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Chromatic graph homology for brace algebras

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Abstract. We prove that chromatic graph homology for commutative dg algebras, due to Helme-Guizon and Rong, can be extended to brace algebras, at least when the graph is a planar tree. Examples of brace algebras include the cochain algebra of a simplicial set and the Hochschild cochain complex of an associative algebra.

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Introduction

Let G be a finite graph with the set of vertices $V(G)$ and the set of edges $E(G)$. We assume that G has no loops (edges connecting a vertex with itself) or multiple edges (any pair of vertices is connected by at most one edge). We will also choose and fix a bijection of $V(G)$ with $\{1, \dots, n\}$, where $n = |V(G)|$, i.e a total order on $V(G)$. For a graded commutative unital algebra A which is flat over a coefficient ring k (in applications, $\mathbb{Q}, \mathbb{Z}$ or $\mathbb{F}_p$), we follow [HGR] and define the chromatic graph homology complex $C_G(A)$ in one of the two eqiuvalent ways:

(1) As a quotient of the tensor product of $A^{\otimes n} \otimes \Lambda$ by an ideal of relations (all unlabeled tensor products are over k). Here Λ is the exterior algebra over k generated by odd variables e_α corresponding to the

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edges $\alpha \in E(G)$. The ideal of relations is generated by the elements $(a^{[i]} - a^{[j]})e_\alpha$, where $a^{[i]} := 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)}$ (similarly for $a^{[j]}$) and the edge α connects vertices labeled by i and j. The quotient $C_G(A)$ algebra carries a differential d which descends from the wedge product with $\sum_{\alpha \in E(G)} e_\alpha$.

(2) The same complex can be defined by considering subsets of edges $S \subset E(G)$. For such a subset S denote by G/S the graph obtained by contracting all edges in S. The labeling on G induces one on G/S if we assign to each vertex of G/S the label which is minimal across all vertices of G that contract to it. Then the vertices of G/S are labeled by a subset of $\{1, \dots, n\}$. If $l(S)$ is the cardinality of $V(G/S)$, we can think of an elementary tensor product $a_1 \otimes \dots \otimes a_{l(S)} \in A^{l(S)}$ as built from the elements a_i assigned to the vertices of G/S (due to the total order on vertices of G/S induced from G by above labeling). To emphasize this point of view we will write $A^{\otimes(G/S)}$ for $A^{\otimes l(S)}$.

Fixing also a linear ordering on $E(G)$, we can define the element $e_S \in \Lambda$ as wedge product of all e_α for $\alpha \in S$. Now set $C_G(A)$ to be the complex

$$A^{\otimes n} \to \bigoplus_{\alpha \in E(G)} A^{\otimes(G/\alpha)} \cdot e_\alpha \to \bigoplus_{S \subset E(G), |S|=2} A^{\otimes(G/S)} \cdot e_S \to \bigoplus_{S \subset E(G), |S|=3} A^{\otimes(G/S)} \cdot e_S \to \dots$$

The diffferential d is induced by adding an edge α to a subset S and replacing e_S by $e_\alpha \wedge e_S$, which is nonzero only if $\alpha \notin S$. As for the factors involving tensor powers of A, we have two cases. In the first case, $l(S \cup \alpha) = l(S) = l$, i.e., α projects to a loop in G/S. Then we use the identity map on $A^{\otimes l}$. In the second case, $l(S \cup \alpha) = l(S) - 1$ if the projection of α to G/S connects two distinct vertices i and j. Then we map $A^{\otimes(G/S)} \to A^{\otimes(G/S \cup \alpha)}$ by applying the product of A to the tensor factors corresponding to i and j, and using the Koszul sign rule when a permutation is used to move these terms to the left, then multiply, then return to its appropriate position in $A^{\otimes(G/S \cup \alpha)}$. The Koszul sign rule and $e_\alpha \wedge e_\beta = -e_\beta \wedge e_\alpha$ ensure that $d^2 = 0$.

If $A = \bigoplus_{j \geq 0} A_j$ has a nontrivial grading, the complex $C_G(A)$ acquires a bigrading in which $a \in A_j$ is given bidegree $(j, 0)$, each e_α bidegree $(0, 1)$ and the differential d bidegree $(0, 1)$. If A is a dg algebra with differential δ, we can incorporate it into the complex $C^\bullet_G(A)$ by giving it a total differential $d = d_0 + d_1$ where d_0 is the Leibniz rule extension of δ to the tensor powers of A and d_1 is induced by edge contractions and multiplication as above.

