New York Journal of Mathematics

New York J. Math. **24** (2018) 43–51.

A multivariate generalization of the von Neumann–Wold decomposition

Ameer Athavale

Abstract. Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space. If T is an isometry acting on $\mathcal{H}$, then the von Neumann–Wold decomposition theorem asserts that T can be expressed as a direct sum of the unilateral shift (of some multiplicity) and a unitary operator. We establish a multivariate generalization of the von Neumann–Wold decomposition and explore some of the implications of that generalization. In particular we derive a universal representation theorem for members of a special class of spherical isometries and verify that any member of that class is hyperreflexive.

Contents

1. Introduction

In this note the symbols $\mathbb{N}$ and $\mathbb{Z}_+$ respectively stand for the set of positive integers and for the set of nonnegative integers. If $\mathcal{H}$ is a complex infinite-dimensional separable Hilbert space, then we use $\mathcal{B}(\mathcal{H})$ to denote the algebra of bounded linear operators on $\mathcal{H}$ and use $I_{\mathcal{H}}$ to denote the identity operator on $\mathcal{H}$. If $\{e_p\}_{p\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then the operator $S_{(1)} \in \mathcal{B}(\mathcal{H})$ defined by $S_{(1)}e_p = e_{p+1}$ is referred to as a *unilateral shift*. (Since any two unilateral shifts are unitarily equivalent, one usually employs the expression *the unilateral shift*). For a cardinal k, the k-fold direct sum of $S_{(1)}$ with itself acting on the k-fold orthogonal ampliation of $\mathcal{H}$ is the *unilateral shift of multiplicity* k.

If T is an isometry in $\mathcal{B}(\mathcal{H})$, the classical von Neumann–Wold decomposition theorem asserts that $\mathcal{H}$ can be written as the orthogonal direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where both $\mathcal{H}_1$ and $\mathcal{H}_2$ are reducing for T, where $T|\mathcal{H}_1$ is the

Received June 14, 2017.

2010 *Mathematics Subject Classification.* Primary 47A13.

Key words and phrases. Spherical isometry, q-hypercontraction.

ISSN 1076-9803/2018

unilateral shift of some multiplicity $k(\leq \aleph_0)$, and where $T|\mathcal{H}_2$ is a unitary operator (which, we note, is a normal isometry); in other words,

$$T = S_{(1)}^{[k]} \oplus U \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2$$

with U unitary (refer to [22, Theorem 3.5.17]). (One of the summands may be absent).

Let $\{e_{p_1,\ldots,p_n}\}_{(p_1,\ldots,p_n)\in\mathbb{N}^n}$ be an orthonormal basis for $\mathcal{H}$. The n-tuple $S_{(n)} = (S_1, \ldots, S_n)$ of operators $S_i \in \mathcal{B}(\mathcal{H})$ defined by

$$S_i e_{p_1,\ldots,p_n} = \sqrt{\frac{p_i + 1}{p_1 + \cdots + p_n + n}}\, e_{p_1,\ldots,p_i+1,\ldots p_n}$$

will be referred to as a *spherical shift*. (Since any two spherical shifts are unitarily equivalent in the sense that a single unitary operator intertwines their corresponding operator coordinates, we hereafter employ the expression *the spherical shift*). The tuple $M_z = (M_{z_1}, \ldots, M_{z_n})$ of multiplications by coordinate functions z_i on the Hardy space $H^2(\mathbb{B}^{2n})$ of the unit ball $\mathbb{B}^{2n}$ in $\mathbb{C}^n$, to be referred to as the *Szegö tuple*, is a classical model of the spherical shift (refer to [18, Section 2]). For the spherical shift $S_{(n)} = (S_1, \ldots, S_n)$, one has $S_iS_j = S_jS_i$ for all i and j and also that the equality $I_{\mathcal{H}} - S_1^*S_1 - \cdots - S_n^*S_n = 0$ holds; thus $S_{(n)}$ is a *spherical isometry* (refer to [5]). For a cardinal k, the coordinatewise k-fold direct sum of $S_{(n)}$ with itself acting coordinatewise on the k-fold orthogonal ampliation of $\mathcal{H}$ may be referred to as the *spherical shift of multiplicity* k. For $n = 1$, the spherical shift is the unilateral shift and a spherical isometry is an isometry. It is thus natural to seek a generalization of the von Neumann–Wold decomposition in the context of a spherical isometry and the spherical shift; in other words, one would like to explore for a spherical isometry T the validity of

$$T = S_{(n)}^{[k]} \oplus U$$

for some cardinal k and some $U = (U_1, \ldots, U_n)$ where U is a *spherical unitary*, that is, a spherical isometry consisting of normal operators.

