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Realising the Toeplitz algebra of a higher-rank graph as a Cuntz–Krieger algebra

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Abstract. For a row-finite higher-rank graph Λ, we construct a higher-rank graph $T\Lambda$ such that the Toeplitz algebra of Λ is isomorphic to the Cuntz–Krieger algebra of $T\Lambda$. We then prove that the higher-rank graph $T\Lambda$ is always aperiodic and use this fact to give another proof of a uniqueness theorem for the Toeplitz algebras of higher-rank graphs.

Contents

1. Introduction

Higher-rank graphs and their Cuntz–Krieger algebras were introduced by Kumjian and Pask in [5] as a generalisation of the Cuntz–Krieger algebras of directed graphs. Kumjian and Pask proved an analogue of the Cuntz–Krieger uniqueness theorem for a family of *aperiodic* higher-rank graphs [5, Theorem 4.6]. Aperiodicity is a generalisation of Condition (L) for directed graphs and comes in several forms for different kinds of higher-rank graphs (see [1, 5, 6, 10, 11, 12, 13, 14]).

The Toeplitz algebra of a directed graph is an extension of the Cuntz–Krieger algebra in which the Cuntz–Krieger equations at vertices are replaced by inequalities. An analogous family of Toeplitz algebras for higher-rank graph was introduced and studied by Raeburn and Sims [9]. They

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proved a uniqueness theorem for Toeplitz algebras [9, Theorem 8.1], generalising a previous theorem for directed graphs [3, Theorem 4.1].

For a directed graph E, the Toeplitz algebra of E is canonically isomorphic to the Cuntz–Krieger algebra of a graph TE (see [7, Theorem 3.7] and [15, Lemma 3.5]). Here we provide an analogous construction for a row-finite higher-rank graph Λ. We build a higher-rank graph $T\Lambda$, and show that the Toeplitz algebra of Λ is canonically isomorphic to the Cuntz–Krieger algebra of $T\Lambda$ (Theorem 4.1). Our proof relies on the uniqueness theorem of [9]. However, it is interesting to observe that the higher-rank graph $T\Lambda$ is always aperiodic. Hence our isomorphism shows that the uniqueness theorem of [9] is a consequence of the general Cuntz–Krieger uniqueness theorem of [11] (see Remark 4.3).

2. Higher-rank graphs

Let k be a positive integer. We regard $\mathbb{N}^k$ as an additive semigroup with identity 0. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum.

A *higher-rank graph* or *k-graph* is a pair (Λ, d) consisting of a countable small category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu v$ and $d(\mu) = m$, $d(\nu) = n$. We then write $\lambda(0, m)$ for μ and $\lambda(m, m+n)$ for ν. We regard elements of Λ^0 as *vertices* and elements of Λ as *paths*. For detailed explanation and examples, see [8, Chapter 10].

For $v \in \Lambda^0$ and $E \subseteq \Lambda$, we define $vE := \{\lambda \in E : r(\lambda) = v\}$ and $m \in \mathbb{N}^k$, we write $\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m\}$.We use term *edge* to denote a path $e \in \Lambda^{e_i}$ where $1 \leq i \leq k$, and write

$$\Lambda^1 := \bigcup_{1 \leq i \leq k} \Lambda^{e_i}$$

for the set of all edges. We say that Λ is *row-finite* if for every $v \in \Lambda^0$, the set $v\Lambda^{e_i}$ is finite for $1 \leq i \leq k$. Finally, we say $v \in \Lambda^0$ is a *source* if there exists $m \in \mathbb{N}^k$ such that $v\Lambda^m = \emptyset$.

For a row-finite k-graph Λ, we shall construct a k-graph $T\Lambda$ which is row-finite and always has sources. Our k-graph $T\Lambda$ is typically not *locally convex* in the sense of [10, Definition 3.9] (see Remark 3.3), so the appropriate definition of Cuntz–Krieger Λ-family is the one in [11]. For detailed discussion about row-finite k-graphs and their generalisations, see [16, Section 2].

From now on, we focus on a row-finite k-graph Λ. For $\lambda, \mu \in \Lambda$, we say that τ is a *minimal common extension* of λ and μ if

$$d(\tau) = d(\lambda) \vee d(\mu), \; \tau(0, d(\lambda)) = \lambda \text{ and } \tau(0, d(\mu)) = \mu.$$

Let $\mathrm{MCE}(\lambda,\mu)$ denote the collection of all minimal common extensions of λ and μ. Then we write

$$\Lambda^{\min}(\lambda,\mu) := \left\{ \left(\lambda',\mu'\right) \in \Lambda \times \Lambda : \lambda\lambda' = \mu\mu' \in \mathrm{MCE}(\lambda,\mu) \right\}.$$

A set $E \subseteq v\Lambda^1$ is *exhaustive* if for all $\lambda \in v\Lambda$, there exists $e \in E$ such that $\Lambda^{\min}(\lambda,e) \neq \emptyset$.

A *Toeplitz–Cuntz–Krieger* Λ*-family* is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra B satisfying:

(TCK1) $\left\{t_v : v \in \Lambda^0\right\}$ is a collection of mutually orthogonal projections.
(TCK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$.
(TCK3) $t_\lambda^* t_\mu = \sum_{(\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)} t_{\lambda'} t_{\mu'}^*$ for all $\lambda,\mu \in \Lambda$.

Remark 2.1. In [9, Lemma 9.2], Raeburn and Sims required also that "for all $m \in \mathbb{N}^k \backslash \{0\}$, $v \in \Lambda^0$, and every set $E \subseteq v\Lambda^m$, $t_v \geq \sum_{\lambda \in E} t_\lambda t_\lambda^*$". However, by [11, Lemma 2.7 (iii)], this follows from (TCK1)–(TCK3), and hence our definition is basically same as that of [9].

