

# On approximate Connes-amenability of enveloping dual Banach algebras

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ABSTRACT. For a Banach algebra  $\mathcal{A}$ , we introduce various approximate virtual diagonals such as approximate WAP-virtual diagonal and approximate virtual diagonal.

For the enveloping dual Banach algebra  $F(\mathcal{A})$  of  $\mathcal{A}$ , we show that  $F(\mathcal{A})$  is approximately Connes-amenable if and only if  $\mathcal{A}$  has an approximate WAP-virtual diagonal.

Further, for a discrete group  $G$ , we show that if the group algebra  $\ell^1(G)$  has an approximate WAP-virtual diagonal, then it has an approximate virtual diagonal.

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## 1. Introduction and preliminaries

The notion of approximate Connes-amenability for a dual Banach algebra was first introduced by Esslamzadeh *et al.* (2012). A dual Banach algebra  $\mathcal{A}$  is called approximately Connes-amenable if for every normal dual  $\mathcal{A}$ -bimodule  $X$ , every  $w^*$ -continuous derivation  $D : \mathcal{A} \rightarrow X$  is approximately inner, that is, there exists a net  $(x_\alpha) \subseteq X$  such that  $D(b) = \lim_\alpha b \cdot x_\alpha - x_\alpha \cdot b$  ( $b \in \mathcal{A}$ ).

Let  $\mathcal{A}$  be a Banach algebra and let  $X$  be a Banach  $\mathcal{A}$ -bimodule. An element  $x \in X$  is called weakly almost periodic if the module maps

$$\mathcal{A} \rightarrow X; \quad a \mapsto a \cdot x \quad \text{and} \quad a \mapsto x \cdot a$$

are weakly compact. The set of all weakly almost periodic elements of  $X$  is denoted by  $\text{WAP}(X)$ , which is a norm-closed subspace of  $X$ .

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For a Banach algebra  $\mathcal{A}$ , one of the most important subspaces of  $\mathcal{A}^*$  is  $\text{WAP}(\mathcal{A}^*)$ , which is left introverted in the sense of [L197, §1]. Runde (2004) observed that, the space  $F(\mathcal{A}) = \text{WAP}(\mathcal{A}^*)^*$  is a dual Banach algebra with the first Arens product inherited from  $\mathcal{A}^{**}$ . He also showed that  $F(\mathcal{A})$  is a *canonical* dual Banach algebra associated to  $\mathcal{A}$  (see [L197] or [Run04, Theorem 4.10] for more details).

Choi *et al.* (2014) called  $F(\mathcal{A})$  the enveloping dual Banach algebra associated to  $\mathcal{A}$  and they studied Connes-amenability of  $F(\mathcal{A})$ . Indeed, they introduced the notion of WAP-virtual diagonal for a Banach algebra  $\mathcal{A}$  and they showed that for a given Banach algebra  $\mathcal{A}$ , the dual Banach algebra  $F(\mathcal{A})$  is Connes-amenable if and only if  $\mathcal{A}$  admits a WAP-virtual diagonal. They also showed that for a group algebra  $L^1(G)$ , the existence of a virtual diagonal is equivalent to the existence of a WAP-virtual diagonal. As a consequence  $L^1(G)$  is amenable if and only if  $F(L^1(G))$  is Connes-amenable, a fact that previously was shown by Runde (2004).

Motivated by these results, we investigate an *approximate* analogue of WAP-virtual diagonal related to approximate Connes-amenability of  $F(\mathcal{A})$ . Indeed, we introduce a notion of an approximate WAP-virtual diagonal for  $\mathcal{A}$  and we show that  $F(\mathcal{A})$  is approximately Connes-amenable if and only if  $\mathcal{A}$  has an approximate WAP-virtual diagonal. We also introduce various notions of approximate-type virtual diagonals for a Banach algebra -such as approximate virtual diagonal- and we show that for a discrete group  $G$ , if  $\ell^1(G)$  has an approximate WAP-virtual diagonal, then it has an approximate virtual diagonal.

Given a Banach algebra  $\mathcal{A}$ , its unitization is denoted by  $\mathcal{A}^\#$ . For a Banach  $\mathcal{A}$ -bimodule  $X$ , the topological dual space  $X^*$  of  $X$  becomes a Banach  $\mathcal{A}$ -bimodule via the following actions

$$(1.1) \quad \begin{aligned} \langle x, a \cdot \varphi \rangle &= \langle x \cdot a, \varphi \rangle, \\ \langle x, \varphi \cdot a \rangle &= \langle a \cdot x, \varphi \rangle \quad (a \in \mathcal{A}, x \in X, \varphi \in X^*). \end{aligned}$$

**Definition 1.1.** A Banach algebra  $\mathcal{A}$  is called dual, if it is a dual Banach space with a predual  $\mathcal{A}_*$  such that the multiplication in  $\mathcal{A}$  is separately  $\sigma(\mathcal{A}, \mathcal{A}_*)$ -continuous [Run01, Definition 1.1]. Equivalently, a Banach algebra  $\mathcal{A}$  is dual if it has a (not necessarily unique) predual  $\mathcal{A}_*$  which is a closed submodule of  $\mathcal{A}^*$  [Run02, Exercise 4.4.1].

