New York Journal of Mathematics

New York J. Math. **24** (2018) 702–738.

Trajectorial martingale transforms. Convergence and integration

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Abstract. Starting with a trajectory space, providing a non-stochastic analogue of a discrete time martingale process, we use the notion of super-replication to introduce definitions for null and full sets and the associated notion of a property holding almost everywhere (a.e.). The latter providing what can be seen as the worst case analogue of sets of measure zero in a stochastic setting. The a.e. notion is used to prove the pointwise convergence, on a full set of the original trajectory space, of the limit of a trajectorial transform sequence. The setting also allows to construct a natural integration operator which we study in detail.

Contents

Received September 6, 2017.

2010 *Mathematics Subject Classification.* Primary: 60G42, 60G48; Secondary: 60G17.

Key words and phrases. Trajectorial martingales, worst case uncertainty, superhedging, non-lattice integration.

ISSN 1076-9803/2018

1. Introduction

A recent trend in the literature incorporates uncertainty in the distribution and on the support of a modelling stochastic process by minimizing or dispensing all together with probabilistic assumptions. An example is given by sublinear expectations and their associated stochastic calculus ([11]). Some results in financial mathematics weaken, or eliminate entirely, stochastic assumptions; as examples, we mention robust versions of the Fundamental theorem of Asset Pricing [3], [4] and [5]. Along this line of research, our framework does not make use of any prior stochastic assumptions.

Martingales are a fundamental class of stochastic processes; in particular, they play a crucial role in defining stochastic integrals, modeling gambling games and providing no arbitrage models of financial markets. The paper investigates traces of the martingale notion that remain after the removal of the apriori given probability space. We ask the question: what charateristics of the path space and/or gambling strategies, associated to a martingale, are responsible for the uncertainty properties inherent in the process? Definition 6, provides a setting to develop some trajectorial martingale theory that remains after the removal of the defining measure. The stochastic point of view places the uncertainty on random events occurring accordingly to a probability law. The point of view pursued here pays attention to individual trajectories and, because of this characteristic, could be labelled a worst case point of view. Even though an apriori measure is not assumed, there is a natural notion of outer functional (superhedging, or super-replication, in a financial interpretation) that leads to the definitions of null and full sets.

To show that we have captured a useful trajectorial analogue of a martingale process we prove a trajectorial version of Doob's pointwise convergence theorem. Our integral operator can also be constructed conditional on a given trajectory segment $S_0, \ldots, S_k$ leading to (conditional) integral operators which are the substitute of conditional expectations. The said conditional integrals can be used to extend our results from trajectorial martingale transforms to more general trajectorial martingales. These developments are left for future research.

Our work grew independently of the much related work [12] (see further references therein) that develops an outer measure and the analogue of an stochastic integral in a non probabilistic setting. [12] works in continuous time while we assume a discrete time setting leading to a different approach and constructions. The work of [12] is extended and further developed in [10]. The basic ingredient in this line of research, as well as in ours, is the notion of superhedging that can be seen as the replacement of the original stochastic assumptions by a worst case point of view. Other work, rather unrelated, connecting trajectory based results and martingales are [2] and [1].

Daniell and Lebesgue integration heavily rely on working with a vector lattice of functions. By necessity, our setting precludes the lattice property. Despite this fact, a well defined, but weaker, integral is still available. We follow the original developments in [8] (see also [9] and [7]) but are forced to provide alternative definitions and proofs to the ones from [8] given our specific motivations and setting.

Thus, our setting starts with a trajectory space $\mathcal{S}$, $S \in \mathcal{S}$ being a sequence of real numbers S_i with common initial value $S_0 = s_0$. No apriori topology, measure structure or cardinality constraints are placed on $\mathcal{S}$. The main object of study are the trajectorial transforms, i.e. a sequence of functionals $\Pi_n^{V,H}(S) \equiv V + \sum_{i=0}^{n-1} H_i(S)(S_{i+1} - S_i)$ where V is a real number and $H_i(S) = H_i(S_0, \ldots, S_i)$; these Riemann sums will dictate several definitions in the paper. Even though the results of the paper are purely mathematical and do not require an interpretation for $\Pi_n^{V,H}$, it is useful to provide them with a financial meaning so as to motivate the developments to come. Under such perspective, consider a portfolio that holds shares of a risky asset and cash in a riskless bank account that pays no interest: $H_i(S)$ represents the number of shares of asset S when its value is S_i, $H_i(S)(S_{i+1} - S_i)$ is the profit/loss resulting from holding $H_i(S)$ shares and the asset changing value from S_i to S_{i+1}. V is the initial investment for setting up a portfolio with $H_0(S)$ shares and a deposit of $V - H_0(S)S_0$ in a riskless bank account. No additional funds, besides the original investment, are inputted or withdrawn from the portfolio (i.e. it is a self financing portfolio). Therefore, $\Pi_n^{V,H}(S)$ is the total value of a portfolio that results from performing the trades H_i, $i = 0, \ldots, n-1$. We rely on these interpretations when describing related notions below.

Several of our results rely on the existence of *contrarian trajectories* (CT), one such trajectory will move in a contrarian way to a given investing portfolio (as per Definition 12). These trajectories have the effect of making potential profits arbitrarily small. This is a key local trajectorial property that we use; it holds for discrete time martingales and we provide sufficient conditions for its validity, in our general setting. There are closely related conditions which are used at some points in our developments.

Sets of measure zero appear in stochastic models because their reliance on measure theory, while the use of sets of measure zero in the latter theory is due to a variety of reasons (e.g. to incorporate functions that take infinite values). The conceptual role of such sets in stochastic modeling is quite ambiguous; below we provide an informal/heuristic discussion of null sets in the proposed setting and make precise comments about their role elsewhere in the paper.

$A \subseteq \mathcal{S}$ is called a *null set* if betting on the event of its occurrence can be done with an arbitrarily small investment. Essentially, an event holds a.e. if it may only fail on a null set A and its complement A^c contains a CT (but see the remark after the introduction of the a.e. notion). The latter implies that potential profits could be arbitrarily small in case A^c occurs. What is the likelihood of a null event? Our definition of null set implies that investors betting on the associated event will see arbitrarily large returns relative to an arbitrarily small investment. Therefore, for realistic models, a probability assignation to a null set should be zero (see remark in [10] indicating that Vovk's outer measure dominates simultaneously all local martingale measures).

We describe next the contents of the paper. Knowledge of finance is not used nor required but, for the interested reader, we refer to [6] for financial background material associated to the present paper. The brief Section 2 defines the trajectory setting. Section 3 defines the outer functional, describes some of its basic properties and the notion of a property holding a.e. Section 4 proves the convergence of trajectorial martingale transforms. Section 5 defines an integral operator and a space of integrable functions and proves the Beppo-Levi convergence theorem. Section 6 identifies the integral providing a useful characterization. Section 7 shows that existence of contrarian trajectories (CT) imply the crucial property of continuity from below, the latter property is needed for the construction of the integral and to establish convergence in a full set for the martingale transform sequence. Section 8 provides two approaches establishing existence of CT under a variety of conditions. Appendix A describes an alternative integral that satisfies the monotone convergence theorem, Appendix B remarks on the case when the trajectory space is given by the paths of a martingale process. Appendix C makes some basic comments on some financial implications. Appendix D collects, for the reader's convenience, the well known results on upcrossing inequalities that we use.

2. Trajectorial setting

Definition 1 (Trajectory Set)**.** *Given a real number s_0, a trajectory set, denoted by $\mathcal{S} = \mathcal{S}(s_0)$, is a subset of*

$$\mathcal{S}_\infty(s_0) = \{S = (S_i)_{i\geq 0} : S_i \in \mathbb{R}, S_0 = s_0\}.$$

A set $\mathcal{H}$ consists of sequences $H = (H_i)_{i\geq 0}$, where H_i is a function $H_i : \mathcal{S} \to \mathbb{R}$, which are assumed to be non-anticipative in the following sense: for all $S, S' \in \mathcal{S}$, if $S_j = S'_j, 0 \leq j \leq i$, then $H_i(S) = H_i(S')$ (i.e. $Hi(S) = H_i(S_0, \dots, S_i)$). $H \in \mathcal{H}$ may, occasionally, be referred to as a portfolio. The null portfolio is assumed to belong to $\mathcal{H}$. The pair $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ may, occasionally, be referred to as a market.

The notation $\Delta_i S = S_{i+1} - S_i$, $i \geq 0$, will be used.

2.1. Hypothesis on $\mathcal{H}$. Several assumptions on $\mathcal{H}$ are needed for different results in the paper. It is relevant to keep hypothesis on $\mathcal{H}$ minimal but, for simplicity, we assume them all at once and list them in this short section.

We assume that for any $f : \mathcal{S} \to \mathbb{R}$, where $f(S) = V^f + \sum_{i=0}^{n^f-1} H_i^f(S)\, \Delta_i S$ for some $V^f \in \mathbb{R}, H^f \in \mathcal{H}, n^f \geq 0$, there exists $H^* \in \mathcal{H}$ such that $H_i^* = H_i^f$ for $0 \leq i < n^f$ and $H_i^* = 0$ for $i \geq n^f$, such function could be H^f itself. Therefore f can be written as $f(S) = V^f + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i^f(S)\, \Delta_i S$.

The following simple portfolios are also assumed to be in $\mathcal{H}$. Given a sequence of (trajectory based) stopping times $\{\tau_k\}_{k=0}^{\infty}$, $0 \leq \tau_0 \leq \tau_1 \leq \ldots \leq \tau_k \leq \tau_\infty = \infty$ (stopping times are introduced in Definition 16) and constants $d_k,\ k \geq 0$; set: $H_i(S) \equiv \sum_{k=0}^{\infty} d_k \mathbf{1}_{[\tau_k(S), \tau_{k+1}(S))}(i)$ for $i \geq 0$. We then assume such $H \equiv \{H_i\}_{i\geq 0} \in \mathcal{H}$; in particular $\{H_i \equiv 1\}_{i \geq 0} \in \mathcal{H}$.

Finally, we assume $\mathcal{H} + \alpha\mathcal{H} \subseteq \mathcal{H}$ for all $\alpha \in \mathbb{R}$.

3. Daniell-Leinert outer functional and null sets

The following notation will be used,

$$\mathcal{C} \equiv \{f : \mathcal{S} \to [-\infty, \infty]\},\ \mathcal{P} = \{f : \mathcal{S} \to [0, \infty]\}.$$

We define $(f+g)(S) = 0$ whenever $f(S) + g(S)$ is undefined. We follow Leinert's integration framework [8] but with some needed variations. When the context makes it clear, we may write an expression like $\Pi_n^{V^m, H^m}(S) \geq 0$ (and neglect to explicitly add $\forall S \in \mathcal{S}, \forall m, \forall n$).

Definition 2 (Trajectorial Transforms)**.** *Given $V \in \mathbb{R}$ and $H \in \mathcal{H}$, set*

$$\Pi_n^{V,H}(S) = V + \sum_{i=0}^{n-1} H_i(S)\Delta_i S,\ V \in \mathbb{R},\ H \in \mathcal{H},\ n \geq 0,$$

notice that $\Pi_n^{S_0,1} = S_n$. The sequence of functions $\Pi^{V,H} \equiv \{\Pi_n^{V,H}\}$ is called a trajectorial transform.

A trajectorial transform, when augmented with certain hypothesis on $\mathcal{S}$ or $\mathcal{M}$, will be our analogue of martingale transforms. Such hypothesis are studied in Section 7. We will also use the notation $\Pi_n^{V,F}$ for a given $F = \{F_i\}_{i\geq 0}$ sequence of non-anticipative functions (not necessarily in $\mathcal{H}$.) The following functional plays the analogue role to the outer measure in Caratheodory's approach to Lebesgue integral.

Definition 3. *For $f \in \mathcal{P}$, define*

$$\overline{I}(f) = \inf\{\sum_{m=1}^{\infty} V^m : f \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty}\ \Pi_n^{V^m,H^m},\ \Pi_n^{V^m,H^m} \geq 0\}. \tag{3.1}$$

Definition 4. *For $g : \mathcal{S} \to [-\infty, \infty]$, define $\|g\| = \overline{I}(|g|)$. A function g is a* null function *if $\|g\| = 0$, a subset $E \subset \mathcal{S}$ is a* null set *if $\|\mathbf{1}_E\| = 0$. Similarly, a subset $E \subset \mathcal{S}$ is a* full set *if $\|\mathbf{1}_E\| = 1$. A property that holds in the complement of a null set, is said to hold "almost everywhere" (a.e.)*

Proposition 3 provides conditions that guarantee that complements of null sets are full sets. The notion of a.e., introduced above, follows the usual definition even though in our setting one needs to check separately that the complement of a null set is full. Including this latter property into the a.e. definition would be natural but non standard.

Based on $\|.\|$, we will construct an integral operator on a complete space $\mathcal{L}^1$ of real integrable functions defined on $\mathcal{S}$. This will be done in Section 5. Less structure is required to prove convergence of trajectorial martingale transforms and hence, we prove that result first. We collect needed definitions and intermediate results in the remaining of the present section.

We leave out the simple proofs of the following results.

Proposition 1. *$\overline{I}$ is isotone, positive homogeneous, countable subadditive and $\overline{I}(1_{\mathcal{S}}) \leq 1$.*

The next result is a proposition in [8, p 259]. Leinert's proof is valid given that our $\overline{I}$ satisfies the properties in Proposition 1.