Meanwhile, based on [11, Proposition C.3], a *Cuntz–Krieger* Λ*-family* is a Toeplitz–Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ which satisfies

(CK) $\prod_{e \in E} (t_v - t_e t_e^*) = 0$ for all $v \in \Lambda^0$ and exhaustive $E \subseteq v\Lambda^1$.

Raeburn and Sims proved in [9, Section 4] that there is a C^*-algebra $TC^*(\Lambda)$ generated by a universal Toeplitz–Cuntz–Krieger Λ-family

$$\{t_\lambda : \lambda \in \Lambda\}.$$

If $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ-family in a C^*-algebra B, we write ϕ_T for the homomorphism of $TC^*(\Lambda)$ into B such that $\phi_T(t_\lambda) = T_\lambda$ for $\lambda \in \Lambda$. The quotient of $TC^*(\Lambda)$ by the ideal generated by

$$\left\{ \prod_{e \in E} (t_v - t_e t_e^*) : v \in \Lambda^0, E \subseteq v\Lambda^1 \text{ is exhaustive} \right\}$$

is generated by a universal family of the Cuntz–Krieger Λ-family

$$\{s_\lambda : \lambda \in \Lambda\},$$

and hence we can identify it with the C^*-algebra $C^*(\Lambda)$. For a Cuntz–Krieger Λ-family $\{S_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B, we write π_S for the homorphism of $C^*(\Lambda)$ into B such that $\pi_S(s_\lambda) = S_\lambda$ for $\lambda \in \Lambda$. Furthermore, we have $s_v \neq 0$ for $v \in \Lambda^0$ [11, Proposition 2.12].

As for directed graphs, we have uniqueness theorems for the Toeplitz algebra [9, Theorem 8.1] and the Cuntz–Krieger algebra [6, Theorem 4.7]. The former does not need any hypothesis on the k-graph as stated in the following theorem.

Theorem 2.2. *Let Λ be a row-finite k-graph. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a Toeplitz–Cuntz–Krieger Λ-family in a C^*-algebra B. Suppose that for every $v \in \Lambda^0$,*

$$\prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \neq 0 \qquad (*)$$

(where this includes $T_v \neq 0$ if $v\Lambda^1 = \emptyset$). Suppose that $\phi_T : TC^*(\Lambda) \to B$ is the homomorphism such that $\phi_T(t_\lambda) = T_\lambda$ for $\lambda \in \Lambda$. Then

$$\phi_T : TC^*(\Lambda) \to B$$

is injective.

Remark 2.3. Every k-graph Λ gives a product system of graphs over $\mathbb{N}^k$ and a Toeplitz–Cuntz–Krieger Λ-family gives a Toeplitz Λ-family of the product system [9, Lemma 9.2]. Lemma 9.3 of [9] shows that, if the Toeplitz–Cuntz–Krieger Λ-family satisfies $(*)$, then the Toeplitz Λ-family satisfies the hypothesis of [9, Theorem 8.1].

Remark 2.4. In the actual hypothesis, we need to verify whether

$$\prod_{1\leq i\leq k}\left(T_v - \sum_{e\in G_i} T_e T_e^*\right) \neq 0$$

for every $v \in \Lambda^0$, $1 \leq i \leq k$, and finite set $G_i \subseteq v\Lambda^{e_i}$. However, since we only consider row-finite k-graphs, then for every $v \in \Lambda^0$ and $1 \leq i \leq k$, the set $v\Lambda^{e_i}$ is finite. Thus for a row finite k-graph, we can simplify Lemma 9.3 of [9] as Theorem 2.2.

On the other hand, Lewin and Sims in [6, Theorem 4.7] proved that the Cuntz–Krieger uniqueness theorem only holds for k-graphs which satisfy the following *aperiodicity* condition: for every pair of distinct paths $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, there exists $\eta \in s(\lambda)\Lambda$ such that $\mathrm{MCE}(\lambda\eta, \mu\eta) = \emptyset$ [6, Definition 3.1]. (For discussion about the equivalence of various aperiodicity definitions, see [6, 12, 13, 14].) Now we state the uniqueness theorem as follows:

Theorem 2.5 ([6, Theorem 4.7])**.** *Suppose that Λ is an aperiodic row-finite k-graph and $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family in a C^*-algebra B such that $S_v \neq 0$ for $v \in \Lambda^0$. Suppose that $\pi_S : C^*(\Lambda) \to B$ is the homomorphism such that $\pi_S(s_\lambda) = S_\lambda$ for $\lambda \in \Lambda$ for $\lambda \in \Lambda$. Then π_S is an injective homomorphism.*

3. The k-graph $T\Lambda$

Suppose that Λ is a row-finite k-graph. In this section, we define a k-graph $T\Lambda$; later we show that $TC^*(\Lambda) \cong C^*(T\Lambda)$ (Theorem 4.1). Interestingly, our k-graph $T\Lambda$ is always aperiodic (Proposition 3.5).