It is our plan to characterize those operator tuples that admit such a decomposition and then capture the classical von Neumann–Wold decomposition for the case $n = 1$. For that purpose, we find it convenient to use some of the notation and terminology from [6]. Let $T = (T_1, \ldots, T_n)$ be a tuple of commuting operators T_i in $\mathcal{B}(\mathcal{H})$. We use T^* to denote the tuple $(T_1^*, \ldots, T_n^*)$. Also, for any polynomial $p(z, w) = \sum_{s,t\in Z_+^n} a_{s,t} z^s w^t$ in the variables $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ with real coefficients $a_{s,t}$, we interpret $p(T, T^*)$ to be the operator $\sum_{s,t\in Z_+^n} a_{s,t} {T^*}^t T^s$. For $q \in \mathbb{N}$, a tuple $T = (T_1, \ldots, T_n)$ of commuting operators in $\mathcal{B}(\mathcal{H})$ is said to be a *q-hypercontraction* if

$$(1 - z_1w_1 - \cdots - z_nw_n)^r (T, T^*) \geq 0 \text{ for all } r \in \mathbb{N} \text{ such that } 1 \leq r \leq q.$$

We say that T *extends to* W if there exist a Hilbert space $\mathcal{K}$, a tuple $W = (W_1, \ldots, W_n)$ of commuting operators W_i in $\mathcal{B}(\mathcal{K})$, and an isometry V from $\mathcal{H}$ into $\mathcal{K}$ such that Range(V) is invariant for each i and $T_i = V^*W_iV$ for each i. (It is customary to think of V as the inclusion of $\mathcal{H}$ into a larger Hilbert space $\mathcal{K}$, identify $\mathcal{H}$ with $V(\mathcal{H})$, and to think of V^* as the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$).

We state for the reader's convenience that the spherical shift $S_{(n)}$ as defined here is to be identified with the tuple M_{m+p} of [6] by choosing $m = n$ and $p = 0$ and is to be identified with the tuple $S^{(n)*}$ of [23] (as acting coordinatewise on $l^2(\mathbb{Z}^n_+, \mathbb{C}) = l^2(\mathbb{Z}^n_+)$). It is known that the Taylor spectrum $\sigma(S_{(n)})$ of $S_{(n)}$ (as well as the Taylor spectrum $\sigma(S^*_{(n)})$ of $S^*_{(n)}$) equals the closure of the unit ball $\mathbb{B}^{2n}$ in $\mathbb{C}^n$ (refer to [12]). The following result plays a crucial role in the sequel and is a special consequence of [6, Theorem 4.2].

Theorem 1.1. *Let S be an n-tuple of commuting operators in $\mathcal{B}(\mathcal{H})$ such that $\sigma(S)$ is contained in the closure of $\mathbb{B}^{2n}$. The following statements are equivalent.*

(i) *S is an n-hypercontraction.*

(ii) *There exist a Hilbert space $\mathcal{K}$ and a unital representation (that is, a $*$-homomorphism) $\pi : \mathcal{B}(H^2(\mathbb{B}^{2n})) \to \mathcal{B}(\mathcal{K})$ such that S extends to $\pi(M^*_z) \equiv (\pi(M^*_{z_1}), \ldots, \pi(M^*_{z_n}))$ (where $M_z = (M_{z_1}, \ldots, M_{z_n})$ is the Szegö tuple).*

In Section 2 we prove our main result (which is Theorem 2.1) and in Section 3 we demonstrate a couple of its applications.