Let  $\mathcal{A}$  be a dual Banach algebra and let a dual Banach space  $X$  be an  $\mathcal{A}$ -bimodule. An element  $x \in X$  is called normal, if the module actions  $a \mapsto a \cdot x$  and  $a \mapsto x \cdot a$  are  $w^*$ - $w^*$ -continuous. The set of all normal elements in  $X$  is denoted by  $X_\sigma$ . We say that  $X$  is normal if  $X = X_\sigma$ .

Recall that for a Banach algebra  $\mathcal{A}$  and a Banach  $\mathcal{A}$ -bimodule  $X$ , a bounded linear map  $D : \mathcal{A} \rightarrow X$  is called a bounded derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for every  $a, b \in \mathcal{A}$ . A derivation  $D : \mathcal{A} \rightarrow X$  is called inner if there exists an element  $x \in X$  such that for every  $a \in \mathcal{A}$ ,

$$D(a) = \text{ad}_x(a) = a \cdot x - x \cdot a.$$

A dual Banach algebra  $\mathcal{A}$  is called Connes-amenable if for every normal dual  $\mathcal{A}$ -bimodule  $X$ , every  $w^*$ - $w^*$ -continuous derivation  $D : \mathcal{A} \rightarrow X$  is inner [Run04, Definition 1.5].

Following [CSS15], for a given Banach algebra  $\mathcal{A}$  and a Banach  $\mathcal{A}$ -bimodule  $X$ , the  $\mathcal{A}$ -bimodule  $\text{WAP}(X^*)$  is denoted by  $F_{\mathcal{A}}(X)_*$  and its dual space  $\text{WAP}(X^*)^*$  is denoted by  $F_{\mathcal{A}}(X)$ . Choi *et al.* [CSS15, Theorem 4.3] showed that if  $X$  is a Banach  $\mathcal{A}$ -bimodule, then  $F_{\mathcal{A}}(X)$  is a normal dual  $F(\mathcal{A})$ -bimodule. Since  $\text{WAP}(X^*)$  is a closed subspace of  $X^*$ , we have a quotient map  $q : X^{**} \twoheadrightarrow F_{\mathcal{A}}(X)$ . Composing the canonical inclusion map  $X \hookrightarrow X^{**}$  with  $q$ , we obtain a continuous  $\mathcal{A}$ -bimodule map  $\eta_X : X \rightarrow F_{\mathcal{A}}(X)$  which has a  $w^*$ -dense range. In the special case where  $X = \mathcal{A}$ , we usually omit the subscript and simply use the notation  $F(\mathcal{A})$  and the map  $\eta_{\mathcal{A}} : \mathcal{A} \rightarrow F(\mathcal{A})$  is an algebra homomorphism.

It is well-known that for a given Banach algebra  $\mathcal{A}$ , the projective tensor product  $\mathcal{A} \hat{\otimes} \mathcal{A}$  is a Banach  $\mathcal{A}$ -bimodule through

$$a \cdot (b \otimes c) := ab \otimes c, \quad (b \otimes c) \cdot a := b \otimes ca, \quad (a, b, c \in \mathcal{A})$$

and the map  $\Delta : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$  defined on elementary tensors by  $\Delta(a \otimes b) = ab$  and extended by linearity and continuity, is an  $\mathcal{A}$ -bimodule map with respect to the module structure of  $\mathcal{A} \hat{\otimes} \mathcal{A}$ . Let  $\Delta_{\text{WAP}} : F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A}) \rightarrow F(\mathcal{A})$  be the  $w^*$ - $w^*$ -continuous  $\mathcal{A}$ -bimodule map induced by  $\Delta : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$  (see [CSS15, Corollary 5.2] for more details).

**Definition 1.2.** [CSS15, Definition 6.4] An element  $M \in F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  is called a WAP-virtual diagonal for  $\mathcal{A}$  if

$$a \cdot M = M \cdot a \quad \text{and} \quad \Delta_{\text{WAP}}(M) \cdot a = \eta_{\mathcal{A}}(a) \quad (a \in \mathcal{A}).$$

Recall that an  $\mathcal{A}$ -bimodule  $X$  is called neo-unital if every  $x \in X$  can be written as  $a \cdot y \cdot b$  for some  $a, b \in \mathcal{A}$  and  $y \in X$ .

**Remark 1.3.** Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate identity and let  $X$  be a Banach  $\mathcal{A}$ -bimodule. Then by [Joh72, Proposition 1.8], the subspace  $X_{\text{ess}} := \text{lin}\{a \cdot x \cdot b : a, b \in \mathcal{A}, x \in X\}$  is a neo-unital closed sub- $\mathcal{A}$ -bimodule of  $X$ . Moreover,  $X_{\text{ess}}^{\perp}$  is complemented in  $X^*$ .

## 2. Approximate Connes-amenability of $F(\mathcal{A})$

In this section we find some conditions under which  $F(\mathcal{A})$  is approximately Connes-amenable. For an arbitrary dual Banach algebra  $\mathcal{A}$ , approximate Connes-amenability of  $\mathcal{A}$  is equivalent to approximate Connes-amenability of  $\mathcal{A}^{\sharp}$  [ESM12, Proposition 2.3], so without loss of generality throughout of this section we may suppose that  $F(\mathcal{A})$  has an identity element  $e$ .

