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t-Reductions and t-integral closure of ideals in Noetherian domains

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Abstract. This paper studies t-reductions and t-integral closure of ideals in Noetherian domains. The main objective is to establish satisfactory t-analogues for well-known results in the literature on reductions and integral closure of ideals in Noetherian rings. Namely, Section 2 investigates t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. Section 3 investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. Section 4 studies the t-analogue of Hays' classic notion of C-ideal and its correlation to the integral closure.

Contents

1. Introduction

Throughout, all rings considered are commutative with identity. Let R be a ring and I a proper ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called a basic ideal [13, 28]. The notion of reduction was introduced by Northcott and Rees to contribute to the analytic theory of ideals in Noetherian (local) rings via minimal reductions. In [13, 14], Hays investigated reductions of ideals in more general settings of commutative rings (i.e., not necessarily local or Noetherian); particularly, Noetherian rings and Prüfer domains. He provided several sufficient conditions for an

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ideal to be basic. For instance, in Noetherian rings, an ideal is basic if and only if it is locally basic. He also introduced and studied the dual notion of a basic ideal; namely, an ideal is a C-ideal if it is not a reduction of any larger ideal. Several results about C-ideals are proved; including the fact that this notion is local for regular ideals in Noetherian rings.

It is well-known that an element $x \in R$ is integral over I if and only if I is a reduction of $I + Rx$; and if I is finitely generated, then $J \subseteq I \subseteq \overline{J}$ if and only if J is a reduction of I, where $\overline{J}$ denotes the integral closure of J. This correlation allowed to prove a number of crucial results in the theory including the fact that the integral closure of an ideal is an ideal. For a full treatment of this topic, we refer the reader to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [21].

Let R be a domain, K its quotient field, I a nonzero fractional ideal of R, and $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. The ideal I is a v-ideal (or divisorial) if $I_v = I$ and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$ the set $F_t(R)$ of fractional t-ideals of R is a semigroup with unit R. Ideal t-multiplication converts notions such as principal, Dedekind, Bézout, and Prüfer domains to factorial domains, Krull domains, GCDs, and PvMDs, respectively. We also recall the w-operation: for a nonzero fractional ideal I of R, $I_w = \bigcup(I : J)$, where the union is taken over all finitely generated ideals J of R that satisfy $J_v = R$; equivalently, $I_w = \bigcap IR_M$, where M ranges over the set of all maximal t-ideals of R. We always have $I \subseteq I_w \subseteq I_t \subseteq I_v$. We shall be using the v-, t-, and w-operations freely, and for more details, the reader may consult Gilmer's book [12] and also [1, 2, 3, 4, 6, 8, 10, 19, 27, 29, 30].

Let I be a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$. An element $x \in R$ is t-integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. In [22], the authors investigated the t-reductions and t-integral closure of ideals with the aim of establishing satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Two of their main results assert that "the t-integral closure of an ideal is an integrally closed ideal which is not t-integrally closed in general" and " the t-integral closure coincides with the t-closure in the class of integrally closed domains." In [17], the authors investigated $\star$-reductions of ideals in Prüfer v-multiplication domains (PvMDs). One of their main results asserts that "a domain has the finite w-basic ideal property (resp., w-basic ideal property) if and only if it is a PvMD (resp., a PvMD of t-dimension one)." In [23], the authors investigated t-reductions of ideals in pullback constructions, where the main result established the transfer of the finite t-basic ideal property to pullbacks in line with Fontana-Gabelli's result on PvMDs [9, Theorem 4.1] and Gabelli-Houston's result on v-domains [11, Theorem 4.15]. They also solved an open problem on whether the finite t-basic and v-basic ideal properties are distinct; they proved indeed that these two notions coincide in any arbitrary domain.

This paper studies t-reductions and t-integral closure of ideals in Noetherian domains. The main objective is to establish satisfactory t-analogues for well-known results in the literature on reductions and integral closure of ideals in Noetherian rings. Namely, Section 2 investigates t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. Section 3 investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. Section 4 studies the t-analogue of Hays' classic notion of C-ideal and its correlation to the integral closure.

2. *t*-reductions subject to *t*-invertibility and localization

This section investigates t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. The first objective is to establish a t-analogue for Hays' result on the correlation between invertible reductions and the Krull dimension of a Noetherian domain [13, Theorem 4.4]. The second objective is to reach a satisfactory t-analogue for Hays' global-local result on the basic property in Noetherian rings [13, Theorem 3.6].

Definition 2.1 ([17, 22, 23])**.** Let R be a domain and I a nonzero ideal of R.

(1) An ideal $J \subseteq I$ is a *t-reduction* of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$. The ideal J is a *trivial t-reduction* of I if $J_t = I_t$.
(2) I is *t-basic* if it has no t-reduction other than the trivial t-reductions.
(3) R has the t-basic ideal property if every nonzero ideal of R is t-basic.

Clearly, the notion of t-reduction extends naturally to fractional ideals. Also, notice that a reduction is necessarily a t-reduction; and the converse is not true, in general. Each of [22, Example 2.2] and [17, Example 1.5] exhibits a Noetherian domain R with two t-ideals $J \subsetneqq I$ such that J is a t-reduction but not a reduction of I.

In 1973, Hays proved the following result:

Theorem 2.2 ([13, Theorem 4.4])**.** *Let R be a Noetherian domain such that R/M is infinite for every maximal ideal M of R. Then, each nonzero ideal has an invertible reduction if and only if* $\dim(R) \leq 1$.

