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Sets and mappings in βS which are not Borel

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Abstract. We extend theorems proved in [4] by showing that, if S is a countably infinite left cancellative semigroup and there is a finite bound on the size of sets of the form $\{x \in S : xa = b\}$ for $a, b \in S$, then the following subsets of βS are not Borel: the set of idempotents, the smallest ideal, any semiprincipal right ideal defined by an element of S^*, and S^*S^*. This has the imediate corollary that, if S is any infinite semigroup which either has the cancellation properties just described or has infinitely many cancellable elements, then the set of idempotents in βS is not Borel. We extend a theorem proved in [1], which states that for any infinite discrete group G and any $p \in G^*$, $\lambda_p : \beta G \to \beta G$ is not Borel, by showing that this theorem holds for all infinite semigroups which are right cancellative and very weakly left cancellative. We show that continuous maps between compact spaces map Baire sets to universally measurable sets, although this is far from being the case for Borel sets.

Contents

1. Introduction

Let $(S, \cdot)$ be a discrete semigroup. We take the Stone-Čech compactification βS of S to be the set of ultrafilters on S with the points of S identified with the principal ultrafilters. Given $A \subseteq S$, we let $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets of βS as well as a basis for the closed sets. And, as the notation suggests, $\overline{A}$ is the closure of A in βS. The operation on S extends to βS so that the function ρ_p defined by $\rho_p(x) = x \cdot p$ is continuous for each $p \in \beta S$. Furthermore, S is contained in the topological center of βS, meaning that the function λ_y defined by

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$\lambda_y(x) = y \cdot x$ is continuous for each $y \in S$. Given $p \in \beta S$ and an indexed family $\langle x_s \rangle_{s\in S}$ and a point y in a topological space X, $p\text{–}\lim_{s\in S} x_s = y$ if and only if for every neighborhood U of y, $\{s \in S : x_s \in U\} \in p$. If X is compact and Hausdorff, then $p\text{–}\lim_{s\in S} x_s$ is guaranteed to exist uniquely and, if $\varphi : S \to X$ is defined by $\varphi(s) = x_s$ and $\widetilde{\varphi} : \beta S \to X$ is its continuous extension, then $\widetilde{\varphi}(p) = p\text{–}\lim_{s\in S} x_s$. For $p, q \in \beta S$, $pq = p\text{–}\lim_{s\in S} q\text{–}\lim_{t\in S} st$. For $A \subseteq S$, $A \in pq$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$ where $s^{-1}A = \{t \in S : st \in A\}$.

If $A \subseteq S$, A^* will denote $c\ell_{\beta S}(A) \setminus A$. We write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Every compact Hausdorff right topological semigroup T has important algebraic properties, including the fact that it has at least one idempotent. If V is a subset of T, $E(V)$ will denote the set of idempotents in V. T has a smallest two sided ideal, $K(T)$, which is the union of all of the minimal right ideals and the union of all of the minimal left ideals of T. Every right ideal of T contains a minimal right ideal, and every left ideal of T contains a minimal left ideal. The intersection of a minimal right ideal and a minimal left ideal is a group; and all the subgroups of T which arise in this way are algebraically isomorphic and are homeomorphic if they lie in the same minimal right ideal. See [3, Part I] for the facts mentioned here, and any other unfamiliar assertions encountered. We remark that the maximal groups in $K(T)$ need not be homeomorphic in general. In fact, if S is an infinite cancellative and commutative semigroup, then by [3, Lemma 6.40 and Theorem 7.42] the maximal groups contained in any minimal left ideal of βS lie in $2^{\mathfrak{c}}$ homeomorphism classes.

We shall use $\mathbb{N}$ to denote the set of positive integers, ω to denote the set of non-negative integers, $\mathbb{Z}$ to denote the set of all integers and $\mathbb{R}$ to denote the set of real numbers. We also take ω to be the first infinite cardinal. $\mathbb{H}$ will denote $\bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. This is a subsemigroup of $\beta\mathbb{N}$ which contains all the idempotents.

Anyone who has worked with $\beta\mathbb{N}$, will not be surprised to learn that some of the algebraically defined subsets of $\beta\mathbb{N}$ are not topologically simple, even though they are very simple to define algebraically. It was shown in [4] that the following subsets of $\beta\mathbb{N}$ are not Borel: the set of idempotents; any semiprincipal right ideal of $\mathbb{N}^*$; the smallest ideal of $\beta\mathbb{N}$; the set of idempotents in any left ideal of $\beta\mathbb{N}$; $\mathbb{N}^* + \mathbb{N}^*$; and $\mathbb{H} + \mathbb{H}$. These results were extended to infinite countable semigroups which can be algebraically embedded in compact Hausdorff topological groups.

A subset X of a semigroup S is a *left solution set* if and only if there exist $a, b \in S$ such that $X = \{x \in S : ax = b\}$. A semigroup S is *weakly left cancellative* provided that all left solution sets in S are finite. If $|S| = \kappa \geq \omega$, then S is *very weakly left cancellative* provided the union of any set of fewer than κ left solution sets has cardinality less than κ. Similarly X is a *right* *solution set* if there exist $a, b \in S$ such that $X = \{x \in S : xa = b\}$, and S is weakly right cancellative if every right solution set is finite.

In Section 2 in the present paper, we extend some of the results of [4] by showing that, if S is any countably infinite left cancellative semigroup and there exists $k \in \mathbb{N}$ such that every right solution set in S has at most k elements, then the following subsets of βS are not Borel: the set of idempotents; any semiprincipal right ideal of S^*; the smallest ideal of βS; and S^*S^*. As an immediate corollary, we obtain the result that, if S is an arbitrary infinite semigroup which either is left cancellative with a finite bound on the size of right solution sets, or has infinitely many cancelable elements, then the set of idempotents in βS is not Borel.