**Remark 2.1.** Let  $\mathcal{A}$  be a Banach algebra with an identity element  $e$ . Let  $X$  be a normal dual  $\mathcal{A}$ -bimodule and let  $D : \mathcal{A} \rightarrow X$  be a bounded derivation. By [GL04, Lemma 2.3], there exists a bounded derivation  $D_1 : \mathcal{A} \rightarrow e \cdot X \cdot e$  such that  $D = D_1 + \text{ad}_x$  for some  $x \in X$ . Furthermore,  $D$  is inner derivation if and only if  $D_1$  is inner.

**Lemma 2.2.** *Let  $\mathcal{A}$  be a Banach algebra. Then  $F(\mathcal{A})$  is approximately Connes-amenable if and only if every bounded derivation from  $\mathcal{A}$  into a unit-linked, normal dual  $F(\mathcal{A})$ -bimodule is approximately inner.*

**Proof.** Suppose that  $F(\mathcal{A})$  is approximately Connes-amenable. Let  $N$  be a unit-linked, normal dual  $F(\mathcal{A})$ -bimodule and let  $D : \mathcal{A} \rightarrow N$  be a bounded derivation. We show that  $D$  is approximately inner. By [ESM12, Theorem 4.4], there is a  $w^*$ - $w^*$ -continuous derivation  $\tilde{D} : F(\mathcal{A}) \rightarrow F_{\mathcal{A}}(N)$  such that  $\tilde{D}\eta_{\mathcal{A}} = \eta_N D$ , that is, the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{D} & N \\ \eta_{\mathcal{A}} \downarrow & & \downarrow \eta_N \\ F(\mathcal{A}) & \xrightarrow{\tilde{D}} & F_{\mathcal{A}}(N). \end{array}$$

Since  $F_{\mathcal{A}}(N)$  is a normal dual  $F(\mathcal{A})$ -bimodule and  $F(\mathcal{A})$  is approximately Connes-amenable, the derivation  $\tilde{D}$  is approximately inner. Since  $\eta_{\mathcal{A}}$  is an algebra homomorphism, the derivation  $\tilde{D}\eta_{\mathcal{A}}$  is also approximately inner. Hence the derivation  $\eta_N D$  is approximately inner, that is, there is a net  $(\xi_i) \subseteq F_{\mathcal{A}}(N)$  such that

$$\eta_N D(a) = \lim_i a \cdot \xi_i - \xi_i \cdot a \quad (a \in \mathcal{A}).$$

By [ESM12, Corollary 5.3] there is a  $w^*$ - $w^*$ -continuous  $\mathcal{A}$ -bimodule map  $\epsilon_N : F_{\mathcal{A}}(N) \rightarrow N$  such that  $\epsilon_N \eta_N = \text{id}_N$ . Now by setting  $x_i = \epsilon_N(\xi_i) \in N$  for each  $i$  we have

$$D(a) = \epsilon_N \eta_N D(a) = \epsilon_N(\lim_i a \cdot \xi_i - \xi_i \cdot a) = \lim_i a \cdot x_i - x_i \cdot a,$$

so  $D : \mathcal{A} \rightarrow N$  is approximately inner.

Conversely, suppose that every bounded derivation from  $\mathcal{A}$  into a unit-linked, normal dual  $F(\mathcal{A})$ -bimodule is approximately inner. Let  $N$  be a normal dual  $F(\mathcal{A})$ -bimodule and let  $D : F(\mathcal{A}) \rightarrow N$  be a  $w^*$ - $w^*$ -continuous derivation. By Remark 2.1 without loss of generality we may assume that  $N$  is a unit-linked, normal dual  $F(\mathcal{A})$ -bimodule. We shall show that  $D$  is approximately inner. Since  $\eta_{\mathcal{A}}$  is an algebra homomorphism, the map  $D' := D\eta_{\mathcal{A}} : \mathcal{A} \rightarrow N$  is a bounded derivation, so by assumption  $D'$  is approximately inner. Since  $N$  is normal,  $D'$  extends to a  $w^*$ - $w^*$ -continuous approximately inner derivation  $D'' : F(\mathcal{A}) \rightarrow N$ . The derivations  $D$  and  $D''$  are  $w^*$ - $w^*$ -continuous and they agree on the  $w^*$ -dense subset  $\eta_{\mathcal{A}}(\mathcal{A}) \subseteq F(\mathcal{A})$ ,

so they agree on  $F(\mathcal{A})$ . Hence  $D$  is approximately inner, that is,  $F(\mathcal{A})$  is approximately Connes-amenable.  $\square$

**Remark 2.3.** Since  $F(\mathcal{A})$  has an identity element  $e$ , by [CSS15, Theorem 6.8] there exists an element  $s \in F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  such that  $\Delta_{\text{WAP}}(s) = e$ . We use this fact in the following lemma several times.

