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A_2 colored polynomials of rigid vertex graphs

Wataru Yuasa

Abstract. The Kauffman–Vogel polynomials are three variable polynomial invariants of 4-valent rigid vertex graphs. A one-variable specialization of the Kauffman–Vogel polynomials for unoriented 4-valent rigid vertex graphs was given by using the Kauffman bracket and the Jones-Wenzl idempotent with the color 2. Bataineh, Elhamdadi and Hajij generalized it to any color with even positive integers. We give another generalization of the one-variable Kauffman–Vogel polynomial for oriented and unoriented 4-valent rigid vertex graphs by using the A_2 bracket and the A_2 clasps. These polynomial invariants are considered as the $\mathfrak{sl}_3$ colored Jones polynomials for singular knots and links.

Contents

1. Introduction

Kauffman considered an isotopy for embeddings of graphs in the 3-space in [Kau89]. It is called the vertex isotopy and he proved that the vertex isotopy for a 4-valent graph is generated by a generalization of the Reidemeister moves. Kauffman and Vogel defined polynomial invariants for

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regular vertex isotopy classes of 4-valent graphs in [KauV92]. These polynomial invariants are three-variable generalizations of polynomial invariants of knots: the Homfly polynomials for oriented knots and the Kauffman polynomials [Kau90] for unoriented knots. A one-variable specialization of the Kauffman–Vogel polynomial of a regular vertex isotopy class of an unoriented 4-valent graph was given by using the Kauffman bracket and the Jones-Wenzl idempotents (see, for example, Sect. 4.4 in [KauL94]). For an unoriented 4-valent graph G, this polynomial is defined by the value of the Kauffman bracket of a skein element obtained by coloring each edge of G with 2 and replacing 4-valent vertices by a certain type of skein element. In [BEH16], Bataineh, Elhamdadi and Hajij generalized the one-variable Kauffman–Vogel polynomials by changing the coloring from 2 to any even positive integer $2n$. There are many invariants of vertex isotopy classes of graphs, for example, Kauffman and Mishra [KauM13], Juyumaya and Lambropoulou [JL09] as an invariant of singular knots, Yamada [Yam89] as an invariant of spatial graphs, and so on. In [Wu12], Wu showed a relationship between the Kauffman–Vogel polynomial and the MOY graph polynomial [MOY98]. By using linear skein theory, some invariants of topological graphs were constructed, for example, in Yokota [Yok96] and Kawagoe [Kaw16].

In this paper, we will define one-variable polynomial invariants of the regular vertex isotopy classes of oriented and unoriented 4-valent graphs. These polynomial invariants are a skein theoretical generalization of the one-variable Kauffman–Vogel polynomials. We will construct the A_2 colored polynomials using the A_2 bracket and the A_2 clasps instead of the Kauffman bracket and the Jones-Wenzl idempotents. These invariants are generalizations of the colored $\mathfrak{sl}_3$ Jones polynomials. In fact, the rigid vertex isotopy class of an oriented (resp. unoriented) 4-valent graph G with no 4-valent vertices is an oriented (resp. unoriented) framed link. In this case, the A_2 colored polynomials of G is the $\mathfrak{sl}_3$ colored Jones polynomial of the oriented (resp. unoriented) framed link colored with one-row Young diagrams of even length (resp. two-row Young diagrams of the same length).

This paper is organized as follows. We introduce the definition of a 4-valent rigid vertex graph by diagrams on S^2 and the generalized Reidemeister moves in Sect. 2. Next, we define the A_2 bracket, the A_2 clasps and show some useful formulas in Sect. 3. In Sect. 4, we define the polynomial invariants of oriented and unoriented 4-valent rigid vertex graphs. In Sect. 5, we compute these invariants for some 4-valent rigid vertex graphs.

2. Rigid vertex graphs

We will treat diagrammatically regular vertex isotopy classes of embeddings of oriented and unoriented 4-valent graphs in S^3 through an equivalence class of 4-valent graph diagrams on S^2. We briefly explain the geometric definition of the rigid vertex graphs (see Kauffman [Kau89] for details.)

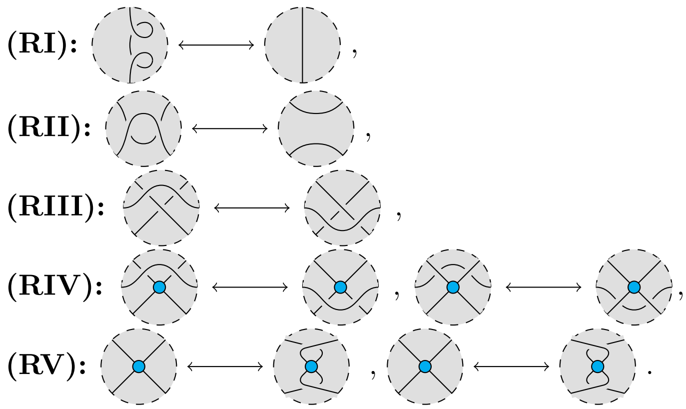
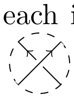


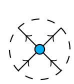
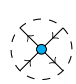
FIGURE 1. The Reidemeister moves for 4-valent graph diagrams

The rigid vertex means that the half-edges attaching to the vertex have a cyclic ordering. An embedding of a 4-valent rigid vertex graphs into S^3 is an embedding of the underlying 4-valent graph into S^3 with the following condition. Each embedded vertex v can be replaced by a small disk D_v in S^3 and the half-edges at v are attached to ∂D_v such that the cyclic ordering coincides with the orientation of ∂D_v.

We deal with the regular isotopy classes of the above graphs in S^3 as diagrams on S^2 with an equivalence relation generated by Reidemeister moves (RI) – (RV).

Definition 2.1.

- A 4-valent graph diagram on S^2 is an immersion of 4-valent graph into S^2 whose intersection points are only transverse double points of edges. At each intersection point, two edges are equipped with crossing data  .
- Two 4-valent graph diagrams G and G' are equivalent if G is related to G' by a finite sequence of Reidemeister moves (RI) – (RV) as in Fig. 2. This equivalence relation is called *regular vertex isotopy* in [Kau89].
- An oriented 4-valent graph diagram (see, for example [KauV92]) is a 4-valent graph diagram whose edges are oriented as one of the following:

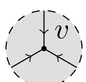
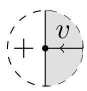
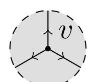
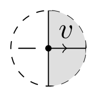
 and .

We call equivalence classes of 4-valent graph diagrams 4-*valent rigid vertex graphs*.