In Section 3 we extend a theorem due to E. Glasner [1] by showing that, if S is an arbitrary infinite cancellative semigroup and if $p \in S^*$, then the map $\lambda_p : \beta S \to \beta S$ is not Borel. E. Glasner proved this theorem in the case in which S is a group, and the methods that we use are based on his.

In Section 4 we discuss continuous images of Borel sets. An elegant example, due to D. Fremlin, shows that continuous functions from $\beta\mathbb{N}$ to metric spaces, need not map Borel sets to universally measurable sets. However, any continuous function from a compact Hausdorff space to a compact Hausdorff space, does map Baire sets to universally measurable sets.

2. Subsets of βS which are not Borel

Throughout this section we will let S be a countably infinite discrete semigroup which is at least weakly left cancellative. We will prove that if S is left cancellative and has a finite bound on the size of right solution sets, then the following subsets of βS are not Borel: the set of idempotents; the smallest ideal; any semiprincipal right ideal defined by an element of S^*; and S^*S^*. The proof is based on the following lemma.

Lemma 2.1. *Every Borel subset of βS is the union of a family of compact subsets of βS of cardinality at most $\mathfrak{c}$.*

Proof. The proof is identical to the proof of [4, Lemma 3.1], where it was stated for $\beta\mathbb{N}$. □

Definition 2.2. We enumerate S as a sequence and write $s \prec t$ if s precedes t in this sequence.

Lemma 2.3. *There is a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for each $n \in \mathbb{N}$,*

(1) *$s_n \prec s_{n+1}$;*
(2) *if $a, b \preceq s_n$, then $ab \prec s_{n+1}$; and*
(3) *if $a \preceq s_n$ and $ab \preceq s_n$, then $b \prec s_{n+1}$.*

Proof. We construct $\langle s_n \rangle_{n=1}^{\infty}$ inductively. One can do this because, given n, $\{s_n\} \cup \{ab : a, b \preceq s_n\}$ is finite and since S is weakly left cancellative, given $a, c \preceq s_n$, $\{b \in S : ab = c\}$ is finite. □

We shall assume that we have fixed a sequence $\langle s_n\rangle_{n=1}^{\infty}$ as guaranteed by Lemma 2.3.

Definition 2.4. We define $\phi : S \to \mathbb{N}$ by $\phi(t) = \min\{n \in \mathbb{N} : t \preceq s_n\}$.

The function ϕ extends to a continuous mapping from βS to $\beta\mathbb{N}$, which we shall also denote by ϕ.

Lemma 2.5. *For every $x \in \beta S$ and every $y \in S^*$, $\phi(xy) \in \{\phi(y) - 1, \phi(y), \phi(y) + 1\}$.*

Proof. We claim that, for every $a, b \in S$ and every $n > 2$ in $\mathbb{N}$, if $a \preceq s_{n-2}$ and $s_{n-1} \prec b \preceq s_n$, then $s_{n-2} \prec ab \prec s_{n+1}$ and hence that $\phi(ab) \in \{\phi(b) - 1, \phi(b), \phi(b)+1\}$. By condition (2) we have directly that $ab \prec s_{n+1}$. Suppose that $ab \preceq s_{n-2}$. Then by condition (3) with n replaced by $n-2$, we have $b \prec s_{n-1}$, a contradiction. So we have for every $a \in S$ and all sufficiently large $b \in S$, $\phi(ab) \in \{\phi(b) - 1, \phi(b), \phi(b) + 1\}$ If $x \in S$ and $y \in S^*$, then $\phi(xy) = y\text{–}\lim_{b\in S} \phi(xb)$. If $x, y \in S^*$, $\phi(xy) = x\text{–}\lim_{a\in S} y\text{–}\lim_{b\in S} \phi(ab)$. Therefore for any $x \in \beta S$ and $y \in S^*$, $\phi(xy) \in \{\phi(y) - 1, \phi(y), \phi(y) + 1\}$. □

Lemma 2.6. *Assume that S is left cancellative and $k \in \mathbb{N}$ such that for any $a, b \in S$, $|\{x \in S : xa = b\}| < k$. Then for any $p, q \in \beta S$, $|\{x \in S : xp = q\}| < k$.*

Proof. Let $p, q \in \beta S$ and suppose that $|\{x \in S : xp = q\}| \geq k$. Pick distinct $x_1, x_2, \ldots, x_k$ in S such that $x_i p = q$ for each $i \in \{1, 2, \ldots, k\}$. Define $f : S \to S$ as follows.

(1) If $v \in S \setminus x_1 S$, then $f(v) = x_1^2$.
(2) Assume that $v = x_1 u$ for some $u \in S$ and note that since S is left cancellative, there is only one such u. Let $f(v) = x_i u$ where i is the first member of $\{2, 3, \ldots, k\}$ such that $x_i u \neq x_1 u$.