**Lemma 2.4.** *Let  $\mathcal{A}$  be a Banach algebra. Then the following are equivalent:*

- (i)  $F(\mathcal{A})$  is approximately Connes-amenable.
- (ii) There exists a net  $(M_{\alpha})_{\alpha} \subseteq F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  such that for every  $a \in \mathcal{A}$ ,  $a \cdot M_{\alpha} - M_{\alpha} \cdot a \rightarrow 0$  and  $\Delta_{\text{WAP}}(M_{\alpha}) = e$  for all  $\alpha$ .

**Proof.** (i) $\Rightarrow$ (ii) Fix  $s \in F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  as mentioned in Remark 2.3. Note that  $\ker \Delta_{\text{WAP}}$  is a normal dual  $F(\mathcal{A})$ -bimodule. Define a  $w^* - w^*$ -continuous derivation  $D : F(\mathcal{A}) \rightarrow \ker \Delta_{\text{WAP}}$  by  $D(\mathbf{a}) = s \cdot \mathbf{a} - \mathbf{a} \cdot s$  ( $\mathbf{a} \in F(\mathcal{A})$ ). Since  $F(\mathcal{A})$  is approximately Connes-amenable, the derivation  $D$  is approximately inner. Thus there is a net  $(N_{\alpha}) \subseteq \ker \Delta_{\text{WAP}}$  such that for every  $a \in \mathcal{A}$

$$(2.1) \quad s \cdot a - a \cdot s = \lim_{\alpha} (a \cdot N_{\alpha} - N_{\alpha} \cdot a).$$

If we set  $M_{\alpha} = N_{\alpha} + s$  for every  $\alpha$ , then we have  $\Delta_{\text{WAP}}(M_{\alpha}) = \Delta_{\text{WAP}}(s) = e$  and for every  $a \in \mathcal{A}$  using (2.1) we have

$$a \cdot M_{\alpha} - M_{\alpha} \cdot a \rightarrow 0,$$

as required.

(ii) $\Rightarrow$ (i) The hypothesis in (ii) ensures that each  $M_{\alpha} - s$  is in  $\ker \Delta_{\text{WAP}}$  and for every  $a \in \mathcal{A}$

$$(2.2) \quad a \cdot (M_{\alpha} - s) - (M_{\alpha} - s) \cdot a \rightarrow s \cdot a - a \cdot s.$$

Let  $N$  be a unit-linked normal dual  $F(\mathcal{A})$ -bimodule and let  $D : \mathcal{A} \rightarrow N$  be a bounded derivation. We show that  $D$  is approximately inner, so by Lemma 2.2,  $F(\mathcal{A})$  is approximately Connes-amenable. Using the terminology of [CSS15, Theorem 6.8], if we define  $d(a) = s \cdot a - a \cdot s$  for all  $a \in \mathcal{A}$ , then  $d : \mathcal{A} \rightarrow \ker \Delta_{\text{WAP}}$  is weakly universal for derivation  $D$  with coefficient in  $N$ , that is, there exists a  $w^* - w^*$ -continuous (and so norm continuous)  $\mathcal{A}$ -bimodule map  $f : \ker \Delta_{\text{WAP}} \rightarrow N$  such that  $fd = D$ . Set  $y_{\alpha} = f(M_{\alpha} - s)$  for every  $\alpha$ . Using (2.2) for every  $a \in \mathcal{A}$  we have

$$\begin{aligned} D(a) &= fd(a) = f(s \cdot a - a \cdot s) \\ &= f(\lim_{\alpha} a \cdot (M_{\alpha} - s) - (M_{\alpha} - s) \cdot a) \\ &= \lim_{\alpha} f(a \cdot (M_{\alpha} - s) - (M_{\alpha} - s) \cdot a) \\ &= \lim_{\alpha} a \cdot y_{\alpha} - y_{\alpha} \cdot a. \end{aligned}$$

Hence  $D$  is approximately inner and this completes the proof.  $\square$

Now we introduce a notion of an approximate WAP-virtual diagonal for  $\mathcal{A}$ .

**Definition 2.5.** Let  $\mathcal{A}$  be a Banach algebra and let  $F(\mathcal{A})$  has an identity element  $e$ . An *approximate WAP-virtual diagonal* for  $\mathcal{A}$  is a net  $(M_\alpha)_\alpha \subseteq F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  such that

$$(2.3) \quad a \cdot M_\alpha - M_\alpha \cdot a \rightarrow 0 \quad (a \in \mathcal{A}) \quad \text{and} \quad \Delta_{\text{WAP}}(M_\alpha) \rightarrow e$$

in the norm topology.

**Theorem 2.6.** *Let  $\mathcal{A}$  be a Banach algebra. Then the following are equivalent:*

- (i)  $F(\mathcal{A})$  is approximately Connes-amenable;
- (ii)  $\mathcal{A}$  has an approximate WAP-virtual diagonal.

**Proof.** (i) $\Rightarrow$ (ii) It is immediate by Lemma 2.4.

(ii) $\Rightarrow$ (i) Suppose that  $\mathcal{A}$  has an approximate WAP-virtual diagonal. Then there exists a net  $(M_\alpha)_\alpha \subseteq F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  such that for every  $a \in \mathcal{A}$ ,

$$(2.4) \quad a \cdot M_\alpha - M_\alpha \cdot a \rightarrow 0 \quad \text{and} \quad \Delta_{\text{WAP}}(M_\alpha) \rightarrow e.$$

Let  $N$  be a unit-linked normal dual  $F(\mathcal{A})$ -bimodule and let  $D : \mathcal{A} \rightarrow N$  be a bounded derivation. We show that  $D$  is approximately inner and then by Lemma 2.2,  $F(\mathcal{A})$  is approximately Connes-amenable.

