

Statistical convergence and operators on Fock space

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ABSTRACT. In this paper, we introduce the notion of discrete statistical Borel convergence. Also, we give necessary and sufficient condition under which a series with bounded sequence of complex numbers is discrete statistically Borel convergent. Moreover, we present in terms of Berezin symbols some characterization Schatten–von Neumann class operators.

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1. Introduction

The concept of statistical convergence was first defined for real numbers by Fast [Fas51] and Steinhaus [Ste51], as a generalization of ordinary convergence. In what follows statistical convergence has been discussed by many authors (see, for example, [GüY15, Fas51, Fri85, Ste51]). In the paper [PK04] the notion of discrete statistical Abel convergence was introduced and obtained important results. Recently, solving of some problems in operator theory has been studied by using the concepts of statistical convergence and Berezin symbols [GüY15]. Namely, they show that under which conditions the weak statistical limit of compact operators is compact.

Let $K \subset \mathbb{N}$ and $\delta(K) := \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in K : k \leq n\}|$ denote the natural density of set $K_n = \{k \in K : k \leq n\}$, where the vertical bars denote number of elements of K_n . A sequence $x = (x_k)_{k \in \mathbb{N}}$ of real (or complex) numbers is said to be statistically convergent to α provided that for every $\varepsilon > 0$, natural density of the set $\{k \in \mathbb{N} : |x_k - \alpha| \geq \varepsilon\}$ is zero. If $(x_k)_{k \in \mathbb{N}}$ is statistically convergent to α we write $st\text{-}\lim x_k = \alpha$.

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We recall that (see [Fri85]) a sequence $x = (x_k)_{k \in \mathbb{N}}$ of real (or complex) numbers is said to be statistically Cauchy provided that for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \varepsilon\}| = 0.$$

The notion “ $x_k = y_k$ for almost all k ” for two sequences $x = \{x_k\}$ and $y = \{y_k\}$ means that the natural density of set $\{k \in \mathbb{N} : x_k \neq y_k\}$ is zero [Fri85].

The following theorem was proved by Fridy [Fri85].

Theorem 1. *The following conditions are equivalent:*

- (i) $\{x_n\}$ is a statistically convergent sequence.
- (ii) $\{x_n\}$ is a statistically Cauchy sequence.
- (iii) $\{x_n\}$ is a sequence for which there is a convergent sequence $\{y_n\}$ such that $x_n = y_n$ for almost all n .

The immediate result of this theorem is following.

Corollary 1. *If $\{x_n\}$ is a statistically convergent to α , then $\{x_n\}$ has a subsequence $\{y_n\}$ such that $\lim_n y_n = \alpha$.*

A series $\sum_{n=0}^{\infty} x_n$ is said to be statistically convergent to α provided that the sequence of its partial sums (s_k) converges statistically to α , that is, for every $\varepsilon > 0$, the natural density of set $\{k \in \mathbb{N} : |s_k - \alpha| \geq \varepsilon\}$ is zero. In this case we abbreviate $st\text{-}\lim s_k = \alpha$.

Following by [Str97], note that the Fock space (or Segal–Bargmann space) is the space of entire functions that are square-integrable with respect to Gaussian measure on the complex plane, that is, the space of all analytic functions f on \mathbb{C} for which

$$\int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty,$$

where $d\mu(z) = e^{-\frac{|z|^2}{2}} \frac{dm(z)}{2}$. The space $\mathcal{F} := \mathcal{F}(\mathbb{C})$ is closed subspace of the space $L^2(\mathbb{C}, d\mu)$ with inner product given by

