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Nonhyperbolic free-by-cyclic and one-relator groups

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Abstract. We show that the free-by-cyclic groups of the form $F_2 \rtimes \mathbb{Z}$ act properly cocompactly on CAT(0) square complexes. We also show using generalized Baumslag–Solitar groups that all known groups defined by a 2-generator 1-relator presentation are either SQ-universal or are cyclic or isomorphic to a soluble Baumslag–Solitar group.

Contents

1. Introduction

The recent far reaching work of Agol and Wise proves that a word hyperbolic group G acting properly and cocompactly on a CAT(0) cube complex must be virtually special, implying that G has a finite index subgroup which embeds in a right angled Artin group (RAAG). A host of very strong conclusions then apply to the group G, for instance G will be linear (over $\mathbb{C}$ or even over $\mathbb{Z}$) and (if G is not elementary) also large, namely G has a finite index subgroup surjecting to a nonabelian free group.

However, if G is a finitely presented nonhyperbolic group acting properly and cocompactly on a CAT(0) cube complex, then the above consequences need no longer hold, indeed G can even be simple [10]. Therefore suppose we have a class of finitely presented groups which is believed to be a well behaved class, but which contains both word hyperbolic and non-word-hyperbolic examples. We can ask: first, do all examples in this class have a nice geometric action, namely a proper cocompact action on a CAT(0) cube

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complex, and second: do they all enjoy these strong group theoretic properties which are a consequence of being a virtually special group. Note that for non-word-hyperbolic groups, satisfying this geometric condition need not imply such group theoretic properties.

In this paper the classes of groups we will be interested in are the following three: groups of the form $F_k \rtimes_\alpha \mathbb{Z}$ for F_k a free group of finite rank k and α an automorphism of F_k (we refer to these as "free-by-cyclic groups"); the more general class of ascending HNN extensions $F_k *_\theta$ of finite rank free groups, where rather than θ having to be an automorphism, as in the free-by-cyclic case, we allow θ to be any injective endomorphism of F_k; and finally the class of groups admitting a presentation with 2 generators and 1 relator, which we refer to as 2-generator 1-relator groups. This last class neither contains nor is contained in either of the other two classes but there is considerable overlap.

In the free-by-cyclic case, it was recently shown in [20] that such word hyperbolic groups do act properly and cocompactly on a CAT(0) cube complex, and therefore are virtually special groups. However, Gersten in [18] gave an example of a free-by-cyclic group which cannot act properly and cocompactly on any CAT(0) space, so this result cannot hold in general in the non-word-hyperbolic case. In Gersten's paper the free group has rank 3 but in Section 2 we consider free-by-cyclic groups of the form $F_2 \rtimes_\alpha \mathbb{Z}$, none of which are word hyperbolic. Therefore it is of interest to show directly that they act properly and cocompactly on CAT(0) cube complexes, which we do in Section 2. This work is based on unpublished work of Bridson and Lustig. Those authors give us the method of changing the natural topological model of the standard 2 complex (shown in Figure 1) to get rid of a pocket of positive curvature, and in this way they go on to show that these groups act on 2-dimensional CAT(0) complexes. We expand on this by showing that one can build these complexes from squares. This also strengthens a result of T. Brady [5] who showed that there is a 2-complex of nonpositive curvature made from equilateral triangles with fundamental group $F_2 \rtimes_\alpha \mathbb{Z}$.

In Section 3 we consider 2-generator 1-relator groups. It is conceivable, but very definitely open, that a word hyperbolic 2-generator 1-relator group always acts properly and cocompactly on a CAT(0) cube complex (for instance see [33] Conjecture 1.9). However on moving to the non-word-hyperbolic case we see a different picture emerging because a group acting properly and cocompactly on any CAT(0) space cannot contain a Baumslag–Solitar group $BS(m,n)$ where $|m| \neq |n|$. Thus the examples of such nasty Baumslag–Solitar groups as $BS(2,3)$ mean that we need not always have largeness nor linearity, or even residual finiteness. However there is a property akin to but weaker than largeness which might be held by all 2-generator 1-relator groups except for the soluble groups $BS(1,m)$ and $\mathbb{Z}$, even in the absence of residual finiteness. This is the property of a group G being SQ-universal: namely that every countable group embeds in a quotient of G.

It was conjectured by P. M. Neumann in [28] back in 1973 that a noncyclic 1-relator group is either SQ-universal or isomorphic to $BS(1,m)$. Now it was shown in [32] that a group having a 1-relator presentation with at least 3 generators is SQ-universal, leaving the 2-generator 1-relator case. Also [30] from 1995 showed that all nonelementary word hyperbolic groups are SQ-universal and this was generalized to nonelementary groups which are hyperbolic relative to any collection of proper subgroups in [1] from 2007.

Recently the concept of a group being acylindrically hyperbolic, which is more general than being hyperbolic with respect to a collection of proper subgroups and which implies SQ-universality, was introduced in [31] and studied in [27] where one application was to 2-generator 1-relator groups. The authors divided these groups into three classes with the first consisting of groups that they could show were acylindrically hyperbolic. We prove in Theorem 3.2 that all the groups in their second case, which they show are not acylindrically hyperbolic, are indeed SQ-universal unless equal to $BS(1,m)$. In fact every group here is formed by taking an HNN extension with base equal to a quotient of some free-by-cyclic group $F_k \rtimes_\alpha \mathbb{Z}$ for α finite order, along with infinite cyclic edge groups. The proof proceeds by identifying them as generalized Baumslag–Solitar groups, whereupon we also show in Theorem 3.2 that any generalized Baumslag–Solitar group either is SQ-universal or is isomorphic to $BS(1,m)$ or $\mathbb{Z}$.

