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On approximate Connes-amenability of enveloping dual Banach algebras

Ahmad Shirinkalam and Abdolrasoul Pourabbas

Abstract. For a Banach algebra $\mathcal{A}$, we introduce various approximate virtual diagonals such as approximate WAP-virtual diagonal and approximate virtual diagonal.

For the enveloping dual Banach algebra $F(\mathcal{A})$ of $\mathcal{A}$, we show that $F(\mathcal{A})$ is approximately Connes-amenable if and only if $\mathcal{A}$ has an approximate WAP-virtual diagonal.

Further, for a discrete group G, we show that if the group algebra $\ell^1(G)$ has an approximate WAP-virtual diagonal, then it has an approximate virtual diagonal.

Contents

1. Introduction and preliminaries

The notion of approximate Connes-amenability for a dual Banach algebra was first introduced by Esslamzadeh *et al.* (2012). A dual Banach algebra $\mathcal{A}$ is called approximately Connes-amenable if for every normal dual $\mathcal{A}$-bimodule X, every w^*-continuous derivation $D : \mathcal{A} \to X$ is approximately inner, that is, there exists a net $(x_\alpha) \subseteq X$ such that $D(b) = \lim_\alpha b \cdot x_\alpha - x_\alpha \cdot b$ $(b \in \mathcal{A})$.

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. An element $x \in X$ is called weakly almost periodic if the module maps

$$\mathcal{A} \to X; \quad a \mapsto a \cdot x \quad \text{and} \quad a \mapsto x \cdot a$$

are weakly compact. The set of all weakly almost periodic elements of X is denoted by $\mathrm{WAP}(X)$, which is a norm-closed subspace of X.

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For a Banach algebra $\mathcal{A}$, one of the most important subspaces of $\mathcal{A}^*$ is $\mathrm{WAP}(\mathcal{A}^*)$, which is left introverted in the sense of [Ll97, §1]. Runde (2004) observed that, the space $F(\mathcal{A}) = \mathrm{WAP}(\mathcal{A}^*)^*$ is a dual Banach algebra with the first Arens product inherited from $\mathcal{A}^{**}$. He also showed that $F(\mathcal{A})$ is a *canonical* dual Banach algebra associated to $\mathcal{A}$ (see [Ll97] or [Run04, Theorem 4.10] for more details).

Choi *et al.* (2014) called $F(\mathcal{A})$ the enveloping dual Banach algebra associated to $\mathcal{A}$ and they studied Connes-amenability of $F(\mathcal{A})$. Indeed, they introduced the notion of WAP-virtual diagonal for a Banach algebra $\mathcal{A}$ and they showed that for a given Banach algebra $\mathcal{A}$, the dual Banach algebra $F(\mathcal{A})$ is Connes-amenable if and only if $\mathcal{A}$ admits a WAP-virtual diagonal. They also showed that for a group algebra $L^1(G)$, the existence of a virtual diagonal is equivalent to the existence of a WAP-virtual diagonal. As a consequence $L^1(G)$ is amenable if and only if $F(L^1(G))$ is Connes-amenable, a fact that previously was shown by Runde (2004).

Motivated by these results, we investigate an *approximate* analogue of WAP-virtual diagonal related to approximate Connes-amenability of $F(\mathcal{A})$. Indeed, we introduce a notion of an approximate WAP-virtual diagonal for $\mathcal{A}$ and we show that $F(\mathcal{A})$ is approximately Connes-amenable if and only if $\mathcal{A}$ has an approximate WAP-virtual diagonal. We also introduce various notions of approximate-type virtual diagonals for a Banach algebra -such as approximate virtual diagonal- and we show that for a discrete group G, if $\ell^1(G)$ has an approximate WAP-virtual diagonal, then it has an approximate virtual diagonal.

Given a Banach algebra $\mathcal{A}$, its unitization is denoted by $\mathcal{A}^\sharp$. For a Banach $\mathcal{A}$-bimodule X, the topological dual space X^* of X becomes a Banach $\mathcal{A}$-bimodule via the following actions

$$\begin{aligned} &\langle x, a \cdot \varphi\rangle = \langle x \cdot a, \varphi\rangle, \\ &\langle x, \varphi \cdot a\rangle = \langle a \cdot x, \varphi\rangle \qquad (a \in \mathcal{A}, x \in X, \varphi \in X^*). \end{aligned} \tag{1.1}$$

Definition 1.1. A Banach algebra $\mathcal{A}$ is called dual, if it is a dual Banach space with a predual $\mathcal{A}_*$ such that the multiplication in $\mathcal{A}$ is separately $\sigma(\mathcal{A}, \mathcal{A}_*)$-continuous [Run01, Definition 1.1]. Equivalently, a Banach algebra $\mathcal{A}$ is dual if it has a (not necessarily unique) predual $\mathcal{A}_*$ which is a closed submodule of $\mathcal{A}^*$ [Run02, Exercise 4.4.1].