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z) \overline{g(z)} d\mu(z), \quad f, g \in L^2(\mathbb{C}, d\mu),$$

and thus is a Hilbert space. The functions z^n ($n \geq 0$) are orthogonal in \mathcal{F} and their linear span is clearly dense in \mathcal{F} . We get by using polar coordinates

$$\|z^n\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |z|^{2n} e^{-\frac{|z|^2}{2}} dA(z) / 2 = \int_0^{\infty} \int_0^{2\pi} r^{2n+1} e^{-r^2/2} dr d\theta / 2\pi$$

$$= \int_0^\infty x^n e^{-\frac{x}{2}} d\frac{x}{2} = 2^n \int_0^\infty t^n e^{-t} dt = n!2^n,$$

which shows that the sequence $\left\{ \frac{z^n}{(n!2^n)^{1/2}} : n \geq 0 \right\}$ is an orthonormal basis in \mathcal{F} . Let P denote orthogonal projection of $L^2(\mathbb{C}, d\mu)$ onto \mathcal{F} . For a function $f \in L^\infty(\mathbb{C})$, the Toeplitz operator $T_f : \mathcal{F} \rightarrow \mathcal{F}$ is defined by

$$T_f(g) = P(fg), \quad g \in \mathcal{F}.$$

By a reproducing kernel Hilbert space (shortly, RKHS) we mean a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of functions on some set Ω such that the linear functional (evaluation functional) $f \rightarrow f(\lambda)$ is bounded on \mathcal{H} for every $\lambda \in \Omega$. If \mathcal{H} is RKHS on set Ω , then by the classical Riesz Representation Theorem there is a function $k_\lambda : \Omega \times \Omega \rightarrow \mathbb{C}$ with defining property $f(\lambda) = \langle f, k_{\mathcal{H},\lambda} \rangle$ for all $\lambda \in \Omega$ and $f \in \mathcal{H}$. We call the family $\{k_\lambda : \lambda \in \Omega\}$ the reproducing kernel of the space \mathcal{H} . As is well known (see [Aro50]) if $\{e_n(z)\}_{n \geq 0}$ is an orthonormal basis for $\mathcal{H}(\Omega)$, then the reproducing kernel of $\mathcal{H}(\Omega)$ is defined as

$$(*) \quad k_{\mathcal{H},\lambda}(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z).$$

We denote the normalized reproducing kernel of the space \mathcal{H} by

$$\widehat{k}_{\mathcal{H},\lambda} = \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}.$$

The prototypical RKHSs are, for example, Hardy–Hilbert space $H^2(\mathbb{D})$ over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, Bergman–Hilbert space $L_a^2(\mathbb{D})$ and Fock–Hilbert space $\mathcal{F}(\mathbb{C})$. An extensive information of the theory of RKHSs is given, for example, in Aronzajn [Aro50], Saitoh [Sai88], Stroethoff [Str97], and Guediri et al. [GuGS15].

For a bounded linear operator T on the RKHS \mathcal{H} , its Berezin symbol \widetilde{T} is defined as

$$\widetilde{T}(\lambda) := \left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle_{\mathcal{H}} \quad (\lambda \in \Omega).$$

Since $|\widetilde{T}(\lambda)| \leq \|T\|$ for all $\lambda \in \Omega$, Berezin symbol of T is a bounded function. Further developments about reproducing kernels and Berezin symbols can be found in the literature [Ber72, KS05, K12, Zhu90].

In this article, by using Berezin symbols, we give some conditions for *st*-(*DB*) convergence of a series of complex numbers. Also, we characterize the Schatten–von Neumann operator classes in terms of their Berezin symbols.

2. New Results

Note that diagonal operator D_x on $\mathcal{F}(\mathbb{C})$ for any bounded sequence $x = (x_n)_{n \geq 0}$ of complex numbers is defined by the formula

$$D_x \frac{z^n}{(n!2^n)^{1/2}} := x_n \frac{z^n}{(n!2^n)^{1/2}}, \quad n = 0, 1, 2, \dots,$$

with respect to the orthonormal basis $\left(\frac{z^n}{(n!2^n)^{1/2}} \right)_{n \geq 0}$ of $\mathcal{F}(\mathbb{C})$.

Definition 1. Let $x = \{x_n\}_{n \geq 0}$ be a sequence of complex numbers. A series $\sum_n x_n$ is discretely statistically Borel convergent to α , provide that for all $t \in \mathbb{R}^+$, the series

$$f(t) = \sum_n x_n \frac{t^n}{n!}$$

is statistically convergent and $\alpha = st\text{-}\lim_m f(t_m)$ whenever a sequence $\{t_m\}$ is statistically convergent to 1 in $(0, +\infty)$.