Then f has no fixed points so by [3, Lemma 3.33], pick A_0, A_1, A_2 such that $S = A_0 \cup A_1 \cup A_2$ and for each $i \in \{0, 1, 2\}$, $A_i \cap f[A_i] = \emptyset$. Pick $i \in \{0, 1, 2\}$ such that $A_i \in x_1 p$. For $j \in \{2, 3, \ldots, k\}$, let $B_j = \{u \in S : f(x_1 u) = x_j u\}$ and pick $j \in \{2, 3, \ldots, k\}$ such that $B_j \in p$. Let $\widetilde{f} : \beta S \to \beta S$ denote the continuous extension of f. Then for $u \in B_j$, $f(x_1 u) = x_j u$ so $\widetilde{f} \circ \lambda_{x_1}$ and λ_{x_j} agree on a member of p so $\widetilde{f}(x_1 p) = x_j p$. Since $A_i \in x_1 p$, $f[A_i] \in \widetilde{f}(x_1 p) = x_j p = x_1 p$ while $f[A_i] \cap A_i = \emptyset$, a contradiction. □

Lemma 2.7. *Assume that S is left cancellative and there is a finite bound on the size of right solution sets in S. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S^* on which ϕ is injective. Then $c\ell\{x_n : n \in \mathbb{N}\}$ meets $S^* \setminus (S^* S^*)$.*

Proof. We may suppose that $\{\phi(x_n) : n \in \mathbb{N}\}$ is discrete, because any infinite subset of a Hausdorff space has an infinite (strongly) discrete subset.

We claim that ϕ is injective on $c\ell\{x_n : n \in \mathbb{N}\}$. To see this, suppose that p and q are distinct elements of $c\ell\{x_n : n \in \mathbb{N}\}$ and $\phi(p) = \phi(q)$. Pick

$A \in p$ and $B \in q$ such that $A \cap B = \emptyset$. Then $\phi(p) \in c\ell(\{\phi(x_n) : x_n \in \overline{A}\})$ and $\phi(q) \in c\ell(\{\phi(x_n) : x_n \in \overline{B}\})$. So, by [3, Theorem 3.40], without loss of generality there exists $m \in \mathbb{N}$ such that $\phi(x_m) \in c\ell(\{\phi(x_n) : n \in \mathbb{N} \setminus \{m\}\})$ – contradicting the assumption that $\{\phi(x_n) : n \in \mathbb{N}\}$ is discrete.

Let x be a point of accumulation of $\langle x_n \rangle_{n=1}^{\infty}$. We claim that $x \notin S^*S^*$. To see this suppose, on the contrary, that $x = yz$ for some $y, z \in S^*$. By Lemma 2.5, ϕ assumes at most three values on βSz. So, if $M = \{n \in \mathbb{N} : \phi(x_n) \notin \phi[\beta Sz]\}$, then $x \in c\ell\{x_n : n \in M\}$. Also, for every $a \in S$, $x \in c\ell\{bz : b \in S \text{ and } a \prec b\}$. It follows from [3, Theorem 3.40] that $x_n \in c\ell\{bz : b \in S\} = \beta Sz$ for some $n \in M$, or for each $a \in S$, there exists $b_a \in S$ such that $a \prec b_a$ and $b_a z \in c\ell\{x_n : n \in \mathbb{N}\}$. The first possibility contradicts the definition of M, and so the second possibility must hold for each $a \in S$. By Lemma 2.5, for each $a \in S$, $\phi(b_a z) \in \{\phi(z) - 1, \phi(z), \phi(z) + 1\}$. Since ϕ is injective on $c\ell\{x_n : n \in \mathbb{N}\}$, we have $|\{b_a z : a \in S\}| \leq 3$. However, since each $b_a \succ a$, $\{b_a : a \in S\}$ is infinite. If every right solution set in S has fewer than k elements, then by Lemma 2.6, $\big|\{s \in S : sz \in \{b_a z : a \in S\}\}\big| < 3k$, a contradiction. □

Corollary 2.8. *Assume that S is left cancellative and there is a finite bound on the size of right solution sets in S. On any Borel subset B of S^*S^*, ϕ assumes at most $\mathfrak{c}$ values.*

Proof. Let B be a Borel subset of S^*S^*. By Lemma 2.1 pick a family $\mathcal{D}$ of compact subsets of βS such that $B = \bigcup \mathcal{D}$ and $|\mathcal{D}| \leq \mathfrak{c}$. Since $B \subseteq S^*S^*$, if $D \in \mathcal{D}$, then $D \subseteq S^*S^*$. By Lemma 2.7, if $D \in \mathcal{D}$, then ϕ assumes only finitely many values on D. □

We put $P = \{s_n : n \in \mathbb{N}\}$. We observe that $\phi(s_n) = n$ for every $n \in \mathbb{N}$, and so $\phi[P^*] = \mathbb{N}^*$ and hence $|\phi[P^*]| = 2^{\mathfrak{c}}$.

Theorem 2.9. *Assume that S is left cancellative and there is a finite bound on the size of right solution sets in S. The following subsets of βS are not Borel: the set of idempotents; the smallest ideal; S^*S^*; and any principal right ideal of βS defined by an element of S^*.*

Proof. We shall show that ϕ assumes $2^{\mathfrak{c}}$ values on the intersection of each of these sets with S^*S^*. This will be sufficient, because $E(\beta S) \cap S^* = E(\beta S) \cap S^*S^*$, for any $q \in S^*$, $q\beta S \setminus S^*S^*$ is countable, and $K(\beta S) \subseteq S^*S^*$. (To verify the latter assertion, by [3, Theorem 4.36] $K(\beta S) \subseteq S^*$ and since $K(\beta S)$ is the union of groups, $K(\beta S) \subseteq S^*S^*$.) So, if any of these sets were Borel, their intersections with S^*S^* would also be Borel. We define an equivalence relation $\equiv$ on βS by stating that $x \equiv y$ if $\phi(x) \in \mathbb{Z} + \phi(y)$. Then the elements of P^* belong to $2^{\mathfrak{c}}$ distinct equivalence classes. For every $p \in P^*$, there is an idempotent e_p in the left ideal βSp of βS. Since $\phi(e_p) \in \phi(p) + \{-1, 0, 1\}$, $e_p \equiv p$. So the elements of $E(\beta S)$ belong to $2^{\mathfrak{c}}$ distinct equivalence classes, and hence $|\phi(E(\beta S) \cap S^*S^*)| = 2^{\mathfrak{c}}$. Similarly, each left ideal βSp meets $K(\beta S)$. So $K(\beta S)$ is a subset of S^*S^* on which ϕ assumes