Let $\mathcal{A}$ be a dual Banach algebra and let a dual Banach space X be an $\mathcal{A}$-bimodule. An element $x \in X$ is called normal, if the module actions $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^*- w^*-continuous. The set of all normal elements in X is denoted by X_σ. We say that X is normal if $X = X_\sigma$.

Recall that for a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule X, a bounded linear map $D : \mathcal{A} \to X$ is called a bounded derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for every $a, b \in \mathcal{A}$. A derivation $D : \mathcal{A} \to X$ is called inner if there exists an element $x \in X$ such that for every $a \in \mathcal{A}$,

$$D(a) = \mathrm{ad}_x(a) = a \cdot x - x \cdot a.$$

A dual Banach algebra $\mathcal{A}$ is called Connes-amenable if for every normal dual $\mathcal{A}$-bimodule X, every w^*– w^*-continuous derivation $D : \mathcal{A} \to X$ is inner [Run04, Definition 1.5].

Following [CSS15], for a given Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule X, the $\mathcal{A}$-bimodule $\mathrm{WAP}(X^*)$ is denoted by $F_{\mathcal{A}}(X)_*$ and its dual space $\mathrm{WAP}(X^*)^*$ is denoted by $F_{\mathcal{A}}(X)$. Choi *et al.* [CSS15, Theorem 4.3] showed that if X is a Banach $\mathcal{A}$-bimodule, then $F_{\mathcal{A}}(X)$ is a normal dual $F(\mathcal{A})$-bimodule. Since $\mathrm{WAP}(X^*)$ is a closed subspace of X^*, we have a quotient map $q : X^{**} \twoheadrightarrow F_{\mathcal{A}}(X)$. Composing the canonical inclusion map $X \hookrightarrow X^{**}$ with q, we obtain a continuous $\mathcal{A}$-bimodule map $\eta_X : X \to F_{\mathcal{A}}(X)$ which has a w^*-dense range. In the special case where $X = \mathcal{A}$, we usually omit the subscript and simply use the notation $F(\mathcal{A})$ and the map $\eta_{\mathcal{A}} : \mathcal{A} \to F(\mathcal{A})$ is an algebra homomorphism.

It is well-known that for a given Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule through

$$a \cdot (b \otimes c) := ab \otimes c, \quad (b \otimes c) \cdot a := b \otimes ca, \quad (a, b, c \in \mathcal{A})$$

and the map $\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ defined on elementary tensors by $\Delta(a \otimes b) = ab$ and extended by linearity and continuity, is an $\mathcal{A}$-bimodule map with respect to the module structure of $\mathcal{A}\hat{\otimes}\mathcal{A}$. Let $\Delta_{\mathrm{WAP}} : F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A}) \to F(\mathcal{A})$ be the w^*–w^*-continuous $\mathcal{A}$-bimodule map induced by $\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ (see [CSS15, Corollary 5.2] for more details).

Definition 1.2. [CSS15, Definition 6.4] An element $M \in F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ is called a WAP-virtual diagonal for $\mathcal{A}$ if

$$a \cdot M = M \cdot a \quad \text{and} \quad \Delta_{\mathrm{WAP}}(M) \cdot a = \eta_{\mathcal{A}}(a) \qquad (a \in \mathcal{A}).$$

Recall that an $\mathcal{A}$-bimodule X is called neo-unital if every $x \in X$ can be written as $a \cdot y \cdot b$ for some $a, b \in \mathcal{A}$ and $y \in X$.

Remark 1.3. Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let X be a Banach $\mathcal{A}$-bimodule. Then by [Joh72, Proposition 1.8], the subspace $X_{\mathrm{ess}} := \mathrm{lin}\{a \cdot x \cdot b : a, b \in \mathcal{A},\ x \in X\}$ is a neo-unital closed sub-$\mathcal{A}$-bimodule of X. Moreover, $X_{\mathrm{ess}}^{\perp}$ is complemented in X^*.

2. Approximate Connes-amenability of $F(\mathcal{A})$

In this section we find some conditions under which $F(\mathcal{A})$ is approximately Connes-amenable. For an arbitrary dual Banach algebra $\mathcal{A}$, approximate Connes-amenability of $\mathcal{A}$ is equivalent to approximate Connes-amenability of $\mathcal{A}^\sharp$ [ESM12, Proposition 2.3], so without loss of generality throughout of this section we may suppose that $F(\mathcal{A})$ has an identity element e.

