A long exact sequence for homology of FI-modules

Wee Liang Gan

ABSTRACT. We construct a long exact sequence involving the homology of an FI-module. Using the long exact sequence, we give two methods to bound the Castelnuovo–Mumford regularity of an FI-module which is generated and related in finite degree. We also prove that for an FI-module which is generated and related in finite degree, if it has a nonzero higher homology, then its homological degrees are strictly increasing (starting from the first homological degree).

CONTENTS

1. Introduction

This article studies homological aspects of the theory of FI-modules. We begin by recalling a few definitions from [1], [2], and [3].

Let $\mathbb{Z}_+$ be the set of non-negative integers. Let $\Bbbk$ be a commutative ring. Let FI be the category whose objects are the finite sets and whose morphisms are the injective maps. An FI-*module* is a functor from FI to the category of $\Bbbk$-modules. For any FI-module V and finite set X, we shall write V_X for $V(X)$.

Suppose V is an FI-module. For any finite set X, let $(JV)_X$ be the $\Bbbk$-submodule of V_X spanned by the images of the maps $f_* : V_Y \to V_X$ for all injections $f : Y \to X$ with $|Y| < |X|$. Then JV is an FI-submodule of V. Let

$$F(V) := V/JV.$$

Then F is a right exact functor from the category of FI-modules to itself. Following [1] and [2], for any $a \in \mathbb{Z}_+$, the *FI-homology functor* H_a is defined to be the a-th left derived functor of F.

Fix a one-element set $\{\star\}$ and define a functor $\sigma : \mathrm{FI} \to \mathrm{FI}$ by

$$X \mapsto X \sqcup \{\star\}.$$

If $f : X \to Y$ is a morphism in FI, then $\sigma(f) : X \sqcup \{\star\} \to Y \sqcup \{\star\}$ is the map $f \sqcup \mathrm{id}_{\{\star\}}$. Following [3, Definition 2.8], the *shift functor* S from the category of FI-modules to itself is defined by $SV = V \circ \sigma$ for every FI-module V.

Suppose V is an FI-module. For any finite set X, one has

$$(SV)_X = V_{X \sqcup \{\star\}}.$$

There is a natural FI-module homomorphism

$$\iota : V \to SV$$

where the maps $\iota_X : V_X \to (SV)_X$ are defined by the inclusion maps

$$X \hookrightarrow X \sqcup \{\star\}.$$

We denote by DV the cokernel of $\iota : V \to SV$. Following [1], we call the functor $D : V \mapsto DV$ the *derivative functor* on the category of FI-modules.

Our main result is the following.

Theorem 1. *Let V be an* FI-*module. Then there is a long exact sequence:*

The proof of Theorem 1 will be given in Section 2.

As applications of Theorem 1, we give in Section 3 two methods to bound from above the Castelnuovo–Mumford regularity of an FI-module which is

generated and related in finite degree. The first method, using the derivative functor D, gives a new proof of the bound first found by Church and Ellenberg [1, Theorem A]. The second method, using the shift functor S, gives a bound which is always less than or equal to one found recently by Li and Ramos [7, Theorem 5.20]. Along the way, we use Theorem 1 to reprove a few results of Li and Yu [8], and Ramos [10]. We also prove that for an FI-module which is generated and related in finite degree, if it has a nonzero higher homology, then its homological degrees are strictly increasing (starting from the first homological degree).

Although some of the results in Section 3 are known, our proofs based on Theorem 1 seem to be more direct than previous proofs. We also expect that our proofs can be easily generalized to FI_G-modules.

2. The long exact sequence

2.1. A Koszul complex. Let V be an FI-module. The FI-homology of V can be computed from a Koszul complex $\widetilde{S}_{-\bullet}V$ first defined in [3, (11)]. We recall the construction of this complex following [4, Section 2].

For any finite set I, let $\Bbbk I$ be the free $\Bbbk$-module with basis I, and $\det(I)$ the free $\Bbbk$-module $\bigwedge^{|I|}\Bbbk I$ of rank one; by convention, if $I = \emptyset$, then $\det(I) = \Bbbk$. If $I = \{i_1, \ldots, i_a\}$, then $i_1 \wedge \cdots \wedge i_a$ is a basis for $\det(I)$.

Suppose X is a finite set and Y is a subset of X. If $i \in X \setminus Y$ and $v \in V_Y$, we shall write $i(v)$ for the element $f_*(v) \in V_{Y \cup \{i\}}$ where $f : Y \hookrightarrow Y \cup \{i\}$ is the inclusion map. For any $a \in \mathbb{Z}_+$, let

$$(\widetilde{S}_{-a}V)_X := \bigoplus_{\substack{I \subset X \\ |I|=a}} V_{X \setminus I} \otimes_{\Bbbk} \det(I).$$

Suppose X and X' are finite sets and $f : X \to X'$ is an injective map. For any $I \subset X$, the map f restricts to an injective map $f \mid_{X \setminus I} : X \setminus I \to X' \setminus f(I)$. We define

$$f_* : (\widetilde{S}_{-a}V)_X \to (\widetilde{S}_{-a}V)_{X'}$$

by the formula

$$f_*(v \otimes i_1 \wedge \cdots \wedge i_a) := (f \mid_{X \setminus I})_*(v) \otimes f(i_1) \wedge \cdots \wedge f(i_a),$$

where $v \in V_{X \setminus I}$, $(f \mid_{X \setminus I})_*(v) \in V_{X' \setminus f(I)}$, and $I = \{i_1, \ldots, i_a\}$. This defines, for each $a \in \mathbb{Z}_+$, an FI-module $\widetilde{S}_{-a}V$.

The differential $d : (\widetilde{S}_{-a}V)_X \to (\widetilde{S}_{-a+1}V)_X$ is defined on each direct summand by the formula

$$d(v \otimes i_1 \wedge \cdots \wedge i_a) := \sum_{p=1}^{a}(-1)^p\, i_p(v) \otimes i_1 \wedge \cdots \widehat{i_p} \cdots \wedge i_a,$$

where $v \in V_{X \setminus I}$, $I = \{i_1, \ldots, i_a\}$, and $\widehat{i_p}$ means that i_p is omitted in the wedge product. We obtain a complex $\widetilde{S}_{-\bullet}V$ of FI-modules:

$$\cdots \longrightarrow \widetilde{S}_{-2}V \longrightarrow \widetilde{S}_{-1}V \longrightarrow \widetilde{S}_0 V \longrightarrow 0.$$

The following theorem was independently proved in [1] and [4].