Proposition 3.1. *Let $\Lambda = (\Lambda, d, r, s)$ be a row-finite k-graph. Then define sets $T\Lambda^0$ and $T\Lambda$ as follows:*

$$T\Lambda^0 := \left\{\alpha(v) : v \in \Lambda^0\right\} \cup \left\{\beta(v) : v\Lambda^1 \neq \emptyset\right\};$$

$$T\Lambda := \{\alpha(\lambda) : \lambda \in \Lambda\} \cup \left\{\beta(\lambda) : \lambda \in \Lambda, s(\lambda)\Lambda^1 \neq \emptyset\right\}.$$

Define functions $r, s : T\Lambda \backslash T\Lambda^0 \to T\Lambda^0$ *by*

$$r(\alpha(\lambda)) = \alpha(r(\lambda)),\ s(\alpha(\lambda)) = \alpha(s(\lambda)),$$
$$r(\beta(\lambda)) = \alpha(r(\lambda)),\ s(\beta(\lambda)) = \beta(s(\lambda))$$

(r, s *are the identity on* $T\Lambda^0$*). We also define a partially defined product* $(\tau, \omega) \mapsto \tau\omega$ *from*

$$\{(\tau, \omega) \in T\Lambda \times T\Lambda : s(\tau) = r(\omega)\}$$

to $T\Lambda$*, where*

$$(\alpha(\lambda), \alpha(\mu)) \mapsto \alpha(\lambda\mu)$$
$$(\alpha(\lambda), \beta(\mu)) \mapsto \beta(\lambda\mu)$$

and a function $d : T\Lambda \to \mathbb{N}^k$ *where*

$$d(\alpha(\lambda)) = d(\beta(\lambda)) = d(\lambda).$$

Then $(T\Lambda, d)$ *is a* k*-graph.*

Proof. First we claim that $T\Lambda$ is a countable category. Note that $T\Lambda$ is countable since Λ is countable.

Now we show that for all paths η, τ, ω in $T\Lambda$ where $s(\eta) = r(\tau)$ and $s(\tau) = r(\omega)$, we have $s(\tau\omega) = s(\omega)$, $r(\tau\omega) = r(\tau)$, and $(\eta\tau)\omega = \eta(\tau\omega)$. If one of τ, ω is a vertex then we are done. So assume otherwise, and we have $\eta = \alpha(\lambda)$, $\tau = \alpha(\mu)$, and ω is either $\alpha(\nu)$ or $\beta(\nu)$ for some paths λ, μ, ν in Λ. In both cases, we always have $s(\lambda) = r(\mu)$, $s(\mu) = r(\nu)$, and $(\lambda\mu)\nu = \lambda(\mu\nu)$. If $\omega = \alpha(\nu)$, we have

$$\begin{aligned} s(\tau\omega) &= s(\alpha(\mu)\alpha(\nu)) = s(\alpha(\mu\nu)) \\ &= \alpha(s(\mu\nu)) = \alpha(s(\nu)) = s(\alpha(\nu)) = s(\omega), \end{aligned}$$

$$\begin{aligned} r(\tau\omega) &= r(\alpha(\mu)\alpha(\nu)) = r(\alpha(\mu\nu)) \\ &= \alpha(r(\mu\nu)) = \alpha(r(\mu)) = r(\alpha(\mu)) = r(\tau), \end{aligned}$$

and

$$\begin{aligned} (\eta\tau)\omega &= \big(\alpha(\lambda)\alpha(\mu)\big)\alpha(\nu) = \alpha(\lambda\mu)\alpha(\nu) = \alpha((\lambda\mu)\nu) \\ &= \alpha(\lambda(\mu\nu)) = \alpha(\lambda)\alpha(\mu\nu) = \alpha(\lambda)\big(\alpha(\mu)\alpha(\nu)\big) = \eta(\tau\omega). \end{aligned}$$

On the other hand, if $\omega = \beta(\nu)$, then

$$\begin{aligned} s(\tau\omega) &= s(\alpha(\mu)\beta(\nu)) = s(\beta(\mu\nu)) \\ &= \beta(s(\mu\nu)) = \beta(s(\nu)) = s(\beta(\nu)) = s(\omega), \end{aligned}$$

$$\begin{aligned} r(\tau\omega) &= r(\alpha(\mu)\beta(\nu)) = r(\beta(\mu\nu)) \\ &= \alpha(r(\mu\nu)) = \alpha(r(\mu)) = r(\alpha(\mu)) = r(\tau), \end{aligned}$$

and

$$\begin{aligned} (\eta\tau)\omega &= \big(\alpha(\lambda)\alpha(\mu)\big)\beta(\nu) = \alpha(\lambda\mu)\beta(\nu) = \beta((\lambda\mu)\nu) \\ &= \beta(\lambda(\mu\nu)) = \alpha(\lambda)\beta(\mu\nu) = \alpha(\lambda)\big(\alpha(\mu)\beta(\nu)\big) = \eta(\tau\omega). \end{aligned}$$

Thus, $T\Lambda$ is a countable category, as claimed.

Now we show that d is a functor. Note that both $T\Lambda$ and $\mathbb{N}^k$ are categories. First take object $x \in T\Lambda^0$, then $d(x) = 0$ is an object in category $\mathbb{N}^k$. Next take morphisms $\tau, \omega \in T\Lambda$ with $s(\tau) = r(\omega)$. Then by definition of d,

$$d(\tau\omega) = d(\tau) + d(\omega).$$

Hence, d is a functor.

To show that d satisfies the factorisation property, take $\omega \in T\Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\omega) = m + n$. By definition, ω is either $\alpha(\lambda)$ or $\beta(\lambda)$ for some path λ in Λ. In both cases, there exist paths μ, ν in Λ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$. Then, we have $d(\alpha(\mu)) = m$, $d(\alpha(\nu)) = d(\beta(\nu)) = n$, and ω is either equal to $\alpha(\mu)\alpha(\nu)$ or $\alpha(\mu)\beta(\nu)$. Therefore, the existence of factorisation is guaranteed.

