

On the uniqueness of algebraic curves passing through n -independent nodes

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ABSTRACT. A set of nodes in the plane is called n -independent if for arbitrary data at those nodes, there is a (not necessarily unique) polynomial of degree at most n that matches the given information. We proved in a previous paper (Hakopian–Toroyan, 2015) that the minimal number of n -independent nodes determining uniquely the curve of degree $k \leq n$ passing through them equals to $\mathcal{D} := (1/2)(k-1)(2n+4-k) + 2$. In this paper we bring a characterization of the case when at least two curves of degree k pass through the nodes of an n -independent node set of cardinality $\mathcal{D} - 1$. Namely, we prove that the latter set has a very special construction: All its nodes but one belong to a (maximal) curve of degree $k - 1$. We show that this result readily yields the above cited one. At the end, an important application to the Gasca–Maeztu conjecture is presented.

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1. Introduction

Denote the space of all bivariate polynomials of total degree $\leq n$ by Π_n :

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}.$$

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We have that

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Consider a set of s distinct nodes

$$\mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \dots, (x_s, y_s)\}.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$(1.1) \quad p(x_i, y_i) = c_i, \quad i = 1, \dots, s,$$

is called interpolation problem.

A polynomial $p \in \Pi_n$ is called an n -fundamental polynomial for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik}, \quad i = 1, \dots, s,$$

where δ is the Kronecker symbol. We denote this fundamental polynomial by $p_k^* = p_A^* = p_{A, \mathcal{X}_s}^*$. Sometimes we call fundamental also a polynomial that vanishes at all nodes of \mathcal{X}_s but one, since it is a nonzero constant times a fundamental polynomial.

Next, let us consider an important concept of n -independence (see [7], [11]).

Definition 1.1. A set of nodes \mathcal{X} is called n -independent if all its nodes have n -fundamental polynomials. Otherwise, if a node has no n -fundamental polynomial, \mathcal{X} is called n -dependent.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n -independence of \mathcal{X}_s is $s \leq N$.

Suppose a node set \mathcal{X}_s is n -independent. Then by the Lagrange formula we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1):

$$p = \sum_{i=1}^s c_i p_i^*.$$

In view of this, we get readily that the node set \mathcal{X}_s is n -independent if and only if the interpolating problem (1.1) is *solvable*, meaning that for any data (c_1, \dots, c_s) there is a polynomial $p \in \Pi_n$ (not necessarily unique) satisfying the interpolation conditions (1.1).

Definition 1.2. The interpolation problem with a set of nodes \mathcal{X}_s and Π_n is called n -poised if for any data (c_1, \dots, c_s) there is a *unique* polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1).

The conditions (1.1) give a system of s linear equations with N unknowns (the coefficients of the polynomial p). The poisedness means that this system has a unique solution for arbitrary right side values. Therefore a necessary condition of poisedness is $s = N$. If this condition holds then we obtain from the linear system:

Proposition 1.3. *A set of nodes \mathcal{X}_N is n -poised if and only if*

$$p \in \Pi_n \text{ and } p|_{\mathcal{X}_N} = 0 \implies p = 0.$$

Thus, geometrically, the node set \mathcal{X}_N is n -poised if and only if there is no curve of degree n passing through all its nodes.

It is worth mentioning:

Proposition 1.4. *For any set \mathcal{X}_{N-1} , i.e., set of cardinality $N - 1$, there is a curve of degree n passing through all its nodes.*

Indeed, the existence of the curve reduces to a system of $N - 1$ linear homogeneous equations with N unknowns – the coefficients of the polynomial of degree n .

It follows from Proposition 1.3 also that a node set of cardinality N is n -poised if and only if it is n -independent.

Next, let us describe the main result of this paper. Suppose we have an n -poised set \mathcal{X}_N . From what was said above we can conclude readily that through any $N - 1$ nodes of \mathcal{X} there pass a unique curve of degree n . Indeed, this curve is given by the fundamental polynomial of the missing node. Next, through any $N - 2$ nodes of \mathcal{X} there pass more than one curve of degree n , for example the curves given by the fundamental polynomials of two missing nodes. Thus we have that the minimal number of n -independent nodes determining uniquely the curve of degree n equals to $N - 1$.