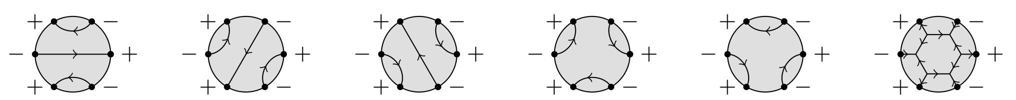
3. The A_2 bracket and some formulas

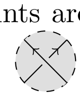
We will construct invariants of oriented and unoriented 4-valent rigid vertex graphs using the linear skein theory corresponding to the quantum A_2 representation. In this section, we introduce the A_2 web spaces, the A_2 bracket, and the A_2 clasps defined by Kuperberg [Kup94, Kup96]. Special skein elements called the A_2 clasps play an important role in construction of the colored A_2 polynomials for 4-valent rigid vertex graphs.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ be an m-tuple of signs $+$ or $-$. Let D_ε denote the unit disk with signed marked points $\{\exp(2\pi\sqrt{-1}/m)^{j-1} \mid j = 1, 2, \dots, m\}$ on its boundary. The sign of $\exp(2\pi\sqrt{-1}/m)^{j-1}$ is given by ε_j for $j = 1, 2, \dots, m$. A *bipartite uni-trivalent graph* G is a directed graph such that each vertex is either trivalent or univalent and the vertices are divided into the sinks and the sources. A sink (resp. source) is a vertex such that all edges adjoining to the vertex point into (resp. away from) it. A *bipartite trivalent graph* G in D_ε is an embedding of a uni-trivalent graph into D_ε such that any vertex v has the following neighborhoods:

 or  if v is a sink,  or  if v is a source.

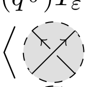
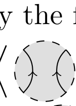
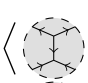
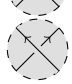
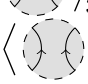
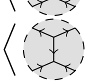
An A_2 *basis web* is the boundary-fixing isotopy class of a bipartite trivalent graph G in D_ε, where any internal face of $D_\varepsilon \setminus G$ has at least six sides. Let us denote B_ε the set of A_2 basis webs in D_ε. For example, $B_{(+,-,+,-,+,-)}$ has the following A_2 basis webs:



The A_2 *web space* W_ε is the $\mathbb{Q}(q^{\frac{1}{6}})$-vector space spanned by B_ε. A *tangled trivalent graph diagram* in D_ε is an immersed bipartite uni-trivalent graph in D_ε whose intersection points are only transverse double points of edges with crossing data  or . Tangled trivalent graph diagrams G and G' are regularly isotopic if G is obtained from G' by a finite sequence of boundary-fixing isotopies and Reidemeister moves, see Figure 2, with some direction of edges.

Tangled trivalent graphs in D_ε are regular isotopy classes of tangled trivalent graph diagrams in D_ε. We denote T_ε the set of tangled trivalent graphs in D_ε.

Definition 3.1 (The A_2 bracket [Kup96])**.** We define a $\mathbb{Q}(q^{\frac{1}{6}})$-linear map $\langle \cdot \rangle_3 \colon \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \to W_\varepsilon$ by the following.

- $\Big\langle$  $\Big\rangle_3 = q^{\frac{1}{3}} \Big\langle$  $\Big\rangle_3 - q^{-\frac{1}{6}} \Big\langle$  $\Big\rangle_3$,
- $\Big\langle$  $\Big\rangle_3 = q^{-\frac{1}{3}} \Big\langle$  $\Big\rangle_3 - q^{\frac{1}{6}} \Big\langle$  $\Big\rangle_3$,

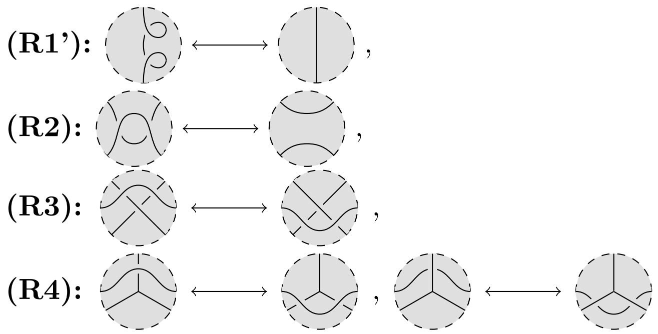
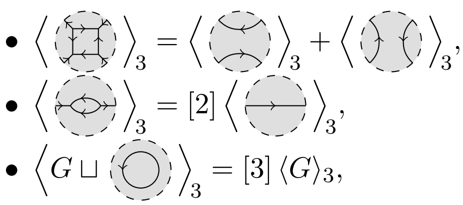
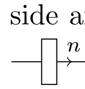


FIGURE 2. The Reidemeister moves for tangled trivalent graph diagrams

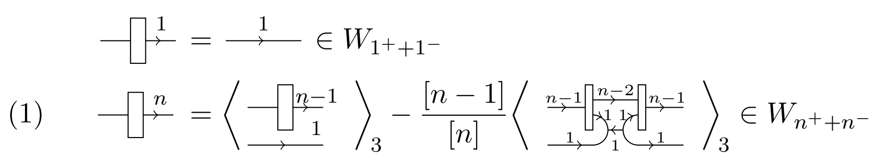


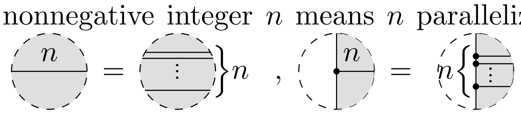
where $[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ is a quantum integer.

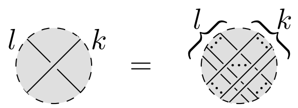
We remark that this map is invariant under the Reidemeister moves for tangled trivalent graphs.

We next consider the A_2 web space $W_{n^+ + n^-} = W_{(+,+,\dots,+,-,-,\dots,-)}$. The n marked points with sign $+$ lie in the right side and the n marked points with sign $-$ in left side. We define A_2 clasps  $\in W_{n^+ + n^-}$ inductively by the following.

Definition 3.2. (The A_2 clasps)

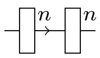


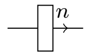
A strand decorated by a nonnegative integer n means n parallelization of the strand. For example,  , and

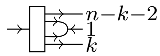
 .

A_2 clasps have the following properties.

Lemma 3.3 (Properties of A_2 clasps)**.** *For any positive integer n,*

- $\Big\langle$ 

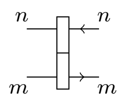
$\Big\rangle_3 =$ 

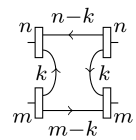
,
- $\Big\langle$ 

$\Big\rangle_3 = 0 \quad (k = 0, 1, \dots, n-2).$

We also define the A_2 clasp of type (n, m) according to Ohtsuki and Yamada [OY97].

Definition 3.4 (the A_2 clasp of type (n, m))**.**

$$\Big\langle$$ 

$$\Big\rangle_3 = \sum_{k=0}^{\min\{m,n\}} (-1)^k \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix}}{\begin{bmatrix} n+m+1 \\ k \end{bmatrix}} \Big\langle$$ 

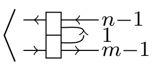
$$\Big\rangle_3$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!} = \frac{\{n\}!}{\{k\}!\{n-k\}!}$$

for $k \leq n$.

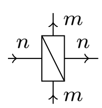
Lemma 3.5 (Property of A_2 clasps of type (m, n))**.**

$$\Big\langle$$ 

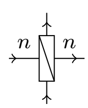
$$\Big\rangle_3 = 0.$$

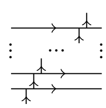
We use the following graphical notations to represent certain A_2 webs.