Next, we establish a t-analogue for this result. To this end, recall that the t-dimension of a domain R, denoted t-$\dim(R)$, is the supremum of the lengths of chains of prime t-ideals in R (and, for the purpose of this definition, (0) is considered as a prime t-ideal although technically it is not); and we always have t-dim$(R) \leq \dim(R)$ [16]. Throughout, $\text{Max}_t(R)$ will denote the set of maximal t-ideals of R.

Theorem 2.3. *Let R be a Noetherian domain such that the residue field of each maximal t-ideal of R is infinite. Then, the following statements are equivalent:*

(1) *Each t-ideal of R has a t-invertible t-reduction;*
(2) *Each maximal t-ideal of R has a t-invertible t-reduction;*
(3) t-dim$(R) \leq 1$.

The following lemma proves the implication (2) $\Rightarrow$ (3) without the infinite residue field assumption.

Lemma 2.4. *Let R be a Noetherian domain. If every maximal t-ideal of R has a t-invertible t-reduction, then t-dim$(R) \leq 1$.*

Proof. Assume that every maximal t-ideal has a t-invertible t-reduction. We may suppose that R is not a field and will prove that t-dim$(R) = 1$. Let $M \in \text{Max}_t(R)$ and let $J = J_t$ be a t-invertible t-reduction of M. Then $(M^{n+1})_t = (JM^n)_t$ for some positive integer n and hence $M^{n+1} \subseteq J \subseteq M$. Now If D is a Noetherian domain and P is a prime t-ideal of D, then PD_P is a prime t-ideal of D_P. This follows from the discussion after Proposition 1.4 of [31]. Thus MR_M is a t-ideal of R_M. Therefore, JR_M is invertible and hence principal in R_M. Moreover, M is minimal over J, and so is MR_M over JR_M. Since R_M is Noetherian, $\text{ht}(M) = \text{ht}(MR_M) = 1$ by the Principal Ideal Theorem. Consequently, t-dim$(R) = 1$, as desired. ☐

The converse of Lemma 2.4 is not true in general. For, let R be an almost Dedekind domain which is not Dedekind. Then R is a one-dimensional locally Noetherian Prüfer domain (i.e., the d- and t-operations coincide). Hence R has the basic ideal property [13, Theorem 6.1]. But R is not Dedekind, so it posses a non-invertible maximal ideal M which has no reduction other than itself.

Proof of Theorem 2.3. (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) is handled by Lemma 2.4. It remains to prove (3) $\Rightarrow$ (1). Suppose that t-dim$(R) = 1$ and let I be a t-ideal of R. Clearly, $\text{ht}(I) = 1$. Since R is Noetherian, it is a TV-domain and hence has finite t-character by [19, Theorem 1.3]. Let $M_1, \dots, M_n$ be all the maximal t-ideals of R containing I. Let $i \in \{1, \dots, n\}$. Since R_{M_i} is a one-dimensional Noetherian domain, by [13, Theorem 4.4], IR_{M_i} has an invertible (so principal) reduction, say $a_i R_{M_i}$. Clearly, $\sqrt{a_i R_{M_i}} = \sqrt{IR_{M_i}} = M_i R_{M_i}$, and so ${M_i}^r R_{M_i} \subseteq a_i R_{M_i}$ for some integer r. Let $A_i := a_i R_{M_i} \cap R$. We have

$$M_i^r \subseteq {M_i}^r R_{M_i} \cap R \subseteq a_i R_{M_i} \cap R = A_i \subseteq M_i.$$

Hence M_i is the only maximal t-ideal of R containing A_i. It follows that $A_i R_M = R_M$ for any $M \in \text{Max}_t(R) \setminus \{M_i\}$. Let $J := \prod_{i=1}^n A_i$. Then, we claim that J is a t-invertible t-reduction of I. First, we show that $J \subseteq I$.

Indeed, one can check that $M_1, \dots, M_n$ are the only maximal t-ideals of R containing J and let $\mathfrak{M} := \mathrm{Max}_t(R) \setminus \{M_1, \dots, M_n\}$. So

$$\begin{array}{rcl} J_w & = & \bigcap_{M\in \mathrm{Max}_t(R)} JR_M \\ & = & \left(\bigcap_{1\leq i\leq n} A_iR_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \left(\bigcap_{1\leq i\leq n} a_iR_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & \subseteq & \left(\bigcap_{1\leq i\leq n} IR_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \bigcap_{M\in \mathrm{Max}_t(R)} IR_M \\ & = & I \end{array}$$

and thus $J \subseteq I$. Second, we show that J is a t-reduction of I. Indeed, let m be a positive integer such that $a_iI^mR_{M_i} = I^{m+1}R_{M_i}$ for all $i = 1, \dots, n$. Notice also that $M_1, \dots, M_n$ are the only maximal t-ideals of R containing JI^m and I^{m+1}. So

$$\begin{array}{rcl} (JI^m)_w & = & \bigcap_{M\in \mathrm{Max}_t(R)} (JI^m)R_M \\ & = & \left(\bigcap_{1\leq i\leq n} a_iI^mR_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & \subseteq & \left(\bigcap_{1\leq i\leq n} I^{m+1}R_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \bigcap_{M\in \mathrm{Max}_t(R)} I^{m+1}R_M \\ & = & (I^{m+1})_w \end{array}$$