Remark 2.1. Let $\mathcal{A}$ be a Banach algebra with an identity element e. Let X be a normal dual $\mathcal{A}$-bimodule and let $D : \mathcal{A} \to X$ be a bounded derivation. By [GL04, Lemma 2.3], there exists a bounded derivation $D_1 : \mathcal{A} \to e \cdot X \cdot e$ such that $D = D_1 + \mathrm{ad}_x$ for some $x \in X$. Furthermore, D is inner derivation if and only if D_1 is inner.

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra. Then $F(\mathcal{A})$ is approximately Connes-amenable if and only if every bounded derivation from $\mathcal{A}$ into a unit-linked, normal dual $F(\mathcal{A})$-bimodule is approximately inner.*

Proof. Suppose that $F(\mathcal{A})$ is approximately Connes-amenable. Let N be a unit-linked, normal dual $F(\mathcal{A})$-bimodule and let $D : \mathcal{A} \to N$ be a bounded derivation. We show that D is approximately inner. By [ESM12, Theorem 4.4], there is a w^*– w^*-continuous derivation $\tilde{D} : F(\mathcal{A}) \to F_{\mathcal{A}}(N)$ such that $\tilde{D}\eta_{\mathcal{A}} = \eta_N D$, that is, the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ D\ } & N \\ {\scriptstyle \eta_{\mathcal{A}}}\big\downarrow & & \big\downarrow{\scriptstyle \eta_N} \\ F(\mathcal{A}) & \xrightarrow[\tilde{D}]{} & F_{\mathcal{A}}(N). \end{array}$$

Since $F_{\mathcal{A}}(N)$ is a normal dual $F(\mathcal{A})$-bimodule and $F(\mathcal{A})$ is approximately Connes-amenable, the derivation $\tilde{D}$ is approximately inner. Since $\eta_{\mathcal{A}}$ is an algebra homomorphism, the derivation $\tilde{D}\eta_{\mathcal{A}}$ is also approximately inner. Hence the derivation $\eta_N D$ is approximately inner, that is, there is a net $(\xi_i) \subseteq F_{\mathcal{A}}(N)$ such that

$$\eta_N D(a) = \lim_i a \cdot \xi_i - \xi_i \cdot a \quad (a \in \mathcal{A}).$$

By [ESM12, Corollary 5.3] there is a w^*– w^*-continuous $\mathcal{A}$-bimodule map $\epsilon_N : F_{\mathcal{A}}(N) \to N$ such that $\epsilon_N \eta_N = \mathrm{id}_N$. Now by setting $x_i = \epsilon_N(\xi_i) \in N$ for each i we have

$$D(a) = \epsilon_N \eta_N D(a) = \epsilon_N(\lim_i a \cdot \xi_i - \xi_i \cdot a) = \lim_i a \cdot x_i - x_i \cdot a,$$

so $D : \mathcal{A} \to N$ is approximately inner.

Conversely, suppose that every bounded derivation from $\mathcal{A}$ into a unit-linked, normal dual $F(\mathcal{A})$-bimodule is approximately inner. Let N be a normal dual $F(\mathcal{A})$-bimodule and let $D : F(\mathcal{A}) \to N$ be a w^*– w^*-continuous derivation. By Remark 2.1 without loss of generality we may assume that N is a unit-linked, normal dual $F(\mathcal{A})$-bimodule. We shall show that D is approximately inner. Since $\eta_{\mathcal{A}}$ is an algebra homomorphism, the map $D' := D\eta_{\mathcal{A}} : \mathcal{A} \to N$ is a bounded derivation, so by assumption D' is approximately inner. Since N is normal, D' extends to a w^*– w^*-continuous approximately inner derivation $D'' : F(\mathcal{A}) \to N$. The derivations D and D'' are w^*–w^*-continuous and they agree on the w^*-dense subset $\eta_{\mathcal{A}}(\mathcal{A}) \subseteq F(\mathcal{A})$, so they agree on $F(\mathcal{A})$. Hence D is approximately inner, that is, $F(\mathcal{A})$ is approximately Connes-amenable. □

Remark 2.3. Since $F(\mathcal{A})$ has an identity element e, by [CSS15, Theorem 6.8] there exists an element $s \in F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ such that $\Delta_{\mathrm{WAP}}(s) = e$. We use this fact in the following lemma several times.