In [14] we considered this problem in the case of arbitrary degree k , $k \leq n - 1$. We proved that the minimal number of n -independent nodes determining uniquely the curve of degree $k \leq n - 1$ equals

$$\mathcal{D} := (1/2)(k - 1)(2n + 4 - k) + 2.$$

Or, more precisely, for any n -independent set of cardinality \mathcal{D} there is at most one curve of degree $k \leq n - 1$ passing through its nodes, while there are n -independent node sets of cardinality $\mathcal{D} - 1$ through which pass at least two such curves. Let us mention that the case $k = n - 1$ of the above described problem is considered in [2].

In this paper we bring a characterization of the sets of cardinality $\mathcal{D} - 1$ through which pass at least two curves of degree k . Namely, we prove that in this case all the nodes of \mathcal{X} but one belong to a curve of degree $k - 1$. Moreover, this latter curve is a maximal curve meaning that it passes through maximal possible number of n -independent nodes (see Section 3).

As we will see in Section 5, this result readily yields the above mentioned result of [14].

At the end let us bring a well-known Berzolari–Radon construction of n -poised sets (see [3], [15]).

Definition 1.5. A set of $N = 1 + \dots + (n + 1)$ nodes is called Berzolari–Radon set for degree n , or briefly BR_n set, if there exist lines $\ell_1, \ell_2, \dots, \ell_{n+1}$, such that the sets $\ell_1, \ell_2 \setminus \ell_1, \ell_3 \setminus (\ell_1 \cup \ell_2), \dots, \ell_{n+1} \setminus (\ell_1 \cup \dots \cup \ell_n)$ contain exactly $(n + 1), n, n - 1, \dots, 1$ nodes, respectively.

2. Some properties of n -independent nodes

Let us start with the following simple (see, e.g., [12], Lemma 2.3; [11] Lemma 2.2) lemma.

Lemma 2.1. *Suppose that a node set \mathcal{X} is n -independent and a node $A \notin \mathcal{X}$ has n -fundamental polynomial with respect to the set $\mathcal{X} \cup \{A\}$. Then the latter node set is n -independent, too.*

Indeed, one can get readily the fundamental polynomial of any node $B \in \mathcal{X}$ with respect to the set $\mathcal{Y} := \mathcal{X} \cup \{A\}$ by using a linear combination of the given fundamental polynomial p_A^* and the fundamental polynomial of B with respect to the set \mathcal{X} .

Evidently, any subset of n -poised set is n -independent. According to the next lemma any n -independent set is a subset of some n -poised set:

Lemma 2.2 (e.g., [9], Lemma 2.1). *Any n -independent set \mathcal{X} with $\#\mathcal{X} < N$ can be enlarged to an n -poised set.*

Proof. It suffices to show that there is a node A such that the set $\mathcal{X} \cup \{A\}$ is n -independent. By Proposition 1.4 there is a nonzero polynomial $q \in \Pi_n$ such that $q|_{\mathcal{X}} = 0$. Now, in view of Lemma 2.1, we may choose a desirable node A by requiring only that $q(A) \neq 0$. Indeed, then q is a fundamental polynomial of A with respect to the set $\mathcal{X} \cup \{A\}$. \square

Denote the linear space of polynomials of total degree at most n vanishing on \mathcal{X} by

$$\mathcal{P}_{n,\mathcal{X}} = \{p \in \Pi_n : p|_{\mathcal{X}} = 0\}.$$

The following is well-known.

Proposition 2.3 (e.g., [9], [11]). *For any node set \mathcal{X} we have that*

$$\dim \mathcal{P}_{n,\mathcal{X}} \geq N - \#\mathcal{X}.$$

Moreover, equality takes place here if and only if the set \mathcal{X} is n -independent.