Remark 1.2. Expanding on the ideas of [25], [9, Theorem 1.8] provided in particular a characterization of n-tuples T of operators in $\mathcal{B}(\mathcal{H})$ that admit decompositions of the type $S^{[k]}_{(n)} \oplus U$; such tuples T must of course be spherical isometries. As shown by Theorem 2.1 below, several of the sufficiency conditions in [9, Theorem 1.8] can be replaced by the condition that T be a spherical isometry. Further, it is not clear to the author how the result of Theorem 2.1 can be deduced from that of [9, Theorem 1.8].

2. Main result

Theorem 2.1. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of operators $T_i \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

(1) *H is the orthogonal direct sum of Hilbert spaces $\mathcal{H}_1$ and H_2 where $\mathcal{H}_1$ and $\mathcal{H}_2$ are reducing for each T_i and are such that*

$$T = S^{[k]}_{(n)} \oplus U \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2$$

for some cardinal $k(\leq \aleph_0)$ and some $U = (U_1, \ldots, U_n)$ where U is a spherical unitary. (One of the summands may be absent).

(2) *T is a spherical isometry and T^* is an n-hypercontraction.*

Proof. Suppose (1) holds. That T is a spherical isometry is obvious. Since U is a spherical unitary, one has

$$(1 - z_1 w_1 - \cdots - z_n w_n)^r (U^*, U^{**}) = 0$$

for all $r \in \mathbb{N}$ such that $1 \leq r \leq n$. Considering π to be the identity representation from $\mathcal{B}(H^2(\mathbb{B}^{2n}))$ onto $\mathcal{B}(H^2(\mathbb{B}^{2n}))$ and considering S to be M_z^* in (ii) of Theorem 1.1, it follows that M_z^* (and equivalently $S_{(n)}^*$) is an n-hypercontraction. (That $S_{(n)}^*$ is an n-hypercontraction can also be seen from [23, Corollary 3 and Lemma 7]). It is now clear that T^* is an n-hypercontraction.

Conversely, suppose (2) holds. In view of [23, Remarks. 7^0], the Taylor spectrum $\sigma(T^*)$ of T^* is contained in the closure of the unit ball $\mathbb{B}^{2n}$ in $\mathbb{C}^n$. By Theorem 1.1 there exist a Hilbert space $\mathcal{K}$ and a unital representation

$$\pi : \mathcal{B}(H^2(\mathbb{B}^{2n})) \to \mathcal{B}(\mathcal{K})$$

such that T^* extends to $\pi(M_z^*)$. We let $\pi_1 = \pi | C^*(M_z)$ where $C^*(M_z)$ is the (unital) C^*-subalgebra of $\mathcal{B}(H^2(\mathbb{B}^{2n}))$ generated by all M_{z_i}; clearly, T^* extends to $\pi_1(M_z^*)$. We can thus write, for each i,

$$\pi_1(M_{z_i}^*) = \begin{bmatrix} T_i^* & X_i \\ 0 & Y_i \end{bmatrix}$$

with $X_i : \mathcal{K} \ominus \mathcal{H} \to \mathcal{H}$ and $Y_i : \mathcal{K} \ominus \mathcal{H} \to \mathcal{K} \ominus \mathcal{H}$.

Since π_1 is a unital representation and M_z is a spherical isometry, we have

$$I_{\mathcal{K}} = \sum_{i=1}^n \pi_1(M_{z_i}^*)\pi_1(M_{z_i}) = \begin{bmatrix} \sum_{i=1}^n T_i^* T_i + X_i X_i^* & \sum_{i=1}^n X_i Y_i^* \\ \sum_{i=1}^n Y_i X_i^* & \sum_{i=1}^n Y_i Y_i^* \end{bmatrix}.$$

However, T being a spherical isometry, one has $\sum_{i=1}^n T_i^* T_i = I_{\mathcal{H}}$ and that forces $X_i = 0$ for each i. This shows that $\mathcal{H}$ is reducing for $\pi_1(C^*(M_z))$. We next consider the subrepresentation $\pi_2 : C^*(M_z) \to \mathcal{B}(\mathcal{H})$ of π_1 defined by $\pi_2(A) = \pi_1(A)|\mathcal{H}$, $A \in C^*(M_z)$. Clearly, $\pi_2(M_{z_i}) = T_i$ for each i. We are now in a position to invoke some standard theory related to the splitting of representations as elucidated in [2, Sections 1.3 and 1.4] (refer also to [4]).