Theorem 2. *Let V be an* FI-*module. Then there is an* FI-*module isomorphism*

$$H_a(V) \cong H_a(\widetilde{S}_{-\bullet}V) \quad \text{for each } a \in \mathbb{Z}_+.$$

Proof. See [1, Proposition 5.10 and proof of Theorem B], or [4, Theorem 1 and Remark 4]. ☐

Applying the shift functor S to the complex $\widetilde{S}_{-\bullet}V$, we obtain a complex $S\widetilde{S}_{-\bullet}V$. Since S is an exact functor, it is immediate from Theorem 2 that one has an isomorphism

$$SH_a(V) \cong H_a(S\widetilde{S}_{-\bullet}V) \quad \text{for each } a \in \mathbb{Z}_+.$$

2.2. Proof of Theorem 1. Let V be an FI-module. The homomorphism $\iota : V \to SV$ defines, in the obvious way, a morphism of complexes $\widetilde{\iota} : \widetilde{S}_{-\bullet}V \to \widetilde{S}_{-\bullet}SV$. By a standard result in homological algebra [12, Section 1.5], Theorem 1 is immediate from Theorem 2 and the following lemma.

Lemma 3. *Let V be an* FI-*module. Then the complex $S\widetilde{S}_{-\bullet}V$ is isomorphic to the mapping cone of $\widetilde{\iota} : \widetilde{S}_{-\bullet}V \to \widetilde{S}_{-\bullet}SV$.*

Proof. We shall follow standard notations (found, for example, in [12, Section 1.5]) and write the mapping cone of $\widetilde{\iota}$ as

$$\text{cone}(\widetilde{\iota}) = (\widetilde{S}_{-\bullet}V)[-1] \oplus \widetilde{S}_{-\bullet}SV.$$

To define a homomorphism

$$\phi : \text{cone}(\widetilde{\iota}) \longrightarrow S\widetilde{S}_{-\bullet}V,$$

we need to define, for each finite set X, a homomorphism of complexes

$$\phi_X : \text{cone}(\widetilde{\iota})_X \longrightarrow (S\widetilde{S}_{-\bullet}V)_X.$$

Suppose $a > 0$. The degree a component of $(\widetilde{S}_{-\bullet}V)_X[-1]$ is $(\widetilde{S}_{-(a-1)}V)_X$. For any $I = \{i_1, \ldots, i_{a-1}\} \subset X$ and $v \in V_{X \setminus I}$, we have the element

$$v \otimes i_1 \wedge \cdots \wedge i_{a-1} \in (\widetilde{S}_{-(a-1)}V)_X;$$

let

$$\phi_X(v \otimes i_1 \wedge \cdots \wedge i_{a-1}) := v \otimes ((\star) \wedge i_1 \wedge \cdots \wedge i_{a-1}) \in (\widetilde{S}_{-a}V)_{X \sqcup \{\star\}}.$$

Here, we used $X \setminus I = (X \sqcup \{\star\}) \setminus (I \sqcup \{\star\})$ to see that the element v on the right hand side is an element of $V_{(X \sqcup \{\star\}) \setminus (I \sqcup \{\star\})}$.

Suppose $a \geqslant 0$. For any $I = \{i_1, \ldots, i_a\} \subset X$ and $v \in (SV)_{X \setminus I}$, we have the element $v \otimes i_1 \wedge \cdots \wedge i_a \in (\widetilde{S}_{-a}SV)_X$; let

$$\phi_X(v \otimes i_1 \wedge \cdots \wedge i_a) := v \otimes i_1 \wedge \cdots \wedge i_a \in (\widetilde{S}_{-a}V)_{X \sqcup \{\star\}}.$$

Here, we used $(SV)_{X \setminus I} = V_{(X \sqcup \{\star\}) \setminus I}$ to see that the element v on the right hand side is an element of $V_{(X \sqcup \{\star\}) \setminus I}$.

By a routine verification, the above defines a homomorphism ϕ of complexes of FI-modules; the key calculation here is the following: for any $I = \{i_1, \ldots, i_{a-1}\} \subset X$ and

$$v \otimes i_1 \wedge \cdots \wedge i_{a-1} \in (\widetilde{S}_{-(a-1)}V)_X,$$

one has:

$$\phi_X\left(d_{\mathrm{cone}(\widetilde{\iota})}(v \otimes i_1 \wedge \cdots \wedge i_{a-1})\right)$$

$$= \phi_X\left(-\iota_{X \setminus I}(v) \otimes i_1 \wedge \cdots \wedge i_{a-1} - \sum_{p=1}^{a-1}(-1)^p \, i_p(v) \otimes i_1 \wedge \cdots \widehat{i_p} \cdots \wedge i_{a-1}\right)$$

$$= -\iota_{X \setminus I}(v) \otimes i_1 \wedge \cdots \wedge i_{a-1} - \sum_{p=1}^{a-1}(-1)^p \, i_p(v) \otimes ((\star) \wedge i_1 \wedge \cdots \widehat{i_p} \cdots \wedge i_{a-1})$$

$$= d\left(v \otimes ((\star) \wedge i_1 \wedge \cdots \wedge i_{a-1})\right)$$

$$= d(\phi_X(v \otimes i_1 \wedge \cdots \wedge i_{a-1})).$$

It is plain that ϕ is bijective and hence an isomorphism.   □

A special case of Lemma 3 appeared in [4, proof of Proposition 6].

3. Applications

3.1. Definitions and notations.
We recall some definitions from [1] and [5].

Let V be any FI-module. For any $n \in \mathbb{Z}_+$, we set $\mathbf{n} := \{1, \ldots, n\}$ (in particular, $\mathbf{0} = \emptyset$). We shall use the convention that the supremum and infimum of an empty set are $-\infty$ and ∞, respectively.