From Lemma 2.1 one gets readily:

Corollary 2.4 (e.g., [12], Corollary 2.4). *Let \mathcal{Y} be a maximal n -independent subset of \mathcal{X} , i.e., $\mathcal{Y} \subset \mathcal{X}$ is n -independent and $\mathcal{Y} \cup \{A\}$ is n -dependent for any $A \in \mathcal{X} \setminus \mathcal{Y}$. Then we have that*

$$(2.1) \quad \mathcal{P}_{n,\mathcal{Y}} = \mathcal{P}_{n,\mathcal{X}}.$$

Proof. We have that $\mathcal{P}_{n,\mathcal{X}} \subset \mathcal{P}_{n,\mathcal{Y}}$, since $\mathcal{Y} \subset \mathcal{X}$. Now, suppose that $p \in \Pi_n$, $p|_{\mathcal{Y}} = 0$ and A is any node of \mathcal{X} . Then $\mathcal{Y} \cup \{A\}$ is dependent and therefore, in view of Lemma 2.1, $p(A) = 0$. \square

From (2.1) and Proposition 2.3 (part “moreover”) we have that

$$(2.2) \quad \dim \mathcal{P}_{n,\mathcal{X}} = N - \#\mathcal{Y},$$

where \mathcal{Y} is any maximal n -independent subset of \mathcal{X} . Thus, all the maximal n -independent subsets of \mathcal{X} have the same cardinality, which is denoted by $\mathcal{H}_n(\mathcal{X})$ – the Hilbert n -function of \mathcal{X} . Hence, according to (2.2), we have that

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \mathcal{H}_n(\mathcal{X}).$$

3. Maximal curves

An algebraic curve in the plane is the zero set of some bivariate polynomial of degree at least 1. We use the same letter, say p , to denote the polynomial $p \in \Pi_k \setminus \Pi_{k-1}$ and the corresponding curve p of degree k defined by equation $p(x, y) = 0$.

According to the following well-known statement there are no more than $n + 1$ n -independent points in any line:

Proposition 3.1. *Assume that ℓ is a line and \mathcal{X}_{n+1} is any subset of ℓ containing $n + 1$ points. Then we have that*

$$p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}_{n+1}} = 0 \quad \implies \quad p = \ell r,$$

where $r \in \Pi_{n-1}$.

Denote

$$d := d(n, k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k).$$

The following is a generalization of Proposition 3.1.

Proposition 3.2 ([16], Prop. 3.1). *Let q be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold:*

- (i) *Any subset of q containing more than $d(n, k)$ nodes is n -dependent.*
- (ii) *Any subset \mathcal{X}_d of q containing exactly $d = d(n, k)$ nodes is n -independent if and only if the following condition holds:*

$$(3.1) \quad p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}_d} = 0 \quad \implies \quad p = qr,$$

where $r \in \Pi_{n-k}$.

Suppose that \mathcal{X} is an n -poised set of nodes and q is an algebraic curve of degree $k \leq n$. Then of course any subset of \mathcal{X} is n -independent too. Therefore, according to Proposition 3.2(i), at most $d(n, k)$ nodes of \mathcal{X} can lie in the curve q . Let us mention that a special case of this when q is a set of k lines is proved in [6].

This motivates the following definition.

Definition 3.3 ([16], Def. 3.1). *Given an n -independent set of nodes \mathcal{X}_s , with $s \geq d(n, k)$. A curve of degree $k \leq n$ passing through $d(n, k)$ points of \mathcal{X}_s , is called *maximal*.*

Note that the maximal line, as a line passing through $n + 1$ nodes, is defined in [4]. Let us mention that $q = \ell_1 \cdots \ell_k$ is a maximal curve of degree k of the node set BR_n (see Def. 1.5), where $k = 1, \dots, n$.

We say that a node $A \in \mathcal{X}$ uses a polynomial $q \in \Pi_k$ if the latter divides the fundamental polynomial $p = p_A^*$, i.e., $p = qr$, for some $r \in \Pi_{n-k}$.

Next, we bring a characterization of maximal curves:

Proposition 3.4 ([16], Prop. 3.3). *Let a node set \mathcal{X} be n -poised. Then a polynomial μ of degree k , $k \leq n$, is a maximal curve if and only if it is used by any node in $\mathcal{X} \setminus \mu$.*

Note that one side of this statement follows from Proposition 3.2(ii). In the case of degree one this was proved in [4].

For other properties of maximal curves we refer reader to [16], [13].