This leaves their third case, which is exactly the class of 2-generator 1-relator groups that are ascending HNN extensions of finite rank free groups. Here we are not quite able to establish SQ-universality of all of these groups not equal to $BS(1,m)$ or $\mathbb{Z}$, though it is known to hold for the free-by-cyclic case. However we do show in Corollary 3.4 that the only possible exception would be a 2-generator 1-relator group equal to a strictly ascending HNN extension of a finite rank free group which either fails to be word hyperbolic and contains no Baumslag–Solitar subgroup, or does contain a Baumslag–Solitar subgroup (but does not contain $\mathbb{Z} \times \mathbb{Z}$) and where all finite index subgroups have first Betti number equal to 1. In both cases it is conjectured that no such examples exist, so we have established P. M. Neumann's conjecture for all the known 2-generator 1-relator groups. We also obtain some general unconditional statements, such as Corollary 3.5 which says that if the relator is in the commutator subgroup of F_2 then G is SQ-universal or equal to $\mathbb{Z} \times \mathbb{Z}$. We then finish by considering in Section 4 which free by cyclic groups $F_k \rtimes_\alpha \mathbb{Z}$ for $k \geq 2$ are acylindrically hyperbolic and show that, modulo an unpublished assertion in [17], the answer is exactly when α has infinite order in $\mathrm{Out}(F_k)$.

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2. Square complexes for free-by-cyclic groups in the rank 2 case

In this section we will prove the following:

Theorem 2.1. *Let ϕ be an automorphism of F_2. Then the group $F_2 \rtimes_\phi \mathbb{Z}$ acts freely, cellularly, properly and cocompactly on a CAT(0) square complex.*

As mentioned before there are many nice consequences of word hyperbolic groups acting properly and cocompactly on CAT(0) cube complexes. However, no automorphism of F_2 is hyperbolic since they all fix the conjugacy class of $[a, b]^{\pm 1}$.

However, there are still advantages to having a group act on a CAT(0) cube complex. For instance, abelian subgroups are quasi-isometrically embedded, and such groups are biautomatic [19, 29] and have a deterministic solution to the word problem in quadratic time [16].

Groups which act on CAT(0) square complexes have the further nice property that all of their finitely presented subgroups also act properly and cocompactly on CAT(0) square complexes. This is proved using a tower argument (see [7],p. 217) and the fact that a sub complex of a nonpositively curved square complex is itself a nonpositively curved square complex (this may fail in higher dimensions). The construction also shows that for F_2-by-$\mathbb{Z}$ groups their geometric dimension is equal to their CAT(0) dimension, namely 2.

2.1. Preliminaries.

We assume that the reader is familiar with the basics of CAT(0) geometry for which the standard reference is [7].

Definition 2.2. We say a metric space is *nonpositively curved* if for each point there is a neighbourhood which is CAT(0).

In the following we will study 2 dimensional piecewise euclidean ($\mathbb{PE}$) complexes. These are complexes built from polygonal subsets of $\mathbb{R}^2$ by gluing along edges, whereupon we put the natural path metric on the resulting complexes. For full details see [7].

Square complexes are special examples of $\mathbb{PE}$ complexes where all the cells are squares.

The following theorems of Gromov allow one to check whether a complex is nonpositively curved just by looking at the links of vertices.

Theorem 2.3 ([6]). *A $\mathbb{PE}$ complex with finitely many isometry types of cells is nonpositively curved if and only if the link of each vertex is a* CAT(1) *space.*

In the two dimensional case the link of any vertex is a graph and so this can be reduced to the following.

Lemma 2.4 ([7]). *A graph is* CAT(1) *if it contains no circuits of length less than 2π.*

Definition 2.5. We say that an action is *proper* if for each compact set K the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.

As these groups will be the fundamental groups of nonpositively curved spaces, they have an action on the universal cover. Since the spaces are compact the action will be proper and cocompact and it will also be a free action since these groups are torsion free.

The groups G_ϕ that we shall be concerned with are mapping tori of $F_2 = F(x, y)$ by a single automorphism $\phi \in \mathrm{Aut}(F_2)$. These groups have presentations of the form

$$\langle x, y, t | txt^{-1} = \phi(x), tyt^{-1} = \phi(y)\rangle.$$

We start by considering the case of automorphisms which are of finite order.

Proposition 2.6. *If $\phi \in \mathrm{Aut}(F_n)$ has order q in $\mathrm{Out}(F_n)$ then G_ϕ is the fundamental group of a nonpositively curved 2-complex. Furthermore, this is finitely covered by $\Gamma \times S^1$ where Γ is a graph with fundamental group F_n.*

Proof. Every finite order automorphism ϕ of F_n can be realised as an isometry of a finite graph Γ; see for instance [13] Theorem 2.1. Let

$$X = \Gamma \times [0, 1]/(x, 0) \sim (\phi(x), 1).$$

X is locally isometric to $\Lambda \times (-\epsilon, \epsilon)$ where Λ is a contractible subset of a graph. This will be CAT(0) and so X is nonpositively curved.

If we take the cover corresponding to the kernel of the map to $\mathbb{Z}_q$, taking t to a generator of $\mathbb{Z}_q$, this will be $\Gamma \times S^1$. □

We now want to look at automorphisms of infinite order. It is well known that the isomorphism class of G_ϕ is dependant only on the conjugacy class of ϕ in $\mathrm{Out}(F_n)$.

Theorem 2.7. [5] *The semigroup Ω generated by $-I, F, L$ and R contains a conjugate of every infinite order matrix in $\mathrm{GL}_2(\mathbb{Z})$. Where*

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

As such we will only need to realise the automorphisms corresponding to these in our groups.

2.2. The construction. We will now construct nonpositively curved complexes with G_ϕ as their fundamental groups, when ϕ is in the semigroup generated by

$$\begin{aligned} \lambda : a &\mapsto ba & \rho : a &\mapsto a \\ b &\mapsto b & b &\mapsto ab \\ \iota : a &\mapsto a^{-1} & \sigma : a &\mapsto b \\ b &\mapsto b^{-1} & b &\mapsto a. \end{aligned}$$

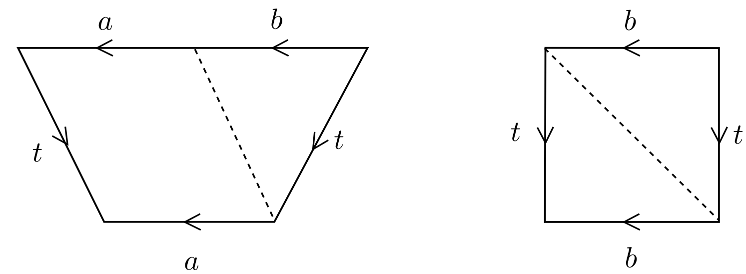

FIGURE 1. The 2-complex associated to λ.