In [BS] we have studied this complex and related it to the topology of the graph configuration space M^G of a compact k-oriented manifold M. This space is the open complement in M^n of the diagonals corresponding to those pairs of vertices which are connected by an edge in G. For the complete graph G on n vertices this gives the usual configuration space of M.

If A is the cohomology algebra of M, the complex $C_G(A)$ is a page of the Bendersky–Gitler spectral sequence that computes the homology of M^G, cf. [BG]. In [BS] higher differentials of this spectral sequence were found by taking A to be a commutative dg algebra that computes the cohomology of M. In characteristic zero one can take A to be the de Rham algebra or Sullivan's cochains. However, for $k = \mathbb{Z}, \mathbb{F}_p$ such a choice may not be possible for a general M. This motivates our attempt to define chromatic graph homology for noncommutative algebras, such as the singular cochain complex of M.

However, if A is just associative with no further structure, then $d_1^2 = 0$ fails already for the connected graph with two edges: one needs at least the identity $abc = acb$. Such algebras do present some interest as the corresponding quadratic operad Perm is the Koszul dual of the operad PreLie, cf. [LV]. But in the case of k-valued cochains we have an associative dg algebra A which satisfies "commutativity up to homotopy".

In more concrete terms, such A is an algebra over the surjection operad $\mathcal{X}$, cf. [MS]. We use only a part of this rich structure, the operations coming from the second or the third piece of a filtration $F_j\mathcal{X}$ on $\mathcal{X}$, cf. *loc. cit.* The suboperad $F_2\mathcal{X}$, isomorphic to the operad of (associative, rather than A_∞) braces Br, also acts on a Hochschild cochain complex of an associative dg-algebra. Our main result extends the construction of graph homology to the case when G is a planar planted tree (we recall the definitions in the next section) and shows that a different choice of the root edge leads to an isomorphic complex, although the isomorphism only preserves the total grading, not the above bigrading.

Theorem 0.1. *Let A be a flat k-algebra over the brace operad* Br *and G a planar planted tree. There exists a sequence of operators d_i, $i \geq 0$ on the bigraded vector space $C_G(A)$, such that:*

1. *d_0 is the differential induced by the differential δ on A and $d_1 = d$ is the map induced by contraction of edges and the multiplication of A, according to the standard orientations on edges of a rooted tree.*
2. *Each d_i has bidegree $(1-i, i)$ and for $i > 0$ it is represented by a sum of operations which contract subtrees in G with i edges.*
3. *The total operator $d = d_0 + d_1 + d_2 + \dots$ has square zero.*
4. *Two complexes obtained from different choices of a root edge in the same planar graph, are isomorphic via an isomorphism*

$$\Phi = 1 + \Phi_1 + \Phi_2 + \cdots$$

where each Φ_i has bidegree $(-i, i)$, thus preserving the total grading but not the bigrading.

We expect that $C_G(A)$ can be defined for a general graph G with a fixed cyclic order of edges at every vertex. One possible strategy is to use maximal (spanning) subtrees of G as in [CK], but at the moment we cannot resolve the issues related to the choice of a root edge on a planar tree. The resulting complex would compute an appropriate truncated version of factorization homology, [AFT], of the fat graph (or ribbon graph) associated to G.

One expected application is to the case when G comes from a knot diagram on a surface, although we may have to assume that A is an E_3 algebra to ensure good behaviour under Reidemeister moves. Another case of interest is, of course, $A = C^*(M, k)$ when it should provide a complex computing the homology $H_*(M^G, k)$ of the graph configuration space M^G (this is where the truncated version is needed rather than full factorization homology). An appropriate extension to the case of a"homotopy Frobenius" algebra would provide homology groups similar to Khovanov homology.