Proposition 2. *Consider $f, g : \mathcal{S} \to [-\infty, \infty]$, then*

(1) $\|g\| = 0$ *iff* $g = 0$ *a.e.*
(2) $\|g\| < \infty$ *then* $|g| < \infty$ *a.e.*
(3) $f = g$ *a.e. then* $\|f\| = \|g\|$.
(4) *The countable union of null sets is a null set.*

3.1. Continuity from below. The next definition is the analogue, in our framework, to Leinert's continuity from below requirement in [8], which, in turn, takes over the role of continuity at 0 in the case of Daniell's integration on a a vector lattice.

Definition 5 (Continuity from Below Property)**.** *$\mathcal{M} = (\mathcal{S}, \mathcal{H})$ is said to satisfy the* continuity from below *property if for any $H \in \mathcal{H}, V \in \mathbb{R}$ and $n^* \geq 0$*

$$V + \sum_{i=0}^{n^*-1} H_i(S)\Delta_i S \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} \ \Pi_n^{V^m,H^m}(S), \implies V \leq \sum_{m=1}^{\infty} V^m,$$

where

$$\Pi_n^{V^m,H^m}(S) = V^m + \sum_{i=0}^{n-1} H_i^m(S)\Delta_i S \geq 0, \ H^m \in \mathcal{H}, \ V^m \in \mathbb{R}. \tag{3.2}$$

Conditions on $\mathcal{M}$ implying the continuity from below property are given in Theorem 6 in Section 7.

Remark 1. *Since $\overline{I}(0) \leq 0$, If $\mathcal{M}$ satisfies the continuity from below property, then $\overline{I}(0) = 0$.*

Proposition 3. *Assume $\mathcal{M}$ satisfies the continuity from below property, then:*

$$\|\mathbf{1}_{\mathcal{S}}\| = 1.$$

Moreover, for any $A \subset \mathcal{S}$,

$$\|\mathbf{1}_A\| = 0 \implies \|\mathbf{1}_{A^c}\| = 1. \tag{3.3}$$

Proof. The inequality $\overline{I}(\mathbf{1}_{\mathcal{S}}) \leq 1$ is immediate from the definition without any additional assumptions. Consider that $\mathbf{1}_{\mathcal{S}} \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} \Pi_n^{V^m,H^m}$, with conditions as in (3.2); then, by continuity from below, $1 \leq \sum_{m=1}^{\infty} V^m$ which implies $1 \leq \overline{I}(\mathbf{1}_{\mathcal{S}})$. The implication in (3.3) follows by subadditivity of $\overline{I}$. □

4. Convergence

Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, we present conditions that imply the pointwise convergence of $\Pi_n^{H,V}(S)$ as $n \to \infty$ in the sense that possible divergence takes place in a null set and convergence in a full set. The word convergence, for the present section, means convergence in $\mathbb{R}$. In particular convergence to ∞ or $-\infty$ is treated as a divergent limit.

We rely on the usual notion of upcrossings of the sequence $\Pi_n^{V,H}(S)$, $n = 0, 1\ldots$, through a band $[a, b]$. Once H and V are fixed, $U_n(S) \equiv U_n^{[a,b]}(S)$ will be the notation for the number of upcrossings, of the sequence $\Pi_j^{V,H}(S)$ through the interval $[a, b]$ up to time n, to alleviate notation, the interval $[a, b]$ may be kept implicit. We refer to Appendix D for notation and some basic results we use, such as the usual upcrossing inequality.

In the next developments we assume $S_n \geq 0$ but clearly this can be weakened as indicated in Appendix D.1.

Theorem 1. *Given* $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, *assume that* $S_n \geq 0$, *for all* n *and* $S \in \mathcal{S}$. *Then, the set of divergence:*
$\mathcal{S}_{div}(\Pi^{S_0,1}) \equiv \{S \in \mathcal{S} : \lim_{n\to\infty} \Pi_n^{S_0,1}(S) = \lim_{n\to\infty} S_n$ *diverges*$\}$, *is a null set.*

Proof. Fix an interval $[a, b]$ and $k \geq 1$, define:

$$A_n^k \equiv \{S : \ U_n(S) \geq k\}, \quad A^k \equiv \cup_{n\geq 1} A_n^k \quad \text{and} \quad A \equiv A_{[a,b]} = \cap_{k\geq 1} A^k.$$

From the upcrossing inequality (D.3), obtained in the Appendix D, it follows that

$$k\ (b-a)\mathbf{1}_{A_n^k}(S) \leq \ a + \sum_{i=0}^{n-1} D_i(S)\ \Delta_i S, \quad \text{for any } n \geq 1.$$

Moreover, for $m \geq n$, if $S \in A_n^k$, then $S \in A_m^k$, while if $S \notin A^k$, $\mathbf{1}_{A^k}(S) = 0 \leq a + \sum_{i=0}^{m-1} D_i(S)\ \Delta_i S$, so we get

$$k\ (b-a)\mathbf{1}_A(S) \leq k\ (b-a)\mathbf{1}_{A^k}(S) \leq \ \liminf_{n\to\infty}\ (a + \sum_{i=0}^{n-1} D_i(S)\ \Delta_i S),$$

that is,

$$\mathbf{1}_A(S) \leq \ \mathbf{1}_{A^k}(S) \leq \frac{a}{k(b-a)} + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{k(b-a)} D_i(S)\ \Delta_i S, \quad \forall S \in \mathcal{S}.$$

Since $\frac{a}{k(b-a)} + \sum_{i=0}^{n-1} \frac{1}{k(b-a)} D_i(S)\Delta_i S \geq 0$, $\forall S \in \mathcal{S}$ and $D = \{D_i\} \in \mathcal{H}$ (see Lemma 8 and Section 2.1), by definition of $\overline{I}$, we have

$$0 \leq ||\mathbf{1}_A|| = \overline{I}(\mathbf{1}_A) \leq \frac{a}{k(b-a)},$$

and so $||\mathbf{1}_A|| = 0$. It then follows from Proposition 2 that $||\mathbf{1}_{\cup_i A_{[a_i,b_i]}}|| = 0$, where $[a_i, b_i]$ is an arbitrary countable collection of intervals.

From Lemma 9 in Appendix D.1, we observe that $\mathbf{1}_{\mathcal{S}_{div}(\Pi^{S_0,1})} \leq \mathbf{1}_{\cup_i A_{[a_i,b_i]}} + \mathbf{1}_{A_\infty}$ where $A_\infty \equiv \{S \in \mathcal{S} : S \notin \ \cup_i A_{[a_i,b_i]}\ \&\ \lim_{n\to\infty} S_n = \infty\}$. Notice now that for any $\epsilon > 0$,

$$A_\infty \subseteq \{S \in \mathcal{S} :\ \exists\ M = M(S),\ S_n \geq \frac{1}{\epsilon},\ \text{if } n \geq M\} \equiv A_\epsilon.$$

If $S \in A_\epsilon$, then $s_0 + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \Delta_i S = S_n \geq \frac{1}{\epsilon}$, consequently, for all $S \in \mathcal{S}$:

$$\mathbf{1}_{A_\infty}(S) \leq \mathbf{1}_{A_\epsilon}(S) \leq \epsilon s_0 + \liminf_{n\to\infty} \sum_{i=0}^{n-1} \epsilon \Delta_i S.$$

Since $\epsilon\ s_0 + \sum_{i=0}^{n-1} \epsilon\Delta_i S = \epsilon S_n \geq 0$ it follows by definition of $\overline{I}$ that

$$\overline{I}(\mathbf{1}_{A_\infty}) \leq \overline{I}(\mathbf{1}_{A_\epsilon}) \leq \epsilon s_0.$$

So $\overline{I}(\mathbf{1}_{A_\infty}) = 0$. It then follows that $||\mathbf{1}_{\mathcal{S}_{div}(\Pi^{S_0,1})}|| = 0$. □

Corollary 1. *Assume $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ satisfies the continuity from below property and $S_n \geq 0$ for all S, then:*

$$\lim_{n\to\infty} \Pi_n^{S_0,1}(S) = \lim_{n\to\infty} S_n \ \textit{converges on a full set.}$$

Proof. When $\mathcal{M}$ satisfies the continuity from below property, Proposition 3 combined with Theorem 1 shows that $\mathcal{S}_{conv}(\Pi^{S_0,1}) \equiv \mathcal{S} \setminus \mathcal{S}_{div}(\Pi^{S_0,1})$ is a full set, namely $||\mathbf{1}_{\mathcal{S}_{conv}}|| = 1$. □

We prove below convergence of $\Pi_n^{V,H}(S)$ in a full set, to do so we apply the previous results. For $V \in \mathbb{R}$ and $H \in \mathcal{H}$ fixed, such that $\Pi_n^{V,H} \geq C$ for a constant C, define

$$\mathcal{S}^{V,H} = \{S^H = \{S_n^H\} \in \mathcal{S}_\infty(V - C) : \exists\ S \in \mathcal{S},\ S_n^H \equiv \Pi_n^{V,H}(S) - C\},$$

notice that $S_n^H \geq 0$ and $S_0^H = V - C$, $\forall S^H \in \mathcal{S}^{V,H}$ and $\mathcal{S} = \mathcal{S}^{S_0,1}$.
Let $\mathcal{H}^{V,H}$ be *any* portfolio set defined on $\mathcal{S}^{V,H}$ that verifies the assumptions listed in Section 2.1. Note that for fixed $S \in \mathcal{S}$, $\Delta_i S^H = H_i(S)\Delta_i S$; for $G \in \mathcal{H}^{V,H}$ define:

$$F_i(S) = G_i(S^H)H_i(S), \text{ and } F = \{F_i\}_{i\geq 0}. \tag{4.1}$$

Notice that F_i is non-anticipative; indeed, assume $\tilde{S}_j = S_j,\ 0 \leq j \leq i$, then

$$\tilde{S}^H_j = V - C + \sum_{i'=0}^{j-1} H_{i'}(\tilde{S})\Delta_{i'}\tilde{S} = V - C + \sum_{i='0}^{j-1} H_{i'}(S)\Delta_{i'}S = S^H_j, \text{ and}$$

$$F_i(\tilde{S}) = G_i(\tilde{S}^H)\ H_i(\tilde{S}) = G_i(S^H)\ H_i(S).$$

Set $\beta : \mathcal{S} \to \mathcal{S}^{V,H}$ by $\beta(S) = S^H$. To alleviate notation, below we will use

$$\mathcal{S}^{V,H}_{div} \equiv \mathcal{S}^{V,H}_{div}(\Pi^{V,1}) \equiv \{S^H \in \mathcal{S}^{V,H} : \lim_{n\to\infty} \Pi^{V,1}_n(S^H) = \lim_{n\to\infty} S^H_n \text{ diverges}\},$$

the notation $\mathcal{S}^{V,H}_{div}(\Pi^{V,1})$ is consistent with the one introduced in the statement of Theorem 1.

Theorem 2. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, consider $V \in \mathbb{R}$ and $H \in \mathcal{H}$ satisfying:*

$$\Pi^{V,H}_n(S) \geq C,\ \forall S \in \mathcal{S},\ \forall n \geq 0, \tag{4.2}$$

for some constant C. Assume also that for $G \in \mathcal{H}^{V,H}$, F given by (4.1), belongs to $\mathcal{H}$ and that $\mathcal{M}$ satisfies the continuity from below property. Then:

$$\lim_{n\to\infty} \Pi^{V,H}_n(S) \textit{ converges on a full set and may diverge in a null set.}$$

Proof. Assumption (4.2) allows us to apply Theorem 1 to $\mathcal{M} = (\mathcal{S}^{V,H}, \mathcal{H}^{V,H})$; therefore, $\mathcal{S}^{V,H}_{div}$ is a null set and so, for $\epsilon > 0$, $k \geq 1$ there exists $G^k \in \mathcal{H}^{V,H}$ such that for any $S \in \mathcal{S}$

$$\mathbf{1}_{\mathcal{S}^{V,H}_{div}}(S^H) \leq \epsilon + \sum_{k\geq 1} \liminf_{n\to\infty} \sum_i G^k_i(S^H)\Delta_i S^H. \tag{4.3}$$

Notice that $\beta^{-1}(\mathcal{S}^{V,H}_{div}) = \mathcal{S}_{div}(\Pi^{V,H}) \equiv \{S \in \mathcal{S} : \lim_{n\to\infty} \Pi^{V,H}_n(S) \text{ diverges}\}$ and, by hypothesis, F defined by (4.1) belongs to $\mathcal{H}$. Then, (4.3) implies

$$\mathbf{1}_{\mathcal{S}_{div}(\Pi^{V,H})}(S) \leq \epsilon + \sum_{k\geq 1} \liminf_{n\to\infty} \sum_i F_i(S)\Delta_i S.$$

Therefore $||1_{\mathcal{S}_{div}(\Pi^{V,H})}|| = 0$, given that $\mathcal{M}$ satisfies the continuity from below property, Proposition 3 implies that $\mathcal{S}_{conv}(\Pi^{V,H}) \equiv \mathcal{S} \setminus \mathcal{S}_{div}(\Pi^{V,H})$ is a full set. $\square$

5. Integral operator

This section constructs an integral operator acting on a class of (integrable) functions defined on $\mathcal{S}$, the developments rely on the previously introduced notion of a property holding a.e. An alternative integral, with somewhat better properties but requiring stronger hypothesis, is detailed in Appendix A. Set:

(5.1)

$$\mathcal{E} = \{f : \mathcal{S} \to \mathbb{R} : f(S) = V^f + \sum_{i=0}^{n^f-1} H^f_i(S)\Delta_i S,\ H^f \in \mathcal{H},\ V^f \in \mathbb{R},\ n^f \geq 0\},$$

where n^f denotes an integer constant that can depend on f. $H_i^f = 0,\ i \geq n^f$ is assumed. Elements $f \in \mathcal{E}$ are referred to as (finite) *trajectorial martingale transforms* (or trajectorial transforms, for short). We assume the necessary hypothesis for $\mathcal{E}$ being a $\mathbb{R}$-linear space, namely $\mathcal{H} + \alpha\mathcal{H} \subset \mathcal{H}$.