We see that from the above this gives us all F_2-by-$\mathbb{Z}$ groups.

We start with the obvious 2-complex shown in Figure 1 for the automorphism λ. This has a repeated corner which means it cannot support a metric of nonpositive curvature.

To get rid of the repeated corner we cut our building blocks along the dotted line identifying the triangles with the repeated corner, resulting in our basic building blocks shown in Figure 2.

Every element of Ω has the form $\phi = \eta_0 \ldots \eta_{n-1}\theta$ where $\eta_i = \rho$ or λ and θ is one of the following automorphisms of order 2:

$$\begin{aligned}
\psi_1 &: (a,b) \mapsto (a,b)\\
\psi_2 &: (a,b) \mapsto (a^{-1},b^{-1})\\
\psi_3 &: (a,b) \mapsto (b,a)\\
\psi_4 &: (a,b) \mapsto (b^{-1},a^{-1}).
\end{aligned}$$

We can assume that we only apply one of these and we do it at the end. This is because in $\mathrm{Out}(F_2)$ the first and second give central elements whereas ψ_4 is equal to the composition $\psi_2\psi_3$ and $\rho\psi_3 = \psi_3\lambda$.

We glue our blocks together to get a complex which realises any automorphism in Ω, so up to isomorphism we have all groups G_ϕ. We do this in the following way: in Figure 2 let $i = 0$ so that we have the positive and the negative vertices $t_0^{\pm 1}$ at each end of t_0. On performing the given gluing we have that $t_0^{\pm 1}$ are not identified, but if we further stick these two vertices together by identifying a_0, b_0 with a_1, b_1 respectively then the resulting 2-complex has fundamental group $F_2 \rtimes_\lambda \mathbb{Z}$ or $F_2 \rtimes_\rho \mathbb{Z}$. Now suppose that our automorphism $\phi = \eta_0 \ldots \eta_{n-1}\theta$, where θ is one of the four special finite order automorphisms above. For each i between 0 and $n-1$ we have a copy C_i of the 2-complex associated to either λ or ρ in Figure 2 which contains the edge t_i. We then glue C_i to C_{i+1} by the identity between the a_{i+1}, b_{i+1}, which means that the vertex t_i^- is identified with t_{i+1}^+ (the edge t_i is oriented from t_i^- to t_i^+). Finally we glue C_{n-1} back to C_0 by identifying the a_n, b_n with a_0, b_0 so that t_{n-1}^- becomes equal to t_0^+.

We say a vertex is at *time* i if it is the vertex where t_{i-1} and t_i meet.

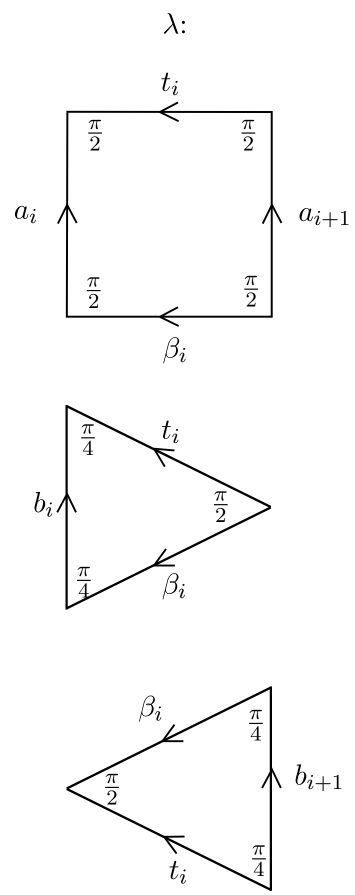

FIGURE 2. The basic building blocks for the construction with angles labelled.

We start with the case of vertices of time not equal to 0, thus these will be where our complexes C_{i-1} and C_i are glued together by the identity between the a_i, b_i.

The link of such a vertex is shown in Figure 3. We now want to assign angles such that there are no circuits of length less than 2π.

If we assign angles as in Figure 2, then we get two types of link as shown in Figure 3. Figure 3 i) corresponds to the automorphisms at the i-th stage both being ρ or both being λ. Figure 3 ii) corresponds to when one automorphism is λ and one is ρ.

We can see in either case that the link has no circuits of length less than 2π.

We now look at the case of the vertex at time 0. This will have to take into account the map θ. We can consider the link as being split into 2

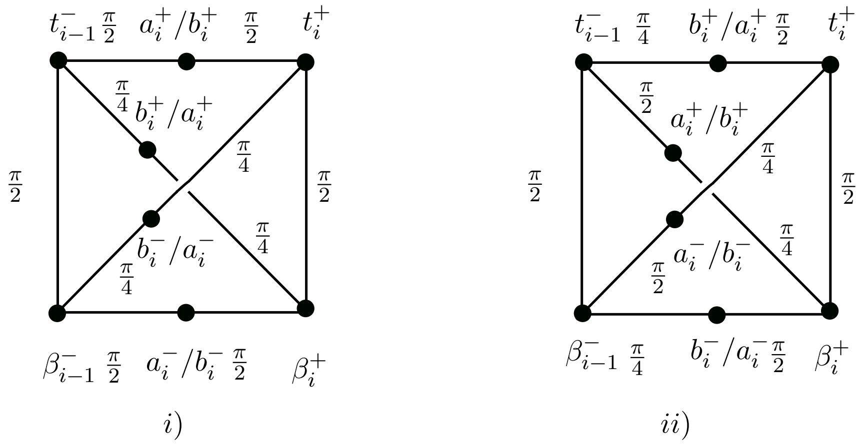

FIGURE 3. The possible links of a vertex.