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

(i) *$F(\mathcal{A})$ is approximately Connes-amenable.*

(ii) *There exists a net $(M_\alpha)_\alpha \subseteq F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ such that for every $a \in \mathcal{A}$, $a \cdot M_\alpha - M_\alpha \cdot a \to 0$ and $\Delta_{\mathrm{WAP}}(M_\alpha) = e$ for all α.*

Proof. (i)⇒(ii) Fix $s \in F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ as mentioned in Remark 2.3. Note that $\ker \Delta_{\mathrm{WAP}}$ is a normal dual $F(\mathcal{A})$-bimodule. Define a w^*– w^*-continuous derivation $D : F(\mathcal{A}) \to \ker \Delta_{\mathrm{WAP}}$ by $D(\mathbf{a}) = s \cdot \mathbf{a} - \mathbf{a} \cdot s \quad (\mathbf{a} \in F(\mathcal{A}))$. Since $F(\mathcal{A})$ is approximately Connes-amenable, the derivation D is approximately inner. Thus there is a net $(N_\alpha) \subseteq \ker \Delta_{\mathrm{WAP}}$ such that for every $a \in \mathcal{A}$

$$s \cdot a - a \cdot s = \lim_\alpha (a \cdot N_\alpha - N_\alpha \cdot a). \tag{2.1}$$

If we set $M_\alpha = N_\alpha + s$ for every α, then we have $\Delta_{\mathrm{WAP}}(M_\alpha) = \Delta_{\mathrm{WAP}}(s) = e$ and for every $a \in \mathcal{A}$ using (2.1) we have

$$a \cdot M_\alpha - M_\alpha \cdot a \to 0,$$

as required.

(ii)⇒(i) The hypothesis in (ii) ensures that each $M_\alpha - s$ is in $\ker \Delta_{\mathrm{WAP}}$ and for every $a \in \mathcal{A}$

$$a \cdot (M_\alpha - s) - (M_\alpha - s) \cdot a \to s \cdot a - a \cdot s. \tag{2.2}$$

Let N be a unit-linked normal dual $F(\mathcal{A})$-bimodule and let $D : \mathcal{A} \to N$ be a bounded derivation. We show that D is approximately inner, so by Lemma 2.2, $F(\mathcal{A})$ is approximately Connes-amenable. Using the terminology of [CSS15, Theorem 6.8], if we define $d(a) = s \cdot a - a \cdot s$ for all $a \in \mathcal{A}$, then $d : \mathcal{A} \to \ker \Delta_{\mathrm{WAP}}$ is weakly universal for derivation D with coefficient in N, that is, there exists a w^*– w^*-continuous (and so norm continuous) $\mathcal{A}$-bimodule map $f : \ker \Delta_{\mathrm{WAP}} \to N$ such that $fd = D$. Set $y_\alpha = f(M_\alpha - s)$ for every α. Using (2.2) for every $a \in \mathcal{A}$ we have

$$\begin{aligned} D(a) = fd(a) &= f(s \cdot a - a \cdot s) \\ &= f(\lim_\alpha a \cdot (M_\alpha - s) - (M_\alpha - s) \cdot a) \\ &= \lim_\alpha f(a \cdot (M_\alpha - s) - (M_\alpha - s) \cdot a) \\ &= \lim_\alpha a \cdot y_\alpha - y_\alpha \cdot a. \end{aligned}$$

Hence D is approximately inner and this completes the proof. □

Now we introduce a notion of an approximate WAP-virtual diagonal for $\mathcal{A}$.

Definition 2.5. Let $\mathcal{A}$ be a Banach algebra and let $F(\mathcal{A})$ has an identity element e. An *approximate* WAP-*virtual diagonal* for $\mathcal{A}$ is a net $(M_\alpha)_\alpha \subseteq F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ such that

$$(2.3)\qquad a \cdot M_\alpha - M_\alpha \cdot a \to 0 \quad (a \in \mathcal{A}) \qquad \text{and} \quad \Delta_{\mathrm{WAP}}(M_\alpha) \to e$$

in the norm topology.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

(i) *$F(\mathcal{A})$ is approximately Connes-amenable;*
(ii) *$\mathcal{A}$ has an approximate* WAP-*virtual diagonal.*

Proof. (i)⇒(ii) It is immediate by Lemma 2.4.

(ii)⇒(i) Suppose that $\mathcal{A}$ has an approximate WAP-virtual diagonal. Then there exists a net $(M_\alpha)_\alpha \subseteq F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ such that for every $a \in \mathcal{A}$,

$$(2.4)\qquad a \cdot M_\alpha - M_\alpha \cdot a \to 0 \quad \text{and} \quad \Delta_{\mathrm{WAP}}(M_\alpha) \to e.$$

Let N be a unit-linked normal dual $F(\mathcal{A})$-bimodule and let $D : \mathcal{A} \to N$ be a bounded derivation. We show that D is approximately inner and then by Lemma 2.2, $F(\mathcal{A})$ is approximately Connes-amenable.