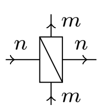
Definition 3.6. For positive integers n and m,

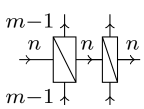


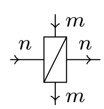
$\in W_{n^+ + m^+ + n^- + m^-}$

is defined as follows: 

$=$ 
$\in W_{n^+ + 1^+ + n^- + 1^-}$ for $m = 1$,

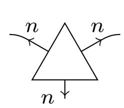


$=$ 

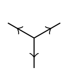
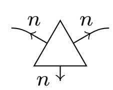
for $m > 1$. We also define 

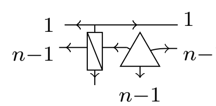
$\in W_{n^+ + m^- + n^- + m^+}$ in the same way.

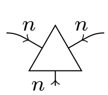
Definition 3.7. For a positive integer n,



$\in W_{n^+ + n^+ + n^+}$

is defined as follows: 
for $n = 1$, 

$=$ 

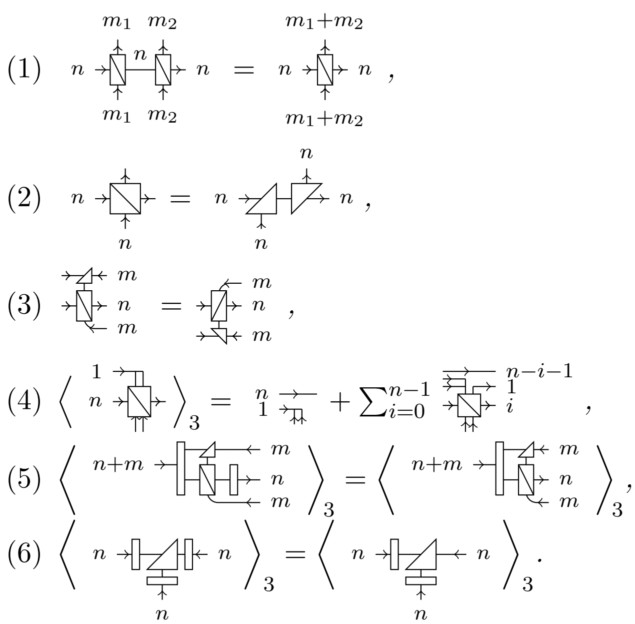
for $n > 1$. We also define 

$\in W_{n^- + n^- + n^-}$ in the same way.

We sometimes omit orientations of A_2 webs when the orientations obviously turn out from the previous A_2 webs.

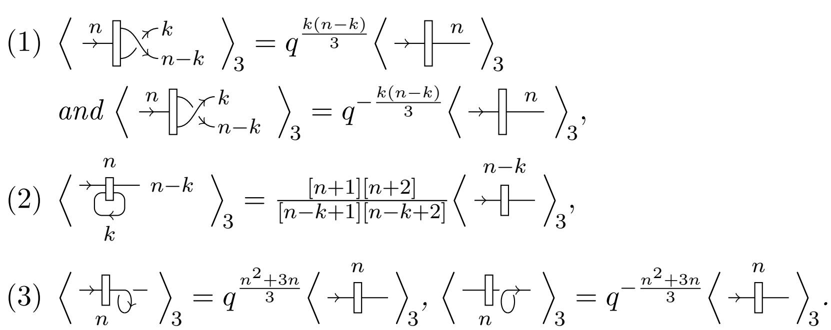
We review some formulas for the A_2 bracket.

Lemma 3.8 ([Yua17])**.**



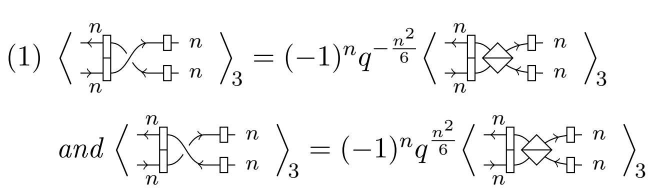
The above equations also hold for the opposite orientations.

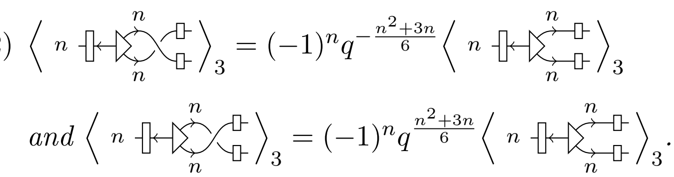
Lemma 3.9. *For $k = 0, 1, \dots, n$,*



Proof. It is easy to prove (1)–(3). See, for example, [OY97]. □

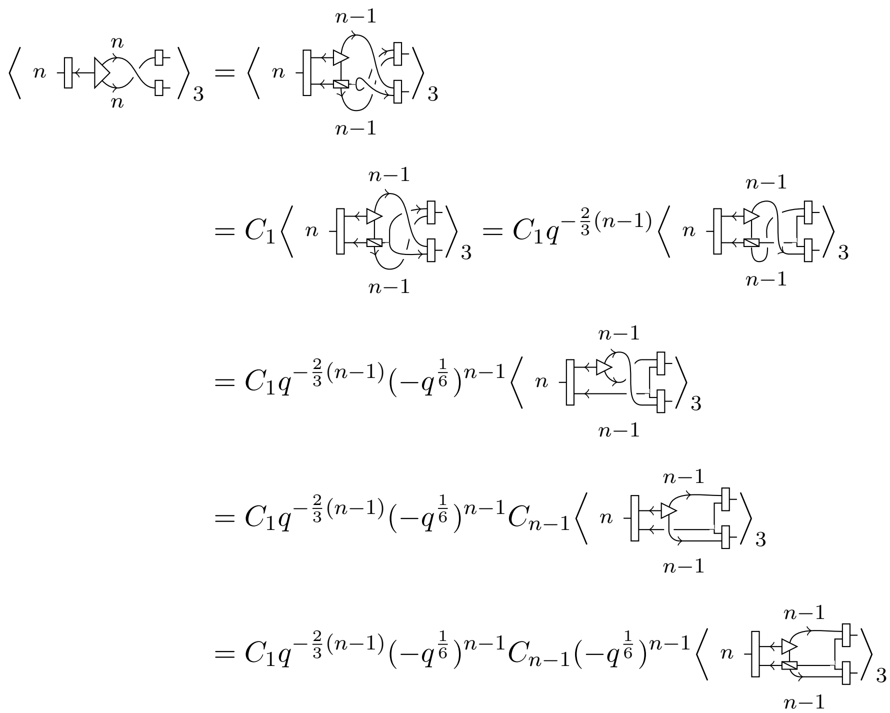
Lemma 3.10.



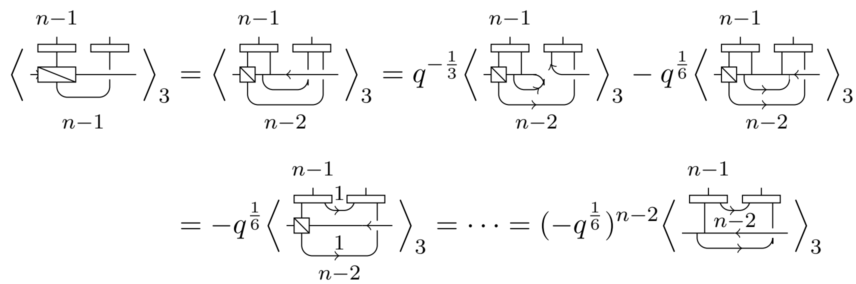
(2) 

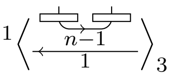
The above equations also hold for the opposite orientations.

