ is generated by the set $\{\bar{x}, \bar{w}, \bar{z}, \bar{v}\}$ where each element has order p, the second paragraph after [3, Definition 8.2] implies that $|G/G^p \cdot A_p| = p^4$. Subsequently, $Z(G/G^p \cdot A_p) \cong \mathbb{F}_p$. For the next statement, we note that $\pi_p(A_p) \cong \langle u\, v^{a_p} \rangle$ and $\pi_p(B_p) \cong \langle u\, v^{b_p} \rangle$. Suppose for a contradiction that there exists a natural number ℓ such that $(u\, v^{a_p})^\ell = u\, v^{b_p} \bmod G^p$. Since $(u\, v^{a_p})^\ell = u^\ell\, v^{\ell\, a_p}$, we must have that $\ell \equiv 1 \bmod p$ and $\ell\, a_p \equiv b_p \bmod p$. Since $\ell\, a_p \equiv a_p \bmod p$, we have that $a_p \equiv b_p \bmod p$ which is a contradiction. In particular, $\pi_p(A_p) \cap \pi_p(B_p) = \{1\}$; hence, $\langle \pi_p(A_p), \pi_p(B_p) \rangle \cong \mathbb{F}_p \times \mathbb{F}_p$. Since $Z(G/G^p) \cong \mathbb{F}_p \times \mathbb{F}_p$, it follows that $\langle \pi_p(A_p), \pi_p(B_p) \rangle \cong Z(G/G^p)$. The remaining two statements are evident. □

Proposition 2.2. $\mathrm{F}_G(n) \precsim (\log(n))^4$.

Proof. Let $g \in G \backslash \{1\}$ such that $\|g\|_S \leq n$. If $\pi_{\gamma_2(G)}(g) \neq 1$, then [1, Corollary 2.3] implies there exists a surjective group morphism $\varphi : G/\gamma_2(G) \to P$ to a finite group such that $|P| \leq C_1 \log(C_1\, n)$ for some constant $C_1 > 0$ and where $\varphi(\pi_{\gamma_2(G)}(g)) \neq 1$. Thus, $\mathrm{D}_G(g) \leq C_1 \log(C_1\, n)$. Hence, we assume that $g = u^{\alpha_u}\, v^{\alpha_v}$. Since $\|u^{\alpha_u}\|, \|v^{\alpha_v}\| \leq n$, [2, 3.B2] implies that there exists a constant $C_2 > 0$ such that $|\alpha_u|, |\alpha_v| \leq C_2\, n^2$. We may without loss of generality assume that $\alpha_u \neq 0$. Chebotarev's Density Theorem and the Prime number theorem imply that there exists a prime $p \in E$ such that $p \nmid \alpha_u$ and where $p \leq C_3 \log(C_3\, n)$ for some constant $C_3 > 0$. Proposition 2.1 implies that either $\pi_{G^p \cdot A_p}(g) \neq 1$ or $\pi_{G^p \cdot B_p}(g) \neq 1$. In either case, we have $\mathrm{D}_G(g) \leq (C_4)^4\, (\log(C_4\, n))^4$. Hence, $\mathrm{F}_G(n) \precsim (\log(n))^4$. □

3. Correct results from [4] and current state of affairs

The following theorems remain correct in [4]. The reason being is that do not in anyway rely on [4, Proposition 4.10]; in fact, they rely on completely different techniques.

Theorem 1.7. *Let N be a finitely generated nilpotent group. Then there exists a $k \in \mathbb{N}$ such that* $\mathrm{Conj}_N(n) \precsim n^k$.

By applying [4, Proposition 4.4] and [4, Proposition 6.1], we have the following theorem.

Theorem 3.1. *Let N be an infinite finitely generated nilpotent group. There exists a constant $\psi_{\mathrm{RF}}(N) \in \mathbb{N}$ such that* $\mathrm{F}_N(n) \precsim (\log(n))^{\psi_{\mathrm{RF}}(N)}$.

We finish by noting that the author was able to recover [4, Theorem 1.8(ii)] and was able to obtain asymptotic lower bounds for $\mathrm{F}_N(n)$ in his thesis in the discussion outline below (see [5] for any unexplained terminology). We provide lower bounds for $\mathrm{F}_N(n)$ with the following theorem (see [5, Theorem 1.2]).