The following conditional spaces play a crucial role. Given $\mathcal{S}$, for $S \in \mathcal{S}$ and $j \geq 0$ set:

$$\mathcal{S}_{(S,j)} \equiv \{\tilde{S} \in \mathcal{S} : \tilde{S}_i = S_i, 0 \leq i \leq j\}.$$

Notice $\mathcal{S}_{(S,0)} = \mathcal{S}$ and that if $S' \in \mathcal{S}_{(S,j)}$, then $\mathcal{S}_{(S',j)} = \mathcal{S}_{(S,j)}$. Pairs (S,j) with $S \in \mathcal{S}$ and $j \geq 0$ will be called nodes, *local* properties are relative to a given node.

Definition 6 (0-Neutral Nodes)**.** *Given a trajectory space $\mathcal{S}$ and a node (S,j):*

- (S,j) *is called a* 0-neutral node *if*

$$\sup_{\tilde{S} \in \mathcal{S}(S,j)} (\tilde{S}_{j+1} - S_j) \geq 0 \text{ and } \inf_{\tilde{S} \in \mathcal{S}(S,j)} (\tilde{S}_{j+1} - S_j) \leq 0. \tag{5.2}$$

$\mathcal{S}$ *is called* locally 0-neutral *if (5.2) holds at each node (S,j).*

The following Lemma, based on local properties of $\mathcal{S}$ gives a basic procedure to construct particular trajectories.

Lemma 1. *Assume $\mathcal{S}$ is locally 0-neutral and let $F = \{F_i\}_{i \geq 0}$ be a sequence of non-anticipative functions and $\epsilon > 0$. Then for any $S^0 \in \mathcal{S}$ and $n_0 \geq 0$ there exists a sequence of trajectories $\{S^n\}_{n \geq 1}$, such that for every $n \geq 1$, $S^n \in \mathcal{S}_{(S^{n-1}, n_0+n-1)}$ and*

$$F_i(S^n)\Delta_i S^n < \frac{\epsilon}{2^{i+1}},\ n_0 \leq i \leq n_0 + n - 1, \tag{5.3}$$

and so,

$$\sum_{i=n_0}^{n_0+n-1} F_i(S^n)\ \Delta_i S^n < \sum_{i=n_0}^{n_0+n-1} \frac{\epsilon}{2^{i+1}}. \tag{5.4}$$

Proof. From local 0-neutrality, there exists $S^1 \in \mathcal{S}_{(S^0, n_0)}$ such that $F_0(S^1)\ \Delta_0 S^1 < \frac{\epsilon}{2^{n_0+1}}$. Inductively, once $S^n \in \mathcal{S}_{(S^{n-1}, n_0+n-1)}$ has been constructed verifying (5.4), there exists $S^{n+1} \in \mathcal{S}_{(S^n, n_0+n)}$ satisfying:

$$F_n(S^{n+1})\ \Delta_n S^{n+1} < \frac{\epsilon}{2^{n+1}},$$

so (5.3) holds; then (5.4) follows by resorting to the non-anticipativity property of F. $\square$

Corollary 2. *Consider $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ with $\mathcal{S}$ locally 0-neutral and $f, g \in \mathcal{E}$ with $f(S) = V^f + \sum_{i=0}^{n^f-1} H_i^f(S)\Delta_i S$ and $g(S) = V^g + \sum_{i=0}^{n^g-1} H_i^g(S)\Delta_i S$. If $f = g$ then $V^f = V^g$.*

Proof. Let $h \equiv (f-g) \in \mathcal{E}$, we can write $h(S) = V + \sum_{i=0}^{n-1} H_i(S)\Delta_i S$ with $V = V^f - V^g$ and $H = H^f - H^g$. Take $\epsilon > 0$, from Lemma 1 with $n_0 = 0$, $F = H$ and $n \equiv \max(n^f, n^g)$, there exists $S^n \in \mathcal{S}$, such that

$$0 = h(S^n) = V + \sum_{i=0}^{n-1} H_i(S^n)\Delta_i S^n \leq V + \epsilon.$$

Given that ϵ is arbitrary, the above gives a contradiction if $V < 0$. In case that $V > 0$ one applies the same reasoning to $-h(S)$, $-V$ and $F = -H$. $\square$

In the general case when $\mathcal{S}$ is locally 0-neutral we can see that $\mathcal{E}$ is not a vector lattice. For example, take $V = 0$ and $H_k = 0$, $k \geq 1$ and $H_0(S) = 1$ so $f(S) = (S_1 - S_0)$. Assume $|f(S)| \in \mathcal{E}$, so $|f(S)| = |S_1 - S_0| = V_G + \sum_{i=0}^{n-1} G_i(S)\ \Delta_i S\ \forall\ S$ for some $G \in \mathcal{H}$ and $n \geq 0$. A similar reasoning as in Corollary 2 implies $|f(S)| = |S_1 - S_0| = V_G + G_0(S)\ (S_1 - S_0)\ \forall\ S$ which is impossible if there exist S, S' such that $S_1 < S_0$ and $S'_1 > S_0$. It follows that $f \in \mathcal{E}$ does not imply $|f| \in \mathcal{E}$ and so the latter is not a vector lattice.

Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ with $\mathcal{S}$ locally 0-neutral, the following operator is well defined by Corollary 2 and is linear. For $f(S) = V^f + \sum_{i=0}^{n^f-1} H_i^f(S)\Delta_i S$, $f \in \mathcal{E}$, define

$$I : \mathcal{E} \to \mathbb{R}, \text{ by } I(f) = V^f. \tag{5.5}$$

Remark 2 (I Continuous from Below)**.** *Whenever $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ satisfies the continuity from bellow property, given by Definition 5, and $\mathcal{S}$ is locally 0-neutral, we will say that I is* continuous from below*. In this case, if $f \in \mathcal{E}$ and $f \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} \Pi_n^{V^m,H^m}$, and the conditions in display (3.2) are in effect, then $I(f) \leq \sum_{m=1}^{\infty} V^m$.*

The next proposition collects some basic properties satisfied by I.

Proposition 4. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume $\mathcal{S}$ is locally 0-neutral. Let $f \in \mathcal{E}$, $f \geq 0$, then*

$$I(f) \geq 0. \tag{5.6}$$

Moreover I is isotone (i.e. order preserving).

Proof. Let $f \in \mathcal{E}$, $f(S) = V^f + \sum_{i=0}^{n^f-1} H_i^f(S)\Delta_i S \geq 0$. Consider $\epsilon > 0$ then by Lemma 1 there exist S^{n^f} such that

$$0 \leq V^f + \sum_{i=0}^{n^f-1} H_i^f(S^{n^f})\Delta_i S^{n^f} < \epsilon + V^f.$$

Which leads to $I(f) = V^f \geq 0$.

Let now $g(S) = V^g + \sum_{i=0}^{n^f-1} H_i^g(S)\Delta_i S$, and $g \leq f$, so $0 \leq (f-g) \in \mathcal{E}$, then (5.6) implies $I(f-g) = V^f - V^g \geq 0$ and so $I(g) \leq I(f)$. $\square$

5.1. Integrable functions. Let $\mathcal{F} \equiv \{f : \mathcal{S} \to [-\infty, \infty] : \|f\| < \infty\}$, where the functions that are equal a.e. are identified. $(\mathcal{F}, \|.\|)$ with pointwise operations, defining $[f+g](S) = 0$ if $f(S) + g(S)$ does not exist, becomes a linear normed space thanks to Propositions 1 and 2 (which do not require any hypothesis).

Item (2) in the next proposition can be considered to be a generalized version of the Beppo-Levi theorem (usually considered in a context of integration).

Proposition 5. *Let $f_n : \mathcal{S} \to [-\infty, \infty]$, $n \geq 1$.*

(1) *If $\sum\limits_{n\geq 1} f_n$ converges pointwise a. e., then*

$$\|\sum_{n\geq 1} f_n\| \leq \sum_{n\geq 1} \|f_n\|.$$

(2) *If $\{f_n\} \subseteq \mathcal{F}$ and $\sum_{n\geq 1} \|f_n\| < \infty$, then $\sum_{n=1}^k f_n$ converges pointwise a.e. and in the norm of $\mathcal{F}$ to $f \equiv \sum\limits_{n\geq 1} f_n$ and $f \in \mathcal{F}$.*

Proof. (1) From our hypothesis, $|\sum\limits_{n\geq 1} f_n|$ defines a function on $\mathcal{P}$ (extended by 0, if necessary). From isotony and countable subadditivity of $\overline{I}$,

$$\|\sum_{n\geq 1} f_n\| = \overline{I}(|\sum_{n\geq 1} f_n|) \leq \sum_{n\geq 1} \overline{I}(|f_n|) = \sum_{n\geq 1} \|f_n\|.$$

(2) Since $\sum\limits_{n\geq 1} |f_n| \in \mathcal{P}$, from countable subadditivity of $\overline{I}$,

$$\|\sum_{n\geq 1} |f_n|\| = \overline{I}(\sum_{n\geq 1} |f_n|) \leq \sum_{n\geq 1} \overline{I}(|f_n|) = \sum_{n\geq 1} \|f_n\| < \infty.$$

Then from Proposition 2, item (2), $\sum\limits_{n\geq 1} |f_n| < \infty$ a. e., in particular $f \equiv \sum\limits_{n\geq 1} f_n$ exists as pointwise limit a.e. From our hypothesis and (1), it follows that $f \in \mathcal{F}$. Finally

$$\|\sum_{n\geq 1} f_n - \sum_{n=1}^{k} f_n\| \leq \sum_{n\geq k+1} \|f_n\| \to_{k\to\infty} 0.$$

□

Theorem 3. *If $\{g_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{F}$, then there exist $g \in \mathcal{F}$ and a subsequence $\{g_{n_k}\}_{k\geq 1}$ which converges a.e. and in norm to g. In particular $\mathcal{F}$ is complete.*

Proof. We select a subsequence $\{g_{n_k}\}_{k\geq 1}$ satisfying $\|g_{n_k} - g_{n_{k+1}}\| < 2^{-k}$, $k \geq 1$. We proceed as follows: choose $n_1 \geq 1$ such that

$\|g_{n_1} - g_n\| < 2^{-1}$, $\forall n \geq n_1$; once n_k has been selected, there exists $n_{k+1} > n_k$ such that $\|g_{n_{k+1}} - g_n\| < 2^{-(k+1)}$, $\forall n \geq n_{k+1}$.

Let $f_k = g_{n_k} - g_{n_{k+1}}$, then $\sum_{k\geq 1} \|f_k\| \leq 1$, so $\{f_k\}_{k\geq 1}$ satisfies the hypothesis of Proposition 5, item (2), and consequently $\sum_{k=1}^{m} f_k$ converges pointwise a.e. and in norm to $\sum_{k=1}^{\infty} f_k$. Thus $g_{n_k} = g_{n_1} - \sum_{m=1}^{k-1} f_m$ converges pointwise a.e, and in norm to $g \equiv g_{n_1} - \sum_{k=1}^{\infty} f_k$. □

Definition 7. *Let $\mathcal{E}' \equiv \{f \in \mathcal{E} : ||f|| < \infty\} \subset \mathcal{F}$ and denote with $\mathcal{L}^1$ its norm closure. $f \in \mathcal{L}^1$ is referred to as an integrable function.*

Remark 3. *(a) $\mathcal{L}^1$ is complete since it is closed in $\mathcal{F}$, which is complete.*

(b) For $g : \mathcal{S} \to [-\infty, \infty]$, $g \in \mathcal{L}^1$ if and only if for every $\epsilon > 0$, there exists $f \in \mathcal{E}'$ such that $\|g - f\| < \epsilon$.

(c) $\mathcal{E}'$ and then $\mathcal{L}^1$ are non trivial spaces, since the functions $\Pi_n^{a,D}$ of $\mathcal{E}$ given by (D.3), in Appendix D, belong to $\mathcal{E}'$.

The following theorem is similar to a result in [8], pg 260.

Theorem 4. *$\mathcal{E}'$ is a subspace of $\mathcal{F}$ and if $\mathcal{S}$ is locally 0-neutral, I is linear on $\mathcal{E}'$. Moreover, if I is continuous from below (as per Remark 2) then:*

$$|I(f)| \leq \|f\|, \ \forall \ f \ \in \mathcal{E}'.$$

Proof. If $f, g \in \mathcal{E}'$, and $\alpha \in \mathbb{R}$, then $\alpha f + g \in \mathcal{E}$, and $\|\alpha f + g\| \leq |\alpha|\|f\| + \|g\| < \infty$. So $\mathcal{E}'$ is a subspace of $\mathcal{E}$ and $\mathcal{F}$, consequently I is linear on $\mathcal{E}'$. Finally, for $f \in \mathcal{E}'$ assume $|f| \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} \Pi_n^{V^m,H^m}$, where $\Pi_n^{V^m,H^m}$ satisfies the conditions in display (3.2). Then, since $f \leq |f|$, by continuity from below of I, $I(f) \leq \sum_{m=1}^{\infty} V^m$, thus

$$I(f) \leq \overline{I}(|f|) = \|f\|.$$

Noticing that $-f \leq |f|$, the above analysis implies $I(-f) \leq ||f||$ and so $I(f) \geq -||f||$. □

Under the assumption of continuity from below, Theorem 4 shows that I can be extended to $\mathcal{L}^1$ by continuity.