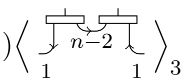
Proof. (1) is derived by Lemma 3.5 and the colored A_2 skein relation in [Yua17] (see Theorem 5.1 in Sect. 5). We only prove the first equation of (2) by induction. It is proven by straightforward calculation for $n = 1$. Set $C_n = (-1)^n q^{-\frac{n^2+3n}{6}}$,

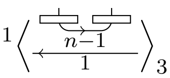
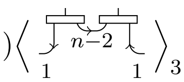


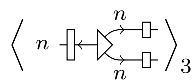
The last equation is easily derived by using the A_2 skein relation $n-1$ times at the crossing. We applied the following calculation to the second line of the above equation.



$= (-q^{\frac{1}{6}})^{n-1}$ 

$+ ((-q^{\frac{1}{6}})^{n-2}q^{-\frac{1}{3}}[2] + (-q^{\frac{1}{6}})^{n-1})$ 

$= (-q^{\frac{1}{6}})^{n-1}$  $- (-q^{\frac{1}{6}})^{n-1}(q^{-1})$  .

We can confirm that the coefficient turns out to be $C_n = (-1)^n q^{-\frac{n^2+3n}{6}}$ and the A_2 web in the last term is the same to  by definition.

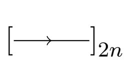
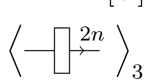
□

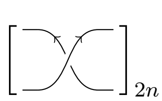
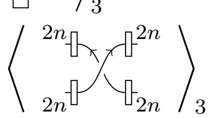
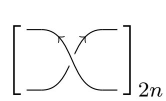
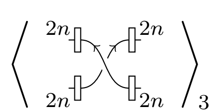
4. The A_2 colored Kauffman–Vogel polynomial

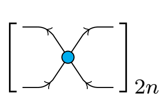
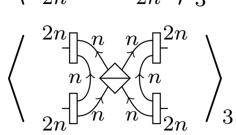
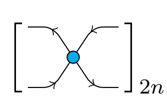
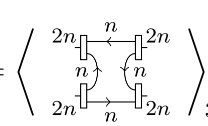
In this section, we will give the definitions of invariants of oriented and unoriented 4-valent rigid vertex graphs using clasped A_2 webs. These invariants are a generalization of Kauffman–Vogel polynomials. Kauffman and Vogel defined three variables polynomial invariants of 4-valent rigid vertex graphs. A one-variable specialization of the Kauffman–Vogel polynomial was given by using the Kauffman bracket and the Jones-Wenzl idempotents (see Chapter 4.3 in [KauL94]). The one variable Kauffman–Vogel polynomial was generalized by Elhamdadi and Hajij [EH17b]. This polynomial is colored by positive even integers. Our invariants, the A_2 colored Kauffman–Vogel polynomials, for oriented and unoriented 4-valent rigid vertex graphs are an A_2 version of the colored one-variable Kauffman–Vogel polynomials. The colored one-variable Kauffman–Vogel polynomial of a framed link with an even color $2n$ coincides with the colored Jones polynomial of it with the color $2n$. The A_2 colored Kauffman–Vogel polynomial of an oriented (resp. unoriented) framed link with color $2n$ (resp. (n, n)) coincides with the colored $\mathfrak{sl}_3$ Jones polynomials of it colored with a Young diagram $(2n, 0)$ (resp. (n, n)).

4.1. Invariants of oriented 4-valent rigid vertex graphs. Let G be an oriented 4-valent rigid vertex graph diagram.

Definition 4.1. We define $[G]_{2n}$ by the following rules:

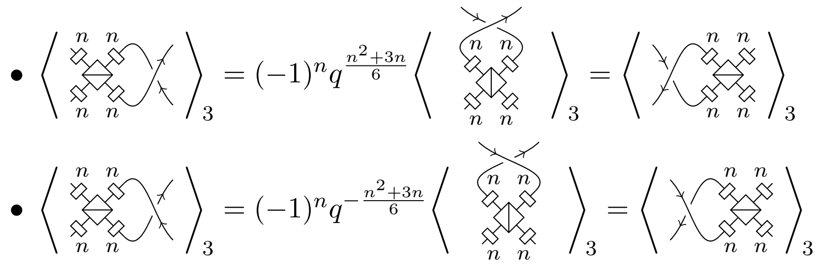
(1)  $=$ ,

(2)  $=$  and  $=$ ,

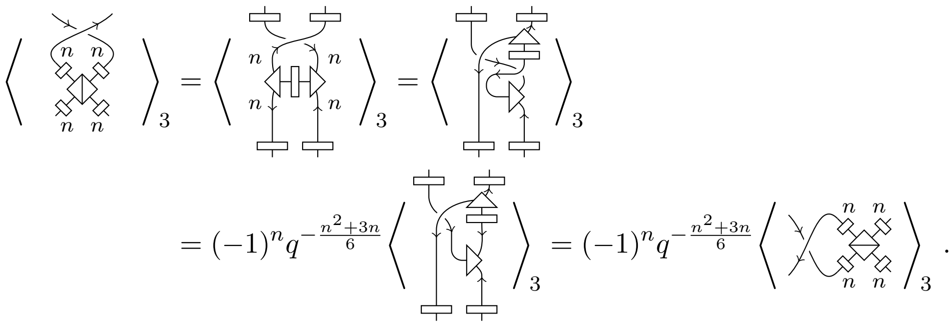
(3)  $=$  and  $=$ .

Theorem 4.2. $[G]_{2n}$ *is invariant under the Reidemeister moves (RI) – (RV).*

Lemma 4.3.

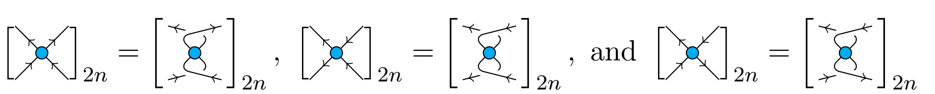


Proof. By using the Reidemeister moves for tangled trivalent graph diagrams and Lemma 3.10 (2),

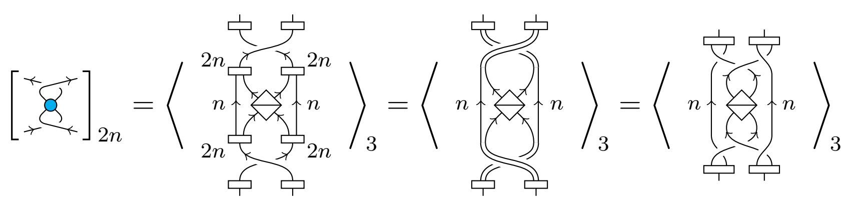


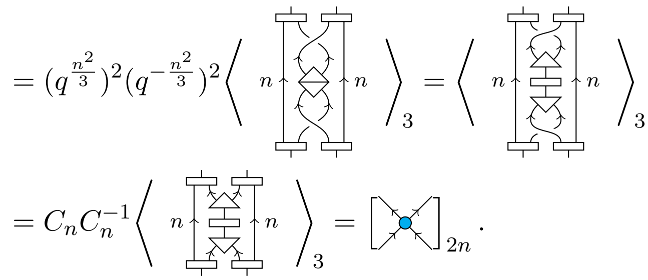
The other identities are also proven in the same way. □