$2^{\mathfrak{c}}$ values. Finally, let $q \in S^*$. Since $qP^* \subseteq S^*S^*$ and ϕ assumes $2^{\mathfrak{c}}$ distinct values on qP^*, S^*S^* is not Borel. Similarly, because $qP^* \subseteq q\beta S$, $q\beta S$ is not Borel. $\square$

We remark that the hypothesis used in the following lemma, that a semigroup has an infite set of cancelable elements, holds in many familiar semigroups which satisfy none of our cancellativity conditions. Obvious examples are provided by $(\omega, \cdot)$ or the $m \times m$ matrices over $\mathbb{R}$, where m denotes a given positive integer.

Corollary 2.10. *Let R be an arbitrary semigroup which is either left cancellative and has a finite bound on the size of right solution sets or which contains an infinite set of cancelable elements. Then the set of idempotents in βR is not Borel.*

Proof. If R is left cancellative and has a finite bound on the sze of left solutions sets, let T be any countably infinite subsemigroup of R. If R has an infite set of cancelable elements, let X be a countably infinite set of cancelable elements of R and let T be the subsemigroup of R generated by X. Then $c\ell_{\beta R}(T)$ is a compact subsemigroup of βR which is a copy of βT and $E(\beta T)$ is not Borel. If $E(\beta R)$ were a Borel subset of βR, $E(\beta R) \cap (\overline{T}) = E(\overline{T})$ would be a Borel subset of βT. $\square$

Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a semigroup R, we say that $\langle x_n \rangle_{n=1}^{\infty}$ has *distinct finite products* provided that whenever $F, G \in \mathcal{P}_f(\mathbb{N})$ and $\prod_{n \in F} x_n = \prod_{n \in G} x_n$ one must have $F = G$, where the products are computed in increasing order of indices. Given $m \in \mathbb{N}$, we let $FP(\langle x_n \rangle_{n=1}^{m}) = \big\{ \prod_{n \in F} x_n : \emptyset \neq F \subseteq \{1, 2, \ldots, n\} \big\}$. We remind the reader that a semigroup R of cardinality κ is very weakly left cancellative if the union of fewer than κ left solution sets has cardinality less than κ.

Lemma 2.11. *Let R be a semigroup with cardinality $\kappa \geq \omega$ and assume that R is very weakly left cancellative and has κ right cancelable elements. There is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of right cancelable elements in R which has distinct finite products.*

Proof. Let $T = \{s \in R : s \text{ is right cancelable in } R\}$, let $I = \{s \in R : s \text{ is a left identity for } R\}$, and for $u, v \in R$, let $A_{u,v} = \{s \in R : us = v\}$. Given $u \in R$, $I \subseteq A_{u,u}$ so $|I| < \kappa$. We construct $\langle x_n \rangle_{n=1}^{\infty}$ in T inductively. Pick $x_1 \in T \setminus I$. Now let $n \in \mathbb{N}$ and assume we have chosen $\langle x_t \rangle_{t=1}^{n}$ in T such that for each $m \in \{1, 2, \ldots, n\}$

(1) $FP(\langle x_t \rangle_{t=1}^{m}) \cap I = \emptyset$;
(2) if $m > 1$, then $x_m \notin FP(\langle x_t \rangle_{t=1}^{m-1})$; and
(3) if $m > 1$ and $u, v \in I \cup FP(\langle x_t \rangle_{t=1}^{m-1})$, then $ux_m \neq v$.

Let $H = I \cup FP(\langle x_t \rangle_{t=1}^{n})$ and let $K = \bigcup \{A_{u,v} : u, v \in H\}$. Then $|H| < \kappa$ so K is the union of fewer than κ left solution sets and thus $|K| < \kappa$. Pick $x_{n+1} \in T \setminus (H \cup K)$.

We have that $x_{n+1} \notin I$. If $\emptyset \neq F \subseteq \{1,2,\ldots,n\}$, $u = \prod_{t\in F} x_t$, and $v \in I$, then $x_{n+1} \notin A_{u,v}$ so $\prod_{t\in F\cup\{n+1\}} x_t \notin I$. Thus hypohesis (1) holds. Since $FP(\langle x_t\rangle_{t=1}^n) \subseteq H$, hypothesis (2) holds. To verify hypothesis (3), let $u,v \in I \cup FP(\langle x_t\rangle_{t=1}^n)$. Then $x_{n+1} \notin A_{u,v}$ as required.

The construction being complete, suppose we have $F \neq G$ in $\mathcal{P}_f(\mathbb{N})$ such that $\prod_{t\in F} x_t = \prod_{t\in G} x_t$ and pick such F and G with $|F\cup G|$ as small as possible. Assume without loss of generality that $\max F \leq \max G = m$. Suppose first that $\max F < m$. If $G = \{m\}$ we contradict hypothesis (2) so $|G| > 1$. Let $u = \prod_{t\in G\setminus\{m\}} x_t$ and let $v = \prod_{t\in F} x_t$. Then $ux_m = v$, contradicting hypothesis (3).