Definition 8. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ such that $\mathcal{S}$ is locally 0-neutral, if I is continuous from below, its linear and continuous extension to $\mathcal{L}^1$ is denoted by $\int f$, for $f \in \mathcal{L}^1$, and called the $\mathcal{M}$-integral.*

We assume, in the remainder of the paper, the implicit convention that every time that we rely on the existence of the $\mathcal{M}$-integral, the hypothesis that $\mathcal{S}$ is locally 0-neutral and I is continuous from below are in effect.

Lemma 2. *Let $f \in \mathcal{L}^1$, and $g \in \mathcal{C}$ such that $g = f$ a.e., then $g \in \mathcal{L}^1$, and $\int g = \int f$.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a sequence in $\mathcal{E}'$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$; by item (3) in Proposition 2, $g \in \mathcal{F}$, because $\|g\| = \|f\| < \infty$, and $\lim_{n\to\infty} \|f_n - g\| = 0$; therefore $g \in \mathcal{L}^1$ and so

$$\int g = \lim_{n\to\infty} I(f_n) = \int f.$$

□

The classical Beppo-Levi theorem holds for this integral.

Proposition 6. *Let $\{f_n\}_{n\geq 1} \subseteq \mathcal{L}^1$ such that $\sum_{n\geq 1} \|f_n\| < \infty$; then $\sum_{n\geq 1}^{m} f_n$ converges a.e. and in norm to $f \equiv \sum_{n\geq 1}^{\infty} f_n$. Moreover, $f \in \mathcal{L}^1$ and*

$$\int \sum_{n\geq 1} f_n = \sum_{n\geq 1} \int f_n.$$

Proof. From hypothesis, since $\mathcal{L}^1 \subset \mathcal{F}$, item (2) of Proposition 5 gives that $f \equiv \sum_{n\geq 1} f_n$ exists pointwise a.e., and converges to $f \in \mathcal{F}$ in the norm. So completeness of $\mathcal{L}^1$ and $f_n \in \mathcal{L}^1$, $n \geq 1$, implies $f \in \mathcal{L}^1$.

Linearity and continuity of the $\mathcal{M}$-integral imply,

$$|\int f - \sum_{n\geq 1}^{m} \int f_n| = |\int (f - \sum_{n\geq 1}^{m} f_n)| \leq \|f - \sum_{n\geq 1}^{m} f_n\| \to 0.$$

□

6. $\mathcal{M}$-integral characterization

The present section characterizes the $\mathcal{M}$-integral in terms of the operator $\overline{W}$, given by Definition 10 below . We use the fact that $\overline{W}$ coincides with I on $\mathcal{E}$, and acts on functions defined on the space $\mathcal{C} \equiv \{f : \mathcal{S} \to [-\infty, \infty]\}$, through an extension $\overline{\mathcal{H}}$ of the portfolio set $\mathcal{H}$. In particular this operator is not considered as acting on classes of functions (e.g. elements of $\mathcal{F}$). Some intermediate results are postponed to Appendix A where an alternative integral operator is developed. Such integral has its own notion of null sets based on $\overline{W}$ instead of the norm given by $\overline{I}$.

Definition 9. *$\overline{\mathcal{H}}$ will be a linear space of non-anticipative sequences $H = \{H_i\}_{i\geq 0}$ of functions $H_i : \mathcal{S} \to \mathbb{R}$, with the following properties:*

(1) $\mathcal{H} \subset \overline{\mathcal{H}}$,

(2) *For any sequence* $H^m \in \overline{\mathcal{H}}$, *such that* $\sum_{m=1}^{\infty} H_i^m(S)$ *is convergent, for any* $i \geq 0$, *and any* $S \in \mathcal{S}$, *then* H *defined by*

$$H_i(S) = \sum_{m=1}^{\infty} H_i^m(S) \quad \textit{and} \quad H = \{H_i\}_{i \geq 0},$$

belongs to $\overline{\mathcal{H}}$.

Definition 10. *Define the operator* $\overline{W} : \mathcal{C} \to \mathbb{R}$ *by*
(6.1)

$$\overline{W} f = \inf\{V \in \mathbb{R} : f(S) \leq V + \liminf_{n \to \infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S,\ S \in \mathcal{S},\ \textit{with}\, H \in \overline{\mathcal{H}}\}.$$

Also define $\underline{W} f = -\overline{W}[-f]$.

The notion of up-down node, used in the rest of this section, is given by Definition 11 in Section 7.

Proposition 7. *Assume each node of* $\mathcal{S}$ *is up-down then*

$$\overline{W}|f| \leq \|f\|,\ \forall f \in \mathcal{C}.$$

Proof. It is enough to assume that $\|f\| < \infty$. For $\epsilon > 0$, let $|f| \leq \sum_{m=1}^{\infty} \liminf_{n \to \infty} \Pi_n^{V^m, H^m}$ with $V^m \in \mathbb{R},\ H^m \in \mathcal{H}, \Pi_n^{V^m, H^m} \geq 0\ \forall\ n \geq 0$, and $\sum_{m=1}^{\infty} V^m < \|f\| + \epsilon$. Then, by Fatou's lemma,

$$|f(S)| \leq \sum_{m=1}^{\infty} V^m + \liminf_{n \to \infty} \sum_{i=0}^{n-1} [\sum_{m=1}^{\infty} H_i^m(S)] \Delta_i S,\ \forall S \in \mathcal{S}.$$

Since, by Lemma 3, $\sum_{m=1}^{\infty} H_i^m(S) \equiv H_i(S)$ is well defined and so
$H = \{H_i\}_{i \geq 0}$ belongs to $\overline{\mathcal{H}}$. Therefore, $\overline{W}|f| \leq \sum_{m=1}^{\infty} V^m$, and consequently $\overline{W}|f| \leq \|f\|$. □

Conditions guaranteeing $\overline{W}0 = 0$, appearing below, is given in Corollary 7 in Appendix A.

Corollary 3. *Assume each node of* $\mathcal{S}$ *is up-down and* $\overline{W}0 = 0$. *If* $f \in \mathcal{L}^1$ *and* $f > -\infty$, *then*

$$\int f = \overline{W} f.$$

If $f < \infty$ *then* $\int f = \underline{W} f$ *as well.*

Proof. Let $\{f_n\}_{n \geq 1}$ a sequence in $\mathcal{E}'$ such that $\lim_{n \to \infty} \|f_n - f\| = 0$. Since $f, f_n > -\infty$ and $|\overline{W} f_n| = |I(f_n)| < \infty$, for all $n \geq 1$, Theorem 7 in Appendix

A and Proposition 7 above imply $|\overline{W}f - \overline{W}f_n| \leq \overline{W}|f - f_n| \leq \|f - f_n\|$ and so,

$$\int f = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} \overline{W}f_n = \overline{W}f.$$

Where the intermediate equality holds by Proposition 16 in Appendix A. On the other hand, if $f < \infty$,

$$|\underline{W}f - \underline{W}f_n| = | - \overline{W}[-f] + \overline{W}[-f_n]| \leq \|f - f_n\|.$$

Thus $\underline{W}f = \lim_{n\to\infty} \underline{W}(f_n) = \lim_{n\to\infty} I(f_n) = \int f$, also by Proposition 16. $\square$

Remark 4. *The $\mathcal{M}$-integral is positive on $\mathcal{L}^1$, if each node of $\mathcal{S}$ is up-down and $\overline{W}0 = 0$. Indeed for $0 \leq f \in \mathcal{L}^1$, under the hypothesis of existence of the integral, Corollary 3 applies and so*

$$\int f = \overline{W}f \geq 0.$$

Where the last inequality follows from the isotony of $\overline{W}$ and $\overline{W}0 = 0$.

The characterization given by Corollary 3 is not valid for general $f \in \mathcal{L}^1$ as $f > -\infty$, or $f < \infty$ is required. We provide now another characterization removing such restriction.

Proposition 8. *Assume each node of $\mathcal{S}$ is up-down and $\overline{W}0 = 0$. For $f \in \mathcal{L}^1$, define*

$$\tilde{f}(S) = \begin{cases} f(S) & if \quad |f(S)| < \infty \\ 0 & if \quad |f(S)| = \infty \end{cases}. \tag{6.2}$$

Then

$$\int f = \int \tilde{f} = \overline{W}\tilde{f} = \underline{W}\tilde{f}.$$

Proof. $\tilde{f} \in \mathcal{F}$, because $|\tilde{f}| \leq |f|$ so $\|\tilde{f}\| \leq \|f\| < \infty$. From item (2) in Proposition 2, it is known that $\{|f(S)| = \infty\}$ is a null set, which implies that $f = \tilde{f}$, *a.e.* Then from Lemma 2

$$\int f = \int \tilde{f} = \overline{W}\tilde{f} = \underline{W}(\tilde{f}),$$

where the last two equalities follow from Corollary 3 given that $-\infty < \tilde{f} < \infty$. $\square$

We provide, yet, another characterization that will require the hypothesis $\underline{W}f \leq \overline{W}f$. For this reason we give first a result that provides sufficient conditions for the validity of such inequality.

Proposition 9. *Let $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, with $\overline{W}0 = 0$ and assume that for any $H \in \overline{\mathcal{H}}$,*

$$\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S > -\infty, \ \forall S \in \mathcal{S}.$$

Then for $f \in \mathcal{C}$, such that $\overline{W} f > -\infty$, and $\overline{W}[-f] > -\infty$, it follows that

$$\underline{W} f \leq \overline{W} f. \tag{6.3}$$

Proof. In order to establish (6.3), it is enough to assume $\overline{W} f < \infty$ and $\overline{W}[-f] < \infty$. Since $\overline{W} f$ and $\overline{W}[-f]$ are finite, let $H, G \in \overline{\mathcal{H}}$, $V, U \in \mathbb{R}$ be such that for all $S \in \mathcal{S}$

$$f(S) \leq V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S, \; -f(S) \leq U + \liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\Delta_i S. \tag{6.4}$$

We proceed by cases.
I. $f(S) \neq \pm\infty$ in (6.4) then

$$0 \leq V + U + \liminf_{n\to\infty} \sum_{i=0}^{n-1} [H_i(S) + G_i(S)]\Delta_i S, \; \forall S \in \mathcal{S}. \tag{6.5}$$

II. $f(S) = \infty$, then $\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S = \infty$, thus

$$0 \leq V + U + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S =$$

$$V + U + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S + \liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\Delta_i S.$$

Therefore, (6.5) is valid given that the last sum is well defined by hypothesis.
III. $f(S) = -\infty$, then $\liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\Delta_i S = \infty$. By symmetry with case II, (6.5) is valid.

Finally $0 = \overline{W} 0 \leq V + U$, by taking infimum on V, and then on U. Since the following sum is well defined, then

$$0 \leq \overline{W} f + \overline{W}[-f].$$

□

Proposition 10. *Assume each node of $\mathcal{S}$ is up-down and $\overline{W} 0 = 0$. Consider $f \in \mathcal{L}^1$ and assume $\underline{W} f \leq \overline{W} f$. Let $\tilde{f}$ be as introduced in display (6.2) in Proposition 8. Then, $\underline{W} \tilde{f} = \underline{W} f = \overline{W} f = \overline{W} \tilde{f}$, and so*

$$\int f = \overline{W} f = \underline{W} f.$$

Proof. Define $h \in \mathcal{C}$ by $h(S) = f(S) - \tilde{f}(S)$. By (2) of Proposition 2, $h = 0$ *a.e.*, then $\|h\| = 0$ by item (1) of Proposition 2. In consequence by Proposition 7

$$\overline{W} h \leq \|h\| = 0, \quad \text{and} \quad \underline{W} h = -\overline{W}[-h] \geq -\|h\| = 0.$$

Since $f = \tilde{f} + h$, with the sum well defined, and $|\overline{W} \tilde{f}| < \infty$, by (A.2) and Proposition 14, both in Appendix A,

$$\overline{W} f \leq \overline{W} \tilde{f} + \overline{W} h \leq \overline{W} \tilde{f}$$

On the other hand $|\underline{W}\tilde{f}| = |\overline{W}[-\tilde{f}]| < \infty$ then by Corollary 6

$$\underline{W}\tilde{f} \le \underline{W}\tilde{f} + \underline{W}h \le \underline{W}f \le \overline{W}f \le \overline{W}\tilde{f}.$$

The proof concludes by using Proposition 8. □

6.1. Further convergence properties of the $\mathcal{M}$-integral. Leinert ([8]) provides an in-depth study of the conditions needed for integrals of the type introduced in our paper to have further convergence properties as well as supporting a σ-algebra of integrable subsets. Apparently, without some kind of lattice property it is not possible to go beyond the Beppo-Levi and the monotone convergence theorems. In Appendix A we define an alternative integral $\int' f$ that satisfies all the properties of the $\mathcal{M}$-integral plus some more while still remaining in our non lattice setting. We discuss here some of the implications of those additional properties for the $\mathcal{M}$-integral.

The next two propositions make use of results valid for $\overline{W}$ that are developed in Appendix A. The corresponding statements for the $\mathcal{M}$-integral represent weaker versions. The main point being that the function f, appearing in each of the statements, need to be *assumed* to be integrable a fact that is *derived* in the classical version of the results valid for $\overline{W}$. The notion of Contrarian Trajectories (CT) used in the next proposition is introduced in Definition 12.