Now we show that the factorisation is unique. First suppose

$$\omega = \alpha(\mu)\alpha(\nu) = \alpha(\mu')\alpha(\nu')$$

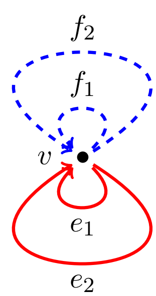
where $d(\alpha(\mu)) = d(\alpha(\mu'))$ and $d(\alpha(\nu)) = d(\alpha(\nu'))$. We consider paths $\lambda = \mu\nu$ and $\lambda' = \mu'\nu'$. Since $\alpha(\lambda) = \omega = \alpha(\lambda')$, then $\lambda = \lambda'$. This implies $\mu = \mu'$ and $\nu = \nu'$ based on the uniquness of factorisation in Λ. Then $\alpha(\mu) = \alpha(\mu')$ and $\alpha(\nu) = \alpha(\nu')$. For the case $\omega = \alpha(\mu)\beta(\nu)$, we get the same result by using the same argument. The conclusion follows. □

Remark 3.2. For a directed graph E (that is, for $k = 1$), the graph TE was constructed by Muhly and Tomforde [7, Definition 3.6] (denoted E_V), and by Sims [15, Section 3] (denoted $\widetilde{E}$). Our notation follows that of Sims because we want to distinguish between paths in $T\Lambda$ (denoted $\alpha(\lambda)$ and $\beta(\lambda)$) and those in Λ (denoted λ).

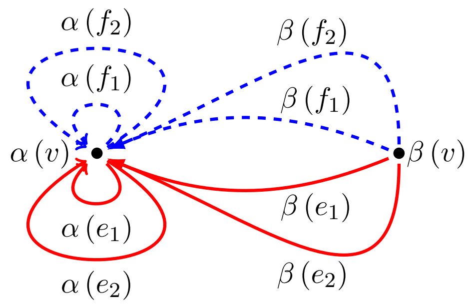
Remark 3.3. Every vertex $\beta(v)$ satisfies $\beta(v)T\Lambda^1 = \emptyset$. Then if Λ has a vertex v which receives edges e, f with $d(e) \neq d(f)$, then there is no edge $g \in \beta(s(e))T\Lambda^{d(f)}$ (or $g \in \alpha(s(e))T\Lambda^{d(f)}$ if $s(e)\Lambda = \emptyset$), and hence $T\Lambda$ is not locally convex.

To give an illustration how we construct the k-graph $T\Lambda$ from a k-graph Λ, we first recall coloured graphs of [4]. By choosing k-different colours $c_1, \ldots, c_k$, we can view paths in Λ^{e_i} as edges of colour c_i. For a k-graph Λ, we call its corresponding coloured graph the *skeleton* of Λ. For further discussion about k-graphs and their skeletons, see [4].

Example 3.4. Consider the 2-graph Λ which has skeleton



where $e_i f_j = f_i e_j$ for all $i, j \in \{1, 2\}$, the solid edges have degree $(1, 0)$ and the dashed edges have degree $(0, 1)$. Then the 2-graph $T\Lambda$ has skeleton



where $\alpha(e_i)\,\alpha(f_j) = \alpha(f_i)\,\alpha(e_j)$ and $\alpha(e_i)\,\beta(f_j) = \alpha(f_i)\,\beta(e_j)$ for all $i, j \in \{1, 2\}$, the solid edges have degree $(1, 0)$ and the dashed edges have degree $(0, 1)$.

The following lemma tells about properties of the k-graph $T\Lambda$.

Proposition 3.5. *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. Then,*

(a) *$T\Lambda$ is row-finite.*
(b) *$T\Lambda$ is aperiodic.*

Proof. To show part (a), take $x \in T\Lambda^0$. If $x = \beta(v)$ for some $v \in \Lambda^0$, then $xT\Lambda^1 = \emptyset$ by Remark 3.3. Suppose $x = \alpha(v)$ for some $v \in \Lambda^0$. If $v\Lambda^1 = \emptyset$, then $xT\Lambda^1 = \emptyset$. Otherwise, for $1 \le i \le k$ such that $v\Lambda^{e_i} \neq \emptyset$, we have

$$|xT\Lambda^{e_i}| \le 2\,|v\Lambda^{e_i}|\,,$$

which is finite.

For part (b), take $\tau, \omega \in T\Lambda$ such that $\tau \neq \omega$ and $s(\tau) = s(\omega)$. We have to show there exists $\eta \in s(\tau)\,T\Lambda$ such that $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. If $s(\tau) = \beta(v)$ for some $v \in \Lambda^0$, then choose $\eta = \beta(v)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. So suppose $s(\tau) = \alpha(v)$ for some $v \in \Lambda^0$. If $v\Lambda^1 = \emptyset$, then choose $\eta = \alpha(v)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. Suppose $v\Lambda^1 \neq \emptyset$. Take $e \in v\Lambda^1$. If $s(e)\,\Lambda^1 = \emptyset$, then choose $\eta = \alpha(e)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. Otherwise, we have $s(e)\,\Lambda^1 \neq \emptyset$. Then choose $\eta = \beta(e)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. Hence, $T\Lambda$ is aperiodic. □

4. Realising $TC^*(\Lambda)$ as a Cuntz–Krieger algebra

Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. In this section, we show that $TC^*(\Lambda)$ is isomorphic to $C^*(T\Lambda)$.