It is well-known (see, for example, [11]) that $C^*(M_z)$ contains the C^*-algebra $\mathcal{K}(H^2(\mathbb{B}^{2n}))$ of compact operators on $H^2(\mathbb{B}^{2n})$ and that the following exact sequence obtains:

$$0 \to \mathcal{K}(H^2(\mathbb{B}^{2n})) \xrightarrow{i} C^*(M_z) \xrightarrow{\phi} C(\mathbb{S}^{2n-1}).$$

Here $C(\mathbb{S}^{2n-1})$ is the C^*-algebra of continuous functions on the topological boundary $\mathbb{S}^{2n-1}$ of $\mathbb{B}^{2n}$, i is the inclusion map, and ϕ is the symbol map (sending in particular the operator M_f of multiplication by a continuous function f on $\mathbb{S}^{2n-1}$ to f).

Let $\mathcal{H}_1$ be the closed linear span of $\{\pi_2(A)f : f \in \mathcal{H},\ A \in \mathcal{K}(H^2(\mathbb{B}^{2n}))\}$ and let $\mathcal{H}_2$ be the orthocomplement of $\mathcal{H}_1$ in $\mathcal{H}$. Since $\mathcal{K}(H^2(\mathbb{B}^{2n}))$ is an ideal of $C^*(M_z)$, it is clear that $\mathcal{H}_1$ (and hence $\mathcal{H}_2$) is reducing for $\pi_2(C^*(M_z))$. We consider the subrepresentations

$$\pi_3 : C^*(M_z) \to \mathcal{B}(\mathcal{H}_1),$$
$$\pi_4 : C^*(M_z) \to \mathcal{B}(\mathcal{H}_2)$$

of π_2 defined by $\pi_3(A) = \pi_2(A)|\mathcal{H}_1$ and $\pi_4(A) = \pi_2(A)|\mathcal{H}_2$, $A \in C^*(M_z)$.

Suppose $\mathcal{H}_2 \neq \{0\}$. Since π_4 annihilates compact operators (as is clear from the definition of $\mathcal{H}_2$), π_4 can be thought of as a representation of $C(\mathbb{S}^{2n-1}) \cong \frac{C^*(M_z)}{\mathcal{K}(H^2(\mathbb{B}^{2n}))}$ on $\mathcal{H}_2$. Thus the tuple

$$(\pi_4(M_{z_1}), \ldots, \pi_4(M_{z_n})) = (T_1|\mathcal{H}_2, \ldots, T_n|\mathcal{H}_2)$$

is a spherical unitary.

Suppse $\mathcal{H}_1 \neq \{0\}$. Let

$$\pi_3' = \pi_3|_{\mathcal{K}(H^2(\mathbb{B}^{2n}))} : \mathcal{K}(H^2(\mathbb{B}^{2n})) \to \mathcal{B}(\mathcal{H}_1).$$

Using the notion of an *approximate identity* (refer, for example, to [2, Proposition 1.3.1]), it is easy to see that the representation π_3' is *nondegenerate* in the sense that the closed linear span of $\pi_3'(\mathcal{K}(H^2(\mathbb{B}^{2n})))(\mathcal{H}_1)$ equals $\mathcal{H}_1$. Appealing to [2, Theorem 1.4.4] (and Corollary 2 thereof), one notes that π_3' can be written as a sum $\sum_\lambda \pi_\lambda'$ of representations π_λ' where, for each λ, there exist a subspace $\mathcal{K}_\lambda$ of $\mathcal{H}_1$ that is reducing for $\pi_3'(\mathcal{K}(H^2(\mathbb{B}^{2n})))$ and a unitary operator U_λ from $H^2(\mathbb{B}^{2n})$ onto $\mathcal{K}_\lambda$ such that the following hold:

(i) $\mathcal{H}_1$ is the orthogonal sum $\oplus_\lambda \mathcal{K}_\lambda$.
(ii) $\pi_\lambda'(B) = \pi_3'(B)|\mathcal{K}_\lambda$, $\pi_\lambda'(B) = U_\lambda B U_\lambda^*$ for each B in $\mathcal{K}(H^2(\mathbb{B}^{2n}))$ and for each λ.