For every  $x \in N_*$  consider the functional  $f_x \in F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})^*$  defined by  $f_x(a \otimes b) = (a \cdot D(b))(x)$  ( $a, b \in \mathcal{A}$ ). Note that for every  $m \in \mathcal{A} \hat{\otimes} \mathcal{A}$  we have

$$f_{x \cdot a - a \cdot x}(m) = (f_x \cdot a - a \cdot f_x)(m) + (\Delta(m) \cdot D(a))(x).$$

Consider the  $\mathcal{A}$ -bimodule map  $\Delta : \mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$ . Then by [CSS15, Corollary 5.2] there exists a unique  $w^*$ - $w^*$ -continuous linear map

$$\Delta_{\text{WAP}} : F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A}) \rightarrow F(\mathcal{A})$$

making the following diagram

$$\begin{array}{ccc} \mathcal{A} \hat{\otimes} \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \\ \eta_{\mathcal{A} \hat{\otimes} \mathcal{A}} \downarrow & & \downarrow \eta_{\mathcal{A}} \\ F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A}) & \xrightarrow{\Delta_{\text{WAP}}} & F(\mathcal{A}) \end{array}$$

commute. Since  $\eta_{\mathcal{A} \hat{\otimes} \mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$  is  $w^*$ -dense in  $F_{\mathcal{A}}(\mathcal{A} \hat{\otimes} \mathcal{A})$ , there is a net  $(m_\beta^\alpha)_\beta \subseteq \mathcal{A} \hat{\otimes} \mathcal{A}$  such that  $\eta_{\mathcal{A} \hat{\otimes} \mathcal{A}}(m_\beta^\alpha) \xrightarrow{w^*} M_\alpha$  for every  $\alpha$ .

Set  $\lambda_\alpha(x) = \langle M_\alpha, f_x \rangle$  for each  $\alpha$ , we have

$$\begin{aligned} & \langle a \cdot \lambda_\alpha - \lambda_\alpha \cdot a, x \rangle \\ &= \langle \lambda_\alpha, x \cdot a - a \cdot x \rangle \\ &= \langle M_\alpha, f_{x \cdot a - a \cdot x} \rangle \\ &= \lim_\beta \langle \eta_{\mathcal{A} \hat{\otimes} \mathcal{A}}(m_\beta^\alpha), f_{x \cdot a - a \cdot x} \rangle \\ &= \lim_\beta \langle f_x \cdot a - a \cdot f_x, m_\beta^\alpha \rangle + \lim_\beta \langle \Delta(m_\beta^\alpha) \cdot D(a), x \rangle \\ &= \lim_\beta \langle f_x \cdot a - a \cdot f_x, m_\beta^\alpha \rangle + \lim_\beta \langle x, \eta_{\mathcal{A}} \Delta(m_\beta^\alpha) \cdot D(a) \rangle \\ &= \lim_\beta \langle \eta_{\mathcal{A} \hat{\otimes} \mathcal{A}}(m_\beta^\alpha), f_x \cdot a - a \cdot f_x \rangle + \lim_\beta \langle x, \Delta_{\text{WAP}} \eta_{\mathcal{A} \hat{\otimes} \mathcal{A}}(m_\beta^\alpha) \cdot D(a) \rangle \\ &= \langle a \cdot M_\alpha - M_\alpha \cdot a, f_x \rangle + \langle x, \Delta_{\text{WAP}}(M_\alpha) \cdot D(a) \rangle. \end{aligned}$$

Because  $\|f_x\| \leq \|D\| \|x\|$ ,

$$\begin{aligned} \|(a \cdot \lambda_\alpha - \lambda_\alpha \cdot a)(x) - D(a)(x)\| &\leq \|a \cdot M_\alpha - M_\alpha \cdot a\| \|D\| \|x\| \\ &\quad + \|x\| \|\Delta_{\text{WAP}}(M_\alpha) - e\| \|D(a)\|. \end{aligned}$$

This together with equation (2.4) imply that  $D(a) = \lim_\alpha \text{ad}_{\lambda_\alpha}(a)$  for every  $a \in \mathcal{A}$ , so  $D$  is approximately inner.  $\square$

### 3. Approximate-type virtual diagonals

Suppose that  $\mathcal{A}$  is a Banach algebra. Consider the  $\mathcal{A}$ -bimodule map  $\Delta^* : \mathcal{A}^* \rightarrow (\mathcal{A} \hat{\otimes} \mathcal{A})^*$ . Let  $V \subseteq (\mathcal{A} \hat{\otimes} \mathcal{A})^*$  be a closed subspace and let

$$E \subseteq (\Delta^*)^{-1}(V) \subseteq \mathcal{A}^*.$$

Then we obtain a map  $\Delta^*|_E : E \rightarrow V$ . We denote the adjoint of  $\Delta^*|_E$  by  $\Delta_E : V^* \rightarrow E^*$ .