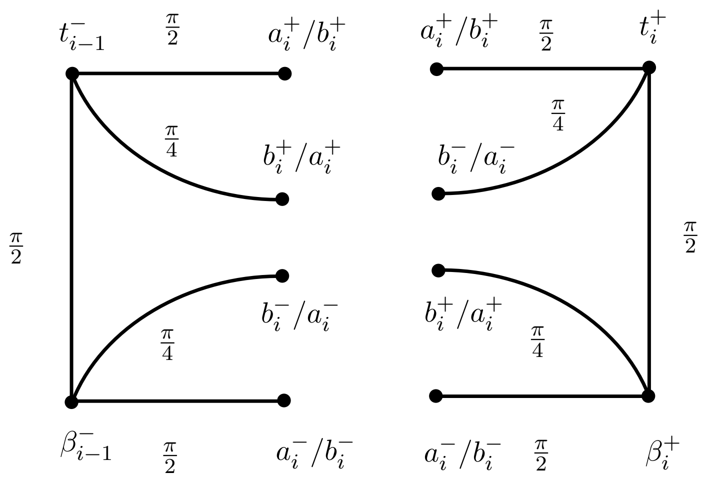

FIGURE 4. The 2 halves of a link.

halves as shown in Figure 4. The finite order maps defined earlier give vertex identifications. There are 16 possible links we may get in this way, corresponding to which automorphisms meet and to one of the 4 finite order automorphisms. All of these give a link which is homeomorphic to the 1 skeleton of a tetrahedron with the set of angles depicted in Figure 3.

2.2.1. Square Complexes. With a more careful assignment of angles we can see that the complexes above can be made into square complexes.

We split the automorphism ϕ into one of three types depending on its decomposition in the semigroup described earlier:

(1) $\phi = \rho^n$ or λ^n;
(2) $\phi = (\rho\lambda)^n(\rho\theta)^\epsilon$ or $(\lambda\rho)^n(\lambda\theta)^\epsilon$ where $\theta \in \{\psi_3, \psi_4\}$ and $\epsilon \in \{0, 1\}$;
(3) all other automorphisms.

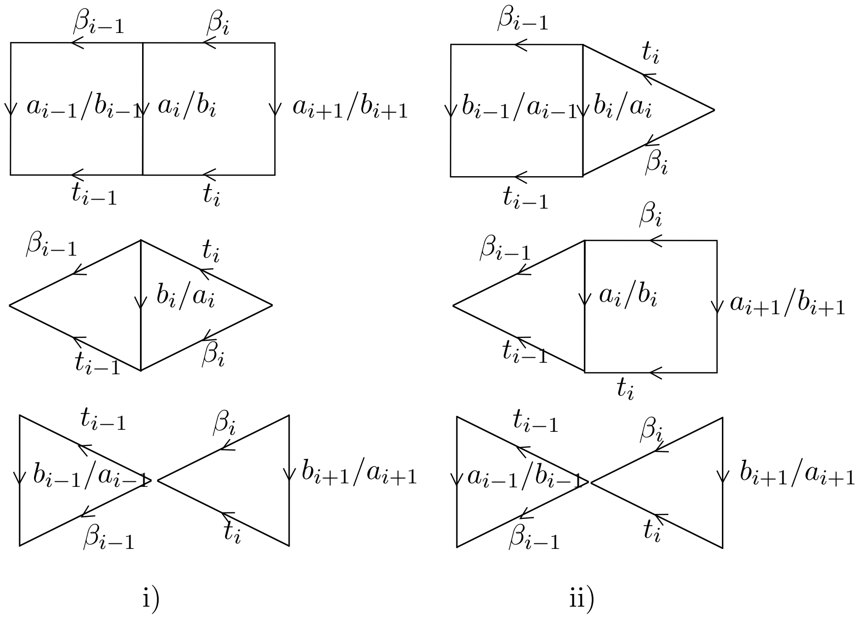

FIGURE 5. The 2 possible cases of automorphisms meeting at a vertex.

We will assign angles to our building blocks based on which meet at time i. We have depicted the two cases of our building blocks meeting at a vertex.

In Case 1 each time our building blocks meet, it will be of the type depicted in Figure 5 i). In this case we keep the angle assignment we had before and make the edges labelled a_j or b_j on the vertical sides of the two rectangles of length 2 and the other edges of length 1. We then make the edge between the 2 triangles of length $\sqrt{2}$ and subdivide the rectangles into squares of edge length 1, thus replacing the two triangles with a new building block which is the square formed by gluing them together. The link of the original vertices in this complex are depicted in Figure 6 i), where the edges corresponding to a_i and b_i have been suppressed as they have valence 2.

In Case 2 each time our building blocks meet, it will be of the type depicted in Figure 5 ii). In this case we collapse the triangles to the vertical line labelled a_i or b_i. Subdivide the resulting rectangles into 2 squares of side length 1. The link of the original vertices in this complex are depicted in Figure 6 ii), where the edges corresponding to a_i and b_i have been suppressed as they have valence 2.

In Case 3 we will have a mix of both Figures 5 i) and ii) and we collapse all the triangles to lines as we did for Case 2. This introduces degenerate squares from building blocks meeting as in Figure 5 i), which could affect the topology of the overall complex if there were a cylinder of such squares.

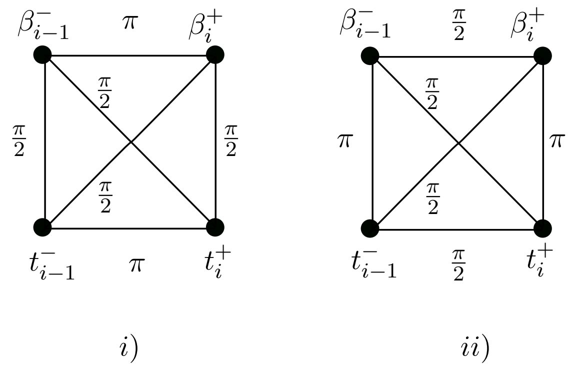

FIGURE 6. The links for automorphisms in Cases 1 and 2.