Proposition 3.5. *Assume that σ is an algebraic curve of degree k , without multiple components, and $\mathcal{X}_s \subset \sigma$ is any n -independent node set of cardinality s , $s < d(n, k)$. Then the set \mathcal{X}_s can be extended to a maximal n -independent set $\mathcal{X}_d \subset \sigma$ of cardinality $d = d(n, k)$.*

Proof. It suffices to show that there is a point $A \in \sigma \setminus \mathcal{X}_s$ such that the set $\mathcal{X}_{s+1} := \mathcal{X}_s \cup \{A\}$ is n -independent. Assume to the contrary that there is no such point, i.e., the set $\mathcal{X}_{s+1} := \mathcal{X}_s \cup \{A\}$ is n -dependent for any $A \in \sigma$. Then, in view of Lemma 2.1, A has no fundamental polynomial with respect to the set \mathcal{X}_{s+1} . In other words we have

$$p \in \Pi_n \text{ and } p|_{\mathcal{X}_s} = 0 \implies p(A) = 0 \text{ for any } A \in \sigma.$$

From here we obtain that

$$\mathcal{P}_{n, \mathcal{X}_s} \subset \mathcal{P}_{n, \sigma} := \{q\sigma : q \in \Pi_{n-k}\}.$$

Now, in view of Proposition 2.3, we get from here

$$N - s = \dim \mathcal{P}_{n, \mathcal{X}_s} \leq \dim \mathcal{P}_{n, \sigma} = N_{n-k}.$$

Therefore $s \geq d(n, k)$, which contradicts the hypothesis. \square

Let us mention that, as it follows from the above proof, the condition (3.1) does not hold if $d < d(n, k)$.

The following lemma follows readily from the fact that the Vandermonde determinant, i.e., the main determinant of the linear system described just after Definition 1.2, is a continuous function of the nodes of \mathcal{X}_N .

Lemma 3.6 (e.g., [8], Remark 1.14). *Suppose that $\mathcal{X}_N = \{(x_i, y_i)\}_{i=1}^N$ is an n -poised set. Then there is a positive number ϵ such that any set*

$$\mathcal{X}'_N = \{(x'_i, y'_i)\}_{i=1}^N,$$

with the property that the distance between (x'_i, y'_i) and (x_i, y_i) is less than ϵ , for each i , is n -poised too.

From here, in view of Lemma 2.2 we get readily:

Corollary 3.7. *Suppose that $\mathcal{X}_s = \{(x_i, y_i)\}_{i=1}^s$ is an n -independent set. Then there is a positive number ϵ such that any set $\mathcal{X}'_s = \{(x'_i, y'_i)\}_{i=1}^s$, with the property that the distance between (x'_i, y'_i) and (x_i, y_i) is less than ϵ , for each i , is n -independent too.*

Finally, let us bring a well-known lemma:

Lemma 3.8. *Suppose that two different curves of degree at most k pass through all the nodes of \mathcal{X} . Then for any node $A \notin \mathcal{X}$ there is a curve of degree at most k passing through A and all the nodes of \mathcal{X} .*

Indeed, if the given curves are σ_1 and σ_2 then the desired curve can be found easily in the form of linear combination $c_1\sigma_1 + c_2\sigma_2$.

4. Main result

In a previous paper we determined the minimal number of n -independent nodes that uniquely determine the curve of degree k , $k \leq n$, passing through them:

Theorem 4.1 ([14], Thm. 1). *Assume that \mathcal{X} is an n -independent set of $d(n, k - 1) + 2$ nodes lying in a curve of degree k with $k \leq n$. Then the curve is determined uniquely by these nodes. Moreover, there is an n -independent set of $d(n, k - 1) + 1$ nodes such that more than one curves of degree k pass through all its nodes.*

Let us mention that this result is obvious in the case $k = n$, while in the case $k = n - 1$ it was established in [2].

In this section we characterize the case when more than one curve of degree k , $k \leq n - 1$, passes through the nodes of an n -independent set \mathcal{X} of cardinality $d(n, k - 1) + 1$.

As we will see later in Section 5 this result yields readily Theorem 4.1.