Thus $\max F = m$. If $|F| > 1$ and $|G| > 1$, then since $x_m \in T$ we get that $\prod_{t\in F\setminus\{m\}} x_t = \prod_{t\in G\setminus\{m\}} x_t$ contradicting the minimality of $|F\cup G|$, so we may assume that $F = \{m\}$ and $|G| > 1$. Let $v = \prod_{t\in G\setminus\{m\}} x_t$. Then $x_m = vx_m$ so for each $s \in R$, $sx_m = svx_m$. Since $x_m \in T$, we have for each $s \in R$, $s = sv$ so that $v \in I$, contradicting hypothesis (1). □

Corollary 2.12. *Let R be a semigroup with cardinality $\kappa \geq \omega$ and assume that R is very weakly left cancellative and has κ right cancelable elements. Then $E(\beta R)$ is not Borel.*

Proof. By Lemma 2.11 and [3, Theorem 6.27], βR contains a subspace L topologically isomorphic to $\mathbb{H}$. In particular L is Borel. Now $E(\mathbb{H}) = E(\beta\mathbb{N})$ is not Borel, and so $E(L)$ is not Borel. If $E(\beta R)$ were Borel, then $E(L) = E(\beta R) \cap L$ would also be Borel. □

Note that the hypotheses of Theorem 2.9 cannot be weakened to left cancellative or right cancellative. If S is a right zero semigroup, then S is left cancellative, βS is a right zero semigroup, and $E(\beta S) = K(\beta S) = \beta S$, $S^*S^* = S^*$ and if $r \in S^*$, then $rS^* = S^*$. If S is a left zero semigroup, then S is right cancellative, βS is a left zero semigroup, and $E(\beta S) = K(\beta S) = \beta S$, $S^*S^* = S^*$ and if $r \in S^*$, then $rS^* = \{r\}$. Nor can they be weakened to weakly right cancellative and weakly left cancellative as shown by the example $(\mathbb{N}, \vee)$, where $x \vee y = \max\{x,y\}$. In this case, for $p,q \in \beta\mathbb{N}$, if $q \in \mathbb{N}^*$, then $p \vee q = q$, while if $q \in \mathbb{N}$ and $p \in \mathbb{N}^*$, then $p \vee q = p$ so $E(\beta\mathbb{N}) = \beta\mathbb{N}$, $\mathbb{N}^* \vee \mathbb{N}^* = K(\beta\mathbb{N}) = \mathbb{N}^*$, and if $r \in \mathbb{N}^*$, then $r \vee \mathbb{N}^* = \mathbb{N}^*$.

Notice that in each of these examples, the specified sets are all compact. This raises the following question.

Question 2.13. *Does there exist a countable semigroup S such that some or all of $E(\beta S)$, $K(\beta S)$, S^*S^*, or rS^* with $r \in S^*$ are not compact and are Borel?*

We remark that the results of Theorem 2.9 are stronger than the statement that the sets considered are not Borel, because they show that they cannot be expressed as the union of $\mathfrak{c}$ or fewer compact subsets. The set of subsets of βS which can be expressed as the union of $\mathfrak{c}$ or fewer compact subsets, is strictly larger than the set of Borel subsets. It contains the analytic subsets of βS, if these are defined as the set of subsets of βS which can be obtained from the Borel sets by applying operation (A). (For a definition of this operation, see, for example, [5, Chapter II, Section 5].)

As shown in the proof of [4, Lemma 3.1], if X is an arbitrary compact Hausdorff space of weight at most $\mathfrak{c}$, the family $\sigma(X)$ of subsets A of X for which A and $X \setminus A$ are unions of $\mathfrak{c}$ or fewer compact subsets, is a σ-algebra which contains the Borel subsets of X. We claim that, if X and Y are compact Hausdorff spaces of weight at most $\mathfrak{c}$ and if $f : X \to Y$ is a continuous open mapping, then $f[\sigma(X)] \subseteq \sigma(Y)$. To see this, let $A \in \sigma(X)$. Clearly, $f[A]$ is the union of $\mathfrak{c}$ or fewer compact subsets of Y. A is also the intersection of a family $\mathcal{U}$ of open subsets of X for which $|\mathcal{U}| \leq \mathfrak{c}$. Let $\mathcal{V} = \{f^{-1}\big[f[U]\big] : U \in \mathcal{U}\}$. Then $Y \setminus f[A] = \bigcup\{Y \setminus f[V] : V \in \mathcal{V}\}$. So $Y \setminus f[A]$ is also the union of $\mathfrak{c}$ or fewer compact subsets of Y. In particular, $\pi_1[\sigma(Y \times X)] \subseteq \sigma(Y)$.

We have therefore shown that the subsets of βS discussed above, are not analytic and are not projective.

We are grateful to D. Saveliev for a very helpful correpondence about these concepts.

3. $\boldsymbol{\lambda_p}$ is not Borel

Throughout this section S will denote an infinite semigroup of cardinality κ which is right cancellative, very weakly left cancellative, and has a designated left identity e. (S may or may not have other left identities.)