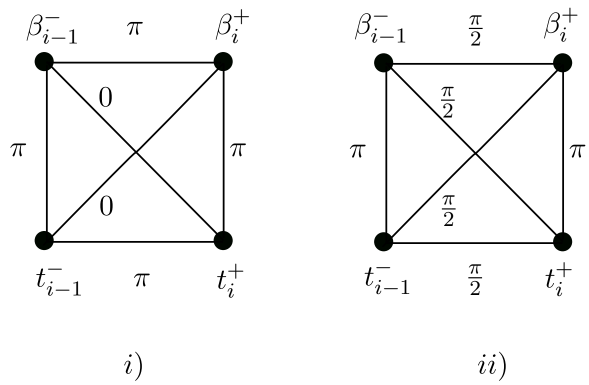

FIGURE 7. The links for automorphisms in Case 3.

However, this will not happen as the only time a cylinder of degenerate squares could occur is if our automorphism is in Case 1.

The links of vertices in these complexes are depicted in Figure 7, where the edges corresponding to a_i and b_i have been suppressed as they have valence 2. The case of edges of 0 length are the degenerate squares where in fact the two edges become identified.

In all the cases we see that the resulting complex will be a nonpositively curved square complex.

3. SQ universal groups

A countable group G is said to be SQ-universal (standing for Subgroup Quotient) if every countable group can be embedded in a quotient of G. This immediately implies that G contains a nonabelian free group and also that G has uncountably many normal subgroups. Moreover SQ-universality is a consequence of G being large (having a finite index subgroup surjecting to a nonabelian free group). However an infinite simple group S containing F_2

would not be SQ-universal, nor would a just infinite group such as $\mathrm{PSL}_n(\mathbb{Z})$ for $n \geq 3$.

As for examples of groups which are SQ-universal, we have all nonelementary word hyperbolic groups by [30]. This means that word hyperbolic groups with property (T) provide lots of further examples of groups which are SQ-universal but not large. Moreover by [1] a finitely generated group which is hyperbolic relative to any collection of proper subgroups is SQ-universal (or virtually cyclic).

An important class of groups in this area is 1-relator groups. It was shown in [2] in 1978 that any group with a presentation of deficiency at least 2 (thus any group having an n-generator 1-relator presentation for $n \geq 3$) is large, leaving 2-generator 1-relator groups. The question of when such a group G contains F_2 has been known for some time: yes, unless G is isomorphic to a Baumslag–Solitar group of the form $BS(1, n)$ (where $n \in \mathbb{Z} - \{0\}$) or is cyclic. Largeness is a different matter; for instance [15] showed that the group $BS(m, n)$ is large if and only if m and n are not coprime. However for the intermediate property of SQ-universality there is a conjecture that appeared in [28] in 1973: a noncyclic 1-relator group is SQ-universal unless it is isomorphic to $BS(1, n)$, thus if true this would be equivalent to containing F_2. Although [32] showed in 1974 that a group with an n-generator 1-relator presentation for $n \geq 3$ is SQ-universal (which was then subsumed by the largeness result mentioned above), for 2-generator 1-relator groups progress was only made in special cases.

However recently the concept of a group being acylindrically hyperbolic was introduced in [31]. We will not need the definition of acylindrical hyperbolicity here, just the fact also in [31] that such a group is SQ-universal. If a group is nonelementary and is relatively hyperbolic with respect to a collection of proper subgroups then it is also acylindrically hyperbolic. This was followed up in [27] where the theory was applied to various situations, including 1-relator groups to obtain the following. Here a subgroup H of a group G is s-normal in G if H is infinite and moreover $H \cap gHg^{-1}$ is infinite for all $g \in G$. The relevance of this is that an s-normal subgroup of an acylindrically hyperbolic group must also be acylindrically hyperbolic, so for instance $H \cong \mathbb{Z}$ being s-normal in G implies that G is not acylindrically hyperbolic (though it could certainly be SQ-universal or even large).

Proposition 3.1 ([27] Proposition 4.20)**.** *Let G be a group with two generators and one defining relator. Then at least one of the following holds:*

(i) *G is acylindrically hyperbolic.*

(ii) *G contains an infinite cyclic s-normal subgroup. More precisely, either G is infinite cyclic or it is an HNN-extension of the form*

$$G = \langle a, b, t \mid a^t = b, w = 1 \rangle$$

of a 2-generator 1-relator group $H = \langle a, b \mid w(a, b) \rangle$ with nontrivial center, so that $a^r = b^s$ in H for some $r, s \in \mathbb{Z} \setminus \{0\}$. In the latter case

H is (finitely generated free)-by-cyclic and contains a finite index normal subgroup splitting as a direct product of a finitely generated free group with an infinite cyclic group.

(iii) *G is isomorphic to an ascending HNN extension of a finite rank free group.*

Moreover, the possibilities (i) *and* (ii) *are mutually exclusive.*

Thus this establishes that groups in class (i) are SQ-universal. We will show the same for groups in (ii) then discuss results for (iii). When considering groups in Case (ii), we will use the class of generalized Baumslag–Solitar, or GBS, groups. These can be defined as those finitely generated groups which act on a tree with infinite cyclic vertex and edge stabilisers. The two recent papers [23] and [24] cover a lot of ground in this area and we now mention the points we will be using, referring to them for more detail.

We can describe a GBS group using the graph of groups theory, where we are given a finite graph (possibly with self loops and/or multiple edges) along with a label, which is a nonzero integer, at each end of each edge. Then every vertex and edge group can be regarded as a copy of the integers $\mathbb{Z}$ generated by $1 \in \mathbb{Z}$, with the two labels on an edge describing the inclusion map of the edge group into each of the neighbouring vertex groups by giving us the image in each vertex group of the generator 1 from the edge group. (Thus the modulus of this label tells us the index of this edge group in the adjacent vertex group and so determines the subgroup of the vertex group uniquely, but there are still two possibilities for the inclusion map, coming from the sign of this label.) In general many different finite labelled graphs can give rise to isomorphic GBS groups. One operation that can be performed without change of the underlying group is an elementary collapse. This is when one end of an edge e next to a vertex v is labelled ± 1 and the edge is not a self loop. We can then contract this edge and multiply all other labels next to v by the label at the other end of e. By doing this repeatedly, we may assume that any edge with an end labelled by ± 1 is a self loop.