Theorem 4.2. *Assume that \mathcal{X} is an n -independent set of $d(n, k - 1) + 1$ nodes with $k \leq n - 1$. Then two different curves of degree k pass through all the nodes of \mathcal{X} if and only if all the nodes of \mathcal{X} but one lie in a maximal curve of degree $k - 1$.*

Proof. Let us start with the inverse implication. Assume that $d(n, k - 1)$ nodes of \mathcal{X} are located in a curve μ of degree $k - 1$. Therefore the curve μ is maximal and the remaining node of \mathcal{X} , which we denote by A , is outside of it: $A \notin \mu$.

Now assume that ℓ_1 and ℓ_2 are two different lines passing through A . Then it is easily seen that $\ell_1\mu$ and $\ell_2\mu$ are two different curves of degree k passing through all the nodes of \mathcal{X} .

Now let us prove the direct implication. Assume that there are two curves of degree k : σ_1 and σ_2 that pass through all the nodes of the n -independent set \mathcal{X} with $\#\mathcal{X} = d(n, k - 1) + 1$. Let us start by choosing a node $B \notin \mathcal{X}$ such that the following three conditions are satisfied:

- (i) The set $\mathcal{X} \cup \{B\}$ is n -independent.
- (ii) B does not lie in any line passing through two nodes of \mathcal{X} .
- (iii) B does not lie in the curves σ_1 and σ_2 .

Let us verify that one can find a such node. Indeed, in view of Lemma 2.2, we can start by choosing a node B' satisfying the condition (i). Then, according to Corollary 3.7, for some positive ϵ all the nodes in ϵ neighborhood of B' satisfy the condition (i). Finally, from this neighborhood we can choose a node B satisfying the conditions (ii) and (iii), too.

Next, in view of Lemma 3.8, there is a curve σ of degree at most k passing through all the nodes of $\mathcal{X}' := \mathcal{X} \cup \{B\}$. According to the condition (iii) σ is different from σ_1 and σ_2 . Then notice that the curve σ passes through more than $d(n, k - 1)$ nodes and therefore its degree equals to k and it has no multiple component.

Now, by using Proposition 3.5, let us extend the set \mathcal{X}' till a maximal n -independent set $\mathcal{X}'' \subset \sigma$. Notice that, since $\#\mathcal{X}'' = d(n, k)$, we need to add $d(n, k) - (d(n, k - 1) + 2) = n - k$ nodes to \mathcal{X}' , denoted by C_1, \dots, C_{n-k} :

$$\mathcal{X}'' := \mathcal{X} \cup \{B\} \cup \{C_i\}_{i=1}^{n-k}.$$

Thus the curve σ becomes maximal with respect to this set.

Then let us consider $n - k - 1$ lines $\ell_1, \ell_2, \dots, \ell_{n-k-1}$ passing through the nodes $C_1, C_2, \dots, C_{n-k-1}$, respectively. We require that each line passes through only one of the mentioned nodes and therefore the lines are distinct. We require also that none of these lines is a component (factor) of σ . Finally let us denote by $\tilde{\ell}$ the line passing through the nodes B and C_{n-k} .

Now notice that the following polynomial

$$\sigma_1 \tilde{\ell} \ell_1 \ell_2 \dots \ell_{n-k-1}$$

of degree n vanishes at all the $d(n, k)$ nodes of $\mathcal{X}'' \subset \sigma$. Consequently, according to Proposition 3.2, σ divides this polynomial:

$$(4.1) \quad \sigma_1 \tilde{\ell} \ell_1 \ell_2 \dots \ell_{n-k-1} = \sigma q, \quad q \in \Pi_{n-k}.$$