Now define, for each λ, $\tau_\lambda : C^*(M_z) \to \mathcal{B}(\mathcal{K}_\lambda)$ by $\tau_\lambda(A) = U_\lambda A U_\lambda^*$, $A \in C^*(M_z)$, and let $\tau = \sum_\lambda \tau_\lambda$. Thus, for $A \in C^*(M_z)$,

$$\tau(A)(\oplus_\lambda h_\lambda) = \oplus_\lambda \tau_\lambda(A) h_\lambda.$$

Clearly, one has $\tau(A)\pi_3'(B) = \pi_3'(AB)$ for $A \in C^*(M_z)$, $B \in \mathcal{K}(H^2(\mathbb{B}^{2n}))$. It follows from the uniqueness considerations present in [2, Section 1.3] that τ must equal π_3. In particular, one has

$$U_\lambda M_{z_i} U_\lambda^* = \tau_\lambda(M_{z_i}) = \tau(M_{z_i})|\mathcal{K}_\lambda = \pi_3(M_{z_i})|\mathcal{K}_\lambda = \pi_2(M_{z_i})|\mathcal{K}_\lambda = T_i|\mathcal{K}_\lambda$$

for each i. If k is the cardinality of the set $\{\mathcal{K}_\lambda\}_\lambda$; then k cannot exceed $\aleph_0$ as $\mathcal{H}$ is separable. We note that $\tilde{U} = \oplus_\lambda U_\lambda$ is a unitary operator for which $(S_{(n)}^{[k]} =)\tilde{U} M_{z_i}^{[k]} \tilde{U}^* = T_i|\mathcal{H}_1$ for each i. □

We record two corollaries of Theorem 2.1.

Corollary 2.2. *The classical von Neumann–Wold decomposition holds.*

Proof. This is the case $n = 1$. If $T = T_1$ is an isometry in $\mathcal{B}(\mathcal{H})$, then $I_{\mathcal{H}} - T_1 T_1^* \geq 0$, that is, $(1 - z_1 w_1)(T_1^*, T_1^{**}) \geq 0$ so that $T^* = T_1^*$ is a 1-hypercontraction perforce; thus Statement (2) of Theorem 2.1 holds and Statement (1) of Theorem 2.1 follows. □

A spherical isometry $T = (T_1, \ldots, T_n)$ for which T^* is an n-hypercontraction will be referred to as a *nice spherical isometry*. A simple example of a spherical isometry that is not a nice spherical isometry is the pair $(S_{(1)}, 0)$.

Corollary 2.3. *Any nice spherical isometry $T = (T_1, \ldots, T_n)$ (with $T_i \in \mathcal{B}(\mathcal{H})$) that is not a spherical unitary has its Taylor spectrum $\sigma(T)$ equal to the closure of the unit ball $\mathbb{B}^{2n}$ in $\mathbb{C}^n$.*

3. Applications

Our first application of Theorem 2.1 is a generalization of a universal representation theorem for an isometry due to Coburn (see [10]; also see [22, Theorem 3.5.18]).

Theorem 3.1. *If $T = (T_1, \ldots, T_n)$ is a nice spherical isometry with T_i in $\mathcal{B}(\mathcal{H})$, then there exists a unique unital representation $\phi : C^*(M_z) \to \mathcal{B}(\mathcal{H})$ such that $\phi(M_{z_i}) = T_i$ for each i; if T moreover is not a spherical unitary then ϕ is isometric.*

Proof. Let T be a nice spherical isometry. Using Theorem 2.1, we have

$$T = S_{(n)}^{[k]} \oplus U \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2$$

for some cardinal k and some $U = (U_1, \ldots, U_n)$ where U is a spherical unitary. For the argument in the next paragraph, we may assume without any harm to generality that both the summands are present.