Ω will denote the set ${}^{S}\{0,1\}$ of functions from S to $\{0,1\}$ with the product topology. We work in the dynamical system $\langle \Omega, \Phi^s \rangle_{s\in S}$ where $\Phi^s : \Omega \to \Omega$ is defined by $\Phi^s(w) = w \circ \rho_s$. That is, for $w \in \Omega$ and $t \in S$, $\Phi^s(w)(t) = w(ts)$. (This is the shift map action in the case in which S is $\mathbb{N}$ or $\mathbb{Z}$). If $p \in S^*$, $\Phi^p : \Omega \to \Omega$ is defined by $\Phi^p(w) = p\text{-}\lim_{s\in S} \Phi^s(w)$. Note that, given $t \in S$ and $w \in \Omega$, $\Phi^p(w)(t) = \big(p\text{-}\lim_{s\in S} \Phi^s(w)\big)(t) = p\text{-}\lim_{s\in S} w(ts)$. If $\overline{w} : \beta S \to \{0,1\}$ denotes the continuous extesnion of w, $\Phi^p(w) = \overline{w}(tp)$. For a given value of p, this is a continuous function of t. We shall say that $w \in \Omega$ is *transitive* if $\{\Phi^s(w) : s \in S\}$ is dense in Ω.

Ω can be given the structure of a compact topological group by noting that $\Omega = {}^{S}\mathbb{Z}_2$. We shall use λ to denote normalised Haar measure on Ω, and shall use $\mathfrak{B}_\lambda$ to denote the σ-algebra of subsets of Ω generated by the Borel sets and the λ-null sets.

The following lemma is well known. We include a proof, however, because the proof is short and simple.

Lemma 3.1. *If $p \in S^*$, then $\{1_A : A \in p\}$ is not $\mathfrak{B}_\lambda$-measurable, where $1_A \in \Omega$ is the characteristic function of A.*

Proof. Suppose that $P = \{1_A : A \in p\}$ is λ-measurable. Then $1_S + P = \{1_{S\setminus A} : A \in p\}$ so P and $1_S + P$ are disjoint subsets of $G = {}^S\mathbb{Z}_2$ whose union is all of G. So $\lambda(P) = \lambda(1_S + P) = \frac{1}{2}$. By [2, Theorem A, Chapter 12, Section 61, and Theorem B, Chapter 12, Section 62], $P + P$ contains a neighborhood of 0 in G. So there exists $F \in \mathcal{P}_f(S)$ such that $\bigcap_{x\in F} \pi_x^{-1}[\{0\}] \subseteq P + P = \{1_{A\Delta B} : A, B \in p\}$. But then $1_{S\setminus F} \in P + P$, so $S \setminus F \notin p$. Consequently p must be a principle ultrafilter, a contradiction. □

Lemma 3.2. *We can choose an element $w_0 \in \Omega$ which is transitive, and so the function $\psi : \beta S \to \Omega$ defined by $\psi(p) = \Phi^p(w_0)$ is a continuous surjection.*

Proof. We enumerate $\mathcal{P}_f(S)$ as a κ-sequence $\langle F_\alpha\rangle_{\alpha<\kappa}$. For $\alpha < \kappa$ let $\tau_\alpha = |F_\alpha|$ and $\delta_\alpha = 2^{\tau_\alpha}$. We note that if $\emptyset \neq A \subseteq S$, $|A| < \kappa$, and $\alpha < \kappa$, then $\{s \in S : F_\alpha s \cap A \neq \emptyset\} = \bigcup_{a\in F_\alpha} \bigcup_{b\in A} \{s \in S : as = b\}$, so is the union of fewer than κ left solution sets and thus, since S is very weakly left cancellative, $|\{s \in S : F_\alpha s \cap A \neq \emptyset\}| < \kappa$. Consequently we may inductively choose $\{s_{\alpha,t} : \alpha < \kappa \text{ and } t \in \{1, 2, \ldots, \delta_\alpha\}\}$ so that $F_\alpha s_{\alpha,t} \cap F_\sigma s_{\sigma,r} = \emptyset$ whenever $\alpha, \sigma < \kappa$, $t \in \{1, 2, \ldots, \delta_\alpha\}$, $r \in \{1, 2, \ldots, \delta_\sigma\}$, and $(\alpha, t) \neq (\sigma, r)$.

For each $\alpha < \kappa$, enumerate the set of functions from F_α to $\{0, 1\}$ as $\langle f_{\alpha,t}\rangle_{t=1}^{\delta_\alpha}$. We define $w_0 \in \omega$ on $\bigcup_{\alpha<\kappa} \bigcup_{t=1}^{\delta_\alpha} F_\alpha s_{\alpha,t}$ by, for $a \in F_\alpha$ and $t \in \{1, 2, \ldots, \delta_\alpha\}$, $w_0(as_{\alpha,t}) = f_{\alpha,t}(a)$. (We are using here the fact that S is right cancellative.) Define $w_0(x)$ at will for $x \in S \setminus \bigcup_{\alpha<\kappa} \bigcup_{t=1}^{\delta_\alpha} F_\alpha s_{\alpha,t}$.

To see that $\{\Phi^s(w_0) : s \in S\}$ is dense in Ω, let U be a nonempty basic open set in Ω. Pick $\alpha < \kappa$ and $t \in \{1, 2, \ldots, \delta_\alpha\}$ such that $U = \bigcap_{a\in F_\alpha} \pi_\alpha^{-1}[\{f_{\alpha,t}(a)\}]$. Then for $a \in F_\alpha$, $\Phi^{s_{\alpha,t}}(w_0)(a) = w_0(as_{\alpha,t}) = f_{\alpha,t}(a)$ and so $\Phi^{s_{\alpha,t}}(w_0) \in U$.

It is routine to verify that ψ is continuous. Since $\psi[\beta S]$ is compact and dense in βS, $\psi[\beta S] = \beta S$.

□

Definition 3.3. We fix $w_0 \in \Omega$ and $\psi : \beta S \to \Omega$ as guaranteed by Lemma 3.2.