The distinct lines $\ell_1, \ell_2, \dots, \ell_{n-k-1}$ do not divide the polynomial $\sigma \in \Pi_k$, therefore all they have to divide $q \in \Pi_{n-k}$. Thus $q = \ell_1 \dots \ell_{n-k-1} \ell'$, where $\ell' \in \Pi_1$. Therefore, we get from (4.1):

$$\sigma_1 \tilde{\ell} = \sigma \ell'.$$

If the lines $\tilde{\ell}, \ell'$ coincide then the curves σ_1, σ coincide, which is impossible. Therefore the line $\tilde{\ell}$ has to divide $\sigma \in \Pi_k$:

$$\sigma = \tilde{\ell} r, \quad r \in \Pi_{k-1}.$$

Now, we are going to derive from this relation that the curve r passes through all the nodes of the set \mathcal{X} but one. Indeed, σ passes through all the nodes of \mathcal{X} . Therefore these nodes are either in the curve r or in the line $\tilde{\ell}$. But the latter line passes through B , and according to the condition (ii), it passes through at most one node of \mathcal{X} . Thus r passes through at least $d(n, k - 1)$

nodes of \mathcal{X} . Since r is a curve of degree $k - 1$ we conclude that r is a maximal curve and passes through exactly $d(n, k - 1)$ nodes of \mathcal{X} . \square

It is worth mentioning that for any n -independent node set \mathcal{X} of cardinality $d(n, k - 1) + 1$, where $k \leq n - 1$, we have that

$$\dim \mathcal{P}_{k, \mathcal{X}} \leq 2,$$

where an equality takes place if and only if all the nodes of \mathcal{X} but one lie in a maximal curve of degree $k - 1$.

Indeed, if

$$\dim \mathcal{P}_{k, \mathcal{X}} \geq 2$$

then according to Theorem 4.2 we have that all the nodes of \mathcal{X} but one lie in a maximal curve μ of degree $k - 1$. Now, according to Proposition 3.2, we have that

$$\mathcal{P}_{k, \mathcal{X}} = \{ \alpha \mu \mid \alpha \in \Pi_1, \alpha(A) = 0 \},$$

where $A \in \mathcal{X}$ is the node outside of μ . Therefore we get readily

$$\dim \mathcal{P}_{k, \mathcal{X}} = \dim \{ \alpha \mid \alpha \in \Pi_1, \alpha(A) = 0 \} = 2.$$

5. A corollary

Here we verify that our main result yields Theorem 4.1, which in view of Theorem 4.2, states that for any n -independent set \mathcal{X} of cardinality

$$d(n, k - 1) + 2$$

there is at most one curve of degree k , $k \leq n$, passing through all its nodes.

Proof of Theorem 4.1. Note that the case $k = n$ is evident, since

$$d(n, n - 1) + 2 = N - 1.$$

Now assume that $k \leq n - 1$. Choose a node $A \in \mathcal{X}$ and consider the set $\mathcal{Y} := \mathcal{X} \setminus \{A\}$. If there is at most one curve of degree k which passes through all the nodes of \mathcal{Y} then the same is true also for the set \mathcal{X} and we are done. Thus assume that there are at least two curves of degree k which pass through all the nodes of the set \mathcal{Y} . Then, according to Theorem 4.2, all the nodes of \mathcal{Y} but one, denoted by B , lie in a maximal curve μ of degree $k - 1$. Therefore, all the nodes of \mathcal{X} but A and B lie in the curve μ . Now, in view of Proposition 3.2, any curve of degree k passing through all the nodes of \mathcal{X} has the following form

$$p = \ell \mu,$$

where $\ell \in \Pi_1$. Finally notice that the line ℓ passes through the nodes A and B and therefore is determined in a unique way. Hence p is determined uniquely, too. \square

6. An application to the Gasca–Maeztu conjecture

Let us recall that a node $A \in \mathcal{X}$ uses a line ℓ means that ℓ is a factor of the fundamental polynomial $p = p_A^*$, i.e., $p = \ell r$, for some $r \in \Pi_{n-1}$.

A GC_n -set in plane is an n -poised set of nodes where the fundamental polynomial of each node is a product of n linear factors. Note that this always takes place in the univariate case.

The Gasca–Maeztu conjecture states that any GC_n -set possesses a subset of $n + 1$ collinear nodes.

It was proved in [5] that any line passing through exactly 2 nodes of a GC_n -set \mathcal{X} can be used at most by one node from \mathcal{X} , provided that the Gasca–Maeztu conjecture is true for all degrees not exceeding n .

Recently, it was announced in [1], that this result holds for any poised set \mathcal{X} , without other restrictions. By the way it follows readily also from Theorem 4.2.