The discretely statistically Borel convergent is denoted by $st\text{-}(DB)$ convergent.

Now we are ready to give main result of this section:

Theorem 2. Let $x = \{x_n\}_{n \geq 0}$ be a bounded sequence of complex numbers.

Then the series $\sum_{n=0}^{\infty} x_n$ is $st\text{-}(DB)$ convergent if and only if

$$st\text{-}\lim_m \frac{\tilde{D}_x(\sqrt{2t_m})}{e^{-t_m}}$$

is finite whenever a sequence $\{t_m\}$ is statistically convergent to 1 in $(0, +\infty)$.

Proof. As $\{x_n\}_{n \geq 0}$ is a bounded sequence, evidently, the diagonal operator D_x is bounded in \mathcal{F} . Now we compute the Berezin symbol of the diagonal operator D_x in the Fock space \mathcal{F} :

$$\begin{aligned} \tilde{D}_x(\lambda) &= \langle D_x \hat{k}_\lambda, \hat{k}_\lambda \rangle \\ &= e^{-|\lambda|^2/2} \langle D_x k_\lambda, k_\lambda \rangle \\ &= e^{-|\lambda|^2/2} \left\langle D_x \sum_{n=0}^{\infty} \frac{(\bar{\lambda}z/2)^n}{n!}, \sum_{n=0}^{\infty} \frac{(\bar{\lambda}z/2)^n}{n!} \right\rangle \\ &= e^{-|\lambda|^2/2} \left\langle \sum_{n=0}^{\infty} x_n \frac{(\bar{\lambda}z/2)^n}{n!}, \sum_{n=0}^{\infty} \frac{(\bar{\lambda}z/2)^n}{n!} \right\rangle \\ &= e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} x_n \frac{(|\lambda|^2/2)^n}{n!}. \end{aligned}$$

Therefore,

$$\tilde{D}_x(\lambda) = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} x_n \frac{(|\lambda|^2/2)^n}{n!}, \quad \lambda \in \mathbb{C}.$$

Let $\frac{|\lambda|^2}{2} = t$. Then

$$(1) \quad \tilde{D}_x(\sqrt{2t}) = e^{-t} \sum_{n=0}^{\infty} x_n \frac{t^n}{n!}.$$

From equality (1),

$$\sum_{n=0}^{\infty} x_n \frac{t^n}{n!} = \frac{\tilde{D}_x(\sqrt{2t})}{e^{-t}}$$

for all $t \in \mathbb{R}^+$. Since $|\tilde{D}_x(\sqrt{2t})| \leq \|D_x\|$, the series $\sum_{n=0}^{\infty} x_n \frac{t_m^n}{n!}$ converges for each $t_m \in \mathbb{R}^+$, and hereby, statistically converges. Beside, we obtain from equality (1) that

$$st\text{-}\lim_m \sum_{n=0}^{\infty} x_n \frac{t_m^n}{n!}$$

is finite, whenever a sequence $\{t_m\}$ is statistically convergent to 1 in $(0, +\infty)$, if and only if

$$st\text{-}\lim_m \frac{\tilde{D}_x(\sqrt{2t_m})}{e^{-t_m}}$$

is finite, whenever a sequence $\{t_m\}$ is statistically convergent to 1 in $(0, +\infty)$. This completes the proof. \square

Before giving the next results, we need some definitions.

For the compact operator A on the Hilbert space H , the n^{th} s -number (or singular value) of A is the n^{th} largest eigenvalue of the operator $(A^*A)^{1/2}$, where each eigenvalue repeats according to its multiplicity. If necessary, the numbers will be appended by 0s to form an infinite sequence. We denote the n^{th} s -number of A by $s_n(A)$ and the set of all compact operators on H by $\mathfrak{G}_\infty = \mathfrak{G}_\infty(H)$. The Schatten-von Neumann class $\mathfrak{G}_p = \mathfrak{G}_p(H)$, $0 < p < +\infty$, is formed by the operators A on H satisfying the condition

$$\sum_{n=0}^{\infty} (s_n(A))^p < +\infty.$$

The space $\mathfrak{G}_p(H)$ ($1 \leq p < +\infty$) is a Banach space with the norm

$$\|A\|_{\mathfrak{G}_p}^2 := \left| \sum_{n=0}^{\infty} (s_n(A))^p \right|^{1/p}.$$

For $p = 2$, the operator $A \in \mathfrak{G}_2(H)$ is called Hilbert–Schmidt operator. It is well known that (see [GoK69])

$$(2) \quad \|A\|_{\mathfrak{G}_2}^2 := \sum_{n=0}^{\infty} \|Ae_n\|_H^2,$$

where $(e_n)_{n \geq 0}$ is any orthonormal basis in H .