Note that all GBS groups have deficiency 1, that is they admit a presentation with one more generator than relator. In particular there always exists a surjective homomorphism from any GBS group to $\mathbb{Z}$.

Theorem 3.2. *If the group G is as in Case* (ii) *of the preceding Proposition then G is a generalized Baumslag–Solitar group. Moreover any generalized Baumslag–Solitar group is either SQ-universal or it is isomorphic to the Baumslag–Solitar group $BS(1, j)$ for some $j \in \mathbb{Z} \setminus \{0\}$ or is infinite cyclic.*

Proof. For the first part we can use Theorem C of [22]. This states that the noncyclic finitely generated groups of cohomological dimension 2 that have an infinite cyclic s-normal subgroup are exactly the generalized Baumslag–Solitar groups. Now a 1-relator group has cohomological dimension 2 if the relator is not a proper power by [25], but a proper power gives rise to a group with torsion, whereas the groups in Case (ii) are all torsion free.

Now let Γ be the underlying graph of our graph of groups that results in the GBS group G. As mentioned above we assume that the only edge labels equal to ± 1 appear on self loops.

It is well known that if Γ contains more than one cycle (here we include self loops as cycles), then G surjects to F_2 and so is SQ-universal, because we introduce a stable letter for each cycle when forming G, and all vertex subgroups can be quotiented out to leave only these stable letters which have no relations between them.

It is also known that if Γ is a tree then G is virtually $F_k \times \mathbb{Z}$ for $k \geq 2$ which is large, hence so is G. This can be seen by quoting Proposition 4.1 of [23] which states that a group is a GBS group with nontrivial centre if and only if it is of the form $F_k \rtimes_\alpha \mathbb{Z}$ with α having finite order in $\mathrm{Out}(F_k)$. Now here G will certainly have a nontrivial centre, namely an element which is a common power of all the generators of the vertex subgroups as these form a generating set for G. Moreover in the case of a tree the surjective homomorphism θ from G to $\mathbb{Z}$ has the property that no nontrivial element of a vertex (or edge) group lies in its kernel, as if so then the whole vertex group does, thus so do the neighbouring edge groups and so on across the whole tree.

We now assume that Γ has exactly one cycle C. First assume this is not a self loop. We pick one edge e lying in C and remove the interior of e to form a tree T and a group H coming from considering T as the corresponding graph of groups. Thus we have our homomorphism $\theta : H \twoheadrightarrow \mathbb{Z}$ as above, with G obtained from H by taking generators h_1, h_2 of the vertex groups at each end v_1, v_2 of e and then adding a stable letter t which results in the presentation

$$G = \langle H, t | \mathrm{rels}(H), t h_1^m t^{-1} = h_2^n \rangle$$

where m, n are the labels at each end of e, neither of which are 0 or ± 1. We now obtain a surjection from G to a Baumslag–Solitar group using the following folklore lemma:

Lemma 3.3. *Let G be an HNN extension of the group H amalgamating the subgroups A, B via the isomorphism $\phi : A \to B$. Suppose we have a homomorphism θ from H onto a quotient Q with $\theta(A)$ isomorphic to $\theta(B)$ such that ϕ descends to an isomorphism $\overline{\phi}$ from $\theta(A)$ to $\theta(B)$, meaning that $\overline{\phi}$ is well defined and bijective with $\overline{\phi}\theta = \theta\phi$. (This occurs if and only if $\phi(K) = L$ for K, L the kernels of the restriction of θ to A, B respectively.) Then on forming the HNN extension R of Q with stable letter s amalgamating $\theta(A)$ and $\theta(B)$ via $\overline{\phi}$, we have that the original HNN extension G has this new HNN extension R as a quotient.*

Proof. We define a homomorphism from the free product $H * \langle t \rangle$ onto R sending t to s and $h \in H$ to $\theta(H) \in Q$. We see that this factors through G because any relation in G of the form $tat^{-1} = \phi(a)$ has the left hand side mapped by θ to $s\theta(a)s^{-1}$ and the right hand side to $\overline{\phi}\theta(a)$, and these two things are equal in R by the HNN construction. $\square$

Consequently in our case we have G surjects to $BS(k_1m, k_2n)$, where $k_1 = \theta(h_1)$ which is not equal to zero as mentioned above because H is formed from a tree, and similarly for k_2. Now a Baumslag–Solitar group $BS(i,j) = \langle t, a | ta^it^{-1} = a^j\rangle$ is known to be SQ-universal if neither of i, j equal ± 1, by Lemma 1.4.3 of [14] if i and j are coprime, and by the well known trick of setting a^d equal to the identity when $d > 1$ divides i and j, to get a surjection to $\mathbb{Z} * \mathbb{Z}_d$ which is virtually free otherwise. As $|m|$ and $|n|$ are both greater than 1, we have that G surjects to an SQ-universal group and so itself is SQ-universal.