Since the Taylor spectrum of the spherical unitary U is contained in $\mathbb{S}^{2n-1}$, one can interpret $f(U)$ for any $f \in C(\mathbb{S}^{2n-1})$ by using the continuous functional calculus for U. As noted in [11, Theorem 1], $C^*(M_z)$ equals $\{M_f + B : f \in C(\mathbb{S}^{2n-1}), B \in \mathcal{K}(H^2(\mathbb{B}^{2n}))\}$. One then has the unital $*$-homomorphism $\phi_U : C^*(M_z) \to \mathcal{B}(\mathcal{H}_2)$ defined by $\phi_U(M_f + B) = f(U)$ ($f \in C(\mathbb{S}^{2n-1})$, $B \in \mathcal{K}(H^2(\mathbb{B}^{2n}))$); clearly, $\phi_U(M_{z_i}) = U_i = T_i/\mathcal{H}_2$ for each i. Further, the unitary operator $\tilde{U}$ in the proof of Theorem 2.1 yields a $*$-homomorphism $\phi_k : C^*(M_z) \to \mathcal{B}(\mathcal{H}_1)$ defined by $\phi_k(A) = \tilde{U} A^{[k]} \tilde{U}^*$ ($A \in C^*(M_z)$) and, as follows from the observations there, $\phi_k(M_{z_i}) = T_i|\mathcal{H}_1$ for each i. Choosing the $*$-homomorphism ϕ to be the sum of ϕ_k and ϕ_U, one sees that $\phi(M_{z_i}) = T_i$ for each i. Since the operators M_{z_i} generate $C^*(M_z)$, ϕ with the property $\phi(M_{z_i}) = T_i$ for each i is unique.

In case T is not a spherical unitary, the summand $S_{(n)}^{[k]}$ is necessarily present and so is the part ϕ_k of ϕ; as ϕ_k is clearly injective, so is ϕ. Since any injective $*$-homomorphism between two C^*-algebras is isometric (see [22, Theorem 3.1.5]), ϕ in this case is isometric. □

Our next application of Theorem 2.1 involves the concept of hyperreflexivity of an operator tuple. If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ and $\mathcal{P}$ is the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is said to be *hyperreflexive* if there exists some positive constant C such that, for all $S \in \mathcal{B}(\mathcal{H})$,

$$d(S, \mathcal{A}) \leq C \sup\{\|(I_{\mathcal{H}} - P)SP\| : P \in \mathcal{P} \text{ with } \mathcal{A}(\mathrm{Ran}(P)) \subset \mathrm{Ran}(P)\}$$

where $d(S, \mathcal{A})$ is the norm distance between S and $\mathcal{A}$; the infimum of such constants C is the *hyperreflexivity constant* $\kappa_{\mathcal{A}}$ (of $\mathcal{A}$). The notion of hyperreflexivity was formally introduced by Arveson in [3] and appears to have its genesis in Arveson's work in [1]. If $T = (T_1, \ldots, T_n)$ is a tuple of operators T_i in $\mathcal{B}(\mathcal{H})$ and $\mathcal{A}(T)$ is the WOT-closed algebra generated by T_i and $I_{\mathcal{H}}$, then T is said to be *hyperreflexive* if $\mathcal{A}(T)$ is hyperreflexive. We note that $\mathcal{A}(T)$ is closed in the weak* topology of $\mathcal{B}(\mathcal{H})$.

A hypereflexive operator tuple is *reflexive* (refer to [24] for the relevant definitions and discussions). It was shown by Didas in [14] that any spherical isometry is reflexive.

For $r \geq 1$, a weak* closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is said to satisfy *property* $(\mathbb{A}_1(r))$ if for every $\epsilon > 0$ and every weak*-continuous functional ϕ on $\mathcal{A}$ there exist vectors x, y in $\mathcal{H}$ such that $\phi(T) = \langle Tx, y\rangle$ for all T in $\mathcal{A}$ and $\|x\|\|y\| < (r + \epsilon)\|\phi\|$.

Theorem 3.2. *Any nice spherical isometry is hyperreflexive.*

Proof. The case $n = 1$ was settled by Davidson in [13]. (As [20, Proposition 3] shows, the corresponding hyperreflexivity constant in this case does not exceed 12). So, let $n > 1$. If T is a nice spherical isometry then we have, by Theorem 2.1,

$$T = S_{(n)}^{[k]} \oplus U \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2.$$