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1. Preliminaries on trees and braces

Let G be a planar tree, i.e., a finite connected contractible graph with a cyclic order on edges incident to any vertex. We also assume that one of the vertices is chosen as a root. This induces an orientation on edges, pointing towards the root. Therefore every nonroot vertex has a number of incoming edges (possibly zero) and one outgoing edge, and we obtain a linear ordering on the incoming edges. For the root vertex we would also like to choose a linear order on incoming edges which is compatible with the cyclic order induced by the planar embedding. Graphically this is denoted by adding a "half edge" or a "root edge" at the root vertex which does not connect it with any of the vertices in $V(G)$. Therefore, G acquires a structure of a planar *planted* tree. We are going to define a complex using this structure but it will turn out later that the complex is independent, up to isomorphism, on the choice of a root vertex and a root edge.

In the example below we choose the vertex marked by 1 as a root, and this gives a linear order on the edges coming into 3 (the edge from 4 is to the left of the edge from 5). However, for the vertex 1 itself similar order appears only from a choice of the root edge, as shown.

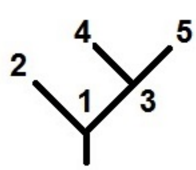


FIGURE 1.

One can define two partial orders on the set of vertices in a planar planted tree. The *vertical order* is induced by the orientation of edges: we write

$j <_v i$ if the oriented path from i to the root passes through j. In particular, the root vertex is minimal with respect to the vertical order. The *horizontal order* is defined on pairs of distinct vertices j, i which are not related by the vertical order. Then the oriented paths from j and i to the root first meet at a third vertex k (distinct from either of j, i, by assumption) and we write $j <_h i$ if the path from j enters k to the left of the path from i.

By definition, two distict vertices are either related by the vertical order or related by the horizontal order.

There is also another, total, ordering on the vertices on G which is a common refinement of the two partial orders. It can be obtained by embedding G into $\mathbb{R}^2$ (in a way compatible with the planted planar structure) and taking its ϵ-neigbourhood with respect to the standard Euclidean metric. Then we walk around the boundary clockwise, starting at the root and writing down the vertices as we first encounter them in this clockwise trip. This total order allows us to given a canonical labeling of vertices in G by the elements of $\{1, \ldots, n\}$, where $n = |V(G)|$. For instance, for the graph on Figure 1 this gives the labeling as shown.

We also need another kind of planar planted trees, related to the operad of (associative) braces Br. One way to describe the dg operad Br is to say that for each n, the complex $\mathrm{Br}(n)$ has in homological degree $-k$ the vector space $\mathrm{Br}(n)_k$ spanned by certain sequences $u = (u(1), \ldots, u(n+k))$ with $u(i) \in \{1, \ldots, n\}$. Thus we can view such a sequence as listing the values of a map $u : \{1, \ldots, n+k\} \to \{1, \ldots, n\}$. The following conditions are imposed on the sequence:

- The induced map u must be surjective, i.e., all elements of $\{1, \ldots, n\}$ appear in the sequence of values,
- nondegenerate in the sense that $u(i) \neq u(i+1)$ for any i,
- and "have complexity ≤ 2" in the sense that they do not contain a subsequence of the form $ijij$ for any pair of distinct values i, j.

Such sequences also admit a description in terms of *brace trees*. These are planar planted trees with vertices colored either black or white. Further, one chooses a bijection between the set of white vertices and $\{1, \ldots, n\}$ and requires that no two black vertices are connected by an edge and that each black vertex has at least two incoming edges (An alternative description, which we do not use here, inserts a black vertex in the middle of each edge in our description. Then black vertices with one incoming edges are allowed and edges can only connect vertices of different colors, i.e., the graph becomes bipartite).

Given such a brace tree, we can form a sequence of integers by starting at the root vertex and going clockwise on boundary of the ϵ-neighborhood as before. This time we ignore the black vertices completely and read the labels off the white vertices any time we approach them, not just the first time. Thus, $i \in \{1, \ldots, n\}$ will appear $l+1$ times if the corresponding vertex of the graph has l incoming edges. This induces a bijection between the set of nondegenerate sequences of complexity ≤ 2 and brace trees. For instance, the brace tree below corresponds to the sequence (123242).

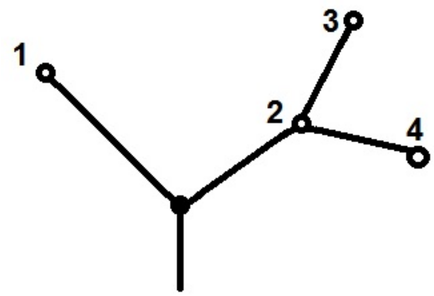


FIGURE 2.