**Definition 3.1.** Let  $V \subseteq (\mathcal{A} \hat{\otimes} \mathcal{A})^*$  be a nonzero closed subspace and let  $E \subseteq (\Delta^*)^{-1}(V) \subseteq \mathcal{A}^*$ . An *approximate  $V$ -virtual diagonal* for  $\mathcal{A}$  is a net  $(M_\alpha) \subseteq V^*$  that satisfies

$$\begin{aligned} a \cdot M_\alpha - M_\alpha \cdot a &\rightarrow 0, \\ \langle \Delta_E(M_\alpha) \cdot a, \varphi \rangle &\rightarrow \langle \varphi, a \rangle, \end{aligned}$$

for  $(a \in \mathcal{A}, \varphi \in E)$ .

We consider the following cases

- (1) Let  $V = (\mathcal{A} \hat{\otimes} \mathcal{A})^*$ . Then we obtain a new concept of a diagonal for  $\mathcal{A}$ , called an *approximate virtual diagonal*, that is, a—*not necessarily bounded*—net  $(M_\alpha) \subseteq (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$  such that

$$\begin{aligned} a \cdot M_\alpha - M_\alpha \cdot a &\rightarrow 0, \\ \langle \Delta^{**}(M_\alpha) \cdot a, \varphi \rangle &\rightarrow \langle \varphi, a \rangle, \end{aligned}$$

for  $(a \in \mathcal{A}, \varphi \in \mathcal{A}^*)$ .

- (2) Let  $V = (\mathcal{A} \hat{\otimes} \mathcal{A})_{\text{ess}}^* \subseteq (\mathcal{A} \hat{\otimes} \mathcal{A})^*$ . Then we obtain an *approximate*  $(\mathcal{A} \hat{\otimes} \mathcal{A})_{\text{ess}}^*$ -virtual diagonal for  $\mathcal{A}$ .

In the following proposition we show that how these two diagonals are related.

**Proposition 3.2.** *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate identity. If  $\mathcal{A}$  has an approximate  $(\mathcal{A} \hat{\otimes} \mathcal{A})_{\text{ess}}^*$ -virtual diagonal, then it has an approximate virtual diagonal.*

**Proof.** Let  $V = (\mathcal{A} \hat{\otimes} \mathcal{A})_{\text{ess}}^*$ . By Remark 1.3,  $V$  is a closed subspace of  $(\mathcal{A} \hat{\otimes} \mathcal{A})^*$  and we have an isomorphism  $\tau : V^* \rightarrow (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}/V^\perp$  defined by  $\tau(T) = x + V^\perp$  for some  $x \in (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$ . Since  $V^\perp$  is complemented in  $(\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$ , there exists an  $\mathcal{A}$ -bimodule projection  $P : (\mathcal{A} \hat{\otimes} \mathcal{A})^{**} \rightarrow V^\perp$  and so there is an  $\mathcal{A}$ -bimodule isomorphism  $\iota : (I - P)(\mathcal{A} \hat{\otimes} \mathcal{A})^{**} \rightarrow V^*$ . For every  $v \in V$  and  $T \in V^*$  by definition of  $\tau$  we have  $\tau(T) = x + V^\perp$  for some  $x \in (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$ , thus

$$\begin{aligned} \langle \iota^{-1}(T), v \rangle &= \langle (I - P)x, v \rangle = \langle x, v \rangle - \langle Px, v \rangle = \langle x, v \rangle = \langle x + V^\perp, v \rangle \\ &= \langle \tau(T), v \rangle. \end{aligned}$$

That is,  $\iota^{-1}(T)|_V = T$  for every  $T \in V^*$ . Now suppose that  $(F_\alpha) \subseteq V^*$  is an approximate  $(\mathcal{A} \hat{\otimes} \mathcal{A})_{\text{ess}}^*$ -virtual diagonal for  $\mathcal{A}$  and set

$$M_\alpha = \iota^{-1}(F_\alpha) \in (\mathcal{A} \hat{\otimes} \mathcal{A})^{**}$$

for every  $\alpha$ . Then

$$(3.1) \quad a \cdot M_\alpha - M_\alpha \cdot a = \iota^{-1}(a \cdot F_\alpha - F_\alpha \cdot a) \rightarrow 0.$$

Moreover, for  $a \in \mathcal{A}$  by Cohen's factorization theorem, there exist  $x, b, y \in \mathcal{A}$  such that  $a = ybx$ . Since for every  $\psi \in \mathcal{A}^*$  we have  $bx \cdot \psi \cdot y \in \mathcal{A}_{\text{ess}}^*$  and since  $\mathcal{A}_{\text{ess}}^* = (\Delta^*)^{-1}(V)$  [CSS15, Lemma 7.7],  $\Delta^*(bx \cdot \psi \cdot y) \in V$ . But  $M_\alpha|_V = \iota^{-1}(F_\alpha)|_V = F_\alpha$  for each  $\alpha$ . Using (3.1)

$$\begin{aligned} (3.2) \quad & \langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle \Delta^{**}(F_\alpha) \cdot b, x \cdot \psi \cdot y \rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle F_\alpha, \Delta^*(bx \cdot \psi \cdot y) \rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle M_\alpha, \Delta^*(bx \cdot \psi \cdot y) \rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle \Delta^{**}(M_\alpha), bx \cdot \psi \cdot y \rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle y \cdot \Delta^{**}(M_\alpha) \cdot bx, \psi \rangle \rightarrow 0. \end{aligned}$$