Theorem 3. *Let H be an infinite dimensional complex Hilbert space, and $A \in \mathfrak{G}_\infty(H)$. Let $\{s_k(A)\}_{k \geq 0}$ be a nonincreasing sequence of its s -numbers. Then the operator A pertains to \mathfrak{G}_p , $0 < p < \infty$, if and only if*

$$\tilde{D}_{\Lambda_p}(\sqrt{2t}) = O(e^{-t}) \quad \text{as } t \rightarrow \infty,$$

where $\Lambda_p = \{s_k(A)^p\}_{k \geq 0}$.

Proof. As is well known, if $x_k \geq 0$ for k adequately large, then usual Borel convergence of the series $\sum_{k=0}^{\infty} x_k$ implies the convergence of the series $\sum_{k=0}^{\infty} x_k$.

As a result, since $s_k(A) \geq 0$, $k \geq 0$, the series $\sum_{k=0}^{\infty} s_k(A)^p$ is convergent if and

only if the series $\sum_{k=0}^{\infty} s_k(A)^p$ is Borel convergent. Beside, by above equality

(1) we obtain

$$\sum_{k=0}^{\infty} s_k(A)^p \frac{t^k}{k!} = \frac{\tilde{D}_{\Lambda_p}(\sqrt{2t})}{e^{-t}}$$

which means that the Borel convergence of $\sum_{k=0}^{\infty} s_k(A)^p$ is equivalent to the assertion that

$$\lim_{t \rightarrow \infty} \frac{\tilde{D}_{\Lambda_p}(\sqrt{2t})}{e^{-t}}$$

is finite. This proves the theorem. \square

3. Characterization of $\mathfrak{G}_p(\mathcal{F})$ -class operators on Fock space

This section was basically motivated by the question posed by Nordgren and Rosenthal [NR94]:

How are Schatten–von Neumann class operators characterized in terms of their Berezin symbols?

Now we study important applications of Berezin symbols in description of Schatten–von Neumann classes of compact operators, and hence, we present some particular answers to this question in the Fock space \mathcal{F} .

Proposition 1. *Assume $A \in \mathfrak{G}_2(\mathcal{F})$. Then*

$$\|A\|_{\mathfrak{G}_2} = \left[\int_{\mathbb{C}} \widetilde{A^*A}(\lambda) d\mu(\lambda) \right]^{1/2} = \left[\int_{\mathbb{C}} \|A\widehat{k}_\lambda\|^2 d\mu(\lambda) \right]^{1/2},$$

where $d\mu(\lambda) = e^{|\lambda|^2/2} dm(\lambda)$.

Proof. Indeed, since $k_{\mathcal{F},\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\bar{\lambda}z/2)^n}{n!} = e^{\bar{\lambda}z/2}$, $\lambda \in \mathbb{C}$, is the reproducing kernel of the Fock space \mathcal{F} , we obtain

