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On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space

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Abstract. The fundamental inequality $w(A^n) \leq w^n(A)$, $(n = 1, 2, ...)$ for the numerical radius is much studied in the literature. But the inverse inequalities for the numerical radius are not well known. By using Hardy–Hilbert type inequalities, we give inverse numerical radius inequalities for reproducing kernel Hilbert spaces. Also, we obtain inverse power inequalities for the Berezin number of an operator.

Contents

1. Introduction

If $p > 1$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, and if a_m, $b_n \geq 0$ are such that

$$0 < \sum_{m=0}^{\infty} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=0}^{\infty} b_n^q < \infty,$$

then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=0}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

An equivalent form is

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=0}^{\infty} a_n^p. \tag{2}$$

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The constant factors

$$\frac{\pi}{\sin\left(\pi/p\right)} \quad \text{and} \quad \left[\frac{\pi}{\sin\left(\pi/p\right)}\right]^{p}$$

are the best possible. Inequality (1) is called the Hardy–Hilbert inequality (see [HaLP52, Yan11]). Hardy–Hilbert type inequalities are important in operator theory and its applications (see [GaGO16, Han09, Kia12]). Moreover, this type of inequality has valuable applications elsewhere in analysis (see [Jin08, KP06])

Inequality (1) may be expressed in terms of operators as follows: Let ℓ^p be the usual sequence space of p-summable complex sequences. Define $T_p : \ell^p \to \ell^p$ to be the linear operator, given by

$$\left(T_p a\right)(n) = \sum_{m=0}^{\infty} \frac{a_m}{m+n+1}, \ \ (n \in \mathbb{N}_0)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any sequence $b = \{b_n\}_{n=1}^{\infty} \in \ell^q$, set

$$\langle T_p a, b\rangle := \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)b_n = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{m+n+1}a_m b_n.$$

Writing the norm of a as $\left\|a\right\|_p = \left(\sum_{n=0}^{\infty}|a|_n^p\right)^{\frac{1}{p}}$, then inequality (1) may be written as:

$$\langle T_p a, b\rangle < \frac{\pi}{\sin\left(\pi/p\right)}\left\|a\right\|_p \left\|b\right\|_q .$$

We call T_p the Hardy–Hilbert operator with the kernel $\dfrac{1}{m+n+1}$.

A reproducing kernel Hilbert space (RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$ on some set Ω is a Hilbert space of functions on Ω such that for every $\lambda \in \Omega$ the linear functional (evaluation functional) $f \to f(\lambda)$ is bounded on $\mathcal{H}$. If $\mathcal{H}$ is RKHS on set Ω, then by the classical Riesz Representation Theorem, for every $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in \mathcal{H}$. We call the family $\{k_\lambda : \lambda \in \Omega\}$ the reproducing kernel of the space $\mathcal{H}$. It is well known that

$$k_{\mathcal{H},\lambda}(z) = \sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z)$$

for any orthonormal basis $\{e_n\}_{n\geq 0}$ of the space $\mathcal{H}(\Omega)$ (see [Aro50, Sai88]).

Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ denotes the normalized reproducing kernel of the space $\mathcal{H}$. For a bounded linear operator A on the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{T}$ is defined by the formula (see [Ber72, NR94])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle_{\mathcal{H}} \ \ (\lambda \in \Omega).$$

It is clear that $\left|\widetilde{A}(\lambda)\right| \leq \|A\|$ for all $\lambda \in \Omega$; that is $\widetilde{A}$ is a bounded function. A thorough study of reproducing kernels and Berezin symbols, can be found in, [Kar06, Kar13].

The Berezin set and Berezin number of an operator A are defined by

$$\operatorname{Ber}(A) := \operatorname{Range}\big(\widetilde{A}\big) = \big\{\widetilde{A}\,(\lambda) : \lambda \in \Omega\big\}$$

and

$$\operatorname{ber}(A) := \sup\big\{\big|\widetilde{A}(\lambda)\big| : \lambda \in \Omega\big\},$$

respectively (see Karaev [Kar06]). Recall that

$$W(A) := \{\langle Af, f\rangle : \|f\|_{\mathcal{H}} = 1\}$$

is the numerical range of the operator A and

$$w(A) := \sup\{|\langle Af, f\rangle| : \|f\|_{\mathcal{H}} = 1\}$$

is the numerical radius of A. It is well known that

$$\operatorname{Ber}(A) \subset W(A) \quad \text{and} \quad \operatorname{ber}(A) \leq w(A) \leq \|A\|$$

for any $A \in \mathcal{B}(\mathcal{H})$. More information about the numerical radius and numerical range can be found, for example, in Halmos [Hal82] and Sattari et al. [SaMY15].