The original brace operations $\{x_1; x_2, \ldots, x_n\}$, see, e.g., [VG], correspond either to the corolla on vertex 1 with edges coming from $2, \ldots, n$ in the natural order or, equivalently, to the sequence $(12131\ldots 1n1)$.

We refer the reader to [MS] for the definition of the differential and the operadic composition of $\mathrm{Br}(n)$. We only note here that the differential is given by erasing values in a sequence, with a certain sign rule, and then omitting those resulting sequences which are either nonsurjective or degenerate.

2. Brace operations from subtree contractions.

The differential of the original chromatic homology complex (with commutative A) was built from edge contractions and multiplication of A. Similarly, the differential of the complex we are about to define will contract subtrees in G and use linear combinations of brace operations that we are about to define.

Let S be a planar planted tree with k vertices (and thus $k-1$ edges). Define an element $m_S \in \mathrm{Br}(k)_{k-2}$ by induction on k.

For $k = 2$ we have a single edge oriented from 1 to 2, and we send this to the product operation, corresponding to the sequence $(12) \in \mathrm{Br}(2)_0$.

Assuming the operations m_S for trees with $< k$ vertices are linear combinations of brace tree operations with coefficients ± 1, consider S with k vertices. As explained in the previous section, these have a canonical labeling by $\{1, \ldots, k\}$ and we can view S as a result of grafting an edge $k \to l$ on the tree S' tree with vertices $\{1, \ldots, k-1\}$. By inductive assumption,

$$m_{S'} = \sum_R (-1)^R R$$

where the sum is over brace trees R in $\mathrm{Br}(k-1)_{k-3}$ and $(-1)^R$ is the sign to be discussed below. Set

$$m_S = \sum_R (-1)^R \sum_{j>j_R} \pm R_{(j,k)}. \tag{1}$$

In the summation above, j_R is the index marking the last occurence of l, the vertex that receives the edge coming out of k. In other words, for the sequence u_R of the tree R we have $u_R(j_R) = l$ and $u_R(j) \neq l$ if $j > j_R$. For such an index j the tree $R_{(j,k)}$ corresponds to a sequence in which the single value $x = u_R(j)$ is replaced by a subsequence xkx. Geometrically this amounts to grafting a edge from k to j in such a way that $l <_h k$ in the horizonal order of the resulting brace tree. Observe that the result is again a signed sum of distinct brace trees, thus allowing a further inductive definition. The sign in the second summation is obtained by the rule similar to the signs in the brace differential, cf. [MS]: we first perform substitutions $2 \mapsto 2k2$ for all occurences of 2 after the last occurence of l, starting with the plus sign and alternating as we move from left to right. Then we perform substitions on 3, 4, and so one. In each case the first occurence of p has the same sign as the last occurence of $(p-1)$ and for all other occurences of p the signs alternate as we move from left to right. The value 1 is excluded since one can show by induction that for all R it only occurs once, as the first value in a sequence, and hence never appears after the last occurence of l.

The simplest interesting case, shown below, is the corrolla with 4 vertices and 3 edges (on the left). For the tree S' on vertices 1, 2 and 3, the element $m_{S'}$ is a single brace tree corresponding to $u_R = (1232)$. We have two substitutions $2 \mapsto 242$ and one substitution $3 \mapsto 343$, corresponding to the brace trees on the right.

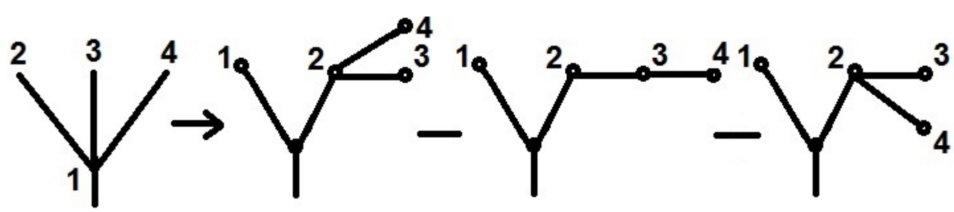


Figure 3.