Proposition 11. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume that $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$ and each node of $\mathcal{S}$ is up-down. Let $\{f_n\} \subseteq \mathcal{L}^1$, $0 \le f_n$, $-\infty < f \in \mathcal{L}^1$ and $f \le \sum_{n\ge 1} f_n$ then:*

$$\int f \le \sum_{n\ge 1} \int f_n. \tag{6.6}$$

Proof. The result follows from the same property satisfied by $\overline{W}$, stated in Proposition 15 Appendix A, after noticing that $\overline{W}f = \int f$ and $\overline{W}f_n = \int f_n$ which hold because Corollary 3. □

We prove below a weaker version of the classical monotone convergence theorem relying on the classical version of the theorem valid for the alternative integral $\int' f$.

Theorem 5. *Assume each node of $\mathcal{S}$ is up-down and $\overline{W}0 = 0$. For $k \ge 1$, let $f_k \in \mathcal{L}^1$, with $-\infty < f_k \uparrow f < \infty$ and $-\infty < \int f_k \le C < \infty$. Then*

$$\overline{W}(f) = \lim_{k\to\infty} \overline{W}(f_k) = \lim_{k\to\infty} \int f_k.$$

If $f \in \mathcal{L}^1$ is further assumed, then $\overline{W}(f) = \int f$.

Proof. We refer to the set of integrable functions $\mathbb{L}^1$ introduced in Appendix A. Our hypothesis allow to apply Proposition 7 which implies $\mathcal{L}^1 \subseteq \mathbb{L}^1$. Therefore $f_n \in \mathbb{L}^1$ and an application of (A.7) gives $\int' f_n = \overline{W}f_n = \int f_n$

where the last equality follows from Corollary 3. We apply Theorem 9 and note that $\int' f = \overline{W} f$ and so we conclude

$$\overline{W} f = \lim_{n\to\infty} \int f_n. \tag{6.7}$$

Finally, if we further assume $f \in \mathcal{L}^1$, Corollary 3 implies $\overline{W} f = \int f$. □

7. Continuity from below and contrarian trajectories

The convergence result and the construction of the $\mathcal{M}$-integral in previous sections relied on the continuity from below property of I. The latter is a crucial analytic property for a Daniell integration approach; the present section provides the key link between general local trajectory properties and continuity from below.

The said properties are introduced in stages starting with Definition 11 below (a refinement of Definition 6) that encodes pathwise properties of discrete time martingales, and continuing in Section 8.

Definition 11 (Up-Down Nodes)**.** *Given a trajectory space $\mathcal{S}$ and a node (S, j):*

- (S, j) *is called an* up-down node *if*

$$\sup_{\tilde{S}\in\mathcal{S}(S,j)} (\tilde{S}_{j+1} - S_j) > 0 \textit{ and } \inf_{\tilde{S}\in\mathcal{S}(S,j)} (\tilde{S}_{j+1} - S_j) < 0.$$

Observe that any up-down node is 0-neutral.

Lemma 1 from section 5, based on the locally 0-neutral property of $\mathcal{S}$, gives a basic procedure to construct contrarian trajectories. Consider the case that there exist $\hat{S} \in \mathcal{S}$ such that $\hat{S}_i = S^i_i,\ i \geq 0$, for the sequence of trajectories $\{S^n\}_{n\geq 1}$ verifying (5.4), as in referred Lemma 1. Such $\hat{S}$ satisfies $\liminf_{n\to\infty} \sum_{i=0}^{n-1} F_i(\hat{S})\ \Delta_i \hat{S} \leq \epsilon$. In that case, $\hat{S}$ will be called an ϵ-*contrarian trajectory* for F. This type of trajectory is crucial to establish the continuity from below property of the operator I. A discussion on existence of these trajectories is given in Section 8.

Definition 12 (Contrarian trajectories, CT)**.** *We will say that a trajectory set $\mathcal{S}$ has the contrarian trajectory (CT) property for $F = \{F_i\}_{i\geq 0}$, a sequence of non-anticipative functions on $\mathcal{S}$, if the following holds: for any $S^* \in \mathcal{S}$, $n^* \geq 0$ and $\epsilon > 0$ there exists $S^\epsilon \in \mathcal{S}_{(S^*,n^*)}$ such that*

$$\liminf_{n\to\infty} \sum_{i=n^*}^{n-1} F_i(S^\epsilon)\ \Delta_i S^\epsilon \leq \epsilon. \tag{7.1}$$

Lemma 3 below is used to establish the continuity from below of the operator I, it requires the up-down property for the nodes of $\mathcal{S}$.

Lemma 3. *Assume each node of $\mathcal{S}$ is up-down. For any $m \geq 1$, let $G^m = \{G_i^m\}_{i\geq 0}$ be sequences of non-anticipative functions on $\mathcal{S}$, and $V^m \in \mathbb{R}$ such that*

$$\Pi_n^{V^m,G^m}(S) = V^m + \sum_{i=0}^{n-1} G_i^m(S)\Delta_i S \geq 0,\ S \in \mathcal{S},\ n \geq 1.\ \textit{If} \sum_{m\geq 1} V^m < \infty,$$

then

$$\sum_{m\geq 1} G_i^m(S) \textit{ is convergent, for any } i \geq 0, \textit{ and } S \in \mathcal{S}.$$

Proof. Assume that $j \geq 0$ is the minimum index such that $\sum_{m\geq 1} G_j^m(S^j)$ is not convergent for some $S^j \in \mathcal{S}$. Then, there exists $\epsilon > 0$, with the property that for any $M \in \mathbb{N}$ there exist $m_2 > m_1 \geq M$ such that

$$|\sum_{m=m_1+1}^{m_2} G_j^m(S^j)| \geq \epsilon. \tag{7.2}$$

Note that m_1, m_2 just depend on M and the conditional space $\mathcal{S}_{(S^j,j)}$. Since by hypothesis the node (S^j, j) is up-down, let

$$\theta^- = \frac{1}{2} \inf_{S\in\mathcal{S}(S^j,j)} (S_{j+1} - S_j^j) < 0 \text{ and } \theta^+ = \frac{1}{2} \sup_{S\in\mathcal{S}(S^j,j)} (S_{j+1} - S_j^j) > 0.$$

Set $\epsilon^* \equiv \epsilon \min\{-\theta^-, \theta^+\}$. If $j > 0$, $\sum_{m\geq 1} G_i^m(S)$ is convergent for any $0 \leq i < j$ and $S \in \mathcal{S}$. Having in mind that, and the convergence of $\sum_{m\geq 1} V^m$, there exist M_0 such that for any $0 \leq i < j$, and $m'' > m' \geq M_0$ implies (recall $V^m \geq 0$ resulting by the $n = 0$ in our assumptions),

$$\sum_{m=m'+1}^{m''} V^m < \frac{\epsilon^*}{2^{j+2}},$$

and

$$|\sum_{m=m'+1}^{m''} G_i^m(S^j)| < \frac{\epsilon^*}{2^{i+2}\rho_i}. \quad (\rho_i = |\Delta_i S^j| \neq 0, \text{ or } \rho_i = 1)$$

By the up-down property, for $M = M_0$ and the corresponding $m_1 < m_2$ as in (7.2), there exists $S^{j+1} \in \mathcal{S}_{(S^j,j)}$ with $\Delta_j S^{j+1} \leq \theta^-$ or $\theta^+ \leq \Delta_j S^{j+1}$, such that

$$\sum_{m=m_1+1}^{m_2} G_j^m(S^{j+1})\Delta_j S^{j+1} \leq -\epsilon|\Delta_j S^{j+1}| \leq -\epsilon^*.$$

Consequently, for $0 \leq i < j$,

$$|\sum_{m=m_1+1}^{m_2} G_i^m(S^{j+1})\Delta_i S^{j+1}| = |\sum_{m=m_1+1}^{m_2} G_i^m(S^j)\Delta_i S^{j+1}| <$$

$$\frac{\epsilon^*}{2^{i+2}\rho_i}|\Delta_i S^{j+1}| \leq \frac{\epsilon^*}{2^{i+2}}.$$

So

$$\sum_{m=m_1+1}^{m_2} V^m + \sum_{m=m_1+1}^{m_2} \sum_{i=0}^{j} G_i^m(S^{j+1})\Delta_i S^{j+1} < -\epsilon^*(1 - \sum_{i=0}^{j} \frac{1}{2^{i+2}}).$$

With the recursive procedure, as in the proof of Lemma 1, for $n: \ j \leq n$, it is possible to find $S^{n+1} \in \mathcal{S}_{(S^n,n)}$, such that $\sum_{m=m_1+1}^{m_2} G_n^m(S^{n+1})\Delta_n S^{n+1} \leq \frac{\epsilon^*}{2^{n+2}}$. Then for S^N, $N \geq j$ fixed, it follows that for any $1 \leq n < N$:

$$\sum_{m=m_1+1}^{m_2} V^m + \sum_{m=m_1+1}^{m_2} \sum_{i=0}^{n} G_i^m(S^N)\Delta_i S^N < -\epsilon^*(1 - \sum_{i=0}^{n} \frac{1}{2^{i+2}}).$$

From which results the contradiction

$$0 \leq \sum_{m=1}^{m_2} \Pi_N^{V^m,G^m}(S^N) - \sum_{m=1}^{m_1} \Pi_N^{V^m,G^m}(S^N) =$$

$$\sum_{m=m_1+1}^{m_2} V^m + \sum_{m=m_1+1}^{m_2} \sum_{i=0}^{N-1} G_i^m(S^N)\Delta_i S^N < -\epsilon^*(1 - \sum_{i=0}^{N-1} \frac{1}{2^{i+2}}) < 0.$$

$\square$

Theorem 6. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume each node in $\mathcal{S}$ is up-down. Moreover, $\mathcal{S}$ has the CT property for any $F = \{F_i = G_i - H_i^f\}_{i\geq 0}$, $H^f \in \mathcal{H}$, $H_i^f = 0$ for all $i \geq n^f$, $G_i \equiv \sum_{m\geq 1} G_i^m$, $G^m \in \mathcal{H}$ and the G^m satisfy the properties in the statement of Lemma 3 for some real numbers V^m. Then, $\mathcal{M}$ and I satisfy the continuity from below property.*

Proof. Let $f \in \mathcal{E}$, $G^m \in \mathcal{H}$, $m \geq 1$, and $V^m \in \mathbb{R}$ such that $\Pi_n^{V^m,G^m} \geq 0$, as in (3.2), and

$$f \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} \Pi_n^{V^m,G^m}. \tag{7.3}$$

Without loss of generality, we may assume that $V = \sum_{m=1}^{\infty} V^m < \infty$ then, by Lemma 3, the functions $G_i(S) \equiv \sum_{m=1}^{\infty} G_i^m(S)$ are well defined for any $S \in \mathcal{S}$ and $i \geq 0$. It can be easily seen that they are non-anticipative. An application of Fatou's Lemma for nonnegative series gives

(7.4)

$$\begin{aligned} V \quad &+ \quad \sum_{m=1}^{\infty} \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i^m(S)\Delta_i S = \sum_{m=1}^{\infty} \liminf_{n\to\infty} (V^m + \sum_{i=0}^{n-1} H_i^m(S)\Delta_i S) \\ &\leq \quad \liminf_{n\to\infty} \sum_{m=1}^{\infty} (V^m + \sum_{i=0}^{n-1} H_i^m(S)\Delta_i S) = V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\Delta_i S. \end{aligned}$$

Letting $f(S) = V^f + \sum_{i=0}^{n^f-1} H_i^f(S)\Delta_i S$, with $H_i^f = 0$ for all $i \geq n^f$; we note in passing that inequalities (7.3) and (7.4) imply

$$V^f + \sum_{i=0}^{n^f-1} H_i^f(S) \leq V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\Delta_i S, \quad \forall S. \tag{7.5}$$

For fixed $\epsilon > 0$, applying the CT property to $F_i \equiv G_i - H_i^f$ with $n^* = 0$, gives

$$\liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S^\epsilon)\Delta_i S^\epsilon \leq \sum_{i=0}^{n^f-1} H_i^f(S^\epsilon)\Delta_i S^\epsilon + \epsilon.$$

Then (7.5) implies $V^f \leq V + \epsilon$ and so $I(f) = V^f \leq V = \sum_{m=1}^{\infty} V^m$. □

8. Existence of contrarian trajectories

Under general conditions, we develop two quite different approaches that lead to the CT property for $\mathcal{S}$ and hence to the validity of the continuity from below property for $\mathcal{M}$ and I (required in several key results of the paper). The said approaches are described in Sections 8.1 and 8.2 below.

8.1. Complete set of trajectories.

Definition 13. *$\mathcal{S}$ is called complete if for all $\{S^n\}_{n\geq 1} \subseteq \mathcal{S}$ satisfying*

$$\forall\ n,\ S_i^n = S_i^{n+1}\ 0 \leq i \leq n, \tag{8.1}$$

there exists $\overline{S} \in \mathcal{S}$ satisfying

$$\overline{S}_i = S_i^n,\ 0 \leq i \leq n,\ \forall\ n \geq 1. \tag{8.2}$$

The following notation will be useful (even when $\mathcal{S}$ may not be complete), given $\{S^n\}_{n\geq 1} \subseteq \mathcal{S}$ obeying (8.1) define

$$\overline{S} = \{\overline{S}_i\}_{i\geq 0} \text{ by } \overline{S}_i \equiv S_i^i, \text{ we will use the notation } \overline{S} = \lim_{n\to\infty} S^n. \tag{8.3}$$

Section 8.1.1 shows how to complete a given space $\mathcal{S}$.