Definition 3.4. If μ is a probability measure on a compact space X, $\mathfrak{B}_\mu$ will denote the σ-algebra of subsets of X generated by the Borel subsets and the μ-null subsets. We shall say that a subset of X is *universally measurable* if it is a member of $\mathfrak{B}_\mu$ for every probability mesaure μ defined on X.

We remind the reader that a subset A of X is in $\mathfrak{B}_\mu$ if and only if $\sup(\{\mu(C) : C \text{ is compact and } C \subseteq A\}) = \inf(\{\mu(U) : U \subseteq X \text{ is open and } A \subseteq U\})$.

We are grateful to E. Glasner for sending us a proof of the following lemma.

Lemma 3.5. *Let X and Y be compact Hausdorf spaces, and let $f : X \to Y$ be a continuous surjection. Then $f[B]$ is universally measurable for every universally measurable subset B of X for which $B = f^{-1}\big[f[B]\big]$.*

Proof. Let μ be a probability measure on Y. It follows from the Hahn Banach Theorem and the Riesz Representation Theorem, that there is a probability measure ν on X for which $\nu(g \circ f) = \mu(g)$ for every continuous $g : Y \to \mathbb{R}$. Let $\varepsilon > 0$. We can choose a compact subset C of B for which $\nu(C) + \varepsilon > \nu(B)$ and a compact subset D of $X \setminus B$ for which $\nu(D) + \varepsilon > \nu(X \setminus B)$. We can then choose disjoint open subsets U and V of Y such that $f[C] \subseteq U$ and $f[D] \subseteq V$, and $\mu(U) < \mu(f[C]) + \varepsilon$ and $\mu(V) < \mu(f[D]) + \varepsilon$. Let g and h be continuous functions from Y to [0,1] such that $g = 1$ on $f[C]$, $g = 0$ on $Y \setminus U$, $h = 1$ on $f[D]$, and $h = 0$ on $Y \setminus V$. Then $\nu(C) \leq \nu(g \circ f) = \mu(g) \leq \mu(f[C]) + \varepsilon$ and $\nu(D) \leq \nu(h \circ f) = \mu(h) < \mu(f[D]) + \varepsilon$. Now $\nu(C) + \nu(D) > 1 - 2\varepsilon$. So $\mu(f[C]) + \mu(f[D]) > 1 - 4\varepsilon$ and hence $\mu(Y \setminus f[D]) < \mu(f[C]) + 4\varepsilon$. Since $Y \setminus f[D]$ is an open set containing $f[B]$ and $f[C]$ is a compact set contained in $f[B]$, it follows that $f[B] \in \mathfrak{B}_\mu$. $\square$

Definition 3.6. Let $p \in S^*$. We put $\mathcal{Q}_p = \{q \in \beta S : \psi(pq)(e) = 1\}$.

Lemma 3.7. *Let $p \in S^*$. Then $\psi[\mathcal{Q}_p] = \{1_A : A \in p\}$.*

Proof. Let $D = \{s \in S : w_0(s) = 1\}$. Note that for any $q \in S^*$,

$$\begin{aligned}
\psi(pq)(e) = 1 \quad &\Leftrightarrow \quad (pq)\text{-}\lim_{s \in S} \Phi^s(w_0)(e) = 1 \\
&\Leftrightarrow \quad (pq)\text{-}\lim_{s \in S} w_0(s) = 1 \\
&\Leftrightarrow \quad \{s \in S : w_0(s) = 1\} \in pq \\
&\Leftrightarrow \quad \{s \in S : s^{-1}D \in q\} \in p\,.
\end{aligned}$$

and

$$\begin{aligned}
\{s \in S : \psi(q)(s) = 1\} \quad &= \quad \{s \in S : \Phi^q(w_0)(s) = 1\} \\
&= \quad \{s \in S : q\text{-}\lim_{t \in S} w_0(st) = 1\} \\
&= \quad \{s \in S : \{t \in S : w_0(st) = 1\} \in q\} \\
&= \quad \{s \in S : s^{-1}D \in q\}\,.
\end{aligned}$$

Consequently, if $q \in \mathcal{Q}_p$ and $A = \{s \in S : \psi(q)(s) = 1\}$, then $\psi(q) = 1_A$ and $A \in p$.

Now assume $A \in p$ and pick, by Lemma 3.2 $q \in \beta S$ such that $\psi(q) = 1_A$. Then $A = \{s \in S : s^{-1}D \in q\} \in p$ so $\psi(pq)(e) = 1$. $\square$

Theorem 3.8. *For each $p \in S^*$, the mapping $\lambda_p : \beta S \to \beta S$ is not Borel.*

Proof. By Lemmas 3.1, 3.5, and 3.7, $\mathcal{Q}_p$ is not a Borel set. Since $\mathcal{Q}_p = \lambda_p^{-1}[\{x \in \beta S : \psi(x)(e) = 1\}]$ and $\{x \in \beta S : \psi(x)(e) = 1\}$ is compact, λ_p is not Borel. $\square$

As in Section 2, we remark that the preceding theorem need not hold if we weaken our hypothesis to left cancellativity, right cancellativity or weak cancellativity. If S is a left zero semigroup, a right zero semigroup or $(\mathbb{N}, \vee)$, then $\lambda_p : \beta S \to \beta S$ is Borel for every $p \in \beta S$.

4. Images of Borel Sets

In this section we address the question of which compact spaces X and Y have the property that, whenever $f : X \to Y$ is continuous, $f[B]$ is a universally measurable subset of Y whenever B is a Borel subset of X. We remark that X and Y have this property if they are metric spaces. However, the following elegant result, due to D. Fremlin in personal correspondence, shows that this property fails dramatically in the case in which $X = \mathbb{N}^*$.