The signs are explained by fact that first 4 is grafted on 2 with alternating signs, then 4 is grafted on 3, and the first sign of grafing on 3 matches the last sign of grafting on 2. We note that the new edge is never grafted on the black vertex since that would give a brace tree of wrong degree.

Remark. It is also possible to give a nonrecursive formula for m_T. It is a signed sum over all brace trees S, such that:

- S has a single black vertex which is its root.
- The left edge coming into the black vertex has the other end labeled by 1, and the vertex 1 is a leaf (has no further incoming edges).
- The right edge coming into the black vertex can be viewed as a root edge of a subtree which only has white vertices and label 2 at the root.
- If $x <_v y$ in T then $x <_h y$ in S.
- If $x <_h y$ in T then it is *not true* that $y <_v x$ in S (which leaves $x <_h y$ or $x <_v y$ or $y <_h x$, all of which occur in Figure 3 for $x = 3$ and $y = 4$).

Now let S be a planar planted tree with n vertices and $T \subset S$ its subtree with k vertices. Note that T has a canonical choice of a root, the minimal vertex in the canonical total ordering of S, and that the total ordering of T is induced by that of S. In particular, T is a planted planar tree as well. The contraction S/T is a tree with $n - k + 1$ vertices and a marked vertex i_T which is the image of T. The following is a key computational lemma in our paper.

Lemma 2.1. *The following equality holds in* $\mathrm{Br}(n)_{n-3}$:

$$d(m_S) = \sum_T m_{S/T} \circ_{i_T} m_T, \tag{2}$$

where the sum runs over all subtrees T in S.

Proof. Denote by $(m_{S'} \leftarrow k)$ the right hand side of (1) and proceed by induction on k. It follows from the definitions that the difference

$$d(m_{S'} \leftarrow k) - (d(m_{S'}) \leftarrow k)$$

is a sum of terms of two types: in the first type a single element $x = u_R(j)$ is replaced by kx and in the second type it is replaced by xk. Looking at the signs we see that the term of the second type for index j cancels out with the term of the first type for index $(j+1)$ (see [GLT] for a very similar definition and exactly the same type of cancellation). Therefore the only two terms that survive are the ones where k is inserted either at the very end, or right after the last occurence of $l = u_R(j_R)$.

In the terms of the right hand side in (2), the first of the surviving terms corresponds to the subtree T on the vertices $\{1, \ldots, k-1\}$ and the second to the subtree on the two vertices $\{k, l\}$. In all other terms on the right hand side of (2), the vertex k is either grafted on the nontrivial subgraph T, or on the contracted graph S/T. The sum of these terms is exactly $(d(m_{S'}) \leftarrow k)$, by inductive assumption. □

3. Chromatic homology complex for a brace algebra A.

Fix an algebra A over the brace operad Br. In particular, A is still an associative dg algebra. The total space of the chromatic homology complex is still defined as

$$C_G(A) = \bigoplus_{S \subset E(G)} A^{\otimes(G/S)} \cdot e_S.$$

Note that in our situation G is a planar planted tree, hence the set of edges has a canonical ordering as explained before and the wedge product e_S of all odd generators e_α over $\alpha \in S$ is well defined (since S has induced ordering). We give $C_G(A)$ the bigrading in which the first component is induced by the grading of A, and the second component by the grading of the exterior algebra on e_α. Hence, each e_α has bidegree $(0, 1)$. We would like to define the differential

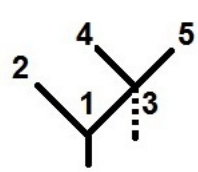
$$d = d_0 + d_1 + d_2 + \dots$$

where each d_i has bidegree $(1 - i, i)$. The operator d_0 is just the natural extension of the differential δ on A to its tensor products. The operator d_i for $i \geq 1$, is given by the sum $\sum m_T \cdot e_T$ where the sum is over (connected) subtrees T with i edges. Each terms $m_T \cdot e_T$ acts as follows: e_T acts by the left wedge product and m_T sends $A^{\otimes(G/S)}$ to $A^{\otimes(G/S \cup T)}$ if the edges of S and T are disjoint, and to zero otherwise. Observe that, since G/S is also a tree, in the first case T projects isomorphically onto its image in G/S. Hence we can apply the signed sum of brace operations m_T from the previous section, to map $A^{\otimes(G/S)}$ to $A^{\otimes(G/S \cup T)}$. This also involves the Koszul sign rules: first the arguments of m_T are brought to the first $(i+1)$ positions by a permutation, then m_T is applied, then its output is returned to the appropriate position, marked by the vertex of $G/S \cup T$ to which T was contracted. The properties (1)–(2) of our main Theorem 0.1 hold by construction and property (3), asserting that $d^2 = 0$, is a reformulation of Lemma 2.1.