We are now only left with the case where there is a single self loop L in our graph Γ. Here we thank the anonymous referee for providing the following simplification of our original argument: if $L = \Gamma$ then G is an actual Baumslag–Solitar group $BS(m, n)$, so we are done as above if $|m|, |n| \neq 1$ and we have a soluble Baumslag–Solitar group and hence the exception in Theorem 3.2 if one of $|m|, |n|$ is equal to 1. Otherwise there will be a separating edge e of Γ such that, on removal of the interior of e, we are left with two components Γ_1 and Γ_2 giving rise to the groups G_1 and G_2, where Γ_1 is a tree (possibly a single vertex) and Γ_2 contains the self loop L. We set v_1, v_2 to be the distinct vertices of e where $v_i \in \Gamma_i$ and h_1, h_2 to be the generators of the respective vertex groups. We also suppose that e is labelled by the integer k at the Γ_1 end and l at the Γ_2 end, whereupon $|k|$ (and also $|l|$) is not equal to 1. Then G is the amalgamated free product $G_1 *_{h_1^k = h_2^l} G_2$. Now we have a surjection of G_2 to $\mathbb{Z}$ obtained from the stable letter of the loop L and this sends h_2 to the identity. Thus on taking any prime p dividing k and any surjective homomorphism $\theta : G_1 \twoheadrightarrow \mathbb{Z}$, whose existence is guaranteed as G_1 is a GBS group, we can compose with the natural quotient map to obtain a homomorphism $\theta_p : G_1 \twoheadrightarrow \mathbb{Z}_p$ which sends h_1^k to the identity. These can now be combined into a homomorphism from G itself onto $\mathbb{Z}_p * \mathbb{Z}$, which is SQ-universal and hence so is G. $\square$

Note: this result can be compared to [23] Theorem 6.7 in which the large GBS groups are determined, but there are cases for which the graph Γ is a single cycle where the group G is SQ-universal but not large.

We now come to Case (iii), that of G being equal to the strictly ascending HNN extension F_k*_θ of F_k, where $\theta : F_k \to F_k$ is injective but need not be surjective. Here we can quote results of the first author in [11]. Theorem 5.4 of that paper states that G will be SQ-universal whenever θ is an automorphism (unless $G \cong \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group when the rank k is 0 or 1). This is proved by showing that $\mathbb{Z} \times \mathbb{Z} \leq G$ implies that G is large, and then invoking Ol'shanskiĭ's theorem on the SQ-universality of word hyperbolic groups and the result in [9] that not containing $\mathbb{Z} \times \mathbb{Z}$ and being word hyperbolic are equivalent in the class of groups F_k*_θ when θ is an automorphism.

In the case where G is a strictly ascending HNN extension of a finite rank free group, we have further results but they are not quite definitive. Again we have that if $\mathbb{Z}\times\mathbb{Z} \leq G$ then G is large (or is equal to $\mathbb{Z}\times\mathbb{Z}$ or the Klein bottle group) by [11] Corollary 4.6. However in the strictly ascending case there are examples where G does not contain $\mathbb{Z}\times\mathbb{Z}$ but does contain a Baumslag–Solitar subgroup, which must be of the form $BS(1,m)$ for $|m| \neq 1$ so that G fails to be word hyperbolic. In [21] it is conjectured that a strictly ascending HNN extension of a finite rank free group is is word hyperbolic if it does not contain a Baumslag–Solitar subgroup and this conjecture seems to be widely believed, but a proof might well require the machinery of train track maps to be developed in full for injective endomorphisms of F_k. Moreover it is an open question whether a 1-relator group (or indeed a group with a finite classifying space) containing no Baumslag–Solitar subgroups is word hyperbolic, so we would be covered in our case if any of these (or their intersection) turned out to be true.

As for when G contains $BS(1,m)$ for $|m| \neq 1$, [11] Theorem 4.7 states that either G is large, or G is itself a Baumslag–Solitar group of the form $BS(1,n)$, or $G \ncong BS(1,n)$ but G has virtual first Betti number equal to 1 and it is conjectured that the last case does not occur. Putting all this together, we have our result on the SQ-universality of 2-generator 1-relator groups:

Corollary 3.4. *If G is a group given by a 2-generator 1-relator presentation that is not $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$ or the Klein bottle group then either G is an SQ-universal group, or G is a strictly ascending HNN extension F_k*_θ of a free group F_k which is not word hyperbolic and such that either:*

(i) *G contains no Baumslag–Solitar subgroup (conjecturally this does not occur), or*

(ii) *G contains a Baumslag–Solitar group $BS(1,m)$ for $|m| \neq 1$ but does not contain $\mathbb{Z}\times\mathbb{Z}$ and the virtual first Betti number of G is 1 (conjecturally this only occurs if $G \cong BS(1,n)$ for $|n| \neq 1$).*

We finish this section with a couple of unconditional results.

Corollary 3.5. *If $G = \langle a,b | w(a,b)\rangle$ and w is in the commutator subgroup of $F(a,b)$ (and without loss of generality cyclically reduced) then G is SQ-universal, except when $G \cong \mathbb{Z}\times\mathbb{Z}$ for w a cyclic permutation of $aba^{-1}b^{-1}$ or its inverse.*

Proof. This proceeds by using the BNS invariant $\Sigma \subseteq S^1$ of G in [3] and the proof is very similar to Theorem D of that paper. The idea is that Σ is an open subset of S^1 and if $\Sigma \cup -\Sigma$ is not all of S^1 then we have a homomorphism $\chi : G \twoheadrightarrow \mathbb{Z}$ that expresses G as a nonascending HNN extension. The Magnus decomposition of this extension is such that G is either in Case (i) or Case (ii) of Proposition 3.1, so that G is SQ-universal by that theorem for Case (i) or by Theorem 3.2 for Case (ii).

Otherwise we have $\Sigma \cap -\Sigma \neq \emptyset$ as S^1 is connected, which means that the kernel of χ is finitely generated and consequently the 1-relator group G can be expressed as $F_k \rtimes_\alpha \mathbb{Z}$, which is SQ-universal if $k \geq 2$. If $k = 1$ then α is the identity as G surjects to $\mathbb{Z} \times \mathbb{Z}$, so $G = \mathbb{Z} \times \mathbb{Z}$ and therefore admits only the above 2-generator 1-relator presentations by [26] Theorem 4.11. □

Finally an SQ-universal group can be thought of as one with many infinite quotients whereas an infinite residually finite group can be thought of as having many finite quotients. We see that all 2-generator 1-relator groups therefore have many quotients of some kind:

Corollary 3.6. *A group with a* 2*-generator* 1*-relator presentation is either SQ-universal or residually finite.*

Proof. This follows from Corollary 3.4 because [4] proved that a strictly ascending HNN extension of a finite rank free group is residually finite. □

4. Acylindrically hyperbolic mapping tori of free groups

For the three cases in the last section, we had that the groups in Case (i) were all acylindrically hyperbolic whereas none in Case (ii) were. However when considering groups in Case (iii) for SQ-universality, we did this independently of results on acylindrically hyperbolic groups. It can therefore be asked which mapping tori of finite rank free groups are acylindrically hyperbolic and indeed this is exactly Problem 8.2 in [27]. Moreover a solution just for the 1-relator groups in this class would then completely determine which 2-generator 1-relator groups are acylindrically hyperbolic, which is Problem 8.1 of [27].