Proof of Theorem 4.2. The invariance under (RI) – (RIV) is guaranteed by the invariance of A_2 webs under the Reidemeister moves (R1) – (R4) for tangled trivalent graph diagrams. Thus we show the invariance under the first move of (RV):

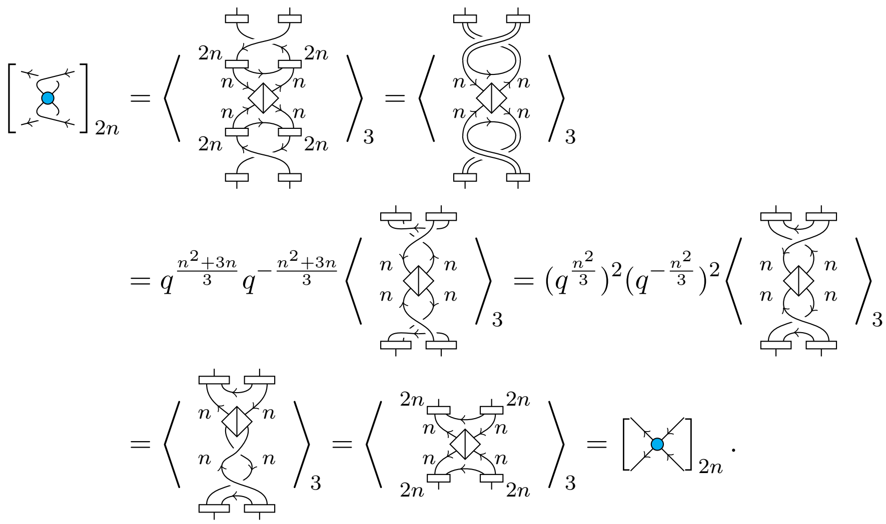


Other cases can be obtained by changing the orientation of the edges or the over/under information at the crossing points in the above diagrams. These cases can be proven in the same way as the proof of the above equations. Therefore, we only show the above three equations. Let us denote the first equation of (2) in Lemma 3.10 by $C_n = (-1)^n q^{-\frac{n^2+3n}{6}}$.

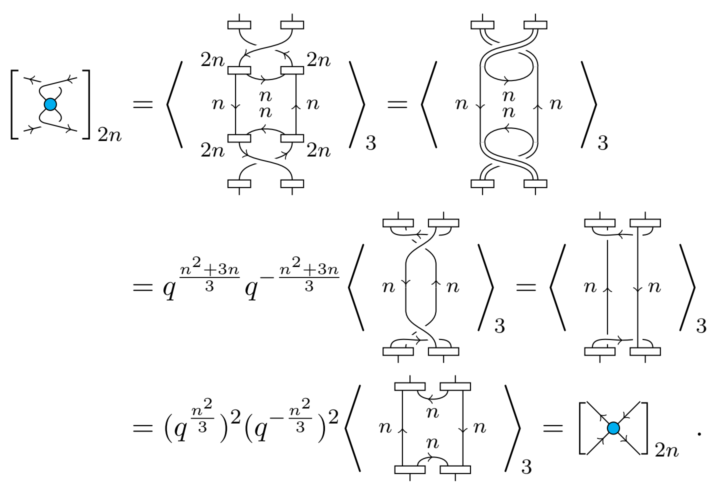




We used Lemma 3.9 (1) substituting n for $2n$ and k for n in the second line of the above identities.



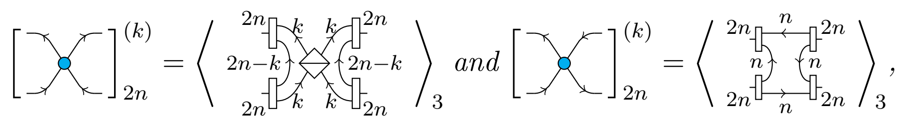
We used Lemma 3.9 (1), (3) and Lemma 4.3 in the second line.



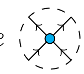
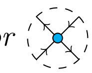
□

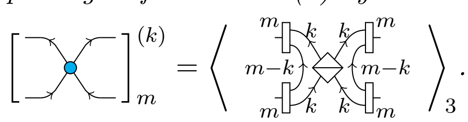
Proposition 4.4.

- *We can also define* $[G]_{2n}^{(k)}$ *by replacing Definition 4.1 (3) with*

(3-k) 

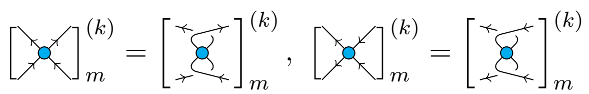
for $k = 0, 1, \dots, 2n$. We note that the definition (3-n) agrees with Definition 4.1 (3).

- *If G is a singular link, that is, all 4-valent vertices of G are* *, then there is no need to define the bracket for* *. In this case, we can define the invariant for any positive integer m and $0 \le k \le m$ by replacing Definition 4.1 (3) by*

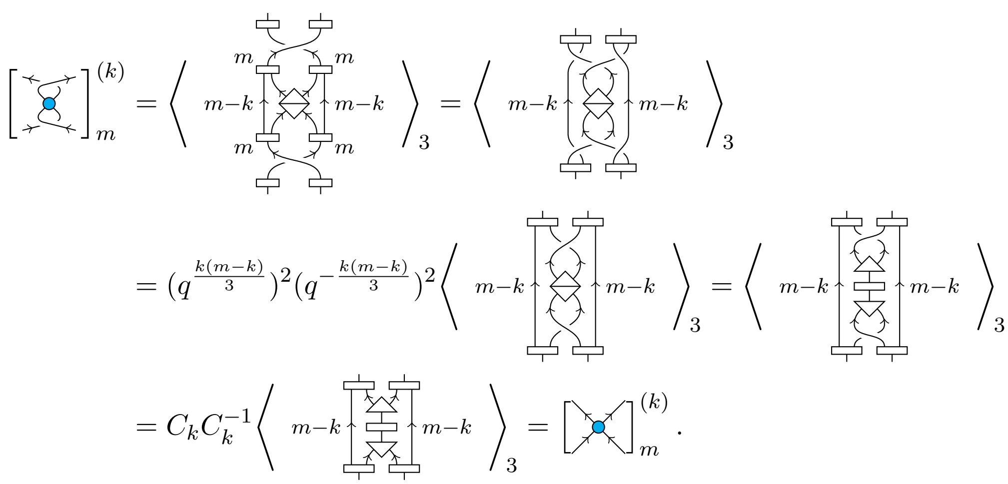
(3-k)' 

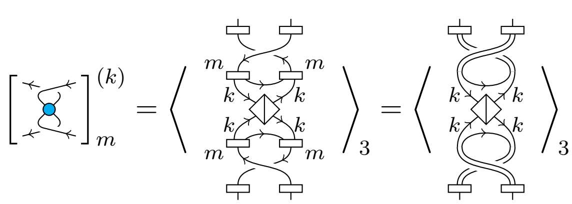
We denote this invariant for a singular link G by also $[G]^{(k)}_m$. This invariant is considered the $\mathfrak{sl}_3$ colored Jones polynomials for singular links.