It follows from [26, Theorem 3.6] that $\mathcal{A}(U)$ is hyperreflexive (with $\kappa_{\mathcal{A}(U)} \leq 3$); also, $\mathcal{A}(U)$ satisfies property $(A_1(1))$ (refer to [8, Proposition 2.05]). It is a consequence of [7, Theorem 3.1] and [15, Corollary 2.12] that the Szegö tuple $M_z (= S_{(n)})$ is hyperreflexive (with $\kappa_{\mathcal{A}(M_z)} \leq 3$); moreover, as follows from [8, Theorem 3.6], $\mathcal{A}(M_z)$ satisfies property $(A_1(1))$. An application of [16, Theorem 4.1] shows that $\mathcal{B} := (\oplus_k \mathcal{A}(S_{(n)})) \oplus \mathcal{A}(U)$ is hyperreflexive. (Actually, [21, Theorem 5.1] yields that $\kappa_{\mathcal{B}} \leq 2{+}3\sup\{\kappa_{\mathcal{A}(U)}, \kappa_{\mathcal{A}(M_z)}\} \leq 11$).

We note that $\mathcal{A}(T)$ is a weak*-closed subalgebra of $\mathcal{B}$. Thus the hyperreflexivity of $\mathcal{A}(T)$ (and hence of T) will follow from [19, Theorem 3.3] if $\mathcal{B}$ is shown to satisfy property $(\mathbb{A}_1(r))$ for some $r \geq 1$. But $\mathcal{B}$ in fact satisfies property $(\mathbb{A}_1(1))$ as a consequence of [17, Proposition 4.1]. (Putting $r = 1$, one further deduces from [19, Theorem 3.3] that $\kappa_{\mathcal{A}(T)} \leq r{+}(1{+}r)\kappa_{\mathcal{B}} \leq 23$). □

References

[1] Arveson, William. Interpolation problems in nest algebras. *J. Functional Analysis* **20** (1975), no. 3, 208–233. MR0383098, Zbl 0309.46053, doi: 10.1016/0022-1236(75)90041-5.

[2] Arveson, William. An invitation to C^*-algebras. Graduate Texts in Mathematics, 39. *Springer-Verlag, New York-Heidelberg*, 1976. x+106 pp. MR0512360, Zbl 0344.46123, doi: 10.1007/978-1-4612-6371-5.

[3] Arveson, William. Ten lectures on operator algebras. CBMS Regional Conference Series in Mathematics, 55. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, 1984. viii+93 pp. ISBN: 0-8218-0705-6. MR0762819, Zbl 0571.47033.

[4] Arveson, William. Subalgebras of C^*-algebras. III. Multivariable operator theory. *Acta Math.* **181** (1998), no. 2, 159–228. MR1668582, Zbl 0952.46035, arXiv:funct-an/9705007, doi: 10.1007/BF02392585.

[5] Athavale, Ameer. On the intertwining of joint isometries. *J. Operator Theory* **23** (1990), no. 2, 339–350. MR1066811, Zbl 0738.47005.

[6] Athavale, Ameer. Model theory on the unit ball in $\mathbb{C}^m$. *J. Opererator Theory* **27** (1992), no. 2, 347–358. MR1249651, Zbl 0804.47005.

[7] Bercovici, Hari. Hyper-reflexivity and the factorization of linear functionals. *J. Funct. Anal.* **158** (1998), no. 1, 242–252. MR1641578, Zbl 0921.47039, doi: 10.1006/jfan.1998.3288.

[8] Bercovici, Hari; Foias, Ciprian; Pearcy, Carl. Dual algebras with applications to invariant subspaces and dilation theory. CBMS Regional Conference Series in Mathematics, 56. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, 1985. xi+108 pp. ISBN: 0-8218-0706-4. MR0787041, Zbl 0569.47007, doi: 10.1112/blms/18.4.427b.

[9] Chavan, Sameer; Kumari, Rani. A Wold type decomposition for a class of row ν-hypercontractions. *J. Opererator Theory* **75** (2016), no. 1, 195–208. MR3474103, Zbl 06658776, doi: 10.7900/jot.2015jan17.2066.

[10] Coburn, Lewis A. The C^*-algebra generated by an isometry. *Bull. Amer. Math. Soc.* **73** (1967), 722–726. MR0213906, Zbl 0153.16603, doi: 10.1090/S0002-9904-1967-11845-7.