The fundamental inequality $w(A^n) \leq w^n(A)$, $(n = 1, 2, ...)$ for the numerical radius is studied in the literature. But the inverse inequalities $w^n(A) \leq C\,(w\,(A^n))$ $(n > 0)$ for numerical radius are not well known. Using Hardy–Hilbert type inequalities, we obtain an inverse numerical radius inequality for reproducing kernel Hilbert spaces. Also, we obtain inverse power inequalities for the Berezin numbers of operators.

2. The main results

Using the Hardy–Hilbert inequality (2), we will obtain the following inequality.

Theorem 1. *Let $p > 1$ and f be a positive continuous function on an interval $\Delta \subset (0, \infty)$. Then*

$$\langle f(A)t, t\rangle^p \left(\left(1 + \frac{1}{2}\right)^p + \left(\frac{1}{2} + \frac{1}{3}\right)^p\right) < 2\left[\frac{\pi}{\sin(\pi/p)}\right]^p \langle f(A)^p t, t\rangle$$

for any positive operator A on $\mathcal{H}$ with spectrum contained in Δ and for all $t \in \mathcal{H}$ with $\|t\|_{\mathcal{H}} = 1$.

Proof. If we put positive scalars a_0, a_1 and $a_i = 0$ $(i = 2, 3, \dots)$ in inequality (2), we get

$$\left[\frac{\pi}{\sin(\pi/p)}\right]^p (a_0^p + a_1^p) > \sum_{n=0}^{\infty}\left(\frac{a_0}{n+1} + \frac{a_1}{n+2}\right)^p \geq \left(a_0 + \frac{a_1}{2}\right)^p + \left(\frac{a_0}{2} + \frac{a_1}{3}\right)^p.$$

If we replace a_0 and a_1 with $\langle f(A)t,t\rangle$, where $t\in\mathcal{H}$ and $\|t\|_{\mathcal{H}}=1$, then we get

$$\left(\langle f(A)t,t\rangle+\frac{\langle f(A)t,t\rangle}{2}\right)^p+\left(\frac{\langle f(A)t,t\rangle}{2}+\frac{\langle f(A)t,t\rangle}{3}\right)^p$$
$$<\left[\frac{\pi}{\sin(\pi/p)}\right]^p\left(\langle f(A)t,t\rangle^p+\langle f\left(A\right)t,t\rangle^p\right).$$

Whence

$$\langle f(A)t,t\rangle^p\left(\left(1+\frac{1}{2}\right)^p+\left(\frac{1}{2}+\frac{1}{3}\right)^p\right)<2\left[\frac{\pi}{\sin(\pi/p)}\right]^p\langle f(A)t,t\rangle^p.$$

Using the McCarty inequality for $p>1$, we have

$$\begin{aligned}\langle f(A)t,t\rangle^p\left(\left(1+\frac{1}{2}\right)^p+\left(\frac{1}{2}+\frac{1}{3}\right)^p\right)&<2\left[\frac{\pi}{\sin(\pi/p)}\right]^p\langle f(A)t,t\rangle^p\\&\le 2\left[\frac{\pi}{\sin(\pi/p)}\right]^p\langle f(A)^pt,t\rangle.\end{aligned}\tag{3}$$

Thus, we reach the desired inequality. □

Corollary 1. *Let f be a positive continuous function on an interval $\Delta\subset(0,\infty)$. If $p>1$, then*

$$w^p\left(f(A)\right)\left(\left(1+\frac{1}{2}\right)^p+\left(\frac{1}{2}+\frac{1}{3}\right)^p\right)<2\left[\frac{\pi}{\sin(\pi/p)}\right]^p w\left(f(A)^p\right).\tag{4}$$

In particular, if $p=2$ we obtain

$$w^2\left(f(A)\right)<\frac{36\pi^2}{53}w\left(f(A)^2\right)\tag{5}$$

for any positive operator A on $\mathcal{H}$ with $\sigma(A)\subset\Delta$.