For every $x \in N_*$ consider the functional $f_x \in F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ defined by $f_x(a \otimes b) = (a \cdot D(b))(x) \quad (a, b \in \mathcal{A})$. Note that for every $m \in \mathcal{A}\hat{\otimes}\mathcal{A}$ we have

$$f_{x\cdot a - a\cdot x}(m) = (f_x \cdot a - a \cdot f_x)(m) + (\Delta(m) \cdot D(a))(x).$$

Consider the $\mathcal{A}$-bimodule map $\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$. Then by [CSS15, Corollary 5.2] there exists a unique w^*– w^*-continuous linear map

$$\Delta_{\mathrm{WAP}} : F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A}) \to F(\mathcal{A})$$

making the following diagram

$$\begin{array}{ccc} \mathcal{A}\hat{\otimes}\mathcal{A} & \xrightarrow{\ \Delta\ } & \mathcal{A} \\ {\scriptstyle \eta_{\mathcal{A}\hat{\otimes}\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{A}}} \\ F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A}) & \xrightarrow[\Delta_{\mathrm{WAP}}]{} & F(\mathcal{A}) \end{array}$$

commute. Since $\eta_{\mathcal{A}\hat{\otimes}\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ is w^*-dense in $F_{\mathcal{A}}(\mathcal{A}\hat{\otimes}\mathcal{A})$, there is a net $(m_\beta^\alpha)_\beta \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\eta_{\mathcal{A}\hat{\otimes}\mathcal{A}}(m_\beta^\alpha) \xrightarrow{w^*} M_\alpha$ for every α.

Set $\lambda_\alpha(x) = \langle M_\alpha, f_x \rangle$ for each α, we have

$$\begin{aligned}
&\langle a \cdot \lambda_\alpha - \lambda_\alpha \cdot a, x \rangle \\
&= \langle \lambda_\alpha, x \cdot a - a \cdot x \rangle \\
&= \langle M_\alpha, f_{x \cdot a - a \cdot x} \rangle \\
&= \lim_\beta \langle \eta_{\mathcal{A}\hat{\otimes}\mathcal{A}}(m_\beta^\alpha), f_{x \cdot a - a \cdot x} \rangle \\
&= \lim_\beta \langle f_x \cdot a - a \cdot f_x, m_\beta^\alpha \rangle + \lim_\beta \langle \Delta(m_\beta^\alpha) \cdot D(a), x \rangle \\
&= \lim_\beta \langle f_x \cdot a - a \cdot f_x, m_\beta^\alpha \rangle + \lim_\beta \langle x, \eta_{\mathcal{A}} \Delta(m_\beta^\alpha) \cdot D(a) \rangle \\
&= \lim_\beta \langle \eta_{\mathcal{A}\hat{\otimes}\mathcal{A}}(m_\beta^\alpha), f_x \cdot a - a \cdot f_x \rangle + \lim_\beta \langle x, \Delta_{\mathrm{WAP}} \eta_{\mathcal{A}\hat{\otimes}\mathcal{A}}(m_\beta^\alpha) \cdot D(a) \rangle \\
&= \langle a \cdot M_\alpha - M_\alpha \cdot a, f_x \rangle + \langle x, \Delta_{\mathrm{WAP}}(M_\alpha) \cdot D(a) \rangle.
\end{aligned}$$

Because $\|f_x\| \le \|D\| \|x\|$,

$$\begin{aligned}
\|(a \cdot \lambda_\alpha - \lambda_\alpha \cdot a)(x) - D(a)(x)\| \le{}& \|a \cdot M_\alpha - M_\alpha \cdot a\| \|D\| \|x\| \\
&+ \|x\| \|\Delta_{\mathrm{WAP}}(M_\alpha) - e\| \|D(a)\|.
\end{aligned}$$

This together with equation (2.4) imply that $D(a) = \lim_\alpha \mathrm{ad}_{\lambda_\alpha}(a)$ for every $a \in \mathcal{A}$, so D is approximately inner. □

3. Approximate-type virtual diagonals

Suppose that $\mathcal{A}$ is a Banach algebra. Consider the $\mathcal{A}$-bimodule map $\Delta^* : \mathcal{A}^* \to (\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Let $V \subseteq (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ be a closed subspace and let

$$E \subseteq (\Delta^*)^{-1}(V) \subseteq \mathcal{A}^*.$$

Then we obtain a map $\Delta^*|_E : E \to V$. We denote the adjoint of $\Delta^*|_E$ by $\Delta_E : V^* \to E^*$.

Definition 3.1. Let $V \subseteq (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ be a nonzero closed subspace and let $E \subseteq (\Delta^*)^{-1}(V) \subseteq \mathcal{A}^*$. An *approximate V-virtual diagonal* for $\mathcal{A}$ is a net $(M_\alpha) \subseteq V^*$ that satisfies

$$a \cdot M_\alpha - M_\alpha \cdot a \to 0,$$
$$\langle \Delta_E(M_\alpha) \cdot a, \varphi \rangle \to \langle \varphi, a \rangle,$$

for ($a \in \mathcal{A}$, $\varphi \in E$).