and thus $(JI^m)_t = (I^{m+1})_t$ since t is coarser than w. Finally, we show that J is t-invertible. Indeed, we have

$$\begin{array}{rcl} (JJ^{-1})_w & = & \bigcap_{M\in \mathrm{Max}_t(R)} (JJ^{-1})R_M \\ & = & \left(\bigcap_{1\leq i\leq n} (JJ^{-1})R_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \left(\bigcap_{1\leq i\leq n} JR_{M_i}J^{-1}R_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \left(\bigcap_{1\leq i\leq n} JR_{M_i}(JR_{M_i})^{-1}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \left(\bigcap_{1\leq i\leq n} JR_{M_i}(a_iR_{M_i})^{-1}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \left(\bigcap_{1\leq i\leq n} a_iR_{M_i}a_i^{-1}R_{M_i}\right) \cap \left(\bigcap_{M\in\mathfrak{M}} R_M\right) \\ & = & \bigcap_{M\in \mathrm{Max}_t(R)} R_M \\ & = & R \end{array}$$

and so J is t-invertible, completing the proof of the theorem. □

Next, we examine the global-local transfer of the t-basic ideal property. Throughout, an ideal I is locally basic (resp., t-locally t-basic) if IR_M is basic (resp., t-basic) for each maximal ideal (resp., maximal t-ideal) M of R containing I. In 1973, Hays proved the following result:

Theorem 2.5 ([13, Theorem 3.6])**.** *In a Noetherian ring, an ideal is basic if and only if it is locally basic.*

Next, we establish a t-analogue for the "if" assertion of this result.

Theorem 2.6. *In a Noetherian domain, if an ideal is t-locally t-basic, then it is t-basic.*

Proof. Let R be a Noetherian domain and let I be a t-locally t-basic ideal of R. Let $J \subseteq I$ be a t-reduction of I; that is, $(JI^n)_t = (I^{n+1})_t$, for some positive integer n. Next, we prove that $J_t = I_t$. Since $(JI^n)_t = (J_tI^n)_t$, we may assume, without loss of generality, that J is a t-ideal. Let $M \in \mathrm{Max}_t(R)$ such that $I \subseteq M$, and let t_M and v_M denote the t- and v- operations with respect to R_M, respectively. By [24, Lemma 2.18], we get

$$\begin{aligned}\big(JR_MI^nR_M\big)_{t_M} &= \big((JI^n)_tR_M\big)_{t_M}\\ &= \big((I^{n+1})_tR_M\big)_{t_M}\\ &= \big(I^{n+1}R_M\big)_{t_M}\end{aligned}$$

and the t-locally t-basic assumption yields

$$\begin{aligned}(JR_M)^{-1} &= ((JR_M)_{v_M})^{-1}\\ &= ((JR_M)_{t_M})^{-1}\\ &= ((IR_M)_{t_M})^{-1}\\ &= ((IR_M)_{v_M})^{-1}\\ &= (IR_M)^{-1}.\end{aligned}$$

Moreover, since $I^{n+1} \subseteq J_t = J \subseteq I$, then a maximal t-ideal contains I if and only if it contains J. It follows that

$$J^{-1}R_M = (JR_M)^{-1} = (IR_M)^{-1} = I^{-1}R_M$$

for all maximal t-ideals of R. Therefore, we obtain

$$\begin{aligned}(J^{-1})_w &= \bigcap_{M\in \mathrm{Max}_t(R)} J^{-1}R_M\\ &= \bigcap_{M\in \mathrm{Max}_t(R)} I^{-1}R_M\\ &= (I^{-1})_w.\end{aligned}$$

Consequently, $J^{-1} = (J^{-1})_v = (I^{-1})_v = I^{-1}$ and thus $J = J_v = I_v = I_t$, as desired. □

It is worthwhile noting that, in his proof of the implication "basic ⇒ locally basic" (Theorem 2.5), Hays used two basic facts; the first of which asserts that $(J \cap I) + IM$ is a reduction of I whenever JR_M is a reduction of IR_M in an arbitrary ring R. A t-analogue for this result is proved below in Proposition 2.7. But, the second fact was Nakayama's lemma, which ensures that $J \subseteq I \subseteq J + IM$ in a local Noetherian ring (R, M) forces $J = I$; and a t-analogue for this Nakayama property is not true in general. For instance, consider the local Noetherian ring $R := k + M^2 \subseteq k[x, y]$, where $M = (x, y)$ and $(M^2)_t = (M^3)_t$ [17, Example 1.5].