Proof. Taking supremum over $t\in\mathcal{H}$, where $\|t\|_{\mathcal{H}}=1$ in inequality (3), we get

$$\begin{aligned}2\left[\frac{\pi}{\sin(\pi/p)}\right]^p w\left(f\left(A\right)^p\right)&=2\left[\frac{\pi}{\sin(\pi/p)}\right]^p\sup_{\|t\|=1}\langle f(A)^pt,t\rangle\\&>\sup_{\|t\|=1}\langle f(A)t,t\rangle^p\left(\left(1+\frac{1}{2}\right)^p+\left(\frac{1}{2}+\frac{1}{3}\right)^p\right)\end{aligned}$$

(since $f\left(t\right)=t^p$ is increasing for $p>1$)

$$=w^p\left(f(A)\right)\left(\left(1+\frac{1}{2}\right)^p+\left(\frac{1}{2}+\frac{1}{3}\right)^p\right).$$

Hence, we obtain the desired inequalities (4) and (5). □

Corollary 2. *Let f be a positive continuous function on an interval $\Delta\subset(0,\infty)$. Then*

$$\operatorname{ber}^2\left(f(A)\right)\le\frac{36\pi^2}{53}\operatorname{ber}\left(f(A)^2\right)$$

for any positive operator T on $\mathcal{H}$ with $\sigma(A) \subset \Delta$.

Proof. By placing $p = 2$ and $t = \widehat{k}_\eta$, we acquire from inequality (3) that

$$\left\langle f(A)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle^2 < \frac{36\pi^2}{53} \left\langle f(A)^2\widehat{k}_\eta, \widehat{k}_\eta \right\rangle$$

for all $\eta \in \Omega$. Therefore

$$\left[\widetilde{f(A)}\,(\eta)\right]^2 \leq \frac{36\pi^2}{53} \widetilde{f(A)^2}\,(\eta)\,,\ \eta \in \Omega,$$

and hence

$$\left[\widetilde{f(A)}\,(\eta)\right]^2 \leq \frac{36\pi^2}{53} \sup_{\eta\in\Omega} \widetilde{f(A)^2}\,(\eta) = \frac{36\pi^2}{53}\operatorname{ber}\left(f(A)^2\right)$$

for all $\eta \in \Omega$. This implies that

$$\left[\sup_{\eta\in\Omega} \widetilde{f(A)}\,(\eta)\right]^2 \leq \frac{36\pi^2}{53}\operatorname{ber}\left(f\,(A)^2\right)$$

and hence

$$\operatorname{ber}^2\left(f(A)\right) \leq \frac{36\pi^2}{53}\operatorname{ber}\left(f(A)^2\right). \qquad \square$$

In the following result, we give an inequality analogous to (1) for operators acting on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Proposition 1. *Let f, g be a positive continuous functions on an interval $\Delta \subset (0, \infty)$. If $p > 1$, then*

$$\widetilde{f(A)}\,(\eta)\,\widetilde{f\,(B)}\,(\xi) + \frac{\widetilde{g(A)}\,(\eta)\,\widetilde{f\,(B)}\,(\xi)}{2} + \frac{\widetilde{f(A)}\,(\eta)\,\widetilde{g\,(B)}\,(\xi)}{2} + \frac{\widetilde{g(A)}\,(\eta)\,\widetilde{g\,(B)}\,(\xi)}{3}$$
$$< \frac{\pi}{\sin(\pi/p)} \left[(f(A)^p + g(A)^p)^{\frac{1}{p}}\right]^{\sim}(\eta) \left[(f\,(B)^q + g\,(B)^q)^{\frac{1}{q}}\right]^{\sim}(\xi)$$

for all positive operators A, B on $\mathcal{H}$ with $\sigma\,(A)\,, \sigma\,(B) \subset \Delta$ and all $\eta, \xi \in \Omega$.