The *degree* $\deg(V)$ of V is

$$\deg(V) := \sup\{n \in \mathbb{Z}_+ \mid V_{\mathbf{n}} \neq 0\}.$$

The *lowest degree* $\mathrm{low}(V)$ of V is

$$\mathrm{low}(V) := \inf\{n \in \mathbb{Z}_+ \mid V_{\mathbf{n}} \neq 0\}.$$

For any $a \in \mathbb{Z}_+$, the *a-th homological degree* $\mathrm{hd}_a(V)$ of V is

$$\mathrm{hd}_a(V) := \deg H_a(V).$$

The *Castelnuovo–Mumford regularity* $\mathrm{reg}(V)$ of V is the infimum of the set of all $c \in \mathbb{Z}$ such that

$$\mathrm{hd}_a(V) \leqslant c + a \quad \text{for every integer } a \geqslant 1.$$

The *torsion degree* $\mathrm{td}(V)$ of V is the supremum of the set of all $n \in \mathbb{Z}_+$ such that there exists a nonzero $v \in V_{\mathbf{n}}$ satisfying $f_*(v) = 0$ for every injection $f : \mathbf{n} \to \mathbf{n} + \mathbf{1}$.

For any $k \in \mathbb{Z}_+$, we say that V is *generated in degree* $\leqslant k$ if $\mathrm{hd}_0(V) \leqslant k$.

For any $k, d \in \mathbb{Z}_+$, we say that V is *generated in degree* $\leqslant k$ *and related in degree* $\leqslant d$ if there exists a short exact sequence

$$0 \longrightarrow W \longrightarrow P \longrightarrow V \longrightarrow 0$$

where P is a projective FI-module generated in degree $\leqslant k$ and W is an FI-module generated in degree $\leqslant d$.

Let KV be the kernel of $\iota : V \to SV$.

Let H_1^D be the first left-derived functor of the right exact functor D.

3.2. Some basic facts. We collect in the following lemma some basic facts which we shall use later.

Lemma 4. *Let V be an FI-module. Then one has the following:*

(i) *There is an isomorphism $KV \cong H_0(KV)$, and $\mathrm{td}(V) = \deg(KV)$.*

(ii) *There is an isomorphism $KV \cong H_1^D(V)$.*

(iii) *If P is a projective FI-module, then DP is a projective FI-module.*

(iv) *If V is generated in degree $\leqslant k$ where $k \in \mathbb{Z}_+$, then DV is generated in degree $\leqslant k - 1$.*

(v) *If V is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$, then DV is generated in degree $\leqslant k - 1$ and related in degree $\leqslant d - 1$.*

(vi) *If V is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$, then $\mathrm{hd}_1(V) \leqslant d$.*

(vii) *There is an isomorphism $S(DV) \cong D(SV)$.*

(viii) *If P is a projective FI-module, then SP is a projective FI-module.*

(ix) *If V is generated in degree $\leqslant k$, then SV is generated in degree $\leqslant k$.*

(x) *If V is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$, then SV is generated in degree $\leqslant k$ and related in degree $\leqslant d$.*

Proof. (i) Trivial.

(ii) See [1, Lemma 4.7(i)].

(iii) See [1, Lemma 4.7(iv)].

(iv) See [1, proof of Proposition 4.6].

(v) Follows from (iii) and (iv).

(vi) Trivial.

(vii) See [10, Lemma 3.5].

(viii) Follows from [3, Proposition 2.12].

(ix) See [3, Corollary 2.13].

(x) Follows from (viii) and (ix). $\square$

The following simple observation is sometimes useful.

Lemma 5. *Let V be an* FI*-module and let $a \in \mathbb{Z}_+$. If $n < \mathrm{low}(V) + a$, then $H_a(V)_{\mathbf{n}} = 0$.*

Proof. If $n < \mathrm{low}(V) + a$, then $(\widetilde{S}_{-a}V)_{\mathbf{n}} = 0$; hence, by Theorem 2, one has $H_a(V)_{\mathbf{n}} = 0$. $\qquad\square$

Corollary 6. *Let V be an* FI*-module and let $a \in \mathbb{Z}_+$. If $H_a(V) \neq 0$, then*
$$\mathrm{hd}_a(V) \geqslant \mathrm{low}(V) + a.$$

Proof. Immediate from Lemma 5. $\qquad\square$

3.3. FI-modules of finite degree. The following result was independently proved by Li [5, Theorem 4.8] and Ramos [10, Corollary 3.11]. Let us give a proof using Theorem 2.

Lemma 7. *Let V be an* FI*-module with $\deg(V) < \infty$. Then*
$$\mathrm{reg}(V) \leqslant \deg(V).$$

Proof. Let $a \in \mathbb{Z}_+$. If $n > \deg(V) + a$, then $(\widetilde{S}_{-a}V)_{\mathbf{n}} = 0$; hence, by Theorem 2, one has $H_a(V)_{\mathbf{n}} = 0$. $\qquad\square$

If V is a finitely generated FI-module with $\deg(V) < \infty$, then Gan and Li [4, Theorem 2] have shown that $\mathrm{reg}(V) = \deg(V)$.

3.4. Bounding regularity using the derivative functor. In bounding the Castelnuovo–Mumford regularity $\mathrm{reg}(V)$ of an FI-module V, our first strategy is to find a bound of $\mathrm{reg}(V)$ in terms of $\mathrm{reg}(DV)$, and then use recurrence to obtain a bound for $\mathrm{reg}(V)$.

Proposition 8. *Let V be an* FI*-module. Then one has:*
$$\mathrm{reg}(V) \leqslant \max\{\mathrm{hd}_1(V) - 1, \mathrm{td}(V), \mathrm{reg}(DV) + 1\}.$$

Proof. Set $c = \max\{\mathrm{hd}_1(V) - 1, \mathrm{td}(V), \mathrm{reg}(DV) + 1\}$. There is nothing to prove if $c = \infty$, so assume $c < \infty$. We need to prove that

$$\mathrm{hd}_a(V) \leqslant c + a \quad \text{for each } a \geqslant 1. \tag{1}$$

When $a = 1$, the inequality (1) holds because $\mathrm{hd}_1(V) - 1 \leqslant c$.