When A is a commutative dg algebra this reduces to the standard chromatic homology complex of [HGR].

4. Dependence on the choice of the root edge.

He we prove the last part of the main result that tells what happens when a root vertex/rood edge of G changes. It suffices to consider the case when the old root a and the new root b are connected by an edge β of G, and the root edges are chosen in such a way that β is maximal in the linear order of edges coming into a, while the same β is minimal in the linear order of edges coming into b. In other words, we are moving the root edge counterclockwise by one edge of G:



In the example above $a = 1$ and $b = 3$ (for comparison purposes we use the labeling induced by the root at a), and the new root edge is marked by a dotted line.

It is clear that composition of such elementary root edge moves allows to compare two arbitrary root edge choices.

To simplify notation, let (C^a, d^a), (C^b, d^b) be the chromatic homology complexes induced by the choices of root at a and b, respectively.

Proposition 4.1. *There exists an isomorphism of complexes*

$$\Phi : (C^a, d^a) \to (C^b, d^b)$$

such that

$$\Phi = 1 + \Phi_1 + \ldots + \Phi_{n-1},$$

the operator Φ_i of bidegree $(-i, i)$ is given, similarly to d, by the sum $\sum_T h_T \cdot e_T$ over subtrees T with i edges, and each h_T is a linear combination of brace operations in $\mathrm{Br}(i+1)_i$ (corresponding to trees with white vertices only).

Proof. The definition of h_T is very similar to that of m_T: we start with a single edge tree on the edge β connecting $a = 1$ and b.

For this initial tree we set m_β to be the brace tree with a single edge and a as a root. This corresponds to the brace operation given by the sequence (aba). Next, we add the other vertices of the graph T, following their canonical order, and use the same substitution rules $x \mapsto xkx$ (and the same sign rules) as for m_T.

We need to show that $\Phi d^a = d^b \Phi$ on each term $A^{\otimes(G/S)} e_S$. Since $h_{T_1} \circ m_{T_2}$ on $A^{\otimes(G/S)}$ is only nonzero when the sets of edges in T_1, T_2, S are pairwise disjoint, it suffices to look at the case when S is empty and $T_1 = G/T_2$. The same consideration applies when looking at the terms of the type $m_{T_1} \circ h_{T_2}$. Since in each degree k we need to show that

$$\sum_i d_i^b \circ \Phi_{k-i} = \sum_j \Phi_j \circ d_{k-j}^a$$

and in degree 0 we have $d_0^a = d_0^b$ (both are just the Leibniz rule extension of the differential on A), the required identity boils down to the equation in the brace operad for any G with k vertices

$$d(h_G) = \sum_{T \subset G} (h_{G/T} \circ_{i_T} m_T - m_{G/T} \circ_{i_T} h_T)$$

where the sum is over all subtrees T with positive number of edges. We note that h_T is zero if T does not contain α_0 and $h_{G/T}$ is zero if T does contain α_0 (then on G/T there is no change of the root edge and root vertex). Hence of the two terms in the parenthesis only at most one will be nonzero, and the identity becomes similar to that of Lemma 2.1. The rest of the proof repeats the one given in that lemma and we omit it. □

5. Further questions and remarks

We outline below our motivation for the main result of the paper and also indicate some related open questions.

(1) It would be desirable to understand the automorphisms Φ induced by a change of planted root structure, from the point of view of higher operads of Batanin or in the equivalent language of ∞-operads. One easy observation is in the case of a tree with two vertices, labeled by 1 and 2. Then $\Phi = 1 + \Phi_1$ with $\Phi_1 = (121)$, when we go from the complex built from 1 as a root, to the complex built from 2 as a root. However, if we go back, then the same recipe tells us to use the sequence (212) instead of (121). Thus, the two isomorphisms are not mutually inverse, although when A is an algebra over the E_3 operad $F_3\mathcal{X}$, they are related by an operation (1212).