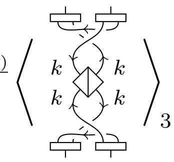
Proof. We only have to show

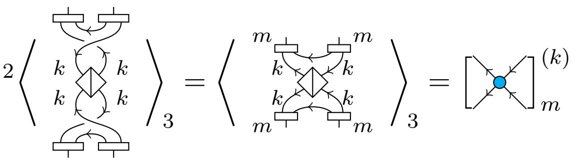


It is proved in a similar way to the proof of Theorem 4.2 as follows.



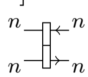
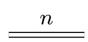
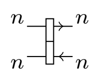
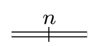


$= q^{\frac{(m-k)^2+3(m-k)}{3}} q^{-\frac{(m-k)^2+3(m-k)}{3}}$ 

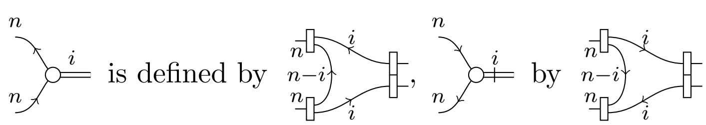
$= (q^{\frac{k(m-k)}{3}})^2 (q^{-\frac{k(m-k)}{3}})^2$ 

We used Lemma 3.9 (1), (3) and Lemma 4.3 in the second line. □

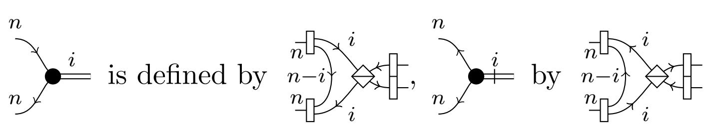
4.2. Invariant of unoriented 4-valent rigid vertex graphs. For unoriented 4-valent rigid vertex graph, we will define the invariant using the colored trivalent graphs. Firstly, we represent two types of clasped A_2 web using colored trivalent graphs with white and black vertices. In general, a diagrammatic expression of a colored trivalent graph for a A_2 web is given by Kim [Kim06].

We denote  by  and  by  .

Definition 4.5. Let n be a nonnegative integer. For $0 \leq i \leq n$, we define two types of trivalent vertices as follows.

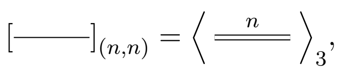


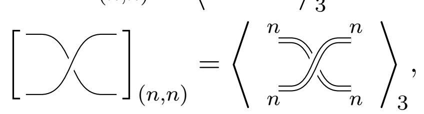
and

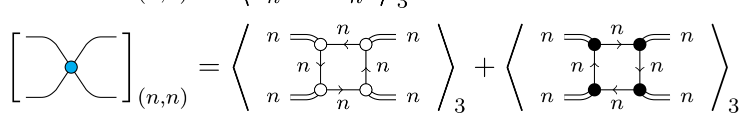


Let $\bar{G}$ be an unoriented 4-valent rigid vertex graph diagram.

Definition 4.6. We define a polynomial $[\bar{G}]_{(n,n)}$ by the following rules:

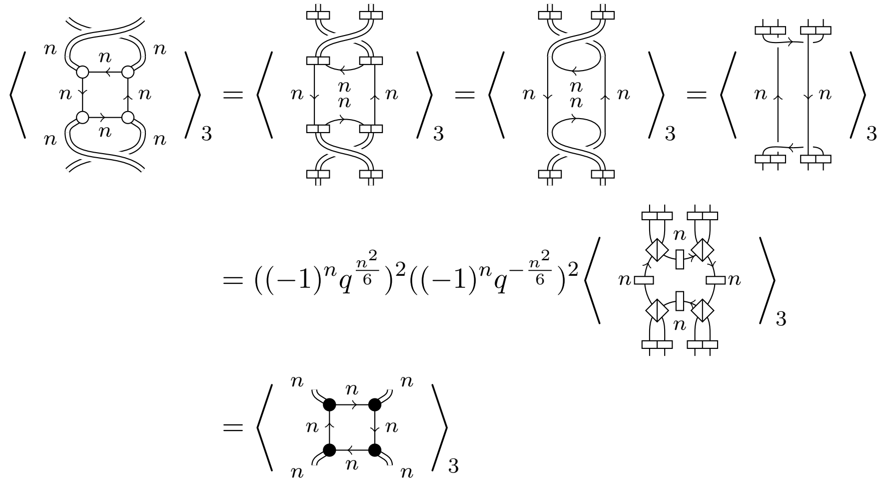
(1) 

(2) 

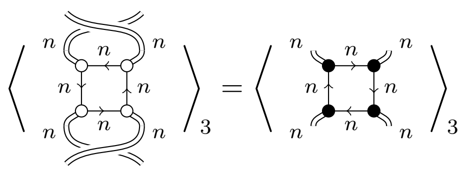
(3) 

Theorem 4.7. $[\bar{G}]_{(n,n)}$ *is invariant under the Reidemeister moves* (RI) – (RV).

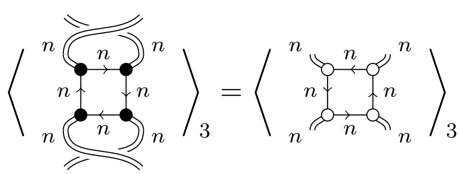
Proof. We show the invariance under the Reidemeister move (RV).



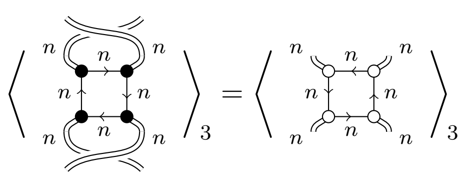
The above calculation is similar to the final calculation of the proof of Theorem 4.2. We used Lemma 3.10 (1) in the second line. In the same way, we can show



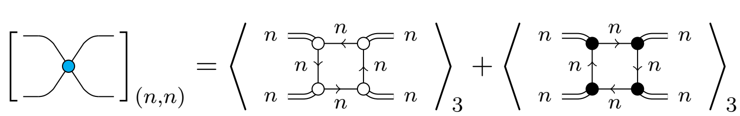
These two identities imply



and



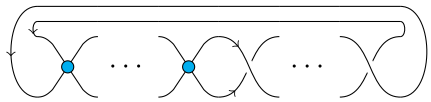
Consequently,



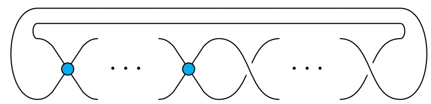
is invariant under the Reidemeister move (RV). □

5. Computing the A_2 colored Kauffman–Vogel polynomials

We define the oriented 4-valent rigid vertex graph $\mathrm{ST}(k,l)$ and the unoriented 4-valent rigid vertex graph $\bar{\mathrm{ST}}(k,l)$ as follows:

$\mathrm{ST}(k,l) =$  ,

k vertices l crossings

$\bar{\mathrm{ST}}(k,l) =$  ,

k vertices l crossings

Elhamdadi and Hajij computed the one-variable Kauffman–Vogel invariant for the Kauffman bracket of $\bar{\mathrm{ST}}(k,l)$ in [EH17a]. We only compute the one-variable Kauffman–Vogel invariant for the A_2 bracket in easy cases.