Suppose, for induction on a, that one has $\mathrm{hd}_{a-1}(V) \leqslant c + a - 1$ for some $a \geqslant 2$. Then
$$H_{a-1}(V)_{\mathbf{n}} = 0 \quad \text{for each } n \geqslant c + a.$$

We have two short exact sequences:
$$0 \longrightarrow KV \longrightarrow V \xrightarrow{\iota_1} V/KV \longrightarrow 0,$$
$$0 \longrightarrow V/KV \xrightarrow{\iota_2} SV \longrightarrow DV \longrightarrow 0.$$

They give two long exact sequences:
$$\cdots \longrightarrow H_a(V)_{\mathbf{n}} \xrightarrow{\iota_{1*}} H_a(V/KV)_{\mathbf{n}} \longrightarrow H_{a-1}(KV)_{\mathbf{n}} \longrightarrow \cdots,$$
$$\cdots \longrightarrow H_a(V/KV)_{\mathbf{n}} \xrightarrow{\iota_{2*}} H_a(SV)_{\mathbf{n}} \longrightarrow H_a(DV)_{\mathbf{n}} \longrightarrow \cdots.$$

Recall that $\deg(KV) = \mathrm{td}(V)$ (see Lemma 4(i)). By Lemma 7 and the inequality $c \geqslant \mathrm{td}(V)$, one has

$$H_{a-1}(KV)_{\mathbf{n}} = 0 \quad \text{for each } n \geqslant c + a.$$

By the inequality $c \geqslant \mathrm{reg}(DV) + 1$, one has

$$H_a(DV)_{\mathbf{n}} = 0 \quad \text{for each } n \geqslant c + a.$$

Since $\iota : V \to SV$ is the composition of $\iota_1 : V \to V/K$ and $\iota_2 : V/K \to SV$, the map $\iota_* : H_a(V)_{\mathbf{n}} \to H_a(SV)_{\mathbf{n}}$ is surjective for each $n \geqslant c + a$.

From Theorem 1, we have an exact sequence:

$$\cdots \longrightarrow H_a(V)_{\mathbf{n}} \xrightarrow{\iota_*} H_a(SV)_{\mathbf{n}} \longrightarrow H_a(V)_{\mathbf{n+1}} \longrightarrow H_{a-1}(V)_{\mathbf{n}} \longrightarrow \cdots .$$

It follows that $H_a(V)_{\mathbf{n+1}} = 0$ for $n \geqslant c + a$, and hence $\mathrm{hd}_a(V) \leqslant c + a.$ $\quad\square$

Finiteness of the Castelnuovo–Mumford regularity for finitely generated FI-modules over a field of characteristic zero was first proved by Sam and Snowden in [11, Corollary 6.3.5]. In the following theorem, the inequalities (2) and (5) were first proved by Church and Ellenberg in [1, Theorem 4.8 and Theorem A] via an intricate combinatorial result [1, Theorem E]. An alternative proof of (2) and (5) was subsequently given by Li in [6, Theorem 2.4] using results from [5] and [8]. (Although the papers [5], [6], and [8] worked with finitely generated FI-modules over a noetherian ring, most of the arguments in there can be adapted to our more general setting.) The proof of (2) we give below follows along the same lines as the argument in [6], and we use the crucial idea in [6] of proving the inequalities (2) and (5) simultaneously by induction on k. However, our proof of (5) via (3) and (4) is quite different from the proofs in [1] and [6].

Theorem 9. *Let V be an FI-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$. Let*

$$\mathrm{hd}_1^D(V) := \max\{\mathrm{hd}_1(D^iV) + i \mid i = 0, 1, \ldots, k\},$$
$$\mathrm{td}^D(V) := \max\{\mathrm{td}(D^iV) + i \mid i = 0, 1, \ldots, k\}.$$

Then one has the following:

$$(2) \qquad\qquad\qquad\qquad \mathrm{td}(V) \leqslant \min\{k, d\} + d - 1,$$

$$(3) \qquad\qquad\qquad\qquad \mathrm{reg}(V) \leqslant \max\{\mathrm{hd}_1^D(V) - 1, \mathrm{td}^D(V)\},$$

$$(4) \qquad \max\{\mathrm{hd}_1^D(V) - 1, \mathrm{td}^D(V)\} \leqslant \min\{k, d\} + d - 1.$$

In particular, one has:

$$(5) \qquad\qquad\qquad\qquad \mathrm{reg}(V) \leqslant \min\{k, d\} + d - 1.$$

Proof. If $V = 0$, then $\mathrm{td}(V), \mathrm{reg}(V), \mathrm{hd}_1^D(V)$, and $\mathrm{td}^D(V)$ are all equal to $-\infty$, so there is nothing to prove.

Suppose $V \neq 0$. We use induction on k. We have a short exact sequence

$$0 \longrightarrow W \longrightarrow P \longrightarrow V \longrightarrow 0$$

where P is a projective FI-module generated in degree $\leqslant k$ and W is an FI-module generated in degree $\leqslant d$. Since $H_1^D(P) = 0$ and $H_1^D(V) = KV$ (see Lemma 4(ii)), we obtain an exact sequence

$$0 \longrightarrow KV \longrightarrow DW \longrightarrow DP \longrightarrow DV \longrightarrow 0,$$

which we break up as two short exact sequences:

$$0 \longrightarrow KV \longrightarrow DW \longrightarrow DW/KV \longrightarrow 0,$$
$$0 \longrightarrow DW/KV \longrightarrow DP \longrightarrow DV \longrightarrow 0.$$

They give two long exact sequences:

$$(6) \qquad \cdots \longrightarrow H_1(DW/KV) \longrightarrow H_0(KV) \longrightarrow H_0(DW) \longrightarrow \cdots,$$
$$(7) \qquad \cdots \longrightarrow H_2(DV) \longrightarrow H_1(DW/KV) \longrightarrow 0 \longrightarrow \cdots,$$

where we used Lemma 4(iii) to see that $H_1(DP) = 0$.