We consider the following cases

(1) Let $V = (\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Then we obtain a new concept of a diagonal for $\mathcal{A}$, called an *approximate virtual diagonal*, that is, a—not necessarily bounded—net $(M_\alpha) \subseteq (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that

$$a \cdot M_\alpha - M_\alpha \cdot a \to 0,$$
$$\langle \Delta^{**}(M_\alpha) \cdot a, \varphi \rangle \to \langle \varphi, a \rangle,$$

for $(a \in \mathcal{A}, \varphi \in \mathcal{A}^*)$.

(2) Let $V = (\mathcal{A}\hat{\otimes}\mathcal{A})^*_{\text{ess}} \subseteq (\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Then we obtain an *approximate* $(\mathcal{A}\hat{\otimes}\mathcal{A})^*_{\text{ess}}$*-virtual diagonal* for A.

In the following proposition we show that how these two diagonals are related.

Proposition 3.2. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. If $\mathcal{A}$ has an approximate $(\mathcal{A}\hat{\otimes}\mathcal{A})^*_{\text{ess}}$-virtual diagonal, then it has an approximate virtual diagonal.*

Proof. Let $V = (\mathcal{A}\hat{\otimes}\mathcal{A})^*_{\text{ess}}$. By Remark 1.3, V is a closed subspace of $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ and we have an isomorphism $\tau : V^* \to (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}/V^{\perp}$ defined by $\tau(T) = x + V^{\perp}$ for some $x \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$. Since $V^{\perp}$ is complemented in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$, there exists an $\mathcal{A}$-bimodule projection $P : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to V^{\perp}$ and so there is an $\mathcal{A}$-bimodule isomorphism $\iota : (I - P)(\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to V^*$. For every $v \in V$ and $T \in V^*$ by definition of τ we have $\tau(T) = x + V^{\perp}$ for some $x \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$, thus

$$\begin{aligned}\langle \iota^{-1}(T), v\rangle = \langle (I-P)x, v\rangle = \langle x, v\rangle - \langle Px, v\rangle = \langle x, v\rangle &= \langle x + V^{\perp}, v\rangle \\ &= \langle \tau(T), v\rangle.\end{aligned}$$

That is, $\iota^{-1}(T)|_V = T$ for every $T \in V^*$. Now suppose that $(F_\alpha) \subseteq V^*$ is an approximate $(\mathcal{A}\hat{\otimes}\mathcal{A})^*_{\text{ess}}$-virtual diagonal for $\mathcal{A}$ and set

$$M_\alpha = \iota^{-1}(F_\alpha) \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$$

for every α. Then

$$a \cdot M_\alpha - M_\alpha \cdot a = \iota^{-1}(a \cdot F_\alpha - F_\alpha \cdot a) \to 0. \tag{3.1}$$

Moreover, for $a \in \mathcal{A}$ by Cohen's factorization theorem, there exist $x, b, y \in \mathcal{A}$ such that $a = ybx$. Since for every $\psi \in \mathcal{A}^*$ we have $bx \cdot \psi \cdot y \in \mathcal{A}^*_{\text{ess}}$ and since $\mathcal{A}^*_{\text{ess}} = (\Delta^*)^{-1}(V)$ [CSS15, Lemma 7.7], $\Delta^*(bx \cdot \psi \cdot y) \in V$. But $M_\alpha|_V = \iota^{-1}(F_\alpha)|_V = F_\alpha$ for each α . Using (3.1)

$$\begin{aligned}&\langle \Delta^{**}(M_\alpha) \cdot a, \psi\rangle - \langle \Delta^{**}(F_\alpha) \cdot b, x \cdot \psi \cdot y\rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi\rangle - \langle F_\alpha, \Delta^*(bx \cdot \psi \cdot y)\rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi\rangle - \langle M_\alpha, \Delta^*(bx \cdot \psi \cdot y)\rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi\rangle - \langle \Delta^{**}(M_\alpha), bx \cdot \psi \cdot y\rangle \\ &= \langle \Delta^{**}(M_\alpha) \cdot a, \psi\rangle - \langle y \cdot \Delta^{**}(M_\alpha) \cdot bx, \psi\rangle \to 0.\end{aligned} \tag{3.2}$$

Also since (F_α) is an approximate $(\mathcal{A}\hat{\otimes}\mathcal{A})^*_{\text{ess}}$-virtual diagonal and since

$$\Delta^*(x \cdot \psi \cdot y) \in V,$$

we have

$$\langle \Delta^{**}(F_\alpha) \cdot b, x \cdot \psi \cdot y\rangle \to \langle x \cdot \psi \cdot y, b\rangle. \tag{3.3}$$