Lemma 1 and completness provide existence of CT for any non-anticipative sequence $F = \{F_i\}$.

Proposition 12. *Assume $\mathcal{S}$ is locally 0-neutral and complete. Then $\mathcal{S}$ satisfies the CT property for any sequence of no-anticipative functions $F = \{F_i\}_{i\geq 0}$.*

Proof. The result follows from (5.4) and completeness. □

Corollary 4. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume each node in $\mathcal{S}$ is up-down. If $\mathcal{S}$ is complete, then $\mathcal{M}$ and I satisfy the continuity from below property.*

Proof. The hypothesis of up-down nodes implies the property of locally 0-neutral for $\mathcal{S}$. Therefore, Proposition 12 is applicable implying the CT property for any non-anticipative sequence $F = \{F_i\}$; the result then follows from Theorem 6. □

8.1.1. Completeness in trajectory spaces. Here we develop the completion of trajectory spaces.

Lemma 4. *Let* $\{S^m\} \subseteq \mathcal{S}$ *satisfy (8.1), then for any* m*:*

$$S_i^i = S_i^m,\ 0 \le i \le m, \tag{8.4}$$

and for $\overline{S}$ *given by (8.3)*

$$\overline{S}_i = S_i^m,\ 0 \le i \le m. \tag{8.5}$$

Proof. For $n = 0$ both equations hold; notice that (8.4) follows directly by induction. (8.5) follows from (8.3) and (8.4). □

Given a trajectory set $\mathcal{S}$ it is trivial to complete it by defining a new trajectory set

$$\overline{\mathcal{S}} \equiv \{\overline{S} : \overline{S} = \{\overline{S}_i = S_i^i\} \text{ where } \{S^i\} \subseteq \mathcal{S} \text{ obeys } (8.1)\},$$

in particular $\mathcal{S} \subseteq \overline{\mathcal{S}}$.

Proposition 13. $\overline{\mathcal{S}}$ *is complete.*

Proof. Take $\{\overline{S}^n\}_{n\ge0} \subseteq \overline{\mathcal{S}}$ satisfying

$$\overline{S}_i^n = \overline{S}_i^{n+1},\ 0 \le i \le n,\ \forall n \ge 0. \tag{8.6}$$

Therefore, for each $n \ge 0$, there exists $\{G^{n,m}\}_{m\ge0} \subseteq \mathcal{S}$ satisfying

$$\overline{S}^n = \lim_{m\to\infty} G^{n,m} \text{ and } G_i^{n,m} = G_i^{n,m+1}\ 0 \le i \le m,\ \forall m \ge 0.$$

From Lemma 4 and (8.6) it follows that

$$G_i^{n+1,i} = G_i^{n+1,n} = \overline{S}_i^{n+1} = \overline{S}_i^n = G_i^{n,m} = G_i^{n,i},\ 0 \le i \le n, m\ \forall n, m \ge 0.$$

Define $S^n \equiv G^{n,n}$. Notice that $\{S^n\} \subseteq \mathcal{S}$ because $G^{n,m} \in \mathcal{S}$ for any pair n, m. Moreover $\{S^n\}$ verifies (8.1), since $\forall n \ge 0$

$$S_i^{n+1} = G_i^{n+1,n+1} = \overline{S}_i^{n+1} = \overline{S}_i^n = G_i^{n,n+1} = G_i^{n,n} = S_i^n,\ 0 \le i \le n.$$

Define $\overline{S}_i = S_i^i$, so $\overline{S} = (\overline{S}_i) \in \overline{\mathcal{S}}$. It remains to check if for all $n \ge 0$: $\overline{S}_i = \overline{S}_i^n,\ 0 \le i \le n$. Indeed it is, because

$$\overline{S}_i = G_i^{i,i} = \overline{S}_i^i = \overline{S}_i^n,$$

where the last equality as in (8.4) again. □

The properties 0-neutral and up-down are defined through $\overline{\mathcal{S}}_{(\overline{S},j)}$, where $\overline{S} = \lim_{n\to\infty} S^n$, and just depend on the $(j+1)-$coordinate. It is enough to observe that for $m > j$,

$$\lim_{n\to\infty} \hat{S}^n = \overline{\hat{S}} \in \overline{\mathcal{S}}_{(\overline{S},j)} \quad \text{if and only if} \quad \hat{S}_j^m = \overline{\hat{S}}_j = \overline{S}_j = S_j^m.$$

Thus

$$\hat{S}^m \in \mathcal{S}_{(S^m,j)} \quad \text{and} \quad \overline{\hat{S}}_{j+1} = \hat{S}_{j+1}^m.$$

8.2. Modified $\overline{I}$. Lemma 5 below establishes existence of CT under a rather weak property on $\mathcal{S}$, the result requires a nonnegative constraint on the sequence $F = \{F_i\}$ though. This extra requirement will force a re-definition of $\overline{I}$ in order for us to be able to conclude continuity from below for $\mathcal{M}$ and I.

Definition 14. *We say that $\mathcal{S}$ satisfies a δ-property if $S_n \geq 0$ for all n and all S and there exists $\delta > 0$ such that for all nodes (S, n):*

(1) *If $S_n < \delta$, then there exists $S' \in \mathcal{S}_{(S,n)}$ and $S'_k = S_n \ \forall\ k \geq n$,*
(2) *If $S_n \geq \delta$, then there exists $S' \in \mathcal{S}_{(S,n)}$ and $S'_{n+1} - S_n \leq -\delta$.*

We assume $S_n \geq 0$ for simplicity, the condition can be extended to $S_n \geq C$ for a constant C.

Lemma 5. *Assume that $\mathcal{S}$ is locally 0-neutral and satisfies a δ-property. Let $F = \{F_i\}_{i \geq 0}$ be a sequence of non-anticipative functions and assume there exists a constant M satisfying $F_i(S) \geq 0$ for all $i \geq M$. Then, for any $\epsilon > 0$ and node (S^*, n^*) there exists $\hat{S} \in \mathcal{S}_{(S^*,n^*)}$ such that*

$$F_i(\hat{S})\ \Delta_i \hat{S} < \frac{\epsilon}{2^{i+1}},\ \forall i \geq n^*. \tag{8.7}$$

It then follows that $\mathcal{S}$ has the CT property w.r.t. F.

Proof. Without loss of generality we can provide a proof for the node $(S^*, 0)$ (otherwise start the construction below at (S^*, n^*)). Using the local 0-neutrality assumption as in Lemma 1, we conclude that there exists S^M satisfying

$$F_i(S^M)\ \Delta_i S^M < \frac{\epsilon}{2^{i+1}},\ \forall i \geq 0 \leq i \leq M-1. \tag{8.8}$$

If $S^M_M < \delta$ we take $\hat{S} \equiv S' \in \mathcal{S}_{(S^M,M)}$ satisfying $S'_k = S^M_M\ \forall k \geq M$. Otherwise, we proceed recursively on $j \geq 0$, set $S^{M,0} \equiv S^M$, if $S^{M,j}_{M+j} < \delta$ set $\hat{S} \equiv S' \in \mathcal{S}_{(S^{M,j},M+j)}$ satisfying $S'_k = S^{M,j}_{M+j}\ \forall k \geq M+j$ and terminate the recursion; otherwise, choose $S^{M,j+1} \in \mathcal{S} \in \mathcal{S}_{(S^{M,j},M+j)}$ satisfying $S^{M,j+1}_{M+j+1} - S^{M,j}_{M+j} \leq -\delta$. Notice that if this recursion continues indefinitely we obtain

$$\sum_{j=0}^{p-1}(S^{M,j+1}_{M+j+1} - S^{M,j}_{M+j}) = S^{M,j+p}_{M+j+p} - S^M_M \leq -p\ \delta$$

with gives a contradiction with $S_n \geq 0$ for large p. Therefore, the recursion terminates at some $J \geq 0$. We remark that by construction

$$F_{M+j}(S^{M,j+1})\ \Delta_{M+j} S^{M,j} \leq 0 < \frac{\epsilon}{2^{M+j+1}},\ 0 \leq j < J. \tag{8.9}$$

The inequalities (8.8) and (8.9) and the fact that $S^{M,J} \in \mathcal{S}_{(S^M,M)}$ imply that $\hat{S} \equiv S^{M,J}$ satisfies (8.7). In turn, that inequality implies the CT property of $\mathcal{S}$ w.r.t. F. $\square$

To make use of the above lemma we are required to change some of our basic definitions.

Definition 15. *For $f \in \mathcal{P}$, define*
(8.10)
$$\overline{\overline{I}}(f) = \inf\{\sum_{m=1}^{\infty} V^m : f \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty}\ \Pi_n^{V^m,H^m},\ \Pi_n^{V^m,H^m} \geq 0, H_i^m \geq 0\ \}.$$

Similarly, the continuity from below in Definition 5 is *modified* so that $H_i^m \geq 0$ is required as well. $\overline{\overline{I}}$ is positive homogeneous, countable subadditive and isotone. This means that all our constructions are valid by replacing $\overline{I}$ with $\overline{\overline{I}}$, it remains to establish the validity of the modified continuity from below property of $\mathcal{M}$ (and so for I).

Corollary 5. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume each node of $\mathcal{S}$ is up-down, and it satisfies a δ-property. Then, $\mathcal{M}$ and I satisfy the (modified) continuity from below property.*

Proof. To avoid repetition, we rely on definitions, notation and the general argument from the proof of Theorem 6. In particular $F_i = G_i - H_i^f$, then if we set $M \equiv n^f$ it follows that $F_i = G_i \geq 0$ for all $i \geq M$. Given $\epsilon > 0$, by an application of Lemma 5, it follows that there exists $S^\epsilon \in \mathcal{S}$ such that

$$\liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S^\epsilon)\Delta_i S^\epsilon \leq \sum_{i=0}^{n^f-1} H_i^f(S^\epsilon)\Delta_i S^\epsilon + \epsilon,$$

which concludes the proof as in Theorem 6. □

Notice that the corresponding space $\mathcal{L}^1$ will be, in general, smaller when defined through $\overline{\overline{I}}$ than when using $\overline{I}$.

Appendix A. Alternative integral operator

This section presents an alternative integral to the $\mathcal{M}$-integral constructed in Section 5. For reasons of space we do not present all the details and rely, in some instances, on obvious extrapolations of the results from Section 5. The main point that will come across is that the new integral allows to prove the classical monotone convergence theorem but requires an extremely large portfolio set $\overline{\mathcal{H}}$. Because of this reason we emphasized the $\mathcal{M}$-integral in the main body of the paper.

The section also presents properties of $\overline{W}$ which are used in Section 6 as well. The next results present some basic properties of $\overline{W}$.

Proposition 14. *Fix $f, g \in \mathcal{C}$.*

(1) *If $f \leq g$ then $\overline{W}f \leq \overline{W}g$.*
(2) *$\overline{W}f \leq \overline{W}|f|$.*
(3) *Assume $f+g$ and $\overline{W}f + \overline{W}g$ are well defined, then*

$$\overline{W}[f+g] \leq \overline{W}f + \overline{W}g.$$

Proof. The first two statements follow directly from the definition of $\overline{W}$. Fix $S \in \mathcal{S}$, it is enough to consider $\overline{W}f(S) < \infty, \overline{W}g(S) < \infty$. Let $V, U \in \mathbb{R}$ and $H, G \in \overline{\mathcal{H}}$ such that for any $S \in \mathcal{S}$,

$$f(S) \leq V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S, \text{ and } \ g(S) \leq \ U + \liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\ \Delta_i S.$$

We need to consider the following cases,

I. If $f(S),\ g(S)$ are finite, then $\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S > -\infty$ and $\liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\ \Delta_i S > -\infty$, consequently its sum is well defined and

$$f(S) + g(S) \leq V + U + \liminf_{n\to\infty} \sum_{i=0}^{n-1} [H_i(S) + G_i(S)]\ \Delta_i S. \tag{A.1}$$

II. If $f(S) = \infty$, then $g(S) > -\infty$, $\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S = \infty$, and $\liminf_{n\to\infty} \sum_{i=0}^{n-1} G_i(S)\ \Delta_i S > -\infty$, so its sum is also well defined and (A.1) holds. By symmetry, (A.1) also holds if $g(S) = \infty$.

III. If $f(S) = -\infty$, then $g(S) < \infty$ so $f(S) + g(S) = -\infty$ and (A.1) holds. Similar if $g(S) = -\infty$.

From (A.1),

$\overline{W}[f+g](S) \leq V+U$, and consequently $\overline{W}[f+g](S) \leq \overline{W}f(S)+\overline{W}g(S)$. $\square$

The next dual property is clear.

Corollary 6. *Fix $f, g \in \mathcal{C}$. Assume $f+g$ and $\underline{W}f + \underline{W}g$ are well defined, then*

$$\underline{W}f + \underline{W}g \leq \underline{W}[f+g].$$

The next lemma is based on Definition 12, given in Section 7.