By Lemma 4(v), the FI-module DV is generated in degree $\leqslant k - 1$ and related in degree $\leqslant d - 1$. Hence, we have:

$\mathrm{td}(V)$

$= \mathrm{hd}_0(KV)$ (Lemma 4(i))

$\leqslant \max\{\mathrm{hd}_0(DW), \mathrm{hd}_1(DW/KV)\}$ (6)

$\leqslant \max\{d - 1, \mathrm{hd}_2(DV)\}$ (Lemma 4(iv) and (7))

$\leqslant \max\{d - 1, \mathrm{reg}(DV) + 2\}$

$\leqslant \max\{d - 1, \min\{k - 1, d - 1\} + (d - 1) - 1 + 2\}$ (induction hypothesis)

$\leqslant \min\{k, d\} + d - 1.$

We also have:

$\mathrm{reg}(V) \leqslant \max\{\mathrm{hd}_1(V) - 1, \mathrm{td}(V), \mathrm{reg}(DV) + 1\}$ (Proposition 8)

$\qquad \leqslant \max\{\mathrm{hd}_1^D(V) - 1, \mathrm{td}^D(V)\}$ (induction hypothesis).

By Lemma 4(v) and Lemma 4(vi), for $i = 0, \ldots, k$, we have:

$$\mathrm{hd}_1(D^i V) + i - 1 \leqslant (d - i) + i - 1 \leqslant \min\{k, d\} + d - 1,$$
$$\mathrm{td}(D^i V) + i \leqslant \min\{k - i, d - i\} + (d - i) - 1 + i \leqslant \min\{k, d\} + d - 1,$$

where we used (2) for V, DV, $\ldots$, and $D^k V$. Hence,

$$\max\{\mathrm{hd}_1^D(V) - 1, \mathrm{td}^D(V)\} \leqslant \min\{k, d\} + d - 1. \qquad \square$$

3.5. Iterated shifts and vanishing of homology. Recall that the FI-homology functor H_a is, by definition, the a-th left derived functor of F. An FI-module V is *F-acyclic* if $H_a(V) = 0$ for every $a \geqslant 1$.

Lemma 10. *Let V be an* FI-*module.*

    (i) *If $a \in \mathbb{Z}_+$ and $H_a(V) = 0$, then $H_a(SV) = 0$.*
    (ii) *If V is F-acyclic, then SV is F-acyclic.*

Proof. Immediate from Theorem 1. □

We say that an FI-module V is *torsion-free* if $\mathrm{td}(V) = -\infty$. By Lemma 4(i), an FI-module V is torsion-free if and only if $KV = 0$.

The following lemma can be deduced from [8, Theorem 3.5 and Lemma 3.12] under some finiteness assumptions. We give a proof using Theorem 1.

Lemma 11. *Let V be a torsion-free* FI-*module. If DV is F-acyclic, then V is F-acyclic.*

Proof. Since $KV = 0$, there is a short exact sequence

$$0 \to V \xrightarrow{\iota} SV \to DV \to 0.$$

From the long exact sequence in homology and the F-acyclicity of DV, we deduce that:

(i) $\iota_* : H_0(V) \longrightarrow H_0(SV)$ is a monomorphism.
(ii) $\iota_* : H_a(V) \longrightarrow H_a(SV)$ is an isomorphism for each $a \geqslant 1$.

Suppose that $a \geqslant 1$. From (i), (ii), and the long exact sequence in Theorem 1, we must have $SH_a(V) = 0$, so $H_a(V)_{\mathbf{n}} = 0$ for each $n \geqslant 1$. By Lemma 5, we have $H_a(V)_{\mathbf{0}} = 0$. Therefore, $H_a(V) = 0$. □

The following result is proved in [6, Corollary 3.3] and [10, Corollary 4.11]; see also [9, Theorem A]. We adapt the argument in [8, Theorem 3.13] using (2) and Lemma 11.

Theorem 12. *Let V be an* FI-*module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$. Then $S^i V$ is F-acyclic for each $i \geqslant \min\{k, d\} + d$.*

Proof. The statement is trivial if $V = 0$.

Suppose $V \neq 0$. We prove the theorem by induction on k. Suppose $i \geqslant \min\{k, d\} + d$. By Theorem 9, the FI-module $S^i V$ is torsion-free. Using Lemma 4(v), one has:

$$
\begin{array}{ll}
S^i(DV) \text{ is } F\text{-acyclic} & \text{(induction hypothesis)} \\
\implies D(S^i V) \text{ is } F\text{-acyclic} & \text{(Lemma 4(vii))} \\
\implies S^i V \text{ is } F\text{-acyclic} & \text{(Lemma 11)}. \qquad \square
\end{array}
$$

Notation 13. If V is an FI-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$, we denote by $\mathrm{N}(V)$ the minimum $i \in \mathbb{Z}_+$ such that $S^i V$ is F-acyclic.

By Theorem 12, one has:

$$\mathrm{N}(V) \leqslant \min\{k, d\} + d.$$

The following result is proved in [8, Theorem 1.3] and [10, Theorem B]. We give another proof using Theorem 1.

Proposition 14. *Let V be an* FI-*module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$. Then V is F-acyclic if and only if there exists $s \geqslant 1$ such that $H_s(V) = 0$.*

Proof. We only have to prove that if s is an integer $\geqslant 1$ such that $H_s(V) = 0$, then V is F-acyclic. We use induction on $\mathrm{N}(V)$ (see Notation 13).

First, if $\mathrm{N}(V) = 0$, then V is F-acyclic. Next, suppose $\mathrm{N}(V) \geqslant 1$ and $H_s(V) = 0$ for some $s \geqslant 1$. By Lemma 10(i), we have $H_s(SV) = 0$. Since $\mathrm{N}(SV) = \mathrm{N}(V) - 1$, by induction hypothesis, the FI-module SV is F-acyclic.