Proof. Let a_0, a_1, b_0, b_1 be positive numbers. In inequality (1), we place $a_n = b_n = 0$, $n \geq 2$. Using (1), we get

$$a_0b_0 + \frac{a_1b_0}{2} + \frac{a_0b_1}{2} + \frac{a_1b_1}{3} < \frac{\pi}{\sin(\pi/p)}\left(a_0^p + a_1^p\right)^{\frac{1}{p}}\left(b_0^q + b_1^q\right)^{\frac{1}{q}}.$$

Let $x, y \in \Delta$. By taking into consideration the inequalities $f, g \geq 0$ and placing $a_0 = f\,(y)$, $a_1 = g\,(y)$, $b_0 = f(x)$, $b_1 = g(x)$, we have

$$\begin{aligned} &f\,(y)\,f(x) + \frac{g\,(y)\,f\,(x)}{2} + \frac{f\,(y)\,g(x)}{2} + \frac{g\,(y)\,g(x)}{3} \\ &< \frac{\pi}{\sin(\pi/p)}\left[f\,(y)^p + g\,(y)^p\right]^{\frac{1}{p}}\left[f(x)^q + g(x)^q\right]^{\frac{1}{q}} \end{aligned} \tag{6}$$

for all $x, y \in \Delta$. Applying the functional calculus for A to inequality (6) we have

$$\begin{aligned}&f(A)f(x)+\frac{g(A)f(x)}{2}+\frac{f(A)g(x)}{2}+\frac{g(A)g(x)}{3}\\&<\frac{\pi}{\sin(\pi/p)}\left[f(A)^p+g(A)^p\right]^{\frac{1}{p}}\left[f(x)^q+g(x)^q\right]^{\frac{1}{q}}\end{aligned}$$

and, consequently,

$$\begin{aligned}&\left\langle f(A)\widehat{k}_\eta,\widehat{k}_\eta\right\rangle f(x)+\frac{\left\langle g(A)\widehat{k}_\eta,\widehat{k}_\eta\right\rangle f(x)}{2}+\frac{\left\langle f(A)\widehat{k}_\eta,\widehat{k}_\eta\right\rangle g(x)}{2}+\frac{\left\langle g(A)\widehat{k}_\eta,\widehat{k}_\eta\right\rangle g(x)}{3}\\&<\frac{\pi}{\sin(\pi/p)}\left\langle\left[f(A)^p+g(A)^p\right]^{\frac{1}{p}}\widehat{k}_\eta,\widehat{k}_\eta\right\rangle\left[f(x)^q+g(x)^q\right]^{\frac{1}{q}}\end{aligned}$$

for all $\eta \in \Omega$ and $x \in \Delta$.

Applying the functional calculus to the positive operator B, we obtain

$$\begin{aligned}&\widetilde{f(A)}(\eta)f(B)+\frac{\widetilde{g(A)}(\eta)f(B)}{2}+\frac{\widetilde{f(A)}(\eta)g(B)}{2}+\frac{\widetilde{g(A)}(\eta)g(B)}{3}\\&<\frac{\pi}{\sin(\pi/p)}\widetilde{(f(A)^p+g(A)^p)^{\frac{1}{p}}}(\eta)\left[f(B)^q+g(B)^q\right]^{\frac{1}{q}}.\end{aligned}$$

Therefore, we get from above inequality that

$$\begin{aligned}&\widetilde{f(A)}(\eta)\widetilde{f(B)}(\xi)+\frac{\widetilde{g(A)}(\eta)\widetilde{f(B)}(\xi)}{2}+\frac{\widetilde{f(A)}(\eta)\widetilde{g(B)}(\xi)}{2}+\frac{\widetilde{g(A)}(\eta)\widetilde{g(B)}(\xi)}{3}\\&<\frac{\pi}{\sin(\pi/p)}\left[(f(A)^p+g(A)^p)^{\frac{1}{p}}\right]^{\sim}(\eta)\left[(f(B)^q+g(B)^q)^{\frac{1}{q}}\right]^{\sim}(\xi)\end{aligned}$$

for all positive operators A, B on $\mathcal{H}$ and all $\eta, \xi \in \Omega$. □

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References

[Aro50] Aronzajn, Nachman. Theory of reproducing kernels. *Trans. Amer. Math. Soc.* **68** (1950), 337–404. MR0051437 (14,479c), Zbl 0037.20701, doi: 10.2307/1990404.