Theorem 3.2. *If N is an infinite, finitely generated nilpotent group such that $N/T(N)$ has step length $c > 1$, then there exists a natural number* $\dim_{\mathrm{RFL}}(N) \geq c+1$ *and where* $(\log(n))^{\dim_{\mathrm{RFL}}(N)} \preceq \mathrm{F}_N(n)$.

To produce a lower asymptotic bound for $\mathrm{F}_N(n)$, we need to construct a sequence of elements $\{x_i\}_{i=1}^\infty$ such that the order of the minimal finite group Q_i where there exists a surjective group morphism $\psi_i : N \to Q_i$ such that $\psi_i(x_i) \neq 1$ has order approximately $(\log(\|x_i\|))^{\dim_{\mathrm{RFL}}(N)}$. In order to find this sequence, we introduce a notion of $\mathbb{F}_p$-dimension associated to any primitive element $x \in \sqrt{\gamma_c(N)}$, denoted $\dim_{\mathbb{F}_p}(N,x)$, which measures the difficulty of separating x from the identity in a finite p-group. If we let $E_{N,x,i} = \{p \in \mathbb{P} \mid \dim_{\mathbb{F}_p}(N,x) = i\}$, we see that there exists a minimal index i_0 such that $|E_{N,x,i_0}| = \infty$. We denote this as $\dim_{\mathrm{RFU}}(N,x)$, and observe that this value captures the complexity of separating powers of x from the identity in finite p-groups as we vary the prime number. By maximizing the value $\dim_{\mathrm{RFU}}(N,x)$ over all such primitive elements, we obtain the value $\dim_{\mathrm{RFL}}(N)$. For any primitive element $x \in \sqrt{\gamma_c(N)}$ where $\dim_{\mathrm{RFL}}(N,x) = \dim_{\mathrm{RFL}}(N)$, there exist a sequence of natural numbers $\{m_i\}_{i=1}^{\infty}$ such that the desired sequence of elements is given by $\{x^{m_i}\}_{i=1}^{\infty}$.

We obtain lower asymptotic bounds for $\mathrm{Conj}_N(n)$ with the following theorem (see [5, Theorem 1.8])

Theorem 3.3. *If N is an infinite, non-virtually abelian, finitely generated nilpotent group where $N^{|T(N)|}$ has step length c, then there exists a natural number $\dim_{\mathrm{Conj}}(N) \geq c+1$ and where $n^{(c-1)\dim_{\mathrm{Conj}}(N)} \precsim \mathrm{Conj}_N(n)$.*

For the lower bounds of $\mathrm{Conj}_N(n)$, we need to find an infinite sequence of non-conjugate elements x_i and y_i such that the minimal finite group Q_i where there exists a surjective group morphism $\psi_i : N \to Q_i$ such that $\psi_i(x_i)$ and $\psi_i(y_i)$ are non-conjugate has order approximately $(\max\{\|x_i\|, \|y_i\|)^{\dim_{\mathrm{Conj}}(N)}$. In order to construct this sequence, we use the concept of admissible 4-tuples. Admissible 4-tuples (g,m,a,b) contain the data of a primitive element in $g \in \sqrt{\gamma_c(N)}$, a natural number m, and elements $a \in \gamma_{c-1}(N)$ and $b \in N$ such that $g^m = [a,b]$. The structure of conjugacy classes in the integral Heisenberg group imply that we may introduce a $\mathbb{F}_p$-dimension to (g,m,a,b), denoted $\dim_{\mathrm{Conj},\mathbb{F}_p}(g,m,a,b)$, that measures the difficulty of separating the conjugacy classes of $a^p\,[a,b]$ and $a^p\,[a,b]^2$ in finite p-groups when $[a,b] \notin N^p$. Observe that there exists a maximal index $1 \leq i_0 \leq h(N)$ such that $|\,\mathrm{LC}_{N,(g,m,a,b),i_0}\,| = \infty$. We denote this value as $\dim_{\mathrm{Conj}}(g,m,a,b)$, and we obtain the value $\dim_{\mathrm{Conj}}(N)$ by maximizing the value $\dim_{\mathrm{Conj}}(g,m,a,b)$ over all such admissible 4-tuples. The admissible 4-tuples (g,m,a,b) which attain this maximum give us the necessary sequence of non-conjugate elements via $a^p\,[a,b]$ and $a^p\,[a,b]^2$ for $p \in \mathrm{LC}_{N,(g,m,a,b),i_0}$.

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