Suppose $1 \leqslant a \leqslant s$. By Theorem 1, there are isomorphisms:

$$H_a(V) \cong SH_{a+1}(V) \cong S^2 H_{a+2}(V) \cong \cdots \cong S^{s-a} H_s(V) = 0.$$

Now suppose $a \geqslant s$. By Theorem 1, there are isomorphisms:

$$0 = H_s(V) \cong SH_{s+1}(V) \cong S^2 H_{s+2}(V) \cong \cdots \cong S^{a-s} H_a(V),$$

so $H_a(V)_\mathbf{n} = 0$ for $n \geqslant a - s$. But by Lemma 5, we also have $H_a(V)_\mathbf{n} = 0$ for $n < a$. Hence, $H_a(V) = 0$.

It follows that V is F-acyclic. $\qquad\square$

A characterization of F-acyclicity in terms of existence of a suitable filtration (called $\sharp$-filtration in [9, Definition 1.10]) is proved in [8, Theorem 1.3] and [10, Theorem B]; we do not need to use this filtration in our present article.

3.6. Bounding regularity using the shift functor. Our second strategy for bounding the Castelnuovo–Mumford regularity $\mathrm{reg}(V)$ of an FI-module V is to find a bound of $\mathrm{reg}(V)$ in terms of $\mathrm{reg}(SV)$, and then use recurrence to obtain a bound for $\mathrm{reg}(V)$. This is similar to the approach used by Li in [5, Section 4].

Proposition 15. *Let V be an* FI-*module. Then*

$$\mathrm{reg}(V) \leqslant \max\{\mathrm{hd}_1(V) - 1, \mathrm{reg}(SV) + 1\}.$$

Proof. Set $c = \max\{\mathrm{hd}_1(V) - 1, \mathrm{reg}(SV) + 1\}$. There is nothing to prove if $c = \infty$, so assume $c < \infty$.

We shall show, by induction on a, that one has:

$$\mathrm{hd}_a(V) \leqslant c + a \quad \text{for each } a \geqslant 1.$$

When $a = 1$, the inequality is immediate from the definition of c.

Assume that one has $\mathrm{hd}_{a-1}(V) \leqslant c + a - 1$ for some $a \geqslant 2$. Then $H_{a-1}(V)_\mathbf{n} = 0$ for $n \geqslant c + a$. By Theorem 1, we have an exact sequence:

$$\cdots \longrightarrow H_a(SV)_\mathbf{n} \longrightarrow H_a(V)_{\mathbf{n+1}} \longrightarrow H_{a-1}(V)_\mathbf{n} \longrightarrow \cdots.$$

Since $c \geqslant \mathrm{reg}(SV) + 1$, we have $H_a(SV)_\mathbf{n} = 0$ for $n \geqslant c + a$. Therefore $H_a(V)_{\mathbf{n+1}} = 0$ for $n \geqslant c + a$, and hence $\mathrm{hd}_a(V) \leqslant c + a$. $\qquad\square$

The following result uses Theorem 12 to ensure the existence of $\mathrm{N}(V)$ (see Notation 13).

Theorem 16. *Let V be an FI-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ where $k, d \in \mathbb{Z}_+$. Let*

$$\mathrm{hd}_1^S(V) := \max\{\mathrm{hd}_1(S^i V) + i \mid i = 0, 1, \ldots, \mathrm{N}(V)\}.$$

Then

$$\mathrm{reg}(V) \leqslant \mathrm{hd}_1^S(V) - 1.$$

Proof. We use induction on $\mathrm{N}(V)$. If $\mathrm{N}(V) = 0$, then $\mathrm{reg}(V) = -\infty$, so there is nothing to prove.

Suppose $\mathrm{N}(V) \geqslant 1$. Since $\mathrm{N}(SV) = \mathrm{N}(V) - 1$, by induction hypothesis, we have $\mathrm{reg}(SV) \leqslant \mathrm{hd}_1^S(SV) - 1$, and hence by Proposition 15, we obtain $\mathrm{reg}(V) \leqslant \mathrm{hd}_1^S(V) - 1$. $\qquad\square$

In the above theorem, one has $\mathrm{hd}_1(S^i V) < \infty$ for each i by Lemma 4.

It was proved by Li and Ramos [7, Theorem 5.18] that, for a finitely generated FI-module V over a noetherian ring, one has:

$$\mathrm{reg}(V) \leqslant \max\{\deg(H_{\mathfrak{m}}^j(V)) + j \mid j = 0, 1, \ldots\},$$

where $H_{\mathfrak{m}}^j(V)$ for $j = 0, 1, \ldots$ are the local cohomology groups of V. It would be too much of a digression for us to review the definition and properties of local cohomology groups of FI-modules; we refer the reader to the paper [7] of Li and Ramos (see [7, Definition 5.11 and Theorem E]). Let us show that the bound in Theorem 16 is always less than or equal to their bound.

Proposition 17. *Suppose that $\Bbbk$ is noetherian and V is a finitely generated FI-module over $\Bbbk$. Then one has:*

$$\mathrm{hd}_1^S(V) - 1 \leqslant \max\{\deg(H_{\mathfrak{m}}^j(V)) + j \mid j = 0, 1, \ldots\}.$$

Proof. For any finitely generated FI-module W, set

$$\varpi(W) := \max\{\deg(H_{\mathfrak{m}}^j(W)) + j \mid j = 0, 1, \ldots.\}.$$

One has $H_{\mathfrak{m}}^j(SW) \cong S H_{\mathfrak{m}}^j(W)$ for each $j \geqslant 0$ (see [7, paragraph before Corollary 5.21]); thus, one has $\varpi(SW) \leqslant \varpi(W) - 1$. Hence, for each $i \in \mathbb{Z}_+$, one has:

$$\mathrm{hd}_1(S^i V) + i - 1 \leqslant \mathrm{reg}(S^i V) + i \leqslant \varpi(S^i V) + i \leqslant \varpi(V),$$

where the second inequality is obtained by applying [7, Theorem 5.18] to $S^i V$. $\qquad\square$

It was conjectured by Li and Ramos [7, Conjecture 5.19] that if V is a finitely generated FI-module over a noetherian ring, and if V is not $\sharp$-filtered, then $\mathrm{reg}(V)$ is equal to $\max\{\deg(H_{\mathfrak{m}}^j(V)) + j \mid j = 0, 1, \ldots\}$.