Proposition 2.7. *Let R be a domain, M a maximal t-ideal of R, and $I \subseteq M$ a nonzero ideal of R. If J is an ideal of R such that JR_M is a t-reduction of IR_M, then $(J \cap I) + IM$ is a t-reduction of I.*

Proof. Let J be an ideal of R such that JR_M is a t-reduction of IR_M, say, $(JR_M I^n R_M)_{t_M} = (I^{n+1}R_M)_{t_M}$, for some positive integer n and where t_M denotes the t-operation with respect to R_M. Let $Q \in \mathrm{Max}_t(R)$ with $Q \neq M$. Then, $(J \cap I + IM)R_Q = IR_Q$ yielding $(J \cap I + IM)I^n R_Q = I^{n+1}R_Q$. Whence, $\big((J \cap I + IM)I^n\big)^{-1} R_Q = \big(I^{n+1}\big)^{-1} R_Q$. On the other hand, we have

$$\begin{array}{rcl} \big((J \cap I + IM)I^n R_M\big)_{t_M} & = & \big((JR_M \cap IR_M + IR_M MR_M)I^n R_M\big)_{t_M} \\ & = & \big((JR_M + IR_M MR_M)I^n R_M\big)_{t_M} \\ & = & \big(JR_M I^n R_M + I^{n+1}R_M MR_M\big)_{t_M} \\ & = & \big(I^{n+1}R_M\big)_{t_M} \end{array}$$

and thus

$$\big((J \cap I + IM)I^n\big)^{-1} R_M = \big(I^{n+1}\big)^{-1} R_M.$$

Therefore, we obtain

$$\begin{array}{rcl} \big((I^{n+1})^{-1}\big)_w & = & \bigcap_{N \in \mathrm{Max}_t(R)} (I^{n+1})^{-1} R_N \\ & = & \bigcap_{N \in \mathrm{Max}_t(R)} \big((J \cap I + IM)I^n\big)^{-1} R_N \\ & = & \big(\big((J \cap I + IM)I^n\big)^{-1}\big)_w \end{array}$$

Consequently, $\big((J \cap I + IM)I^n\big)_t = \big(I^{n+1}\big)_t$. That is, $(J \cap I) + IM$ is a t-reduction of I, completing the proof of the proposition. □

3. t-reductions and t-integral closure in one-dimensional Noetherian domains

This section investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions of ideals in Noetherian rings.

From [22, 23], let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0 \quad \text{with} \quad a_i \in (I^i)_t \ \forall i = 1, ..., n.$$

The set of all elements that are t-integral over I is called the t-integral closure of I, and is denoted by $\widetilde{I}$. If $I = \widetilde{I}$, then I is said to be t-integrally closed. Recall that "*$\widetilde{I}$ is an integrally closed ideal which is not t-integrally closed in general*" [22, Theorem 3.2]. Several ideal-theoretic properties of $\widetilde{I}$ are collected in [22, Remark 3.8], including the basic inclusions

$$I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}.$$

Next, consider the two sets:

$$\widehat{I}^{\,d} := \{x \in R \mid I \text{ is a reduction of } (I, x)\}$$

$$\widehat{I}^{\,t} := \{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\}$$

For the trivial operation, it is well-known that the equality $\overline{I} = \widehat{I}^{\,d}$ always holds [21, Corollary 1.2.2]. This is the very fact which was used to show that $\overline{I}$ is an ideal [21, Corollary 1.3.1]. However, it is still an open problem of whether $\widehat{I}^{\,t}$ is an ideal in general [23, Question 3.5]. We always have

$$I_t \subseteq \widetilde{I} \subseteq \widehat{I}^{\,t}$$

where the second containment is proved in [22, Proposition 3.7] and can be strict as shown by [22, Example 3.10(a)]. Moreover, "*$I_t = \widetilde{I}$ for each nonzero ideal I if and only if R is integrally closed*" [22, Theorem 3.5], and "*$I_t = \widehat{I}^{\,t}$ for each nonzero ideal I if and only if R has the finite t-basic ideal property*" [23, Theorem 3.2].

The class of Prüfer domains is the only known class of domains, so far, where the two notions of reduction and t-reduction coincide (since the t- and trivial operations coincide). The next result shows that such coincidence also occurs in one-dimensional Noetherian domains (where the t- and trivial operations are not necessarily the same).

Theorem 3.1. *In a one-dimensional Noetherian domain, the notions of reduction and t-reduction coincide. Moreover, $\overline{I} = \widetilde{I} = \widehat{I}^{\,t}$ for any nonzero ideal I.*

The proof draws on the following lemma, which is of independent interest.

Recall from [4], an extension of domains $R \subseteq T$ is t-compatible if $I_tT \subseteq (IT)_{t_1}$ for every nonzero ideal I of R, where t_1 denotes the t-operation with respect to T. Throughout, for a domain R, we will denote by $\overline{R}$ the integral closure of R in its quotient field.

Lemma 3.2. *Let R be a domain such that $R \subseteq \overline{R}$ is t-compatible, $\overline{R}$ has the t-basic ideal property, and $\overline{J\overline{R}} = \widetilde{J\overline{R}}$ for any nonzero ideal J of R. Then, the notions of reduction and t-reduction coincide in R.*

Proof. Let $J \subseteq I$ be nonzero ideals of R such that J is a t-reduction of I; say, $(JI^n)_t = (I^{n+1})_t$, for some positive integer n. We need to show that J is a reduction of I. Indeed, by t-compatibility, we have

$$I^{n+1}\overline{R} \subseteq (I^{n+1})_t\overline{R} = (JI^n)_t\overline{R} \subseteq (JI^n\overline{R})_{t_1}$$

yielding $(I^{n+1}\overline{R})_{t_1} \subseteq (JI^n\overline{R})_{t_1}$. The reverse inclusion is obvious. So, $J\overline{R}$ is a t-reduction of $I\overline{R}$. Hence, by hypothesis, $(J\overline{R})_{t_1} = (I\overline{R})_{t_1}$. Therefore, we obtain

$$\begin{aligned} I &\subseteq (I\overline{R})_{t_1} \cap R \\ &= (J\overline{R})_{t_1} \cap R \\ &= \widetilde{J\overline{R}} \cap R \text{ (by [22, Theorem 3.5])} \\ &= \overline{J\overline{R}} \cap R \text{ (by hypothesis)} \\ &= \overline{J} \text{ (by [21, Proposition 1.6.1])}. \end{aligned}$$