Hence (3.2) and (3.3) imply that

$$\begin{aligned}(3.4)\qquad &|\langle \Delta^{**}(M_\alpha)\cdot a, \psi\rangle - \langle \psi, a\rangle| \\ &\le |\langle \Delta^{**}(M_\alpha)\cdot a, \psi\rangle - \langle \Delta^{**}(F_\alpha)\cdot b, x\cdot\psi\cdot y\rangle| \\ &\quad + |\langle \Delta^{**}(F_\alpha)\cdot b, x\cdot\psi\cdot y\rangle - \langle x\cdot\psi\cdot y, b\rangle| \to 0.\end{aligned}$$

We conclude from (3.1) and (3.4) that

$$a\cdot M_\alpha - M_\alpha\cdot a \to 0, \qquad \langle \Delta^{**}(M_\alpha)\cdot a, \psi\rangle \to \langle \psi, a\rangle,$$

which means that (M_α) is an approximate virtual diagonal for $\mathcal{A}$. □

As a corollary of Lemma 2.2, if a Banach algebra $\mathcal{A}$ is approximately amenable, then $F(\mathcal{A})$ is approximately Connes-amenable. For the rest of the paper we consider the converse of this result in the special case, whenever $\mathcal{A} = \ell^1(G)$. Indeed, we show that for a discrete group G, if $\ell^1(G)$ has an approximate WAP-virtual diagonal, then it has an approximate virtual diagonal, although the existence of an approximate virtual diagonal is weaker than approximate amenability of $\ell^1(G)$.

We identify $L^1(G)\hat{\otimes}L^1(G)$ with $L^1(G\times G)$ as $L^1(G)$-bimodules (see [BD73, Example VI. 14] for instance).

Let I be the closed ideal in $L^\infty(G\times G)_{\mathrm{ess}}$ generated by $\Delta^*(C_0(G))$ as in [CSS15, Definition 8.1], that is,

$$I = \overline{\mathrm{lin}\{\Delta^*(f)g : f\in C_0(G), g\in L^\infty(G\times G)_{\mathrm{ess}}\}}.$$

Remark 3.3. The space of uniformly continuous bounded functions on G is denoted by $\mathrm{UCB}(G)$. Since $\Delta^*(\mathrm{UCB}(G)) = \Delta^*(L^\infty(G)_{\mathrm{ess}}) \hookrightarrow L^\infty(G\times G)_{\mathrm{ess}}$, we obtain a map $\Delta_{\mathrm{UCB}} : L^\infty(G\times G)^*_{\mathrm{ess}} \to \mathrm{UCB}(G)^*$ as the adjoint of the inclusion map. By [CSS15, Lemma 8.2],

$$\Delta^*(C_0(G)) \subseteq I \subseteq \mathrm{WAP}(L^\infty(G\times G)),$$

so we obtain a map $\Delta_{C_0} = (\Delta^*|_{C_0(G)})^* : I^* \to C_0(G)^* = M(G)$. By [CSS15, Proposition 8.5] there is a G_d-bimodule map $S : I^* \to L^\infty(G\times G)^*_{\mathrm{ess}}$ such that for every $h\in \mathrm{UCB}(G)$ and $\psi\in I^*$

$$\langle \Delta_{\mathrm{UCB}}(S(\psi)), h\rangle = \langle \iota(\Delta_{C_0}(\psi)), h\rangle,$$

where $\iota : M(G) \to \mathrm{UCB}(G)^*$ is the natural inclusion map defined by $\langle \iota(\mu), h\rangle = \int_G h\, d\mu$ for every $\mu\in M(G)$ and $h\in\mathrm{UCB}(G)$.

Theorem 3.4. *Let G be a discrete group. Consider the following statements:*

(i) $\ell^1(G)$ *has an approximate* WAP*-virtual diagonal.*
(ii) $\ell^1(G)$ *has an approximate* $\ell^\infty(G\times G)_{\mathrm{ess}}$*-virtual diagonal.*
(iii) $\ell^1(G)$ *has an approximate virtual diagonal.*

Then the implications (i)⇒(ii)⇒(iii) hold.