We use the following formulas to calculate the invariants for some examples. Let us denote a q-Pochhammer symbol by

$$(q)_k = \prod_{l=1}^{k}(1-q^l)$$

and a q-binomial coefficient by

$$\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$$

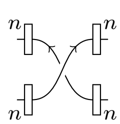
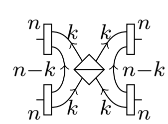
for $k \leq n$. We also define a q-multinomial coefficient as

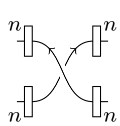
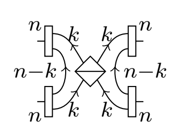
$$\binom{n}{n_1, n_2, \ldots, n_m}_q = \frac{(q)_n}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_m}},$$

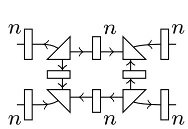
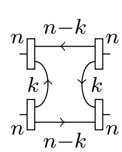
where $n_1, n_2, \ldots, n_m$ are nonnegative integers such that

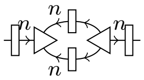
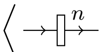
$$n_1 + n_2 + \cdots + n_m = n.$$

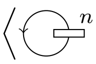
Theorem 5.1 ([Yua17, Theorem 3.11]). *Let n be a positive integer.*

(1) $\Big\langle$  $\Big\rangle_3 = \sum_{k=0}^{n}(-1)^k q^{\frac{2n^2-6nk+3k^2}{6}}\binom{n}{k}_q \Big\langle$  $\Big\rangle_3$,

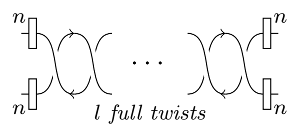
(2) $\Big\langle$  $\Big\rangle_3 = \sum_{k=0}^{n}(-1)^k q^{\frac{-2n^2+3k^2}{6}}\binom{n}{k}_q \Big\langle$  $\Big\rangle_3$,

(3) $\Big\langle$  $\Big\rangle_3 = \sum_{k=0}^{n} \Big\langle$  $\Big\rangle_3$,

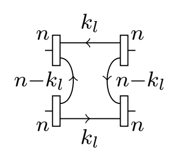
(4) $\Big\langle$  $\Big\rangle_3 = [n+1] \Big\langle$  $\Big\rangle_3$,

(5) $\Big\langle$  $\Big\rangle_3 = \dfrac{[n+1]\,[n+2]}{[2]}\emptyset.$

Theorem 5.2 ([Yua17, Theorem 3.17])**.**

$\Big\langle$  $\Big\rangle_3$

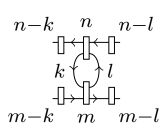
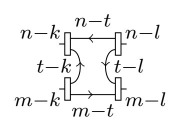
$$= q^{-\frac{2l}{3}(n^2+3n)} \sum_{0\leq k_l\leq\cdots\leq k_1\leq n} q^{n-k_l} q^{\sum_{i=1}^{l}(k_i^2+2k_i)}$$

$\times \dfrac{(q)_n}{(q)_{k_l}} \dbinom{n}{k_1', k_2', \dots, k_l', k_l}_q \Big\langle$  $\Big\rangle_3$,

where k_i, k_i' *are integers such that*

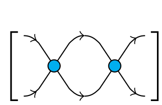
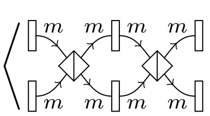
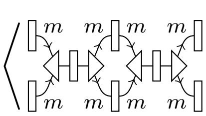
$$k_0 = n, \quad k_{i+1}' = k_i - k_{i+1} \quad \textit{for } i = 0, 1, \dots, l-1.$$

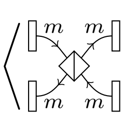
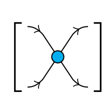
Theorem 5.3. [Yua17, Theorem 4.2]

$\Big\langle$  $\Big\rangle_3 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} \dfrac{\left[{n \atop t}\right]\left[{m \atop t}\right]\left[{t \atop k}\right]\left[{t \atop l}\right]\left[{n+m-t+2 \atop n+m-k-l+2}\right]}{\left[{n \atop k}\right]\left[{m \atop k}\right]\left[{n \atop l}\right]\left[{m \atop l}\right]} \Big\langle$  $\Big\rangle_3$

These formulas work for computations of the one-variable Kauffman–Vogel polynomials for A_2. As easy examples, we compute $[\mathrm{ST}(k,l)]_m^{(m)}$ (see Proposition 4.4) and $\left[\bar{\mathrm{ST}}(1,2l)\right]_{(n,n)}$.

A computation of $[\mathbf{ST}(\boldsymbol{k},\boldsymbol{l})]_{\boldsymbol{m}}^{(\boldsymbol{m})}$. From Theorem 5.1(4) and Lemma 3.8(6),

$\Big[$  $\Big]_m^{(m)} = \Big\langle$  $\Big\rangle_3 = \Big\langle$  $\Big\rangle_3$

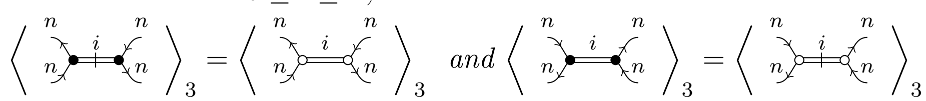
$= [m+1] \Big\langle$  $\Big\rangle_3 = [m+1] \Big[$  $\Big]_m^{(m)}.$

We obtain $[\mathrm{ST}(k,l)]_m^{(m)} = [m+1]^{k-1}\,[\mathrm{ST}(1,l)]_m^{(m)}$. By using Lemma 3.10(2) and Lemma 3.8(6), $[\mathrm{ST}(1,l)]_m^{(m)} = (-1)^{lm} q^{-\frac{l(m^2+3m)}{6}}\,[m+1]$. Therefore,

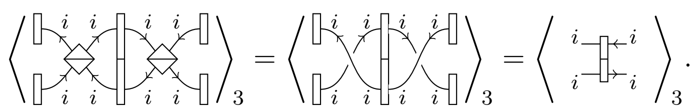
$$[\mathrm{ST}(k,l)]_m^{(m)} = (-1)^{lm} q^{-\frac{l(m^2+3m)}{6}}\,[m+1]^k.$$

A computation of $\left[\bar{\mathbf{S}}\bar{\mathbf{T}}(\mathbf{1}, \mathbf{2}\boldsymbol{l})\right]_{(\boldsymbol{n},\boldsymbol{n})}$. We prepare an easy lemma for colored trivalent graphs.