3.7. Aside on generating degree and relation degree. Let V be an FI-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ for some $k, d \in \mathbb{Z}_+$, that is, there is a short exact sequence

$$0 \longrightarrow W \longrightarrow P \longrightarrow V \longrightarrow 0$$

where P is a projective FI-module generated in degree $\leqslant k$ and W is an FI-module generated in degree $\leqslant d$. It is easy to see that when such a presentation exists, we have $\mathrm{hd}_0(V) \leqslant k$ and $\mathrm{hd}_1(V) \leqslant d$. Moreover, when such a presentation exists, we can find one with $k = \mathrm{hd}_0(V)$. Whence, suppose that $k = \mathrm{hd}_0(V)$; from the long exact sequence in homology, we obtain [5, Lemma 4.4]:

(8) $$\mathrm{hd}_1(V) \leqslant \mathrm{hd}_0(W) \leqslant \max\{\mathrm{hd}_0(V), \mathrm{hd}_1(V)\}.$$

Lemma 18. *Let V be an FI-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ for some $k, d \in \mathbb{Z}_+$. Suppose that $0 \to W \to P \to V \to 0$ is a short exact sequence where P is a projective FI-module generated in degree $\leqslant \mathrm{hd}_0(V)$. If $\mathrm{hd}_0(V) \leqslant \mathrm{hd}_1(V)$, then W is generated in degree $\leqslant \mathrm{hd}_1(V)$.*

Proof. Immediate from (8). $\square$

The following result of Li and Yu [8, Corollary 3.4] says that, where FI-homology is concerned, one can frequently assume that $\mathrm{hd}_0(V) < \mathrm{hd}_1(V)$ and hence apply Lemma 18; see [10, Remark 2.16]. Let us give a proof using Proposition 14.

Lemma 19. *Let V be an FI-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ for some $k, d \in \mathbb{Z}_+$. Suppose that V is not F-acyclic. Let $r = \mathrm{hd}_1(V)$ and let U be the FI-submodule of V generated by $\bigsqcup_{n<r} V_{\mathbf{n}}$. Let $W = V/U$. Then one has the following:*

(i) *W is F-acyclic.*
(ii) *$H_a(U) \cong H_a(V)$ for each $a \geqslant 1$.*
(iii) *$\mathrm{hd}_0(U) < \mathrm{hd}_1(U)$.*

Proof. From the short exact sequence $0 \to U \to V \to W \to 0$, we obtain a long exact sequence in homology:

$$\cdots \longrightarrow H_2(W) \longrightarrow H_1(U) \longrightarrow H_1(V) \longrightarrow H_1(W) \longrightarrow H_0(U) \longrightarrow \cdots.$$

Since $\mathrm{hd}_1(V) = r$ and $\mathrm{hd}_0(U) \leqslant r - 1$, we have $\mathrm{hd}_1(W) \leqslant r$. But $\mathrm{low}(W) \geqslant r$, so by Lemma 5, we have $H_1(W)_{\mathbf{n}} = 0$ for each $n \leqslant r$. Therefore, we must have $H_1(W) = 0$.

By Proposition 14, it follows that W is F-acyclic. Hence, from the long exact sequence, we see that $H_a(U) \cong H_a(V)$ for each $a \geqslant 1$. In particular, $\mathrm{hd}_1(U) = \mathrm{hd}_1(V)$. Thus, $\mathrm{hd}_0(U) < r = \mathrm{hd}_1(U)$. $\square$

The proof of the above lemma in [8] is more elementary than the one we give here. We thought, however, that it might be worthwhile to give a

different explanation of why it is true. As observed in [6] and [10], one can use Lemma 19 to deduce the following.

Corollary 20. *Let V be an* FI-*module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ for some $k, d \in \mathbb{Z}_+$. Then one has:*

$$\operatorname{reg}(V) \leqslant \min\{\operatorname{hd}_0(V), \operatorname{hd}_1(V)\} + \operatorname{hd}_1(V) - 1.$$

If, moreover, V is not F-acyclic, then one has:

$$\operatorname{N}(V) \leqslant \min\{\operatorname{hd}_0(V), \operatorname{hd}_1(V)\} + \operatorname{hd}_1(V).$$

Proof. We may assume that V is not F-acyclic. Let U be the FI-submodule of V defined in Lemma 19 and let $W = V/U$.

By Lemma 18, the FI-module U is generated in degree $\leqslant \operatorname{hd}_0(U)$ and related in degree $\leqslant \operatorname{hd}_1(U)$. We have:

$$\operatorname{reg}(V) = \operatorname{reg}(U) \leqslant \min\{\operatorname{hd}_0(U), \operatorname{hd}_1(U)\} + \operatorname{hd}_1(U) - 1$$
$$\leqslant \min\{\operatorname{hd}_0(V), \operatorname{hd}_1(V)\} + \operatorname{hd}_1(V) - 1,$$

where the first inequality comes from applying Theorem 9 to U.

Since W is F-acyclic, it follows by Lemma 10 that $S^i W$ is F-acyclic for each $i \geqslant 0$. From the long exact sequence in homology associated to the short exact sequence $0 \to S^i U \to S^i V \to S^i W \to 0$, we deduce that:

$$\operatorname{N}(V) = \operatorname{N}(U) \leqslant \min\{\operatorname{hd}_0(U), \operatorname{hd}_1(U)\} + \operatorname{hd}_1(U)$$
$$\leqslant \min\{\operatorname{hd}_0(V), \operatorname{hd}_1(V)\} + \operatorname{hd}_1(V),$$

where the first inequality comes from applying Theorem 12 to U. $\square$

3.8. Homological degrees are strictly increasing. Besides Theorem 1, the proof of the following result also uses Theorem 12 to ensure the existence of $\operatorname{N}(V)$ (see Notation 13).

Theorem 21. *Let V be an* FI-*module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ for some $k, d \in \mathbb{Z}_+$. If V is not F-acyclic, then one has:*

$$\operatorname{hd}_1(V) < \operatorname{hd}_2(V) < \operatorname{hd}_3(V) < \cdots.$$

Proof. We use induction on $\operatorname{N}(V)$. Since V is not F-acyclic, one has $\operatorname{N}(V) > 0$. Moreover, by Proposition 14, one has $H_a(V) \neq 0$ for each $a \geqslant 1$. By Theorem 9, one has $\operatorname{hd}_a(V) < \infty$ for each $a \geqslant 1$.