Theorem 4.1. *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. Let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal Toeplitz–Cuntz–Krieger*

Λ-family and $\{s_\omega : \omega \in T\Lambda\}$ be the universal Cuntz–Krieger $T\Lambda$-family. For $\lambda \in \Lambda$, let

$$T_\lambda := \begin{cases} s_{\alpha(\lambda)} + s_{\beta(\lambda)} & \text{if } s(\lambda)\Lambda^1 \neq \emptyset \\ s_{\alpha(\lambda)} & \text{if } s(\lambda)\Lambda^1 = \emptyset. \end{cases}$$

Then there is an isomorphism $\phi_T : TC^*(\Lambda) \to C^*(T\Lambda)$ satisfying

$$\phi_T(t_\lambda) = T_\lambda$$

for every $\lambda \in \Lambda$.

Furthermore, $s_{\alpha(\lambda)} = \phi_T(t_\lambda)$ if $s(\lambda)\Lambda^1 = \emptyset$. Meanwhile, if $s(\lambda)\Lambda^1 \neq \emptyset$, we have

$$s_{\alpha(\lambda)} = \phi_T\left(t_\lambda - t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*)\right),$$

$$s_{\beta(\lambda)} = \phi_T\left(t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*)\right).$$

Proof that $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ-family. To avoid an argument by cases, for $\lambda \in \Lambda$ with $s(\lambda)\Lambda^1 = \emptyset$, we write

$$s_{\beta(\lambda)} := 0,$$

so that

$$T_\lambda = s_{\alpha(\lambda)} + s_{\beta(\lambda)}.$$

First, we want to show $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ-family in $C^*(T\Lambda)$. For (TCK1), take $v \in \Lambda^0$. Since $\{s_{\alpha(v)}\} \cup \{s_{\beta(v)}\}$ are mutually orthogonal projections, then T_v is a projection. Meanwhile, for $v, w \in \Lambda^0$ with $v \neq w$,

$$T_v T_w = s_{\alpha(v)} s_{\alpha(w)} + s_{\alpha(v)} s_{\beta(w)} + s_{\beta(v)} s_{\alpha(w)} + s_{\beta(v)} s_{\beta(w)} = 0.$$

Next we show (TCK2). Take $\mu, \nu \in \Lambda$ where $s(\mu) = r(\nu)$. Then

$$T_\mu T_\nu = s_{\alpha(\mu)} s_{\alpha(\nu)} + s_{\alpha(\mu)} s_{\beta(\nu)} + s_{\beta(\mu)} s_{\alpha(\nu)} + s_{\beta(\mu)} s_{\beta(\nu)}.$$

If ν is a vertex, the middle terms vanish and we get

$$T_\mu T_\nu = s_{\alpha(\mu)} + s_{\beta(\mu)} = T_\mu,$$

as required. Otherwise, the last two terms vanish and we get

$$T_\mu T_\nu = s_{\alpha(\mu)} s_{\alpha(\nu)} + s_{\alpha(\mu)} s_{\beta(\nu)} = s_{\alpha(\mu\nu)} + s_{\beta(\mu\nu)} = T_{\mu\nu},$$

which is (TCK2).

To show (TCK3), take $\lambda, \mu \in \Lambda$. Then

$$T_\lambda^* T_\mu = s_{\alpha(\lambda)}^* s_{\alpha(\mu)} + s_{\alpha(\lambda)}^* s_{\beta(\mu)} + s_{\beta(\lambda)}^* s_{\alpha(\mu)} + s_{\beta(\lambda)}^* s_{\beta(\mu)}. \tag{4.1}$$

We give separate arguments for $\Lambda^{\min}(\lambda,\mu)=\emptyset$ and $\Lambda^{\min}(\lambda,\mu)\neq\emptyset$. For case $\Lambda^{\min}(\lambda,\mu)=\emptyset$, we have

$$\begin{aligned}\emptyset &= T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu)) = T\Lambda^{\min}(\alpha(\lambda),\beta(\mu))\\ &= T\Lambda^{\min}(\beta(\lambda),\alpha(\mu)) = T\Lambda^{\min}(\beta(\lambda),\beta(\mu)).\end{aligned}$$

Hence, $s^*_{\alpha(\lambda)}s_{\alpha(\mu)} = s^*_{\alpha(\lambda)}s_{\beta(\mu)} = s^*_{\beta(\lambda)}s_{\alpha(\mu)} = s^*_{\beta(\lambda)}s_{\beta(\mu)} = 0$ and then Equation (4.1) becomes

$$T^*_\lambda T_\mu = 0 = \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} T_{\lambda'}T^*_{\mu'}.$$

Now suppose $\Lambda^{\min}(\lambda,\mu)\neq\emptyset$. Take $(a,b)\in\Lambda^{\min}(\lambda,\mu)$. We consider several cases: whether a equals $s(\lambda)$ and/or b equals $s(\mu)$. First suppose $a=s(\lambda)$ and $b=s(\mu)$. So $\lambda=\lambda s(\lambda)=\mu s(\mu)=\mu$. Because $\alpha(\lambda)$ and $\beta(\lambda)$ are paths with the same degree and different sources, then $T\Lambda^{\min}(\alpha(\lambda),\beta(\lambda))=\emptyset$. Thus,

$$s^*_{\beta(\lambda)}s_{\alpha(\lambda)} = 0 = s^*_{\alpha(\lambda)}s_{\beta(\lambda)}$$

and Equation (4.1) becomes

$$\begin{aligned}T^*_\lambda T_\lambda &= s^*_{\alpha(\lambda)}s_{\alpha(\lambda)} + s^*_{\beta(\lambda)}s_{\beta(\lambda)}\\ &= s_{s(\alpha(\lambda))} + s_{s(\beta(\lambda))} = s_{\alpha(s(\lambda))} + s_{\beta(s(\lambda))}\\ &= T_{s(\lambda)} = T_{s(\lambda)}T^*_{s(\lambda)}\\ &= \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\lambda)} T_{\lambda'}T^*_{\mu'} \text{ (since } \Lambda^{\min}(\lambda,\lambda)=\{s(\lambda),s(\lambda)\}\text{).}\end{aligned}$$