Lemma 6. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$. Fix $V \in \mathbb{R}$ and let $H \in \overline{\mathcal{H}}$ be such that*

$$0 \leq V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S, \ \ \forall S.$$

Then for any $n \geq 0$,

$$0 \leq V + \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S, \ \ \forall S.$$

Proof. Assume there exist $n^* \geq 0$, $S^* \in \mathcal{S}$ and $\epsilon > 0$ such that

$$V + \sum_{i=0}^{n^*-1} H_i(S^*)\ \Delta_i S^* < -\epsilon.$$

From the CT assumption, there exists $\hat{S} \in \mathcal{S}$, an ϵ-contrarian trajectory w.r.t. H, starting at (S^*, n^*), such that

$$\liminf_{n\to\infty} \sum_{i=n^*}^{n-1} H_i(\hat{S})\ \Delta_i \hat{S} \leq \epsilon.$$

Therefore $V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(\hat{S})\ \Delta_i \hat{S} < 0$. □

Corollary 7. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$, then $\overline{W}0 = 0$.*

Proof. Clearly $\overline{W}0 \leq 0$, consider $V \in \mathbb{R}$ and $H \in \overline{\mathcal{H}}$ such that $0 \leq V + \liminf_{n\to} \sum_{i=0}^{n-1} H_i(S)\Delta_i S$ for all S. Then Lemma 6 implies $V \geq 0$. □

Proposition 15. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$ and that each node of $\mathcal{S}$ is up-down. Then, $\overline{W}$ is countable subadditive for non-negative functions.*

Proof. Let $g_k \geq 0, \quad k \geq 1$, and $g \leq \sum_{k=1}^{\infty} g_k$. It is enough to assume $\sum_{k=1}^{\infty} \overline{W}(g_k) < \infty$ (is non-negative), which, together with Corollary 7 and Proposition 14 leads to $0 \leq \overline{W}(g_k) < \infty$. So for any $k \geq 1$ there exist $H^k \in \overline{\mathcal{H}}$ such that

$$g_k(S) \leq \overline{W}(g_k) + \epsilon_k + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i^k(S)\ \Delta_i S, \ \ \forall S,$$

for given $\epsilon > 0$ and $\epsilon_k > 0$ with $\sum \epsilon_k = \epsilon$. By Lemma 6, it follows that

$$\overline{W}(g_k) + \epsilon_k + \sum_{i=0}^{n-1} H_i^k(S)\ \Delta_i S \geq 0\ \ \forall S.$$

Thus, Lemma 3 is applicable and so $H_i = \sum_{k=1}^{\infty} H_i^k$ is well defined and $H = \{H_i\}_{i\geq 0}$ belongs to $\overline{\mathcal{H}}$. Then by Fatou's lemma for series

$$g(S) \leq \sum_{k=1}^{\infty} \overline{W}(g_k) + \epsilon + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S,\ \ \forall S.$$

Consequently $\overline{W}(g) \leq \sum_{k=1}^{\infty} \overline{W}(g_k)$. □

Proposition 16. *Assume $\overline{W}0 = 0$ and let $f \in \mathcal{E}$, then*

$$\underline{W}f = V^f = \overline{W}f.$$

Proof. We can write $f(S) = V^f + \sum_{i=0}^{n^f-1} H_i^f(S)\Delta_i S\ \ \forall S \in \mathcal{S}$, with $V^f \in \mathbb{R}$ and $H^f \in \mathcal{H}$ ($H_i^f \equiv 0$ for $i \geq n^f$).

It is clear that $\overline{W}(f) \leq V^f$. Consider $V \in \mathbb{R}$ and $H \in \overline{\mathcal{H}}$ such that

$$f(S) \leq V + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\ \Delta_i S,\ \ \forall S \in \mathcal{S}.$$

Then,

$$0 \leq V - V^f + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(\tilde{S})\ \Delta_i \tilde{S} - \sum_{i=0}^{n^f-1} H_i^f(S)\Delta_i S$$

$$\leq V - V^f \liminf_{n\to\infty} \sum_{i=0}^{n-1} [H_i(S) - H_i^f(S)]\Delta_i S.$$

So $0 = \overline{W}0 \leq V - V^f$, from where $V^f \leq \overline{W}f$. Finally $\underline{W}f = -\overline{W}[-f] = -(-V^f) = \overline{W}f$. □

Proposition 17. *Fix $f \in \mathcal{C}$ such that $f > -\infty$. Then*

$$|\overline{W}f| \leq \overline{W}|f|. \tag{A.2}$$

Proof. Recalling item (2) of Proposition 14, we just need to prove that $-\overline{W}|f| \leq \overline{W}f$. Then it is enough to assume that $\overline{W}|f| < \infty$, which leads to $\overline{W}f < \infty$. Consequently $\overline{W}f + \overline{W}|f|$ is well defined and, by hypothesis $f + |f|$, is well defined as well. Thus Proposition 14 item (3) is applicable, and since $0 \leq f + |f|$ it follows, using Proposition 7, that

$$0 = \overline{W}0 \leq \overline{W}[f + |f|] \leq \overline{W}f + \overline{W}|f|.$$

□

Theorem 7. *Consider $f, g \in \mathcal{C}$ satisfying $f, g > -\infty$ and $\overline{W}f - \overline{W}g$ is well defined. Assume $\overline{W}0 = 0$, then*

$$|\overline{W}f - \overline{W}g| \leq \overline{W}|f - g|. \tag{A.3}$$

Proof. We may assume $\overline{W}|f-g| < \infty$. Using our hypothesis, we can check that:

$$f(S) \leq g(S) + |(f-g)(S)|, \ \forall \ S, \tag{A.4}$$

where we have used our convention $(f-g)(S) = 0$ for the case $g(S) = f(S) = \infty$. The sum in the right hand side in (A.4) is well defined for all S. Notice that $\overline{W}|f-g| \geq 0$ follows from $\overline{W}0 = 0$ and Proposition 14, item (1). Therefore $0 \leq \overline{W}|f-g| < \infty$ and $\overline{W}g + \overline{W}|f-g|$ is well defined, thus by Proposition 14 item (3)

$$\overline{W}f \leq \overline{W}g + \overline{W}|f-g|. \tag{A.5}$$

If both $\overline{W}f$ and $\overline{W}g$ are finite, it follows that $\overline{W}f - \overline{W}g \leq \overline{W}|f-g|$. If $\overline{W}f = \infty$, by hypothesis $\overline{W}g \in [-\infty, \infty)$ and so we contradict (A.5). If $\overline{W}f = -\infty$, by hypothesis $\overline{W}g \in (-\infty, \infty]$ so $\overline{W}f - \overline{W}g = -\infty \leq \overline{W}|f-g|$. We have then established $\overline{W}f - \overline{W}g \leq \overline{W}|f-g|$. The other required inequality: $\overline{W}g - \overline{W}f \leq \overline{W}|f-g|$ follows by symmetry given that our hypothesis are symmetric under the swap of f and g. $\square$

A.1. Alternative norm and integral. Here we introduce a new norm and a related integral, we do so by heavily relying on the operator $\overline{W}$ introduced in Definition 10. Define

$$||f||' \equiv \overline{W}|f| \text{ where } f \in \mathcal{C}.$$

The countable subadditivity property for nonnegative functions of $\overline{W}$ implies the same property for $||\ ||'$ and so, under the required conditions for the validity of Proposition 15, allows to establish the validity of the statements in Proposition 2 but now replacing $||\ ||$ by $||\ ||'$. In the present section the notion of a.e. refers to the one derived from $||\ ||'$.

We can then proceed as we have done with the construction of the $\mathcal{M}$-integral by defining $\mathcal{F}' \equiv \{f \in \mathcal{C} : ||f||' < \infty\}$, which results a completed normed space, and $\mathcal{E}'' \equiv \{f \in \mathcal{E} : ||f||' < \infty\} = \mathcal{E} \cap \mathcal{F}'$ and define $\mathbb{L}^1$ to be the $||\ ||'$-closure of $\mathcal{E}''$ on $\mathcal{F}'$. For completeness we state the most important results.

The following are the analogue of the generalized Beppo-Levi result in Proposition 5 item (2), and Theorem 3.

Proposition 18. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$ and each node of $\mathcal{S}$ is up-down. Let $\{f_n\} \subseteq \mathcal{F}'$ such that $\sum_{n\geq 1} \|f_n\|' < \infty$. Then the limit $\sum_{n\geq 1}^{\infty} f_n \equiv \lim_{m\to\infty} \sum_{n\geq 1}^{m} f_n$ exists a.e. and in the norm $||\ ||'$ and $\sum_{n\geq 1}^{\infty} f_n \in \mathcal{F}$'.*

Theorem 8. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, assume $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$ and each node of $\mathcal{S}$ is up-down. Then $(\mathcal{F}', ||\ ||')$ is complete, and so $\mathbb{L}^1$ is.*

From Propositions 16 and 17 follows that

$$|I(f)| \leq ||f||', \ \forall f \in \mathcal{E}. \tag{A.6}$$

Inequality (A.6) allows to define $\int' f$ to be the continuous linear extension of I from $\mathcal{E}''$ to $\mathbb{L}^1$.

The classical Beppo Levi theorem holds for this integral as well.

Proposition 19. *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ assume $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$, and each node of $\mathcal{S}$ is up-down. Let $f_n \in \mathbb{L}^1$, $n \geq 1$, such that $\sum\limits_{n\geq 1} \|f_n\|' < \infty$. Then $\sum\limits_{n\geq 1}^{m} f_n$ converges a.e. and in the $||\ ||'$-norm to a function in $\mathbb{L}^1$. Moreover,*

$$\int' \sum_{n\geq 1} f_n = \sum_{n\geq 1} \int' f_n.$$

Under the assumption that for $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$, and each node of $\mathcal{S}$ is up-down. Given that $I = \overline{W}$ on $\mathcal{E}$ by Proposition 16, Theorem 7 allows us to conclude

$$\int' f = \overline{W} f \text{ forall } f \in \mathbb{L}^1, \ f > -\infty, \tag{A.7}$$

therefore

$$\int' f = ||f||' \text{ forall } f \in \mathbb{L}^1, \ f \geq 0. \tag{A.8}$$

Following [8], it is (A.8) that allows us to obtain the usual monotone convergence theorem.

Theorem 9 (Monotone Convergence Theorem for $\int' f$)**.** *Given $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ such that $\mathcal{S}$ has the CT property for any $H \in \overline{\mathcal{H}}$, and each node of $\mathcal{S}$ is up-down. Let $\{f_n\}_{n\geq 1} \subseteq \mathbb{L}^1$, $f_n \nearrow f$, $\int' f_n \leq C =$ constant $< \infty$. Then: $||f - f_n||' \to 0$, $f \in \mathbb{L}^1$ and $\int' f = \lim_{n\to\infty} \int' f_n$.*

Proof. Define for $n \geq 1$, $g_n \equiv f_{n+1} - f_n \geq 0$, the result then follows from (A.8) and Proposition 19. □

Appendix B. Connections with martingales

To show a connection with discrete time martingale processes, we rely on the following definitions; let $X = \{X_n\}$ be a martingale on $(\Omega, P, \mathcal{B})$ and filtration $\mathcal{F} = \{\mathcal{F}_n\}$, $\mathcal{F}_n \subseteq \mathcal{B}$ with $\mathcal{F}_0$ trivial. We set $s_0 = X_0(w)$; in this section $\mathcal{S}$ is given by

$$\mathcal{S} = \{S = \{S_n\} \in \mathcal{S}_\infty(s_0) : \ \exists\, w \in \Omega, \ S_n \equiv X_n(w)\}.$$

Where we assumed, without loss of generality, that all the random variables X_n are defined everywhere in Ω. Define $\alpha : \Omega \to \mathcal{S}$ by $\alpha(w) = S$ where $S = \{S_n = X_n(w)\}$.

We do not require any special property of $\mathcal{H}$ but we assume $H = \{H_i\} \in \mathcal{H}$ satisfies $H_i : \mathbb{R}^i \to \mathbb{R}$ are bounded Borel-measurable functions. This will provide the non-anticipative property. Proposition 20 below shows that our definition of a null set does not introduce new sets of measure zero for the case when the trajectories come from a martingale. Similarly sets of probability one will be full sets, according to our definition, and so the approach does not miss such events.

Proposition 20. *Given $(\Omega, P, \mathcal{B})$, assume that H_i, with $H \in \mathcal{H}$ and $H_i : \mathbb{R}^i \to \mathbb{R}$ are bounded Borel-measurable functions. We have the following implications:*

$$\text{(B.1)} \qquad \textit{if } ||\mathbf{1}_A|| = 0 \textit{ and } \alpha^{-1}(A) \in \mathcal{B}, \textit{ then } P(\alpha^{-1}(A)) = 0,$$

$$\text{(B.2)} \qquad \textit{if } P(\alpha^{-1}(A)) = 1, \textit{ then } ||\mathbf{1}_A|| = 1.$$

Proof. To establish (B.1) notice that for a given $\epsilon > 0$ there exists $H^m \in \mathcal{H}$, depending on ϵ and $\epsilon^m > 0$ satisfying $\sum_m \epsilon^m \leq \epsilon$ and

$$\epsilon^m + \sum_{i=0}^{n-1} H_i^m(X_0(w), \ldots, X_i(w)) \; (X_{i+1}(w) - X_i(w)) \geq 0 \;\forall\; w \in \Omega \;\forall\; n \geq 0.$$

Moreover,

$$\mathbf{1}_{\alpha^{-1}(A)}(w) \leq f(w) \equiv$$

$$\sum_m [\epsilon^m + \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i^m(X_0(w), \ldots, X_i(w)) \; (X_{i+1}(w) - X_i(w))]$$

Therefore

$$\alpha^{-1}(A) \subseteq B \equiv \{f(w) \geq 1\}.$$

We remark that $B \in \mathcal{B}$ and

$$P(B) \leq \int f(w) dP(w) \leq \epsilon,$$

where we used: $f \geq 0$ for the first inequality and Fatou's lemma combined with Monotone convergence theorem and the martingale property for the second inequality. Therefore $P(\alpha^{-1}(A)) = 0$ (notice that if $\mathcal{B}$ were complete it would imply the current assumption $\alpha^{-1}(A) \in \mathcal{B}$).