It follows that J is a reduction of I by [21, Corollary 1.2.5], as desired. □

Proof of Theorem 3.1. In order to prove the first statement of the theorem, it suffices to show that R satisfies the three assumptions in Lemma 3.2. Indeed, $R \subseteq \overline{R}$ is t-compatible by [4, Lemma 2.3]. By Mori-Nagata integral closure theorem, $\overline{R}$ is Krull. Therefore, $\overline{R}$ has the t-basic ideal property by [17, Figure 2]. Moreover, since $\dim(\overline{R}) = \dim(R) = 1$ by [21, Theorem 2.2.5], then $\overline{R}$ is Dedekind by [26, Theorem 12.5]. Hence, the t- and trivial operations coincide in $\overline{R}$. Whence, $\overline{J\overline{R}} = \widetilde{J\overline{R}}$ for any nonzero ideal J of R, as desired.

Now, let I be any nonzero ideal I of R. The fact that the two notions of reduction and t-reduction coincide in R combined with [21, Corollary 1.2.2] yields

$$\overline{I} \subseteq \widetilde{I} \subseteq \widehat{I}^{\,t} = \widehat{I}^{\,d} = \overline{I}$$

completing the proof of the theorem. □

As illustrative examples for Theorem 3.1, we consider one-dimensional Noetherian domains which are not divisorial (i.e., t-operation is not trivial), as shown below.

Example 3.3. Let $\mathbb{Q}$ be the field of rational numbers and X an indeterminate over $\mathbb{Q}$. Consider the pseudo-valuation domain (PVD, for short) $R := \mathbb{Q} + X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$. Then, R, as pullback issued from the DVR $\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, is a one-dimensional Noetherian domain. Further, R is not a divisorial domain since, otherwise, V would be a two-generated R-module by [15, Theorem 3.5] or [18, Theorem 2.4], which is absurd since $[V/M : R/M] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

One wonders whether there exist Noetherian domains of dimension > 1 where the notions of reduction and t-reduction coincide. Next, we show this cannot happen in a large class of Noetherian domains.

Proposition 3.4. *Let R be a Noetherian domain with $(R : \overline{R}) \neq 0$. Then, the notions of reduction and t-reduction coincide in R if and only if R has dimension 1.*

Proof. In view of Theorem 3.1, we only need to prove the "only if" assertion. Assume that the notions of reduction and t-reduction coincide in R. Since

R is Noetherian, $\overline{R}$ is a Krull domain (Mori-Nagata theorem). Set $A := (R : \overline{R}) \neq 0$. Clearly, we have

$$\overline{R} = (A : A) = ((R : \overline{R}) : A) = (R : \overline{R}A) = (R : A) = A^{-1}.$$

Suppose, for contradiction, that $\dim(R) = \dim(\overline{R}) \geq 2$ and let N be a maximal ideal of $\overline{R}$ with $\mathrm{ht}(N) \geq 2$. Since $(R : \overline{R}) \neq 0$, $\overline{R}$ is a finitely generated fractional ideal of R, and hence a Noetherian ring. So, by [20, Theorem 3.0 & Proposition 2.3], we have

$$(\overline{R} : N) = (N : N) = \overline{R}$$

and then

$$(R : AN) = ((R : A) : N) = (\overline{R} : N) = \overline{R} = A^{-1}.$$

Hence

$$(AN)_t = (AN)_v = A_v = A.$$

That is, AN is a t-reduction and hence, by hypothesis, a reduction of $(AN)_t = A$. So $A^{n+1}N = (AN)A^n = A^{n+1}$, for some positive integer n. By [25, Theorem 76], $A^{n+1} = 0$, the desired contradiction. □

4. t-C-ideals

This section studies the t-analogue of Hays' classic notion of C-ideal. In a ring, an ideal I is called a *C-ideal* if it is not a reduction of any larger ideal; i.e., if $I \subseteq K$ with $IK^n = K^{n+1}$ for some positive integer n, then $I = K$ [13, 14]. Our aim is to establish satisfactory t-analogues of Hays' results on C-ideals in Noetherian rings.

Definition 4.1. In a domain, a nonzero ideal I is called a *t-C-ideal* if it is not a non-trivial t-reduction of any larger ideal; i.e., if $I \subseteq K$ with $(IK^n)_t = (K^{n+1})_t$ for some positive integer n, then $I_t = K_t$.

Notice that a nonzero ideal I is a t-C-ideal if and only if I_t is a t-C-ideal. This fact will be used in the sequel without explicit mention.

Next, we collect some ideal-theoretic properties of t-C-ideals in an arbitrary domain (i.e, not necessarily Noetherian), as t-analogues of their respective classic counterparts [13, Section 5].