Next suppose $a=s(\lambda)$ and $b\neq s(\mu)$. Then $\lambda=\mu b$ and

$$T\Lambda^{\min}(\alpha(\lambda),\beta(\mu)) = \emptyset = T\Lambda^{\min}(\beta(\lambda),\beta(\mu))$$

since $s(\beta(\mu))T\Lambda^1=\emptyset$. Hence

$$s^*_{\alpha(\lambda)}s_{\beta(\mu)} = 0 = s^*_{\beta(\lambda)}s_{\beta(\mu)}$$

and Equation (4.1) becomes

$$T^*_\lambda T_\mu = s^*_{\alpha(\lambda)}s_{\alpha(\mu)} + s^*_{\beta(\lambda)}s_{\alpha(\mu)}.$$

Every $(\alpha(s(\lambda)),\eta)\in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))$ has $\eta=\alpha(\mu')$ with

$$(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu).$$

Similarly, every $(\beta(s(\lambda)),\eta) \in T\Lambda^{\min}(\beta(\lambda),\alpha(\mu))$ has $\eta = \beta(\mu')$ with $(s(\lambda),\mu') \in \Lambda^{\min}(\lambda,\mu)$. Thus, by using (TCK3) in $C^*(T\Lambda)$,

$$\begin{aligned}
&T_\lambda^* T_\mu \\
&= s_{\alpha(\lambda)}^* s_{\alpha(\mu)} + s_{\beta(\lambda)}^* s_{\alpha(\mu)} \\
&= \sum_{(\alpha(s(\lambda)),\eta)\in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))} s_{\alpha(s(\lambda))} s_\eta^* + \sum_{(\beta(s(\lambda)),\eta)\in T\Lambda^{\min}(\beta(\lambda),\alpha(\mu))} s_{\beta(s(\lambda))} s_\eta^* \\
&= \sum_{(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu)} s_{\alpha(s(\lambda))} s_{\alpha(\mu')}^* + \sum_{(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu)} s_{\beta(s(\lambda))} s_{\beta(\mu')}^* \\
&= \sum_{(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(s(\lambda))} s_{\alpha(\mu')}^* + s_{\beta(s(\lambda))} s_{\beta(\mu')}^*\right) \\
&= \sum_{(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(s(\lambda))} + s_{\beta(s(\lambda))}\right)\left(s_{\alpha(\mu')}^* + s_{\beta(\mu')}^*\right) \\
&= \sum_{(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu)} T_{s(\lambda)} T_{\mu'}^* = \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} T_{\lambda'} T_{\mu'}^*.
\end{aligned}$$

By taking adjoints, we deduce (TCK3) when $a \neq s(\lambda)$ and $b = s(\mu)$.

Now we consider the last case, which is $a \neq s(\lambda)$ and $b \neq s(\mu)$. This means we have neither $\lambda = \mu b$ nor $\mu = \lambda a$. Hence,

$$T\Lambda^{\min}(\alpha(\lambda),\beta(\mu)) = T\Lambda^{\min}(\beta(\lambda),\alpha(\mu)) = T\Lambda^{\min}(\beta(\lambda),\beta(\mu)) = \emptyset$$

since $s(\beta(\lambda))\,T\Lambda^1 = \emptyset = s(\beta(\mu))\,T\Lambda^1 = \emptyset$. Hence,

$$s_{\alpha(\lambda)}^* s_{\beta(\mu)} = s_{\beta(\lambda)}^* s_{\alpha(\mu)} = s_{\beta(\lambda)}^* s_{\beta(\mu)} = 0.$$

On the other hand, we have

$$\begin{aligned}
&T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu)) \\
&\qquad = \left\{\left(\alpha(\lambda'),\alpha(\mu')\right),\left(\beta(\lambda'),\beta(\mu')\right) : (\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)\right\}.
\end{aligned}$$

Therefore, Equation (4.1) becomes

$$\begin{aligned}
T_\lambda^* T_\mu = s_{\alpha(\lambda)}^* s_{\alpha(\mu)} &= \sum_{(\omega,\eta)\in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))} s_\omega s_\eta^* \\
&= \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(\lambda')} s_{\alpha(\mu')}^* + s_{\beta(\lambda')} s_{\beta(\mu')}^*\right) \\
&= \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(\lambda')} + s_{\beta(\lambda')}\right)\left(s_{\alpha(\mu')}^* + s_{\beta(\mu')}^*\right) \\
&= \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} T_{\lambda'} T_{\mu'}^*.
\end{aligned}$$

So for all cases, we have

$$T_\lambda^* T_\mu = \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} T_{\lambda'} T_{\mu'}^*$$

and $\{T_\lambda : \lambda \in \Lambda\}$ satisfies (TCK3). □

Proof that ϕ_T is injective. Now the universal property of $TC^*(\Lambda)$ gives a homomorphism $\phi_T : TC^*(\Lambda) \to C^*(T\Lambda)$ satisfying $\phi_T(t_\lambda) = T_\lambda$ for every $\lambda \in \Lambda$.