Below we consider the case of lines passing through exactly 3 nodes.

Corollary 6.1. *Let \mathcal{X} be an n -poised set of nodes and ℓ be a used line which passes through exactly 3 nodes. Then ℓ is used either by exactly one or by exactly three nodes from \mathcal{X} . Moreover, if it is used by three nodes, then they are noncollinear.*

Proof. Assume that $\ell \cap \mathcal{X} = \{A_1, A_2, A_3\}$. Assume also that there are two nodes $B, C \in \mathcal{X}$ using the line ℓ :

$$p_B^* = \ell q_1, \quad p_C^* = \ell q_2,$$

where $q_1, q_2 \in \Pi_{n-1}$.

Both the polynomials q_1, q_2 vanish at $N - 5$ nodes of the set

$$\mathcal{X}' := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C\}.$$

Hence these $N - 5 = d(n, n - 2) + 1$ nodes do not uniquely determine the curve of degree $n - 1$ passing through them. By Theorem 4.2 there exists a maximal curve μ of degree $n - 2$ passing through $N - 6$ nodes of \mathcal{X}' and the remaining node denoted by D is outside of it. Now, according to Proposition 3.4, the node D uses μ :

$$p_D^* = \mu q, \quad q \in \Pi_2.$$

This quadratic polynomial q has to vanish at the three nodes $A_1, A_2, A_3 \in \ell$. Therefore, in view of Proposition 3.1, we have that $q = \ell \ell'$ with $\ell' \in \Pi_1$. Hence the node D uses the line ℓ :

$$p_D^* = \mu \ell \ell', \quad \ell' \in \Pi_1.$$

Thus if two nodes $B, C \in \mathcal{X}$ use the line ℓ then there exists a third node $D \in \mathcal{X}$ using it and all the nodes of $\mathcal{Y} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D\}$ lie in a maximal curve μ of degree $n - 2$:

$$(6.1) \quad \mathcal{Y} \subset \mu.$$

Next, let us show that there is no fourth node using ℓ . Assume by way of contradiction that except of the nodes B, C, D , there is a fourth node E using ℓ . Of course we have that $E \in \mathcal{Y}$.

Then B and E are using ℓ therefore, as was proved above, there exists a third node $F \in \mathcal{X}$ (which may coincide or not with C or D) using it and all the nodes of $\tilde{\mathcal{Y}} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, E, F\}$ are located in a maximal curve $\tilde{\mu}$ of degree $n - 2$. We have also that

$$(6.2) \quad p_E^* = \tilde{\mu}\tilde{q}, \quad \tilde{q} \in \Pi_2.$$

Now, notice that both μ and $\tilde{\mu}$ pass through all the nodes of the set $\mathcal{Z} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D, E, F\}$ with $\#\mathcal{Z} \geq N - 8$.

Then, we get from Theorem 4.1, with $k = n - 2$, that $N - 8 = d(n, n - 3) + 2$ nodes determine the curve of degree $n - 2$ passing through them uniquely. Thus μ and $\tilde{\mu}$ coincide.

Therefore, in view of (6.1) and (6.2), p_E^* vanishes at all the nodes of \mathcal{Y} , which is a contradiction since $E \in \mathcal{Y}$.

Now, let us verify the last “moreover” statement. Suppose three nodes $B, C, D \in \mathcal{X}$ use the line ℓ . Then, as we obtained earlier, the nodes

$$\mathcal{Y} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D\}$$

are located in a maximal curve μ of degree $n - 2$. Suppose conversely that the nodes B, C and D are lying in a line ℓ_1 . Then we have that all the nodes of the set \mathcal{X} are lying in the curve $\mu\ell\ell_1$ of degree n . This, in view of Proposition 1.3, is a contradiction. \square