Proposition 4.2. *In a domain R, the following assertions hold:*

(1) *Every prime t-ideal is a t-C-ideal.*
(2) *Any intersection of t-C-ideals is a t-C-ideal (cf.* [13, Lemma 5.2]*).*
(3) *If I and J are t-comaximal t-C-ideals, then IJ is a t-C-ideal (cf.* [13, Theorem 5.6]*).*
(4) *Let I be a nonzero ideal and let J be a t-invertible t-C-ideal. Then, IJ is a t-C-ideal if and only if I is a t-C-ideal (cf.* [13, Theorem 5.7]*).*

Proof. (1) Let P be a prime t-ideal of R. Suppose $P \subseteq K$ with $(PK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Then

$$(K_t)^{n+1} \subseteq (K^{n+1})_t = (PK^n)_t \subseteq P$$

which yields $K_t \subseteq P$ and hence $P = K$. So, P is a t-C-ideal.

(2) Let $\{A_\lambda\}$ be a set of t-C-ideals of R and let $B := \cap_\lambda A_\lambda$. Suppose $B \subseteq K$ with $(BK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Then, for each λ, we have

$$(K^{n+1})_t = (K^n(\cap_\lambda A_\lambda))_t \subseteq (K^n A_\lambda)_t$$

yielding

$$((K + A_\lambda)^{n+1})_t = (A_\lambda(K + A_\lambda)^n)_t.$$

It follows that $K_t \subseteq (K + A_\lambda)_t = (A_\lambda)_t$ and thus $B_t = K_t$, as desired.

(3) Let I and J be two t-C-ideals of R and assume $IJ \subseteq K$ with $(IJK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. If $(I + J)_t = R$, then by [7, Lemma 16], $(IJ)_t = (I \cap J)_t$. It follows that $((I \cap J)_t K_t^n)_t = (K_t^{n+1})_t$. Hence $(I \cap J)_t = K_t$ since $I \cap J$ is a t-C-ideal by (2). That is, $(IJ)_t = K_t$.

(4) Let I be a nonzero ideal and J a t-invertible t-C-ideal of R. Suppose IJ is a t-C-ideal and $I \subseteq K$ with $(IK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Composing by J^{n+1} and taking the t-closure, we get

$$(IJ(KJ)^n)_t = ((KJ)^{n+1})_t.$$

Hence, $(IJ)_t = (KJ)_t$. As J is t-invertible, we get $I_t = K_t$. That is, I is a t-C-ideal.

Conversely, suppose I is a t-C-ideal and $IJ \subseteq K$ with $(IJK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Therefore, we have

$$(K^{n+1})_t \subseteq (JK^n)_t \text{ and } (K^{n+1})_t \subseteq (IK^n)_t.$$

So, one can easily check that

$$((J + K)^{n+1})_t = (K^{n+1} + J(K + I)^n)_t = (J(K + I)^n)_t.$$

It follows that $K_t \subseteq J_t$ as J is a t-C-ideal by hypothesis. Next, let $F := KJ^{-1}$. Clearly,

$$I \subseteq F \subseteq K_t J^{-1} \subseteq (JJ^{-1})_t = R.$$

Further, we have

$$(IJ(FJ)^n)_t = ((FJ)^{n+1})_t.$$

The fact that J is t-invertible yields

$$(IF^n)_t = (F^{n+1})_t.$$

Consequently, $F_t = I_t$ as I is a t-C-ideal by hypothesis. That is, $K_t = (IJ)_t$. $\square$

The next theorem completes Hays' result [13, Theorem 5.11] on C-ideals in the context of integrally closed Noetherian domains.

Theorem 4.3. *Let R be a Noetherian domain. The following assertions are equivalent:*

(1) *R is integrally closed;*
(2) *Each invertible ideal is a C-ideal;*
(3) *Each principal ideal is a C-ideal;*
(4) *Each nonzero ideal is a t-C-ideal;*
(5) *Each t-invertible t-ideal is a t-C-ideal;*
(6) *Each principal ideal is a t-C-ideal;*
(7) *$\overline{I} \subseteq I_t$ for each nonzero ideal I of R;*
(8) *$\widetilde{I} = I_t$ for each nonzero ideal I of R;*
(9) *$\widehat{I}^{\,t} = I_t$ for each nonzero ideal I of R;*
(10) *R has the t-basic ideal property.*

The proof of this result draws on the following elementary lemmas.

Lemma 4.4. *A domain D has the t-basic ideal property if and only if every nonzero ideal of D is a t-C-ideal.*

Proof. Straightforward. □

Lemma 4.5. *In a Noetherian domain, every nonzero ideal is a reduction (resp., t-reduction) of its integral closure (resp., t-integral closure).*

Proof. Combine [21, Corollary 1.2.5] and [22, Proposition 3.7(b)] with the assumption that every ideal is finitely generated (and so is the t-integral closure of any nonzero ideal). □

Proof of Theorem 4.3. (1) ⇔ (2) ⇔ (3) is [13, Theorem 5.11]. Moreover, (1) ⇔ (7) ⇔ (8), (9) ⇔ (10), and (10) ⇔ (4) hold in any arbitrary domain (i.e., not necessarily Noetherian) by [22, Theorem 3.5], [23, Theorem 3.2], and Lemma 4.4, respectively. Also, (4) ⇒ (5) ⇒ (6) are trivial.

(1) ⇒ (10) Assume R is integrally closed. Then, R is Krull and hence it has the t-basic ideal property by [17, Figure 2], as desired.