Proof. (i)⇒(ii) Let $(M_\alpha) \subseteq F(\ell^1(G \times G))$ be an approximate WAP-virtual diagonal for $\ell^1(G)$. Then for every $a \in \ell^1(G)$

$$a \cdot M_\alpha - M_\alpha \cdot a \to 0 \quad (a \in \ell^1(G)) \quad \text{and} \quad \Delta_{\mathrm{WAP}}(M_\alpha) \to e,$$

where e denotes the identity element of $F(\ell^1(G))$. Since

$$\Delta^*(c_0(G)) \subseteq I \subseteq \mathrm{WAP}(\ell^\infty(G \times G)),$$

each M_α can be restricted to a functional on I. Let $S : I^* \to \ell^\infty(G \times G)^*_{\mathrm{ess}}$ be as mentioned in Remark 3.3. Set $N_\alpha = S(M_\alpha|_I)$. We show that $(N_\alpha)_\alpha$ is an approximate $\ell^\infty(G \times G)_{\mathrm{ess}}$-virtual diagonal. Since G is discrete and S is a continuous G-bimodule map, it is a continuous $\ell^1(G)$-bimodule map, so

$$a \cdot N_\alpha - N_\alpha \cdot a = S(a \cdot M_\alpha - M_\alpha \cdot a) \to 0 \quad (a \in \ell^1(G)).$$

Since (M_α) is an approximate WAP-virtual diagonal $\Delta_{c_0}(M_\alpha|_I) \cdot a \to a$ in $\ell^1(G)$ and since the map $\iota : \ell^1(G) \to \mathrm{UCB}(G)^*$ is w^*– w^*-continuous, for every $h \in \mathrm{UCB}(G)$ we have

$$\begin{aligned}\langle h, a\rangle &= \lim_\alpha \langle \iota\Delta_{c_0}(M_\alpha|_I \cdot a), h\rangle \\ &= \lim_\alpha \langle \Delta_{\mathrm{UCB}} S(M_\alpha|_I \cdot a), h\rangle \\ &= \lim_\alpha \langle \Delta_{\mathrm{UCB}}(N_\alpha \cdot a), h\rangle.\end{aligned}$$

Hence $(N_\alpha)_\alpha$ is an approximate $\ell^\infty(G \times G)_{\mathrm{ess}}$-virtual diagonal for $\ell^1(G)$.

(ii)⇒(iii) holds by Proposition 3.2. □

Note that, it is not clear for us that the map S in Remark 3.3 is always an $L^1(G)$-bimodule map. If S is an $L^1(G)$-bimodule map, then the previous theorem holds not only for discrete groups but also for each locally compact group G.

References

[BD73] Bonsall, Frank F.; Duncan, John. Complete normed algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80. *Springer-Verlag, New York-Heidelberg*, 1973. x+301 pp. MR0423029, Zbl 0271.46039.

[CSS15] Choi, Yemon; Samei, Ebrahim; Stokke, Ross. Extension of derivations, and Connes-amenability of the enveloping dual Banach algebra. *Math. Scand.* **117** (2015), no. 2, 258–303. MR3437210, Zbl 1328.47041, arXiv:1307.6287, doi: 10.7146/math.scand.a-22870.

[ESM12] Esslamzadeh, Gholam Hossein; Shojaee, Behrouz; Mahmoodi, A. Approximate Connes-amenability of dual Banach algebras. *Bull. Belg. Math. Soc. Simon Stevin* **19** (2012), 193–213. MR2977226, Zbl 06050933, arXiv:0908.3566.

[GL04] Ghahramani, Fereidoun; Loy, Richard J. Generalized notions of amenability. *J. Funct. Anal.* **208** (2004), 229–260. MR2034298, Zbl 1045.46029, doi: 10.1016/S0022-1236(03)00214-3.

[Joh72] Johnson, Barry Edward. Cohomology in Banach algebras. Memoirs of the American Mathematical Society, 127. *American Mathematical Society, Providence, R.I.*, 1972. iii+96 pp. MR374934, Zbl 0256.18014.

[Ll97] Lau, Anthony To-Ming; Loy, Richard J. Weak amenability of Banach algebras on locally compact groups. *J. Funct. Anal.* **145** (1997), 175–204. MR1442165, Zbl 0890.46036, doi: 10.1006/jfan.1996.3002.

[Run01] Runde, Volker. Amenability for dual Banach algebras. *Studia Math.* **148** (2001), 47–66. MR1881439, Zbl 1003.46028, arXiv:math/0203199, doi: 10.4064/sm148-1-5.

[Run04] Runde, Volker. Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule. *Math. Scand.* **95** (2004), 124–144. MR2091485, Zbl 1087.46035, arXiv:math/0305431, doi: 10.7146/math.scand.a-14452.

[Run02] Runde, Volker. Lectures on amenability. Lecture Notes in Mathematics, 1774. *Springer-Verlag, Berlin*, 2002. MR1874893, Zbl 0999.46022, doi: 10.1007/b82937.

(Ahmad Shirinkalam) Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran
shirinkalam_a@aut.ac.ir

(Abdolrasoul Pourabbas) Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran
arpabbas@aut.ac.ir

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