References

- [1] BAYRAMYAN, VAHAGN. On the usage of 2-node lines in n -poised sets, International Conference. *Harmonic Analysis and Approximations, VI*, (Tsaghkadzor, Armenia, 2015), Abstracts, p.18. <http://mathconf.sci.am/haa2015/HAA%20VI%20Abstracts%20Book.pdf>.
- [2] BAYRAMYAN, VAHAGN; HAKOPIAN, HAKOP; TOROYAN, SOFIK. On the uniqueness of algebraic curves. *Proceedings of YSU, Phys. Math. Sci.* **2015**, no. 1, 3–7. Zbl 1327.14150.
- [3] BERZOLARI, LUIGI. Sulla determinazione di una curva o di una superficie algebrica e su alcune questioni di postulazione. *Lomb. Ist. Rend.* **47** (1914), 556–564.
- [4] CARNICER, JESUS MIGUEL; GASCA, MARIANO. A conjecture on multivariate polynomial interpolation. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **95** (2001), no. 1, 145–153. MR1899358 (2003d:41031), Zbl 1013.41001.
- [5] CARNICER, JESUS MIGUEL; GASCA, MARIANO. On Chung and Yao’s geometric characterization for bivariate polynomial interpolation. *Curve and surface design* (Saint-Malo, 2002), 21–30, Mod. Methods Math., *Nashboro Press, Brentwood, TN*, 2003. MR2042470, Zbl 1054.41001.
- [6] CARNICER, JESUS MIGUEL; GASCA, MARIANO. Planar configurations with simple Lagrange interpolation formulae. *Mathematical methods for curves and surfaces* (Oslo, 2000), 55–62, Innov. Appl. Math., *Vanderbilt Univ. Press, Nashville, TN*, 2001. MR1857824, Zbl 1003.65008.

- [7] EISENBUD, DAVID; GREEN, MARK; HARRIS, JOE. Cayley–Bacharach theorems and conjectures. *Bull. Amer. Math. Soc. (N.S.)* **33** (1996), no. 3, 295–324. MR1376653 (97a:14059), Zbl 0871.14024, doi: 10.1090/S0273-0979-96-00666-0.
- [8] HAKOPIAN, HAKOP. On the regularity of multivariate Hermite interpolation. *J. Approx. Theory* **105** (2000), no.1, 1–18, MR1768522, Zbl 0957.41002, doi: 10.1006/jath.1999.3345.
- [9] HAKOPIAN, HAKOP; JETTER, KURT; ZIMMERMANN, GEORG. Vandermonde matrices for intersection points of curves. *Jaen J. Approx.* **1** (2009), no. 1, 67–81. MR2597908, Zbl 1176.41003.
- [10] HAKOPIAN, HAKOP; JETTER, KURT; ZIMMERMANN, GEORG. The Gasca–Maeztu conjecture for $n = 5$. *Numer. Math.* **127** (2014), no. 4, 685–713. MR3229990, Zbl 1304.41002, doi: 10.1007/s00211-013-0599-4.
- [11] HAKOPIAN, HAKOP; MALINYAN, ARGISHTI. Characterization of n -independent sets with no more than $3n$ points. *Jaen J. Approx.* **4** (2012), no. 2, 121–136. MR3203464, Zbl 1295.41035.
- [12] HAKOPIAN, HAKOP; MUSHYAN, GEVORG. On multivariate segmental interpolation problem. *J. Comp. Sci. & Appl. Math.* **1** (2015), 19–29. <http://research-publication.com/articles/JCSAM/2015/JCSAM-Vol11-13.pdf>
- [13] HAKOPIAN, HAKOP; RAFAYELYAN, LEVON. On a generalization of the Gasca–Maeztu conjecture. *New York J. Math.* **21** (2015), 351–367. MR3358548, Zbl 1318.41001.
- [14] HAKOPIAN, HAKOP; TOROYAN, SOFIK. On the minimal number of nodes uniquely determining algebraic curves. *Proc. of YSU, Phys. Math. Sci.* **3** (2015), 17–22. Zbl 1333.41002. <http://mi.mathnet.ru/uzeru31>.
- [15] RADON, JOHANN. Zur mechanischen Kubatur. *Monatsh. Math.* **52** (1948), 286–300. MR0033206, Zbl 0031.31504, doi: 10.1007/BF01525334.
- [16] RAFAYELYAN, LEVON. Poised nodes set constructions on algebraic curves. *East J. Approx.* **17** (2011), no. 3, 285–298. MR2953081, Zbl 06077289.

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