We show the injectivity of ϕ_T by using Theorem 2.2. Take $v \in \Lambda^0$. We show

$$\prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \neq 0.$$

First suppose $v\Lambda^1 \neq \emptyset$. Take $1 \leq i \leq k$ such that $v\Lambda^{e_i} \neq \emptyset$. We claim

$$\prod_{e \in v\Lambda^{e_i}} (T_v - T_e T_e^*) \geq s_{\beta(v)}.$$

Since $v\Lambda^{e_i} \neq \emptyset$, then $\alpha(v) T\Lambda^{e_i} \neq \emptyset$ and by [11, Lemma 2.7 (iii)],

$$\begin{aligned}
s_{\alpha(v)} &\geq \sum_{g \in \alpha(v) T\Lambda^{e_i}} s_g s_g^* \\
&= \sum_{e \in v\Lambda^{e_i}} s_{\alpha(e)} s_{\alpha(e)}^* + \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 \neq \emptyset}} s_{\beta(e)} s_{\beta(e)}^* \\
&= \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 \neq \emptyset}} \left(s_{\alpha(e)} s_{\alpha(e)}^* + s_{\beta(e)} s_{\beta(e)}^* \right) + \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 = \emptyset}} s_{\alpha(e)} s_{\alpha(e)}^* \\
&= \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 \neq \emptyset}} T_e T_e^* + \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 = \emptyset}} T_e T_e^* \\
&= \sum_{e \in v\Lambda^{e_i}} T_e T_e^*.
\end{aligned}$$

Meanwhile, since every $e \in v\Lambda^{e_i}$ has the same degree,

$$\begin{aligned}
\prod_{e \in v\Lambda^{e_i}} (T_v - T_e T_e^*) &= T_v - \sum_{e \in v\Lambda^{e_i}} T_e T_e^* \\
&= \left(s_{\alpha(v)} + s_{\beta(v)} \right) - \sum_{e \in v\Lambda^{e_i}} T_e T_e^* \\
&= s_{\beta(v)} + \left(s_{\alpha(v)} - \sum_{e \in v\Lambda^{e_i}} T_e T_e^* \right) \\
&\geq s_{\beta(v)},
\end{aligned}$$

as claimed. This claim implies

$$\prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \geq \prod_{\{i : v\Lambda^{e_i} \neq \emptyset\}} s_{\beta(v)} = s_{\beta(v)} \neq 0$$

since $v\Lambda^1 \neq \emptyset$, as required.

Finally, for $v \in \Lambda^0$ with $v\Lambda^1 = \emptyset$, we have

$$T_v = s_{\alpha(v)} \neq 0.$$

Hence, by Theorem 2.2, ϕ_T is injective. □

Proof that ϕ_T is surjective. Now we show the surjectivity of ϕ_T. Since $C^*(T\Lambda)$ is generated by $\{s_\tau : \tau \in T\Lambda\}$, then it suffices to show that for every $\tau \in T\Lambda$, $s_\tau \in \operatorname{im}(\phi_T)$. Recall that for every $\tau \in T\Lambda$, s_τ is either $s_{\alpha(\lambda)}$ or $s_{\beta(\lambda)}$ for some $\lambda \in \Lambda$.

Take $v \in \Lambda^0$. First we show $s_{\alpha(v)}$ and $s_{\beta(v)}$ (if it exists) belong to $\operatorname{im}(\phi_T)$. If $v\Lambda^1 = \emptyset$, then

$$s_{\alpha(v)} = T_v \in \operatorname{im}(\phi_T).$$

Next suppose $v\Lambda^1 \neq \emptyset$. First we show that $s_{\beta(v)} = \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*)$. Note that for every $f \in \alpha(v) T\Lambda^1$, the projection $s_{\alpha(v)} - s_f s_f^* \leq s_{\alpha(v)}$ is othogonal to $s_{\beta(v)}$. This implies

$$\begin{aligned}\prod_{f \in \alpha(v) T\Lambda^1} ((s_{\alpha(v)} + s_{\beta(v)}) - s_f s_f^*) &= s_{\beta(v)} + \prod_{f \in \alpha(v) T\Lambda^1} (s_{\alpha(v)} - s_f s_f^*) \\ &= s_{\beta(v)},\end{aligned}$$

since $v\Lambda^1$ is an exhaustive set. Hence,

$$\begin{aligned}s_{\beta(v)} &= \prod_{f \in \alpha(v) T\Lambda^1} ((s_{\alpha(v)} + s_{\beta(v)}) - s_f s_f^*) \\ &= \prod_{e \in v\Lambda^1} (T_v - s_{\alpha(e)} s_{\alpha(e)}^*) \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 \neq \emptyset}} (T_v - s_{\beta(e)} s_{\beta(e)}^*) \\ &= \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 = \emptyset}} (T_v - s_{\alpha(e)} s_{\alpha(e)}^*) \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 \neq \emptyset}} (T_v - s_{\alpha(e)} s_{\alpha(e)}^*)(T_v - s_{\beta(e)} s_{\beta(e)}^*) \\ &= \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 = \emptyset}} (T_v - s_{\alpha(e)} s_{\alpha(e)}^*) \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 \neq \emptyset}} (T_v - (s_{\alpha(e)} s_{\alpha(e)}^* + s_{\beta(e)} s_{\beta(e)}^*)) \\ &= \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 = \emptyset}} (T_v - T_e T_e^*) \prod_{\substack{e \in v\Lambda^1 \\ s(e)\Lambda^1 \neq \emptyset}} (T_v - T_e T_e^*) \\ &= \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*),\end{aligned}$$

as required, and $s_{\beta(v)}$ belongs to $\operatorname{im}(\phi_T)$. Furthermore,

$$s_{\alpha(v)} = T_v - s_{\beta(v)} = T_v - \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \in \operatorname{im}(\phi_T),$$

as required.