[11] Coburn, Lewis A. Singular integral operators and Toeplitz operators on odd spheres. *Indiana Univ. Math. J.* **23** (1973/74), 433–439. MR0322595, Zbl 0271.46052, doi: 10.1512/iumj.1973.23.23036.

[12] Curto, Raul E. Fredholm and invertible n-tuples of operators. The deformation problem. *Trans. Amer. Math. Soc.* **266** (1981), no. 1, 129–159. MR0613789, Zbl 0457.47017, doi: 10.2307/1998391.

[13] Davidson, Kenneth R. The distance to the analytic Toeplitz operators. *Illinois J. Math.* **31** (1987), no. 2, 265–273. MR0882114, Zbl 0599.47034, doi: 10.2206/kyushujm.48.409.

[14] Didas, Michael. Spherical isometries are reflexive. *Integral Equations Operator Theory* **52** (2005), no. 4, 599–604. MR2184607, Zbl 1081.47003, doi: 10.1007/s00020-005-1354-8.

[15] Eschmeier, Jörg. C_{00}-representations of $H^\infty(G)$ with dominating Harte spectrum. *Banach Algebras, '97* (Blaubeuren), 135–151. *de Gruyter, Berlin*, 1998. MR1656603, Zbl 0919.47008.

[16] Hadwin, Donald. A general view of reflexivity. *Trans. Amer. Math. Soc.* **344** (1994), no. 1, 325–360. MR1239639, Zbl 0802.46010, doi: 10.2307/2154719.

[17] Hadwin, Donald W.; Nordgren, Eric A. Subalgebras of reflexive algebras. *J. Opererator Theory* **7** (1982), no. 1, 3–23. MR0650190, Zbl 0483.47023.

[18] Jewell, Nicholas P.; Lubin, A. R. Commuting weighted shifts and analytic function theory in several variables. *J. Opererator Theory* **1** (1979), no. 2, 207–223. MR0532875, Zbl 0431.47016.

[19] Kraus, Jon; Larson, David R. Reflexivity and distance formulae. *Proc. London Math. Soc.* (3) **53** (1986), no. 2, 340–356. MR0850224, Zbl 0623.47046, doi: 10.1112/plms/s3-53.2.340.

[20] Kliś, Kamila; Ptak, Marek. Quasinormal operators are hyperreflexive. *Topological algebras, their applications, and related topics*, 241–244, Banach Center Publ., 67. *Polish Acad. Sci. Inst. Math., Warsaw*, 2005. MR2143929, Zbl 1068.47087, doi: 10.4064/bc67-0-19.

[21] Kliś, Kamila; Ptak, Marek. k-hyperreflexive subspaces. *Houston J. Math.* **32** (2006), no. 1, 299–313. MR2202367, Zbl 1110.47061.

[22] Murphy, Gerard J. C^*-algebras and operator theory. *Academic Press, Inc., Boston, MA*, 1990. x+286 pp. ISBN: 0-12-511360-9. MR1074574, Zbl 0714.46041.

[23] Müller, Vladimír; Vasilescu, Florian-Horia. Standard models for some commuting multioperators. *Proc. Amer. Math. Soc.* **117** (1993), no. 4, 979–989. MR1112498, Zbl 0777.47009, doi: 10.2307/2159525.

[24] Ptak, Marek. On the reflexivity and hyperreflexivity of algebras and subspaces. *Perspectives in operator theory*, 249–259, Banach Center Publ., 75. *Polish Acad. Sci. Inst. Math., Warsaw*, 2007. MR2341350, Zbl 1132.47057, doi: 10.4064/bc75-0-15.

[25] Richter, Stefan; Sundberg, Carl. Joint extensions in families of contractive commuting operator tuples. *J. Funct. Anal.* **258** (2010), no. 10, 3319–3346. MR2601618, Zbl 1195.47006, doi: 10.1016/j.jfa.2010.01.011.

[26] Rosenoer, Shlomo. Distance estimates for von Neumann algebras. *Proc. Amer. Math. Soc.* **86** (1982), no. 2, 248–252. MR0667283, Zbl 0542.47027, doi: 10.2307/2043390.

(Ameer Athavale) Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India
athavale@math.iitb.ac.in

This paper is available via http://nyjm.albany.edu/j/2018/24-3.html.