Also since  $(F_\alpha)$  is an approximate  $(\mathcal{A} \hat{\otimes} \mathcal{A})_{\text{ess}}^*$ -virtual diagonal and since

$$\Delta^*(x \cdot \psi \cdot y) \in V,$$

we have

$$(3.3) \quad \langle \Delta^{**}(F_\alpha) \cdot b, x \cdot \psi \cdot y \rangle \rightarrow \langle x \cdot \psi \cdot y, b \rangle.$$



Hence (3.2) and (3.3) imply that

$$(3.4) \quad \begin{aligned} & |\langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle \psi, a \rangle| \\ & \leq |\langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle - \langle \Delta^{**}(F_\alpha) \cdot b, x \cdot \psi \cdot y \rangle| \\ & \quad + |\langle \Delta^{**}(F_\alpha) \cdot b, x \cdot \psi \cdot y \rangle - \langle x \cdot \psi \cdot y, b \rangle| \rightarrow 0. \end{aligned}$$

We conclude from (3.1) and (3.4) that

$$a \cdot M_\alpha - M_\alpha \cdot a \rightarrow 0, \quad \langle \Delta^{**}(M_\alpha) \cdot a, \psi \rangle \rightarrow \langle \psi, a \rangle,$$

which means that  $(M_\alpha)$  is an approximate virtual diagonal for  $\mathcal{A}$ . □

As a corollary of Lemma 2.2, if a Banach algebra  $\mathcal{A}$  is approximately amenable, then  $F(\mathcal{A})$  is approximately Connes-amenable. For the rest of the paper we consider the converse of this result in the special case, whenever  $\mathcal{A} = \ell^1(G)$ . Indeed, we show that for a discrete group  $G$ , if  $\ell^1(G)$  has an approximate WAP-virtual diagonal, then it has an approximate virtual diagonal, although the existence of an approximate virtual diagonal is weaker than approximate amenability of  $\ell^1(G)$ .

We identify  $L^1(G) \hat{\otimes} L^1(G)$  with  $L^1(G \times G)$  as  $L^1(G)$ -bimodules (see [BD73, Example VI. 14] for instance).

Let  $I$  be the closed ideal in  $L^\infty(G \times G)_{\text{ess}}$  generated by  $\Delta^*(C_0(G))$  as in [CSS15, Definition 8.1], that is,

$$I = \overline{\text{lin}\{\Delta^*(f)g : f \in C_0(G), g \in L^\infty(G \times G)_{\text{ess}}\}}.$$

**Remark 3.3.** The space of uniformly continuous bounded functions on  $G$  is denoted by  $\text{UCB}(G)$ . Since  $\Delta^*(\text{UCB}(G)) = \Delta^*(L^\infty(G)_{\text{ess}}) \hookrightarrow L^\infty(G \times G)_{\text{ess}}$ , we obtain a map  $\Delta_{\text{UCB}} : L^\infty(G \times G)_{\text{ess}}^* \rightarrow \text{UCB}(G)^*$  as the adjoint of the inclusion map. By [CSS15, Lemma 8.2],

$$\Delta^*(C_0(G)) \subseteq I \subseteq \text{WAP}(L^\infty(G \times G)),$$

so we obtain a map  $\Delta_{C_0} = (\Delta^*|_{C_0(G)})^* : I^* \rightarrow C_0(G)^* = M(G)$ . By [CSS15, Proposition 8.5] there is a  $G_d$ -bimodule map  $S : I^* \rightarrow L^\infty(G \times G)_{\text{ess}}^*$  such that for every  $h \in \text{UCB}(G)$  and  $\psi \in I^*$

$$\langle \Delta_{\text{UCB}}(S(\psi)), h \rangle = \langle \iota(\Delta_{C_0}(\psi)), h \rangle,$$

where  $\iota : M(G) \rightarrow \text{UCB}(G)^*$  is the natural inclusion map defined by  $\langle \iota(\mu), h \rangle = \int_G h \, d\mu$  for every  $\mu \in M(G)$  and  $h \in \text{UCB}(G)$ .

**Theorem 3.4.** *Let  $G$  be a discrete group. Consider the following statements:*

- (i)  $\ell^1(G)$  has an approximate WAP-virtual diagonal.
- (ii)  $\ell^1(G)$  has an approximate  $\ell^\infty(G \times G)_{\text{ess}}$ -virtual diagonal.
- (iii)  $\ell^1(G)$  has an approximate virtual diagonal.

Then the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold.