Suppose first that $\operatorname{N}(V) = 1$. Then SV is F-acyclic. From Theorem 1, one has

$$SH_{a+1}(V) \cong H_a(V) \quad \text{for each } a \geqslant 1.$$

This implies $\operatorname{hd}_{a+1}(V) = \operatorname{hd}_a(V) + 1$ for each $a \geqslant 1$.

Next, suppose that $\operatorname{N}(V) > 1$. Let $a \geqslant 1$ and let $n = \operatorname{hd}_a(V)$. We need to show that $\operatorname{hd}_{a+1}(V) \geqslant n + 1$.

Suppose, on the contrary, that $\mathrm{hd}_{a+1}(V) \leqslant n$. Then $H_{a+1}(V)_{\mathbf{n+1}} = 0$. From Theorem 1, we have an exact sequence:

$$\cdots \longrightarrow H_{a+1}(V)_{\mathbf{n+1}} \longrightarrow H_a(V)_{\mathbf{n}} \longrightarrow H_a(SV)_{\mathbf{n}} \longrightarrow \cdots .$$

Since $H_a(V)_{\mathbf{n}} \neq 0$, it follows that $H_a(SV)_{\mathbf{n}} \neq 0$, and so $\mathrm{hd}_a(SV) \geqslant n$. Since $\mathrm{N}(SV) = \mathrm{N}(V) - 1$, by induction hypothesis, one has $\mathrm{hd}_{a+1}(SV) \geqslant n+1$. Thus, there exists $r \geqslant n+1$ such that $H_{a+1}(SV)_{\mathbf{r}} \neq 0$. But from Theorem 1, we have an exact sequence:

$$\cdots \longrightarrow H_{a+1}(V)_{\mathbf{r}} \longrightarrow H_{a+1}(SV)_{\mathbf{r}} \longrightarrow H_{a+1}(V)_{\mathbf{r+1}} \longrightarrow \cdots .$$

Since $r > \mathrm{hd}_{a+1}(V)$, we have $H_{a+1}(V)_{\mathbf{r}} = 0$ and $H_{a+1}(V)_{\mathbf{r+1}} = 0$, so $H_{a+1}(SV)_{\mathbf{r}} = 0$, a contradiction. We conclude that $\mathrm{hd}_{a+1}(V) \geqslant n+1$. $\quad\square$

The following corollary uses Theorem 9 to see that $\mathrm{reg}(V) < \infty$.

Corollary 22. *Let V be an* FI*-module which is generated in degree $\leqslant k$ and related in degree $\leqslant d$ for some $k, d \in \mathbb{Z}_+$. If V is not F-acyclic, then there exists $s \geqslant 1$ such that*

$$\mathrm{hd}_a(V) = \mathrm{reg}(V) + a \quad \text{for each } a \geqslant s.$$

Proof. By Theorem 21, we have:

$$\mathrm{hd}_1(V) - 1 \leqslant \mathrm{hd}_2(V) - 2 \leqslant \mathrm{hd}_3(V) - 3 \leqslant \cdots .$$

But by Theorem 9, we have $\mathrm{reg}(V) < \infty$. The claim is now immediate from the definition of $\mathrm{reg}(V)$. $\quad\square$

References

[1] CHURCH, THOMAS; ELLENBERG, JORDAN S. Homology of FI-modules. Preprint, 2015. arXiv:1506.01022v2.

[2] CHURCH, THOMAS; ELLENBERG, JORDAN S.; FARB, BENSON. FI-modules and stability for representations of symmetric groups. *Duke Math. J.* **164** (2015), no. 9, 1833–1910. MR3357185, Zbl 1339.55004, arXiv:1204.4533v4, doi: 10.1215/00127094-3120274.

[3] CHURCH, THOMAS; ELLENBERG, JORDAN S.; FARB, BENSON; NAGPAL, ROHIT. FI-modules over Noetherian rings. *Geom. Topol.* **18** (2014), no. 5, 2951–2984. MR3285226, Zbl 1344.20016, arXiv:1210.1854v2, doi: 10.2140/gt.2014.18.2951.

[4] GAN, WEE LIANG; LI, LIPING. A remark on FI-module homology. Preprint, 2015. arXiv:1505.01777v4.

[5] LI, LIPING. Homological degrees of representations of categories with shift functors. Preprint, 2015. arXiv:1507.08023v3.

[6] LI, LIPING. Upper bounds of homological invariants of FI_G-modules. *Arch. Math.* (Basel) **107** (2016), no. 3, 201–211. MR3538516, Zbl 06629274, arXiv:1512.05879v3, doi: 10.1007/s00013-016-0921-3.

[7] LI, LIPING; RAMOS, ERIC. Depth and the local cohomology of FI_G-modules. Preprint, 2016. arXiv:1602.04405v3.

[8] LI, LIPING; YU, NINA. Filtrations and homological degrees of FI-modules. Preprint, 2015. arXiv:1511.02977v2.

[9] NAGPAL, ROHIT. FI-modules and the cohomology of modular representations of symmetric groups. Preprint, 2015. arXiv:1505.04294v1.

[10] RAMOS, ERIC. Homological invariants of FI-modules and FI$_G$-modules. Preprint, 2015. arXiv:1511.03964v3.

[11] SAM, STEVEN V.; SNOWDEN, ANDREW. GL-equivariant modules over polynomial rings in infinitely many variables. *Trans. Amer. Math. Soc.* **368** (2016), no. 2, 1097–1158. MR3430359, Zbl 06560453, arXiv:1206.2233v3, doi: 10.1090/tran/6355.

[12] WEIBEL, CHARLES A. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp. ISBN: 0-521-43500-5; 0-521-55987-1. MR1269324 (95f:18001), Zbl 0797.18001, doi: 10.1017/CBO9781139644136.

(Wee Liang Gan) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521, USA
wlgan@math.ucr.edu

This paper is available via http://nyjm.albany.edu/j/2016/22-64.html.