(6) ⇒ (1) By [12, Lemma 24.6], it suffices to show that every principal ideal is integrally closed. Let (a) be a principal ideal of R and let

$$b \in \overline{(a)} \subseteq \widetilde{(a)} \subseteq \widehat{(a)}^{\,t}.$$

So, (a) is a t-reduction of (a, b). Since (a) is a t-C-ideal, $(a) = (a, b)_t$; that is, $b \in (a)$. Thus, (a) is integrally closed. □

Recall that a Krull domain has the t-basic ideal property and the converse is not true in general [17, Example 3.3]. However, the two notions coincide in Noetherian domains as shown by Theorem 4.3, which also provides a t-analogue for Hays' result that "a Noetherian domain is Dedekind if and only if it has the basic ideal property" [13, Corollary 6.6]:

Corollary 4.6. *A Noetherian domain is Krull if and only if it has the t-basic ideal property.*

In [13], Hays proved that the notion of regular C-ideal is local in Noetherian rings (cf. [13, Theorem 5.8 & Theorem 5.9 & Corollary 5.10]). We close this section by establishing a satisfactory t-analogue for this result.

Proposition 4.7. *In a Noetherian domain, if an ideal is t-locally a t-C-ideal, then it is a t-C-ideal.*

Proof. Let I be a nonzero ideal of R which is t-locally a t-C-ideal. Suppose $I \subseteq K$ with $(IK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Localizing at $M \in \mathrm{Max}_t(R)$, we get

$$((IK^n)_t R_M)_{t_M} = ((K^{n+1})_t R_M)_{t_M}$$

where t_M denotes the t- operation in R_M. By [24, Lemma 2.18], we have

$$(IR_M K{R_M}^n)_{t_M} = (K{R_M}^{n+1})_{t_M}.$$

Since I is t-locally a t-C-ideal, $(IR_M)_{t_M} = (KR_M)_{t_M}$. Consequently, as all ideals are finitely generated, $I^{-1}R_M = K^{-1}R_M$, $\forall M \in \mathrm{Max}_t(R)$. It follows that

$$(I^{-1})_w = \bigcap_{M \in \mathrm{Max}_t(R)} (I^{-1})R_M = \bigcap_{M \in \mathrm{Max}_t(R)} (K^{-1})R_M = (K^{-1})_w.$$

Thus, $I_t = K_t$; that is, I is a t-C-ideal. □

The converse of the above result is still elusively open.

References

[1] Anderson, Daniel D.; Anderson, David F.; Costa, Douglas L.; Dobbs, David E.; Mott, Joe L.; Zafrullah, Muhammad. Some characterizations of v-domains and related properties. *Colloq. Math.* **58** (1989), no. 1, 1–9. MR1028154, Zbl 0701.13001, doi: 10.4064/cm-58-1-1-9. 405

[2] Anderson, Daniel D.; Anderson, David F.; Fontana, Marco; Zafrullah, Muhammad. On v-domains and star operations. *Comm. Algebra* **37** (2009), no. 9, 3018–3043. MR2554189, Zbl 1185.13003, arXiv:0809.2947, doi: 10.1080/00927870802502688. 405

[3] Anderson, David F. Integral v-ideals. *Glasgow Math. J.* **22** (1981), no. 2, 167–172. MR0623001, Zbl 0467.13012, doi: 10.1017/S0017089500004638. 405

[4] Anderson, David F.; Baghdadi, Said El; Zafrullah, Muhammad. The v-operation in extensions of integral domains. *J. Algebra Appl.* **11** (2012), no. 1, 1250007, 18 pp. MR2900877, Zbl 1246.13009, doi: 10.1142/S0219498811005312. 405, 411, 412

[5] Bouvier, Alain; Zafrullah, Muhammad. On some class groups of an integral domain. *Bull. Soc. Math. Grèce (N.S.)* **29** (1988), 45–59. MR1039430, Zbl 0754.13017.

[6] Chang, Gyu Whan; Zafrullah, Muhammad. The w-integral closure of integral domains. *J. Algebra* **295** (2006), no. 1, 195–210. MR2188857, Zbl 1096.13008, doi: 10.1016/j.jalgebra.2005.04.025. 405

[7] Dumitrescu, Tiberiu; Zafrullah, Muhammad. t-Shreier domains. *Comm. Algebra* **39** (2011), no. 3, 808–818. MR2782565, Zbl 1221.13003, doi: 10.1080/00927871003597642. 414

[8] El Baghdadi, Said; Gabelli, Stefania. w-divisorial domains. *J. Algebra* **285** (2005), no. 1, 335–355. MR2119116, Zbl 1094.13037, arXiv:math/0412452, doi: 10.1016/j.jalgebra.2004.11.016. 405

[9] Fontana, Marco; Gabelli, Stefania. On the class group and the local class group of a pullback. *J. Algebra* **181** (1996), no. 3, 803–835. MR1386580, Zbl 0871.13006, doi: 10.1006/jabr.1996.0147. 406

[10] Fontana, Marco; Zafrullah, Muhammad. On v-domains: a survey. *Commutative algebra–Noetherian and non-Noetherian perspectives*, 145–179. *Springer, New York*, 2011. MR2762510, Zbl 1227.13013, arXiv:0902.3592, doi: 10.1007/978-1-4419-6990-3_6. 405

[11] Gabelli, Stefania; Houston, Evan. Coherentlike conditions in pullbacks. *Michigan Math. J.* **44** (1997), no. 1, 99–123. MR1439671, Zbl 0896.13007, doi: 10.1307/mmj/1029005623. 406