**Proof.** (i) $\Rightarrow$ (ii) Let  $(M_\alpha) \subseteq F(\ell^1(G \times G))$  be an approximate WAP-virtual diagonal for  $\ell^1(G)$ . Then for every  $a \in \ell^1(G)$

$$a \cdot M_\alpha - M_\alpha \cdot a \rightarrow 0 \quad (a \in \ell^1(G)) \quad \text{and} \quad \Delta_{\text{WAP}}(M_\alpha) \rightarrow e,$$

where  $e$  denotes the identity element of  $F(\ell^1(G))$ . Since

$$\Delta^*(c_0(G)) \subseteq I \subseteq \text{WAP}(\ell^\infty(G \times G)),$$

each  $M_\alpha$  can be restricted to a functional on  $I$ . Let  $S : I^* \rightarrow \ell^\infty(G \times G)_{\text{ess}}^*$  be as mentioned in Remark 3.3. Set  $N_\alpha = S(M_\alpha|_I)$ . We show that  $(N_\alpha)_\alpha$  is an approximate  $\ell^\infty(G \times G)_{\text{ess}}$ -virtual diagonal. Since  $G$  is discrete and  $S$  is a continuous  $G$ -bimodule map, it is a continuous  $\ell^1(G)$ -bimodule map, so

$$a \cdot N_\alpha - N_\alpha \cdot a = S(a \cdot M_\alpha - M_\alpha \cdot a) \rightarrow 0 \quad (a \in \ell^1(G)).$$

Since  $(M_\alpha)$  is an approximate WAP-virtual diagonal  $\Delta_{c_0}(M_\alpha|_I) \cdot a \rightarrow a$  in  $\ell^1(G)$  and since the map  $\iota : \ell^1(G) \rightarrow \text{UCB}(G)^*$  is  $w^*$ - $w^*$ -continuous, for every  $h \in \text{UCB}(G)$  we have

$$\begin{aligned} \langle h, a \rangle &= \lim_\alpha \langle \iota \Delta_{c_0}(M_\alpha|_I \cdot a), h \rangle \\ &= \lim_\alpha \langle \Delta_{\text{UCB}} S(M_\alpha|_I \cdot a), h \rangle \\ &= \lim_\alpha \langle \Delta_{\text{UCB}}(N_\alpha \cdot a), h \rangle. \end{aligned}$$

Hence  $(N_\alpha)_\alpha$  is an approximate  $\ell^\infty(G \times G)_{\text{ess}}$ -virtual diagonal for  $\ell^1(G)$ .

(ii) $\Rightarrow$ (iii) holds by Proposition 3.2.  $\square$

Note that, it is not clear for us that the map  $S$  in Remark 3.3 is always an  $L^1(G)$ -bimodule map. If  $S$  is an  $L^1(G)$ -bimodule map, then the previous theorem holds not only for discrete groups but also for each locally compact group  $G$ .

## References

- [BD73] BONSALL, FRANK F.; DUNCAN, JOHN. Complete normed algebras. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 80. Springer-Verlag, New York-Heidelberg, 1973. x+301 pp. [MR0423029](#), [Zbl 0271.46039](#).
- [CSS15] CHOI, YEMON; SAMEI, EBRAHIM; STOKKE, ROSS. Extension of derivations, and Connes-amenability of the enveloping dual Banach algebra. *Math. Scand.* **117** (2015), no. 2, 258–303. [MR3437210](#), [Zbl 1328.47041](#), [arXiv:1307.6287](#), doi: [10.7146/math.scand.a-22870](#).
- [ESM12] ESSLAMZADEH, GHOLAM HOSSEIN; SHOJAEI, BEHROUZ; MAHMOODI, A. Approximate Connes-amenability of dual Banach algebras. *Bull. Belg. Math. Soc. Simon Stevin* **19** (2012), 193–213. [MR2977226](#), [Zbl 06050933](#), [arXiv:0908.3566](#).
- [GL04] GHAHRAMANI, FEREDOUN; LOY, RICHARD J. Generalized notions of amenability. *J. Funct. Anal.* **208** (2004), 229–260. [MR2034298](#), [Zbl 1045.46029](#), doi: [10.1016/S0022-1236\(03\)00214-3](#).
- [Joh72] JOHNSON, BARRY EDWARD. Cohomology in Banach algebras. *Memoirs of the American Mathematical Society*, 127. American Mathematical Society, Providence, R.I., 1972. iii+96 pp. [MR374934](#), [Zbl 0256.18014](#).

- [Ll97] LAU, ANTHONY TO-MING; LOY, RICHARD J. Weak amenability of Banach algebras on locally compact groups. *J. Funct. Anal.* **145** (1997), 175–204. [MR1442165](#), [Zbl 0890.46036](#), doi: [10.1006/jfan.1996.3002](#).
- [Run01] RUNDE, VOLKER. Amenability for dual Banach algebras. *Studia Math.* **148** (2001), 47–66. [MR1881439](#), [Zbl 1003.46028](#), [arXiv:math/0203199](#), doi: [10.4064/sm148-1-5](#).
- [Run04] RUNDE, VOLKER. Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule. *Math. Scand.* **95** (2004), 124–144. [MR2091485](#), [Zbl 1087.46035](#), [arXiv:math/0305431](#), doi: [10.7146/math.scand.a-14452](#).
- [Run02] RUNDE, VOLKER. Lectures on amenability. Lecture Notes in Mathematics, 1774. *Springer-Verlag, Berlin*, 2002. [MR1874893](#), [Zbl 0999.46022](#), doi: [10.1007/b82937](#).

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