Lemma 5.4. *For $0 \leq i \leq n$,*

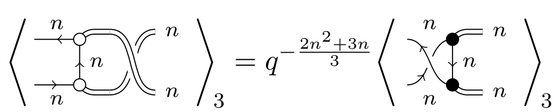


Proof. Thus Lemma follows from

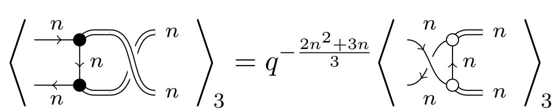


The first equation is obtained by applying Theorem 5.1 (1) and (2) to the center tangle. We expand the clasp of type (i, i) in the center tangle by using Definition 3.4 and use Lemme 3.9 (1) and (3). Thus, we obtain the second equation. □

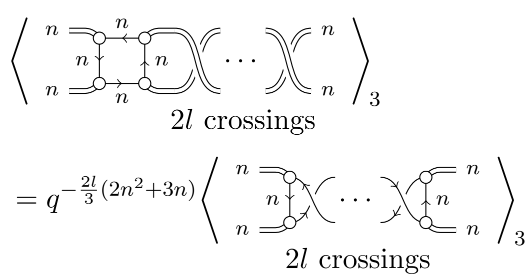
As in the same computation to the proof of Theorem 4.7, we can see



and

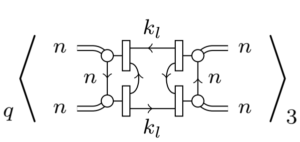


The above equations and Theorem 5.2 derives:

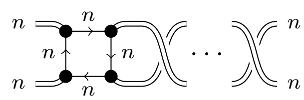


$$= q^{-2l(n^2+2n)} \sum_{0 \leq k_l \leq \cdots \leq k_1 \leq n} q^{n-k_l} q^{\sum_{i=1}^{l}(k_i^2+2k_i)}$$

$$\times \frac{(q)_n}{(q)_{k_l}} \binom{n}{k'_1, k'_2, \ldots, k'_l, k_l}_q$$

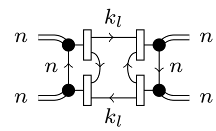


In the same way, we obtain

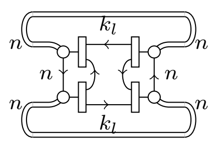
$$\Bigg\langle$$

$$\Bigg\rangle_3$$

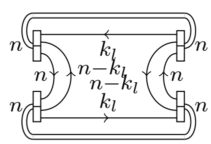
2l crossings

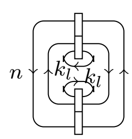
$$= q^{-2l(n^2+2n)} \sum_{0\leq k_l \leq \cdots \leq k_1 \leq n} q^{n-k_l} q^{\sum_{i=1}^{l}(k_i^2+2k_i)}$$

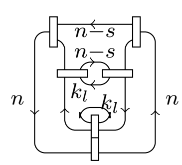
$$\times \frac{(q)_n}{(q)_{k_l}} \binom{n}{k_1', k_2', \ldots, k_l', k_l}_q \Bigg\langle$$

$$\Bigg\rangle_3 .$$

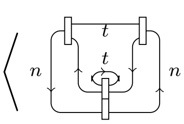
The closure of A_2 webs appearing on the right-hand side of the above two equations are the same planar web with the opposite orientation each other by Lemma 5.4. Therefore, these webs have the same value. We compute the value of the closure. For $0 \leq k_l \leq n$,

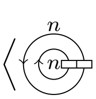
$$\Bigg\langle$$

$$\Bigg\rangle_3$$

$$= \Bigg\langle$$

$$\Bigg\rangle_3$$

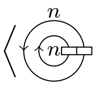
$$= \Bigg\langle$$

$$\Bigg\rangle_3$$

$$= \sum_{s=0}^{n} (-1)^s \frac{{\begin{bmatrix} n \\ s \end{bmatrix}}^2}{\begin{bmatrix} 2n+1 \\ s \end{bmatrix}} \Bigg\langle$$

$$\Bigg\rangle_3$$

$$= \sum_{s=0}^{n} (-1)^s \frac{{\begin{bmatrix} n \\ s \end{bmatrix}}^2}{\begin{bmatrix} 2n+1 \\ s \end{bmatrix}} \left(\sum_{t=\max\{n-s,k_l\}}^{\min\{n-s+k_l,n\}} \frac{{\begin{bmatrix} n \\ t \end{bmatrix}}^2 \begin{bmatrix} t \\ k_l \end{bmatrix} \begin{bmatrix} t \\ n-s \end{bmatrix} \begin{bmatrix} 2n-t+2 \\ k_l+n-s-t \end{bmatrix}}{{\begin{bmatrix} n \\ k_l \end{bmatrix}}^2 {\begin{bmatrix} n \\ n-s \end{bmatrix}}^2} \Bigg\langle \right.$$

$$\left. \Bigg\rangle_3 \right)$$

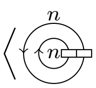
$$= \sum_{n-k_l \leq n-s \leq n} (-1)^s \frac{\begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} n+2 \\ k_l-s \end{bmatrix}}{\begin{bmatrix} 2n+1 \\ s \end{bmatrix} \begin{bmatrix} n \\ k_l \end{bmatrix}} \Big\langle$$

$$\Big\rangle_3 .$$

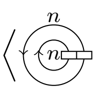
We used Definition 3.4 in the third equation, Theorem 5.3 in the fourth equation, and Lemma 3.5 in the last equation. We remark that

$\Big\langle$  $\Big\rangle_3 = \dfrac{[n+1]^2\,[2n+2]}{[2]}$

(see, for example, Lemma 5.6 in [OY97]). Consequently,

$$\begin{aligned}
&\left[\bar{\mathrm{ST}}(1,2l)\right]_{(n,n)} \\
&= 2q^{-2l(n^2+2n)} \sum_{0\le k_l\le\cdots\le k_1\le n} q^{n-k_l} q^{\sum_{i=1}^{l}(k_i^2+2k_i)} \frac{(q)_n}{(q)_{k_l}} \binom{n}{k_1', k_2', \ldots, k_l', k_l}_q \\
&\quad\times \left(\sum_{0\le s\le k_l} (-1)^s \frac{\begin{bmatrix} n \\ s \end{bmatrix}\begin{bmatrix} n+2 \\ k_l-s \end{bmatrix}}{\begin{bmatrix} 2n+1 \\ s \end{bmatrix}\begin{bmatrix} n \\ k_l \end{bmatrix}} \Big\langle \text{(diagram)} \Big\rangle_3 \right) \\
&= 2q^{-2l(n^2+2n)} \sum_{0\le s\le k_l\le\cdots\le k_1\le n} (-1)^s q^{n-k_l} q^{\sum_{i=1}^{l}(k_i^2+2k_i)} q^{s(n-k_l)-k_l} q^{\frac{s^2+3s}{2}} \\
&\quad\times \frac{(q)_n}{(q)_{k_l}} \binom{n}{k_1', k_2', \ldots, k_l', k_l}_q \frac{\binom{n}{s}_q \binom{n+2}{k_l-s}_q}{\binom{2n+1}{s}_q \binom{n}{k_l}_q} \Big\langle \text{(diagram)} \Big\rangle_3 .
\end{aligned}$$





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(Wataru Yuasa) Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan
yuasa.w.aa@m.titech.ac.jp

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