To establish (B.2) consider

$$\text{(B.3)} \qquad \mathbf{1}_A(S) \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} f_n^m(S), \;\forall\; S,$$

where $f_n^m(S) = V^m + \sum_{i=0}^{n-1} H_i^m(S)\, \Delta_i S \;\forall\, S$, with $H^m \in \mathcal{H}$, $V^m \in \mathbb{R}^+$, as well as $f_n^m \in \mathcal{E}$, $f^m \geq 0$, for $m \geq 1,\ n \geq 0$. (B.3) implies

$$\mathbf{1}_{\alpha^{-1}(A)}(w) \leq \sum_{m=1}^{\infty} \liminf_{n\to\infty} f_n^m(\alpha(w)),\ \forall\, w. \tag{B.4}$$

As $\alpha^{-1}(A) \in \mathcal{B}$, taking expectations in both sides of (B.4) and applying Fatou's lemma combined with Monotone convergence theorem and the martingale property it follows that $\sum_{m=1}^{\infty} V^m \geq 1$, therefore $\overline{I}(\mathbf{1}_A) \geq 1$ but we know that $\overline{I}(\mathbf{1}_A) \leq 1$ as well so the result follows. ☐

Proposition 21. *Given the market $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, $H \in \mathcal{H}$, write $H_i(w) \equiv H_i(X_0(w), \ldots, X_i(w))$ and assume $H_i : \mathbb{R}^i \to \mathbb{R}$ are bounded Borel measurable. Furthermore, there exists $C_H(w)$ integrable satisfying:*

$$\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(w) \Delta_i X(w) \geq C_H(w) \text{ a.e.}, \tag{B.5}$$

where $\Delta_i X(w) \equiv (X_{i+1}(w) - X_i(w))$. Let $\epsilon > 0$, then there exists a set A_ϵ with $P(A_\epsilon) > 0$ and $S^\epsilon \in \alpha(A_\epsilon)$ is an ϵ-CT for H (as per Definition 12 and paragraph prior to that definition).

Proof. Define $A_\epsilon \equiv \{w \in \Omega : \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(w)\, \Delta_i X(w) \leq \epsilon\}$. Noticing that Fatou's lemma is applicable because of (B.5), we obtain

$$0 = \liminf_{n\to\infty} \int_\Omega \sum_{i=0}^{n-1} H_i(w)\ \Delta_i X(w) dP(w) \geq$$

$$\int_\Omega \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(w)\ \Delta_i X(w) dP(w) \geq \int_{A_\epsilon} C_H(w) dP(w) + P(A_\epsilon^c)\epsilon.$$

The obtained inequality implies a contradiction if $P(A_\epsilon) = 0$. ☐

Appendix C. Financial interpretation

We make some informal comments on a financial interpretation; call $A \subseteq \mathcal{S}$ an *arbitrage set* if there exists $H \in \mathcal{H}$ satisfying

$$\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S) \Delta_i S \geq 1_A(S)\ \forall\, S, \tag{C.1}$$

according to our definitions, 1_A is then a null function. The fact that (C.1) holds can be considered to provide an arbitrage strategy, namely H, in a market model of the type $(\mathcal{S}, \mathcal{H})$. Under the assumption of existence of contrarian trajectories in the limit, for each $1 > \epsilon > 0$, there exists such contrarian trajectory $\overline{S}$ and (C.1) implies that $\overline{S} \in A^c$. It then follows that

$$\inf_{S \in A^c} \liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(S)\Delta_i S = 0.$$

This co-existence between contrarian trajectories and arbitrage opportuities is present, under the label 0-neutrality in [6]. Under stronger, but still natural hypothesis, one can eliminate such notion of arbitrage; one only needs to require that for any given $H \in \mathcal{H}$ there exists a contrarian trajectory $\overline{S}$ satisfying

$$\liminf_{n\to\infty} \sum_{i=0}^{n-1} H_i(\overline{S})\Delta_i S < 0,$$

this will eliminate the possibility that an arbitrage set exists.

Finally, our definition of null sets includes sets which are not necessarily arbitrage sets but their characteristic functions can be superhedged with arbitrarily small initial investments.

Appendix D. Upcrossing inequalities

In order to obtain the upcrossing inequality, first we introduce the formal definition of upcrossing through a band. Let $\mathcal{S}$ a trajectory set, for any $n \geq 1$ and $0 < a < b$ real numbers, define for a generic $S \in \mathcal{S}$:

$$\tau_0(S) = \inf\{i \geq 0 : \; S_i < a\}$$
$$\tau_1(S) = \inf\{i \geq \tau_0(S) : \; S_i > b\}$$

and continue recursively for $k \geq 1$:

$$\begin{aligned} \tau_{2k}(S) = &\quad \inf\{i \geq \tau_{2k-1}(S) : \; S_i < a\} \\ \tau_{2k+1}(S) = &\quad \inf\{i \geq \tau_{2k}(S) : \; S_i > b\}. \end{aligned} \tag{D.1}$$

and we use the convention $\inf\ \emptyset = \infty$. Notice that, if τ_k's are finite:

$$S_{\tau_{2k}(S)} < a \text{ and } S_{\tau_{2k+1}(S)} > b \quad \text{then} \quad \tau_{2k+1}(S) \geq \tau_{2k}(S) + 1.$$

The quantities τ_k are actually stopping times, according to the following definition from [6].

Definition 16. *Given a trajectory space $\mathcal{S}$, a trajectory based stopping time is a function $\nu : \mathcal{S} \to \mathbb{N}$ such that if $S, S' \in \mathcal{S}$ and $S_i = S'_i$ for $0 \leq i \leq \nu(S)$ then $\nu(S') = \nu(S)$.*

Lemma 7. *The quantities τ_k defined by (D.1) are trajectory based stopping times.*

Proof. Consider $S, S' \in \mathcal{S}$. If $\tau_0(S) = 0$, it means that $s_0 < a$ and consequently $\tau_0(S') = 0$. Assume now that $S_i = S'_i$ for $0 \leq i \leq \tau_0(S) \neq 0$, then for $0 \leq i < \tau_0(S)$, $S'_i = S_i \geq a$ and $S'_{\tau_0(S)} = S_{\tau_0(S)} < a$, it means that $\tau_0(S') = \tau_0(S)$. Let us finish the proof by induction on k.

Assumed $S_i = S'_i$ for $0 \le i \le \tau_{k+1}(S)$, by inductive hypothesis $\tau_k(S') = \tau_k(S)$. Thus $S_i = S'_i$ for $\tau_k(S') \le i \le \tau_{k+1}(S)$. For $i < \tau_{k+1}(S)$, if $k+1$ is even, as for the case $k = 0$, $S'_i = S_i \ge a$, and if $k+1$ is odd $S'_i = S_i \le b$, it follows that $\tau_{k+1}(S') = \tau_{k+1}(S)$. □

In the sequel, for simplicity, once S is clearly understood, we will write τ_k instead of $\tau_k(S)$ for any $k \ge 0$.

Definition 17. *For $n \ge 1$ and $S \in \mathcal{S}$, denote by $U_n^{[a,b]}(S) = U_n(S)$ the number of upcrossings of the sequence $(S)_{i=1}^n$ through the interval $[a,b]$, it is, $U_n(S) = 0$, or*

$$U_n(S) = \max\{k \in \mathbb{N} : \tau_{2k-1} \le n\}.$$

The total number of upcrossings of S will be denoted by

$$U^{[a,b]}(S) = U(S) = \sup\{n \ge 1 : U_n(S)\}.$$

We are now going to introduce a portfolio that allows to count the upcrossings.

Lemma 8. *For any $i \ge 0$, define $D_i : \mathcal{S} \to \mathbb{R}$ by*

$$D_i(S) = 1, \quad \text{if there exists} \quad k \ge 0 \quad \text{such that} \quad \tau_{2k}(S) \le i < \tau_{2k+1}(S),$$
$$D_i(S) = 0 \quad \text{otherwise.}$$

It follows that $D = (D_i)_{i \ge 0}$ is non-anticipative.

Proof. Fix $i \ge 0$ and let $S, S' \in \mathcal{S}$ with $S_j = S'_j$ for $0 \le j \le i$. Assume there exists $k \ge 0$, such that $\tau_{2k}(S) \le i < \tau_{2k+1}(S)$, then $\tau_{2k}(S') = \tau_{2k}(S)$. Thus, it must be $i < \tau_{2k+1}(S')$, and $D_i(S') = D_i(S) = 1$. On the other hand, if $D_i(S) = 0$, also $D_i(S') = 0$, if not, by symmetry, it would be a contradiction. □

Proposition 22. *For any $S \in \mathcal{S}$ and $n \ge 1$ one of the following inequality holds.*

$$\text{(D.2)} \qquad \begin{aligned} \textstyle\sum_{i=0}^{n-1} D_i(S)\Delta_i S &\ge (b-a)U_n^{[a,b]}(S). \\ \textstyle\sum_{i=0}^{n-1} D_i(S)\Delta_i S &\ge (b-a)U_n^{[a,b]}(S) + (S_n - a)\ . \end{aligned}$$

Therefore, since $S_i \ge 0$ for any $i \ge 0$, then

$$\text{(D.3)} \qquad (b-a)U_{n,a,b}(S) \le a + \sum_{i=0}^{n-1} D_i(S)\Delta_i S.$$

Proof. Fix $S \in \mathcal{S}$ and $n \ge 1$. It is enough to consider $\tau_0 < \infty$. Observe that if $0 \le i < \tau_0$ then $D_i(S) = 0$, so the non null terms in (D.2) starts at $i = \tau_0$. If $n < \tau_0$, it follows that $U_n(S) = 0$ and $\sum_{i=0}^{n-1} D_i(S)\Delta_i S = 0 = (b-a)U_n^{[a,b]}(S)$. We are going to split the rest of the proof in the following to cases:

I) $\tau_{2k-1} \le n < \tau_{2k}$, for some $k \ge 1$.

II) $\tau_{2k} \le n < \tau_{2k+1}$, for some $k \ge 0$.
Observe that in both cases $U_n^{[a,b]}(S) = k$. For case I),

$$\sum_{i=0}^{n-1} D_i(S)\Delta_i S = \sum_{i=0}^{k-1} (S_{\tau_{2i+1}} - S_{\tau_{2i}}) > (b-a)U_n^{[a,b]}(S).$$

For case II)

$$\sum_{i=0}^{n-1} D_i(S)\Delta_i S = S_n - S_{\tau_{2k}} + \sum_{i=0}^{k-1} (S_{\tau_{2i+1}} - S_{\tau_{2i}}) > S_n - a + (b-a)U_n^{[a,b]}(S).$$

□

D.1. Upcrossings: some technical matters. The purpose of this section is to show that in markets $\mathcal{M} = (\mathcal{S}, \mathcal{H})$, where the nodes S_n of any trajectory could be negative, the convergence theorem 1 still holds.

For a real number $K > 0$ consider a market $\mathcal{M} = (\mathcal{S}, \mathcal{H})$ such that $S_i \ge -K$ for any $S \in \mathcal{S}$ and $i \ge 0$. All previous results in that section holds up to equation (D.2), included. Equation (D.3) must be replaced by (D.4) below. Since $S_i \ge -K$ for any $i \ge 0$, then

$$(b-a)U_{n,a,b}(S) \le a + K + \sum_{i=0}^{n-1} D_i(S)\Delta_i S. \tag{D.4}$$

The next lemma is implicit when proving convergence through the use of upcrossings, we include its proof for completeness.

Lemma 9. *If $\{a_n\}_{n\ge1} \subseteq \mathbb{R}$, with $a_n \ge 0$ for $n \ge 1$, does not converge in $\mathbb{R}$ and upcrosses any interval only a finite number of times then, $\{a_n\}_{n\ge1}$ diverges to ∞.*

Proof. Since $\{a_n\}_{n\ge1}$ does not converge in $\mathbb{R}$,

$$\forall r \in \mathbb{R}\ \exists\ \epsilon_r > 0 : \forall n \ge 1\ \exists\ n' > n \ \text{ with } \ |a_{n'} - r| > \epsilon_r.$$

Fix $r > 0$. Let $n_1 > 1$ the first integer such that $|a_{n_1} - r| > \epsilon_r$. Once $n_1, ..., n_i$ were chosen, let $n_{i+1} > n_i$ the first integer such that $|a_{n_{i+1}} - r| > \epsilon_r$.

Assume first that $a_{n_i} > r + \epsilon_r$ for infinitely many integers $i \ge 1$. In this case there can only be a finite number of integers $n \ge n_1$ such that $S_n < r$. Otherwise, there would be an infinite number of upcrosses through $[r, r+\epsilon_r]$. Let n^r be the largest such n then, $a_n \ge r$ for any $n \ge n^r$. We then conclude that $\{a_n\}_{n\ge1}$ diverges to ∞.

On the other hand if there are infinitely many integers $i \ge 1$ such that $a_{n_i} < r - \epsilon_r$, there exists n^r such that $a_n \le r$ for any $n \ge n^r$. This means, in this case, that $\{a_n\}$ converges to 0, which contradicts the hypothesis. □

Acknowledgement

The authors would like to thank Dr. Pedro Catuogno for stimulating conversations on the subject of the paper. S. Ferrando would like to thank Professor König for providing a personal copy of reference [7].

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This paper is available via http://nyjm.albany.edu/j/2018/24-35.html.