$$\begin{aligned}
 \|A\|_{\mathfrak{G}_2}^2 &= \sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{n=1}^{\infty} \langle Ae_n, Ae_n \rangle \\
 &= \sum_{n=1}^{\infty} \langle A^* Ae_n, e_n \rangle = \sum_{n=1}^{\infty} \int_{\mathbb{C}} (A^* Ae_n)(\lambda) \overline{e_n(\lambda)} dm(\lambda) \\
 &= \sum_{n=1}^{\infty} \int_{\mathbb{C}} \langle A^* Ae_n, k_\lambda \rangle \overline{e_n(\lambda)} dm(\lambda) \\
 &= \int_{\mathbb{C}} \left\langle A^* A \sum_{n=1}^{\infty} e_n(z) \overline{e_n(\lambda)}, k_\lambda \right\rangle dm(\lambda) \\
 &= \int_{\mathbb{C}} \langle A^* Ak_\lambda, k_\lambda \rangle dm(\lambda) \\
 &= \int_{\mathbb{C}} \langle A^* A \widehat{k}_\lambda, \widehat{k}_\lambda \rangle e^{|\lambda|^2/2} dm(\lambda) \\
 &= \int_{\mathbb{C}} \widetilde{A^* A}(\lambda) d\mu(\lambda) = \int_{\mathbb{C}} \|A \widehat{k}_\lambda\|^2 d\mu(\lambda)
 \end{aligned}$$

for any orthonormal basis $\{e_n(z)\}_{n \geq 1}$ of \mathcal{F} , which completes the proof. \square

Theorem 4. Suppose A is a compact operator on the Fock space \mathcal{F} . Then A is a Hilbert–Schmidt operator if and only if

$$\sup_{\lambda \in \mathbb{C}} e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left(T \frac{z^n}{e^{\bar{\lambda}z/2}} A^* A T \frac{z^n}{e^{\bar{\lambda}z/2}} \right) (\lambda) < +\infty.$$

Proof. As $\left\{ \frac{z^n}{(n!2^n)^{1/2}} \right\}_{n \geq 0}$ is an orthonormal basis in the Fock space \mathcal{F} and $k_\lambda(z) = e^{\bar{\lambda}z/2}$ is the reproducing kernel of \mathcal{F} , we obtain for any $\lambda \in \mathbb{C}$

$$\begin{aligned}
 \sum_{n=0}^{\infty} \left\| A \frac{z^n}{(n!2^n)^{1/2}} \right\|_{\mathcal{F}}^2 &= \sum_{n=0}^{\infty} \frac{1}{n!2^n} \langle Az^n, Az^n \rangle \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!2^n} \langle A^* Az^n, z^n \rangle \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left\langle A^* A \frac{z^n}{\widehat{k}_\lambda(z)} \widehat{k}_\lambda(z), \frac{z^n}{\widehat{k}_\lambda(z)} \widehat{k}_\lambda(z) \right\rangle
 \end{aligned}$$

$$\begin{aligned}
&= e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left\langle A^* A \frac{z^n}{e^{\bar{\lambda}z/2}} \widehat{k}_\lambda(z), \frac{z^n}{e^{\bar{\lambda}z/2}} \widehat{k}_\lambda(z) \right\rangle \\
&= e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left\langle A^* AT \frac{z^n}{e^{\bar{\lambda}z/2}} \widehat{k}_\lambda(z), T \frac{z^n}{e^{\bar{\lambda}z/2}} \widehat{k}_\lambda(z) \right\rangle \\
&= e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left\langle T \frac{z^n}{e^{\bar{\lambda}z/2}} A^* AT \frac{z^n}{e^{\bar{\lambda}z/2}} \widehat{k}_\lambda(z), \widehat{k}_\lambda(z) \right\rangle \\
&= e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left(T \frac{z^n}{e^{\bar{\lambda}z/2}} A^* AT \frac{z^n}{e^{\bar{\lambda}z/2}} \right)^{\sim}(\lambda).
\end{aligned}$$

Hence, by taking into consideration (2), we obtain

$$\sum_{n=0}^{\infty} (s_n(A))^2 = e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left(T \frac{z^n}{e^{\bar{\lambda}z/2}} A^* AT \frac{z^n}{e^{\bar{\lambda}z/2}} \right)^{\sim}(\lambda)$$

for all $\lambda \in \mathbb{C}$, from which it is concluded that $A \in \mathfrak{G}_2(\mathcal{F})$ if and only if

$$\sup_{\lambda \in \mathbb{C}} e^{|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{1}{n!2^n} \left(T \frac{z^n}{e^{\bar{\lambda}z/2}} A^* AT \frac{z^n}{e^{\bar{\lambda}z/2}} \right)^{\sim}(\lambda) < +\infty.$$

This completes the proof. \square

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