Theorem 4.1. *Let $f : \mathbb{N}^* \to Y$ be a continuous surjection onto a compact metric space. Then, for every subset E of Y, there is an open subset U of $\mathbb{N}^*$ such that $f[U] = E$.*

Proof. For every $y \in Y$, $f^{-1}[\{y\}]$ is a non-empty G_δ subset of $\mathbb{N}^*$. It therefore contains a non-empty open subset U_y of $\mathbb{N}^*$ by [3, Theorem 3.36]. If $U = \bigcup\{U_y : y \in E\}$, then U is open in $\mathbb{N}^*$ and $f[U] = E$. $\square$

We shall show that continuous mappings between compact Hausdorff spaces do map Baire sets to universally measurable sets, where we define the Baire subsets of a compact Hausdorff space X to be the sets in the smallest σ-algebra of subsets of X containing the compact G_δ subsets of X. (Other definitions exist in the literature.)

Definition 4.2. A *determining system* in a space X is a family $\mathfrak{U}$ of subsets of X indexed by the set of finite sequences of positive integers. The nucleus $N(\mathfrak{U})$ of $\mathfrak{U}$ is $\bigcup\{A_{n_1} \cap A_{n_1 n_2} \cap A_{n_1 n_2 n_3} \ldots : \langle n_i\rangle_{i=1}^\infty$ is a sequence in $\mathbb{N}\}$.

We shall call such a system a *compact determing system* if all the sets in the system are compact and $A_{n_1 n_2 \ldots n_k n_{k+1}} \subseteq A_{n_1 n_2 \ldots n_k}$ for all positive integers $n_1, n_2, \ldots, n_{k+1}$.

Determining systems were first defined by Alexandrov in 1916. In any topological space, every determining system of universally measurable sets has a nucleus which is universally measurable by [5, Theorem 5.5].

Lemma 4.3. *The set of nuclei of compact determining systems in a compact Hausdorff space X is closed under countable unions and countable intersections.*

Proof. Suppose that $\mathfrak{U}(m) = \{A_{n_1 n_2 \ldots n_k}(m) : \langle n_i\rangle_{i=1}^k$ is a finite sequence in $\mathbb{N}\}$ is a compact determining system for each $m \in \mathbb{N}$. Let $N(m) = N(\mathfrak{U}(m))$ for each $m \in \mathbb{N}$.

Then $\bigcup_{m=1}^\infty N(m)$ is the nucleus of the system $\{B_{n_1 n_2 \ldots n_k} : \langle n_i\rangle_{i=1}^k$ is a finite sequence in $\mathbb{N}\}$ defined by putting $B_{n_1 n_2 \ldots n_k} = A_{n_2 n_3 \ldots n_k}(n_1)$ if $k > 1$, and $B_n = X$ for every $n \in \mathbb{N}$.

To see that $\bigcap_{m=1}^\infty N(m)$ is the nucleus of a compact determining system

$$\{C_{n_1 n_2 \ldots n_k} : \langle n_i\rangle_{i=1}^k \text{ is a finite sequence in } \mathbb{N}\}\,,$$

choose a partition of $\mathbb{N}$ into a sequence $\langle E_n\rangle_{n=1}^\infty$ of infinite pairwise disjoint subsets. For each finite sequence $\sigma = \langle n_1 n_2 \ldots n_k\rangle$ of positive integers and each $m \in \mathbb{N}$, let σ_m be the subsequence of σ formed by the integers n_i for which $i \in E_m$. Then put $C_\sigma = \bigcap_{m\in\mathbb{N}} A_{\sigma_m}(m)$, with $A_\emptyset(m)$ defined to be X. $\square$

Lemma 4.4. *Let X be a compact Hausdorff space. If B is a compact G_δ subset of X or a σ-compact subset of X, then B is the nucleus of a compact determining system.*

Proof. Note that any compact set C is the nucleus of a compact determining system defined by $A_{n_1 n_2 \ldots n_k} = C$. A compact G_δ is the intersection of a sequence $\langle C_n \rangle_{n=1}^\infty$ of compact sets, so the conclusion follows from Lemma 4.3. $\square$

Lemma 4.5. *Let X be a compact Hausdorff space. Every Baire subset of X is the nucleus of a compact determing system.*

Proof. If B is a compact G_δ subset of X, then both B and $X \setminus B$ are nuclei of compact determining systems, because $X \setminus B$ is σ-compact. Since the set of nuclei of compact determining systems is closed under countable unions and intersections, it contains all the Baire subsets of X. $\square$

Theorem 4.6. *Let X and Y be compact Hausdorff spaces and let $f : X \to Y$ be a continuous surjection. If B is a Baire subset of X, then $f[B]$ is a universally measurable subset of Y.*

Proof. B is the nucleus of a compact determining system

$$\{A_{n_1 n_2 \ldots n_k} : \langle n_i \rangle_{i=1}^k \text{ is a finite sequence in } \mathbb{N}\},$$

and so $f[B]$ is the nucleus of the compact determining system

$$\{f[A_{n_1 n_2 \ldots n_k}] : \langle n_i \rangle_{i=1}^k \text{ is a finite sequence in } \mathbb{N}\},$$

because for every decreasing sequence $\langle C_n \rangle_{n=1}^\infty$ of compact subsets of X,

$$f[\textstyle\bigcap_{n=1}^\infty C_n] = \bigcap_{n=1}^\infty f[C_n].$$

It follows that $f[B]$ is universally measurable by [5, Theorem 5.5]. $\square$

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