(2) Suppose we want change the planar structure on a graph and A is an algebra over $F_3\mathcal{X}$, the nondegenerate surjections of complexity ≤ 3. This corresponds to sequences which are allowed to contain subsequences $ijij$ with distinct i, j but not subsequences $ijiji$. Note that these no longer correspond to any brace trees. In such a setting, we have a strong computational evidence (about 40 examples so far) that one can define similar isomorphisms $\Psi = 1 + \Psi_2 + \Psi_3 + \ldots$ relating the complexes of a rooted tree with different planar structures. It suffices to consider the case when one exchanges the order of two neighboring incoming edges of a vertex and keeps the planar structure elsewhere. Furthermore since we can change the root vertex and the root edge, we can assume that the two edges being swapped are the two leftmost incoming edges of the root vertex. Similarly to the case of changing the planted tree structure, swapping the edges twice does not give an identity isomorphism, but something that is conjecturally homotopic to identity if A has a structure of an algebra over the operad $F_4\mathcal{X}$. This may be another indication of relevance of homotopy operads.

(3) A possible way to extend the construction to arbitrary graphs is to consider a graph with a total ordering on vertices and to orient the edges so they point from the larger vertex to the smaller one. In this case it is possible that contraction of an edge (or a subtree) will reverse orientation on the remaining edges, as one can see in the simple case of a graph with vertices 1, 2, 3 and the two edges contecting 3 with 1 and 3 with 2. An easy computation shows that contraction of edges only will lead to a square zero operator precisely when the graded algebra associative A (without any further structure) satisfies

$$abc = (-1)^{\deg(b)\deg(c)} acb.$$

This condition is certainly observed for commutative A, but it is slightly weaker than commutativity. It corresponds to algebras over a quadratic operad Perm (which in degree n has the standard n-dimensional permutation module over Σ_n). This operad is Koszul dual to the operad PreLie, cf. [LV]. By the above comment, we have the obvious map Perm $\to$ Com to the operad of commutative algebras, since every commutative algebra is also a Perm algebra.

By the general theory, Perm has a minimal resolution

$$\Omega(s\mathrm{PreLie}^c) \to \mathrm{Perm}$$

by the cobar construction on the suspension of the cooperad dual to PreLie. Since the full surjection operad $\mathcal{X}$ resolves Com in characteristic zero, we have a covering morphism $\Omega(s\mathrm{PreLie}^c) \to \mathcal{X}$, which is uniquely determined by a twisting cochain $s\mathrm{PreLie}^c \to \mathcal{X}$, see [LV]. Once we remember that the elements of PreLie^c are represented by nonplanar rooted trees, this becomes very similar to the correspondence $T \mapsto m_T$ proved earlier. Note that Lemma 2.1 is more or less the defining identity of a twisting cochain, but in that setting we choose a planar planted structure on a nonplanar rooted tree. Such as choice allows to select a cochain with values in $\mathrm{Br} \subset \mathcal{X}$.

On the other hand, it is possible that a twisted cochain

$$s\mathrm{PreLie}^c \to \mathcal{X}$$

only exists over $\mathbb{Q}$ (as a result of averaging over different additional structures on a nonplanar rooted tree) and/or takes values in surjections of complexity ≤ 3 rather than Br.

(4) It appears that the nerve of the poset operad on complete graphs is also relevant to our construction, but we were not able to make this connection explicit.
(5) When A is the associative algebra of singular cochains on a topological space M, the complex $C_G(A)$ and its differential can be obtained by a standard homological perturbation theory argument from the Eilenberg–Zilber contraction of the standard simplicial object associated with the graph configuration space $M^G \subset M^{\times n}$. See [BS] for definitions and [BZ] for the precise formulation of the result.
(6) In the commutative case, chromatic graph homology of a general graph can — in some sense — be reduced to the case of a tree by considering spanning trees (i.e., maximal tree subgraphs in a graph). See, for example, the construction of [CK] in the case of an alternating link. At the moment we don't know how do generalize this approach due to the need to select the planted root structure in our approach.

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