[Ber72] Berezin, Felix A. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1134–1167. MR0350504 (50,2996), Zbl 0247.47019, doi: 10.1070/IM1972v006n05ABEH001913.

[GaGO16] Garayev, Mubariz T.; Gürdal, Mehmet; Okudan, Arzu. Hardy-Hilbert's inequality and power inequalities for Berezin numbers of operators. *Math. Inequal. Appl.* **19** (2016), no. 3, 883–891. MR3535209, Zbl 06645323, doi: 10.7153/mia-19-64.

[Hal82] Halmos, Paul R. A Hilbert space problem book. Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, 17. *Springer-Verlag, New York-Berlin*, 1982. xvii+369 pp. ISBN: 0-387-90685-1. MR0675952, Zbl 0496.47001.

[Han09] Hansen, Frank. Non-commutative Hardy inequalities. *Bull. Lond. Math. Soc.* **41** (2009), no. 6, 1009–1016. MR2575332 (2011g:47035), Zbl 1188.47017, arXiv:0806.0782, doi: 10.1112/blms/bdp078.

[HaLP52] Hardy, Godfrey H.; Littlewood, John E.; Pólya, George. Inequalities. Reprint of the 1952 edition. Cambridge Mathematical Library. *Cambridge University Press, Cambridge*, 1988. xii+324 pp. ISBN: 0-521-35880-9. MR0944909 (89d:26016), Zbl 0047.05302.

[Jin08] Jin, Jianjun. On Hilbert's type inequalities. *J. Math. Anal. Appl.* **340** (2008), no. 2, 932–942. MR2390899 (2009a:26027), Zbl 1142.26013, doi: 10.1016/j.jmaa.2007.09.036.

[Kar06] Karaev, Mubariz T. Berezin symbol and invertibility of operators on the functional Hilbert spaces. *J. Funct. Anal.* **238** (2006), no. 1, 181–192. MR2253012 (2007d:47027), Zbl 1102.47018, doi: 10.1016/j.jfa.2006.04.030.

[Kar13] Karaev, Mubariz T. Reproducing kernels and Berezin symbols techniques in various questions of operator theory. *Complex Anal. Oper. Theory* **7** (2013), no. 4, 983–1018. MR3079840, Zbl 1303.47012, doi: 10.1007/s11785-012-0232-z.

[Kia12] Kian, Mohsen. Hardy–Hilbert type inequalities for Hilbert space operators. *Ann. Funct. Anal.* **3** (2012), no. 2, 128–134. MR2948393, Zbl 1268.47026, doi: 10.15352/afa/1399899937.

[KP06] Krnić, Mario; Pečarić. Josip. Extension of Hilbert's inequality. *J. Math. Anal. Appl.* **324** (2006), no. 1, 150–160. MR2262462 (2007e:26027), Zbl 1112.26019, doi: 10.1016/j.jmaa.2005.11.069.

[NR94] Nordgren, Eric; Rosenthal, Peter. Boundary values of Berezin symbols. *Nonselfadjoint operators and related topics* (Beer Sheva, 1992), 362–368, Oper. Theory Adv. Appl., 73. *Birkhäuser, Basel*, 1994. MR1320554 (96b:46036), Zbl 0874.47013, doi: 10.1007/978-3-0348-8522-5_14.

[Sai88] Saitoh, Saburou. Theory of reproducing kernels and its applications. Pitman Research Notes in Mathematics Series, 189. *Longman Scientific and Technical, Harlow; copublished in the United States with John Wiley and Sons, Inc., New York*, 1988. x+157 pp. ISBN: 0-582-03564-3. MR0983117 (90f:46045), Zbl 0652.30003.

[SaMY15] Sattari, Mostafa; Moslahian, Mohammad Sal; Yamazaki, Takeaki. Some generalized numerical radius inequalities for Hilbert space operators. *Linear Algebra Appl.* **470** (2015), 216–227. MR3314313, Zbl 1322.47010, arXiv:1409.0321, doi: 10.1016/j.laa.2014.08.003.

[Yan11] Yang, Bicheng. Discrete Hilbert-type inequalities. *Bentham Science Publishers Ltd.*, 2011. doi: 10.2174/97816080524241110101.

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