[12] Gilmer, Robert. Multiplicative ideal theory. Pure and Applied Mathematics, 12. *Marcel Dekker, Inc., New York*, 1972. x+609pp. MR0427289, Zbl 0248.13001. 405, 415

[13] Hays, James H. Reductions of ideals in commutative rings. *Trans. Amer. Math. Soc.* **177** (1973), 51–63. MR0323770, Zbl 0266.13001, doi: 10.2307/1996583. 404, 406, 407, 408, 413, 414, 415, 416

[14] Hays, James H. Reductions of ideals in Prüfer domains. *Proc. Amer. Math. Soc.* **52** (1975), 81–84. MR0376655, Zbl 0345.13013, doi: 10.2307/2040105. 404, 413

[15] Hedstrom, John R.; Houston, Evan G. Pseudo-valuation domains. *Pacific J. Math.* **75** (1978), no. 1, 137–147. MR0485811, Zbl 0368.13002, doi: 10.2140/pjm.1978.75.137. 412

[16] Houston, Evan G. Prime t-ideals in $R[X]$. *Commutative ring theory* (Fés, 1992), 163–170, Lecture Notes in Pure and Appl. Math., 153. *Dekker, New York*, 1994. MR1261887, Zbl 0798.13001. 407

[17] Houston, Evan; Kabbaj, Salah-Eddine; Mimouni, Abdeslam. $\star$-reductions of ideals and Prüfer v-multiplication domains. *J. Commut. Algebra* **9** (2017), no. 4, 491–505. MR3713525, Zbl 06797096, arXiv:1602.07035, doi: 10.1216/JCA-2017-9-4-491. 405, 406, 409, 412, 415

[18] Houston, Evan; Mimouni, Abdeslam; Park, Mi Hee. Noetherian domains which admit only finitely many star operations. *J. Algebra* **366** (2012), 78–93. MR2942644, Zbl 1262.13041, doi: 10.1016/j.jalgebra.2012.05.015. 412

[19] Houston, Evan; Zafrullah, Muhammad. Integral domains in which each t-ideal is divisorial. *Michigan Math. J.* **35** (1988), no. 2, 291–300. MR0959276, Zbl 0675.13001, doi: 10.1307/mmj/1029003756. 405, 407

[20] Huckaba, James A.; Papick, Ira J. When the dual of an ideal is a ring. *Manuscripta Math.* **37** (1982), no. 1, 67–85. MR0649566, Zbl 0484.13001, doi: 10.1007/BF01239947. 413

[21] Huneke, Craig; Swanson, Irena. Integral closure of ideals, rings, and modules. London Mathematical Society Lecture Note Series, 336. *Cambridge University Press, Cambridge*, 2006. xiv+431 pp. ISBN: 978-0-521-68860-4; 0-521-68860-4. MR2266432, Zbl 1117.13001. 405, 411, 412, 415

[22] Kabbaj, Salah-Eddine; Kadri, Abdulilah. t-Reductions and t-integral closure of ideals. *Rocky Mountain J. Math.* **47** (2017), no. 6, 1875–1899. MR3725248, Zbl 06816574, arXiv:1602.07041, doi: 10.1216/RMJ-2017-47-6-1875. 405, 406, 410, 411, 412, 415

[23] Kabbaj, Salah-Eddine; Kadri, Abdulilah; Mimouni, Abdeslam. On t-reductions of ideals in pullbacks. *New York J. Math.* **22** (2016), 875–889. MR3548128, Zbl 1360.13055, arXiv:1607.06705. 405, 406, 410, 411, 415

[24] Kang, Byung Gyun. $\star$-operations in integral domains. Ph.D. thesis, The University of Iowa. Iowa City, 1987. *ProQuest LLC, Ann Arbor, MI*, 1987. 119 pp. MR2635967. 409, 416

[25] Kaplansky, Irving. Commutative rings. Revised edition. *The University of Chicago Press, Chicago, Ill.-London*, 1974. ix+182 pp. MR0345945, Zbl 0296.13001. 413

[26] Matsumura, Hideyuki. Commutative ring theory. Cambridge Studies in Advanced Mathematics, 8. *Cambridge University Press, Cambridge*, 1986. xiv+320 pp. ISBN: 0-521-25916-9. MR0879273, Zbl 0603.13001, doi: 10.1017/S0013091500006702. 412

[27] Mimouni, Abdeslam. Integral domains in which each ideal is a w-ideal. *Comm. Algebra* **33** (2005), no. 5, 1345–1355. MR2149062, Zbl 1079.13016, doi: 10.1081/AGB-200058369. 405

[28] Northcott, D. G.; Rees, David. Reductions of ideals in local rings. *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158. MR0059889, Zbl 0057.02601. 404

[29] Wang, Fanggui. w-dimension of domains. *Comm. Algebra* **27** (1999), no. 5, 2267–2276. MR1683866, Zbl 0923.13010, doi: 10.1080/00927879908826564. 405

[30] Wang, Fanggui. w-dimension of domains, II. *Comm. Algebra* **29** (2001), no. 6, 2419–2428. MR1845120, Zbl 1016.13007, doi: 10.1081/AGB-100002398. 405

[31] Zafrullah, Muhammad. Well behaved prime t-ideals. *J. Pure Appl. Algebra* **65** (1990), no. 2, 199–207. MR1068255, Zbl 0705.13001, doi: 10.1016/0022-4049(90)90119-3. 407

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