It is clear that an ascending HNN extension $F_k \rtimes_\alpha \mathbb{Z}$ of F_k formed using an automorphism α of finite order in $\mathrm{Out}(F_k)$ will not be acylindrically hyperbolic because of the existence of an infinite order element in the centre. Thus a possible answer to Problem 8.2 is that all other ascending HNN extensions of F_k are acylindrically hyperbolic with the exception of $BS(1,m)$ when $k = 1$. This would imply two mutually exclusive cases for these groups: either they are acylindrically hyperbolic or they are generalized Baumslag–Solitar groups, and it would also imply in answer to Problem 8.1 of [27] that a 1-relator group is acylindrically hyperbolic if and only if it does not contain an infinite cyclic s-normal subgroup. As a partial answer to Problem 8.2 we have:

Proposition 4.1. *If a finitely generated group G of cohomological dimension* 2 *has a finite index subgroup H splitting over $\mathbb{Z}$ then either G is acylindrically hyperbolic or it is a generalized Baumslag–Solitar group.*

Proof. We can apply [22] Theorem C to H as it also has cohomological dimension 2. This implies that if H splits over $A \cong \mathbb{Z} \leq H$ and A is s-normal then H is a generalized Baumslag–Solitar group and therefore so is G by [22] Corollary 3 (ii) as it is torsion free. Otherwise we can apply Corollaries 2.2 and 2.3 of [27] which state that if H is an amalgamated free product or HNN extension over an edge group which is not s-normal and not equal to a vertex group under any inclusion then H is acylindrically hyperbolic and therefore so is G by [27] Lemma 3.8. However if $A \cong \mathbb{Z}$ is equal to a vertex group then we have $H = BS(1, m)$ which is also a generalized Baumslag–Solitar group. □

Corollary 4.2. *An ascending HNN extension $F_k *_\theta$ for $k \geq 2$ that virtually splits over $\mathbb{Z}$ is either acylindrically hyperbolic or is virtually $F_k \times \mathbb{Z}$.*

Proof. All ascending HNN extensions of F_k have geometric and thus cohomological dimension 2, so by Proposition 4.1 we obtain acylindric hyperbolicity unless we have a generalized Baumslag–Solitar group. They are all also residually finite, but by [24] Corollary 7.7 a generalized Baumslag–Solitar group is not residually finite unless it is virtually $F_k \times \mathbb{Z}$ or $BS(1, m)$ when $k = 1$. The latter case gives rise to soluble groups which therefore cannot contain F_k for $k \geq 2$. □

We finish by returning to the case where the mapping torus is formed using an automorphism, so we are back in the class of free-by-cyclic groups $G = F_k \rtimes_\alpha \mathbb{Z}$, and we will consider which of these groups are acylindrically hyperbolic. Of course this will be true if G is word hyperbolic or hyperbolic with respect to a collection of proper subgroups. There are plenty of examples of word hyperbolic free-by-cyclic groups when $k \geq 3$. When $k = 2$ there are none, but most are relatively hyperbolic with respect to the peripheral $\mathbb{Z} \times \mathbb{Z}$ subgroup because they will be the fundamental group of a finite volume hyperbolic 1-punctured torus bundle. The exceptions are when the monodromy has finite order, giving the virtually $F_2 \times \mathbb{Z}$ case which cannot be acylindrically hyperbolic, and parabolic monodromy where all groups will be commensurable with $G = F(a, b) \rtimes_\lambda \mathbb{Z}$.

As for other free-by-cyclic groups, let us now assume that $[\alpha] \in \mathrm{Out}(F_k)$ is a polynomially growing automorphism. We can therefore quote recent results on this in [12] which itself utilises the train track technology of Bestvina, Feighn and Handel. The two facts from [12] Section 5 that we now use are that:

- If $[\alpha]$ is polynomially growing then there is a positive power $[\alpha^j]$ in $\mathrm{UPG}(F_k)$, which is the subgroup of polynomially growing outer automorphisms whose abelianised action on $\mathbb{Z}^k$ has unipotent image.
- If $[\alpha] \in \mathrm{UPG}(F_k)$ then $F_k \rtimes_\alpha \mathbb{Z}$ splits over $\mathbb{Z}$.

Corollary 4.3. *If $[\alpha] \in \mathrm{Out}(F_k)$ is a polynomially growing automorphism then $G = F_k \rtimes_\alpha \mathbb{Z}$ is acylindrically hyperbolic unless $[\alpha]$ has finite order in $\mathrm{Out}(F_k)$.*

Proof. By taking a power of α, which corresponds to a finite index subgroup of G, we have splittings over $\mathbb{Z}$ of this finite index subgroup and so Corollary 4.2 immediately implies this. $\square$

This adds to results in the literature that provide a description of free-by-cyclic groups according to the type of hyperbolicity: for $G = F_k \rtimes_\alpha \mathbb{Z}$ we have that:

- G is word hyperbolic if and only if no positive power of α sends $w \in F_k \setminus \{id\}$ to a conjugate of itself.
- It is claimed that G is relatively hyperbolic if $[\alpha]$ is not of polynomial growth. (This is from the unpublished manuscript [17] where the peripheral subgroups are the mapping tori of the polynomially growing subgroups under $[\alpha]$ of F_k.)

Thus if the above claim is true then we would have that G is not acylindrically hyperbolic if and only if $[\alpha] \in \mathrm{Out}(F_k)$ has finite order.

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