Now take $\lambda \in \Lambda$. We have to show $s_{\alpha(\lambda)}$ and $s_{\beta(\lambda)}$ (if it exists) belong to $\mathrm{im}\,(\phi_T)$. If $s(\lambda)\,\Lambda^1 = \emptyset$, then

$$s_{\alpha(\lambda)} = s_{\alpha(\lambda)} s_{\alpha(s(\lambda))} = T_\lambda T_{s(\lambda)} = T_\lambda \in \mathrm{im}\,(\phi_T)\,.$$

Next suppose $s(\lambda)\,\Lambda^1 \neq \emptyset$. Then $s_{\beta(\lambda)} s_{\alpha(s(\lambda))} = 0$ and $s_{\alpha(\lambda)} s_{\beta(s(\lambda))} = 0$. Hence,

$$\begin{aligned} s_{\alpha(\lambda)} &= s_{\alpha(\lambda)} s_{\alpha(s(\lambda))} = \left(s_{\alpha(\lambda)} + s_{\beta(\lambda)}\right) s_{\alpha(s(\lambda))} \\ &= T_\lambda \left(T_{s(\lambda)} - \prod_{e \in s(\lambda)\Lambda^1} (T_{s(\lambda)} - T_e T_e^*) \right) \\ &= T_\lambda - T_\lambda \prod_{e \in s(\lambda)\Lambda^1} (T_{s(\lambda)} - T_e T_e^*) \in \mathrm{im}\,(\phi_T) \end{aligned}$$

and

$$\begin{aligned} s_{\beta(\lambda)} &= s_{\beta(\lambda)} s_{\beta(s(\lambda))} = \left(s_{\alpha(\lambda)} + s_{\beta(\lambda)}\right) s_{\beta(s(\lambda))} \\ &= T_\lambda \prod_{e \in s(\lambda)\Lambda^1} (T_{s(\lambda)} - T_e T_e^*) \in \mathrm{im}\,(\phi_T)\,. \end{aligned}$$

Therefore, ϕ_T is surjective and an isomorphism. □

Corollary 4.2. *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. Let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal Toeplitz–Cuntz–Krieger Λ-family and $\{s_\omega : \omega \in T\Lambda\}$ be the universal Cuntz–Krieger $T\Lambda$-family. For $\tau \in T\Lambda$, define*

$$S_\tau := \begin{cases} t_\lambda & \text{if } \tau = \alpha(\lambda) \text{ with } s(\lambda)\,\Lambda^1 = \emptyset \\ t_\lambda - t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*) & \text{if } \tau = \alpha(\lambda) \text{ with } s(\lambda)\,\Lambda^1 \neq \emptyset \\ t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*) & \text{if } \tau = \beta(\lambda) \text{ with } s(\lambda)\,\Lambda^1 \neq \emptyset. \end{cases}$$

Suppose that $\phi_T : TC^*(\Lambda) \to C^*(T\Lambda)$ is the isomorphism as in Theorem 4.1 and $\pi_S : C^*(T\Lambda) \to TC^*(\Lambda)$ is the homomorphism such that $\pi_S(s_\tau) = S_\tau$ for $\tau \in T\Lambda$. Then $\phi_T^{-1} = \pi_S$.

Proof. Take $\lambda \in \Lambda$. By Theorem 4.1, we get $\phi_T^{-1}\left(s_{\alpha(\lambda)}\right) = t_\lambda$ if $s(\lambda)\,\Lambda^1 = \emptyset$. Meanwhile, if $s(\lambda)\,\Lambda^1 \neq \emptyset$, by Theorem 4.1, we have

$$\begin{aligned} \phi_T^{-1}\left(s_{\alpha(\lambda)}\right) &= t_\lambda - t_\lambda \prod_{e \in v\Lambda^1} (t_v - t_e t_e^*), \\ \phi_T^{-1}\left(s_{\beta(\lambda)}\right) &= t_\lambda \prod_{e \in v\Lambda^1} (t_v - t_e t_e^*). \end{aligned}$$

Hence, $\phi_T^{-1}(s_\tau) = S_\tau$ for $\tau \in T\Lambda$. This implies that $\{S_\tau : \tau \in T\Lambda\}$ is a Cuntz–Krieger $T\Lambda$-family, and then $\phi_T^{-1} = \pi_S$. □

Remark 4.3. Proposition 3.5 says that $T\Lambda$ is always aperiodic, and hence the Cuntz–Krieger uniqueness theorem always applies to $T\Lambda$. This helps explain why no hypothesis on Λ is required in the uniqueness theorem of [9, Theorem 8.1]. Indeed, we could have deduced that theorem by applying the Cuntz–Krieger uniqueness theorem to $T\Lambda$. With our current proof of Theorem 4.1, this argument would be circular, since we used [9, Theorem 8.1] in the proof of Theorem 4.1. However, we could prove Corollary 4.2 directly by showing that $\{S_\tau : \tau \in T\Lambda\}$ is a Cuntz–Krieger $T\Lambda$-family in $TC^*(\Lambda)$, hence gives a homomorphism $\pi_S : C^*(T\Lambda) \to TC^*(\Lambda)$, and using the Cuntz–Krieger uniqueness theorem to see that π_S is injective. Then we could deduce [9, Theorem 8.1] from Corollary 4.2, and this would be a legitimate proof. We worked out the details of this approach, but it seemed to require an extensive cases argument, and hence became substantially more complicated.

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