

## Crossed products and twisted $k$ -graph algebras

Nathan Brownlowe, Valentin Deaconu,  
Alex Kumjian and David Pask

ABSTRACT. An automorphism  $\beta$  of a  $k$ -graph  $\Lambda$  induces a crossed product  $C^*(\Lambda) \rtimes_{\beta} \mathbb{Z}$  which is isomorphic to a  $(k+1)$ -graph algebra  $C^*(\Lambda \times_{\beta} \mathbb{Z})$ . In this paper we show how this process interacts with  $k$ -graph  $C^*$ -algebras which have been twisted by an element of their second cohomology group. This analysis is done using a long exact sequence in cohomology associated to this data. We conclude with some examples.

### CONTENTS

1. Background	231
1.1. Higher-rank graphs	231
1.2. Cubical cohomology of $k$ -graphs	232
1.3. Crossed product graphs	233
1.4. The long exact sequence of cohomology	234
1.5. Twisted $k$ -graph $C^*$ -algebras	236
2. Main results	237
3. Examples	242
References	248

A higher-rank graph (or  $k$ -graph) is a countable category  $\Lambda$  together with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$  satisfying a factorisation property. For  $k = 1$ ,  $\Lambda$  is the path category of a directed graph  $E_{\Lambda}$ . In general we view a  $k$ -graph as a higher dimensional analog of a directed graph. In [14] it was shown how to associate a  $C^*$ -algebra to a  $k$ -graph in such a way that for  $k = 1$  we have  $C^*(\Lambda) = C^*(E_{\Lambda})$ .

The universal property of a  $k$ -graph  $C^*$ -algebra  $C^*(\Lambda)$  implies that an automorphism  $\beta$  of  $\Lambda$  induces an automorphism  $\beta$  of  $C^*(\Lambda)$  and hence gives rise to a crossed product  $C^*(\Lambda) \rtimes_{\beta} \mathbb{Z}$ . The results of [8] show that there is

---

Received January 14, 2015.

2010 *Mathematics Subject Classification*. Primary 46L05; Secondary 18G60, 55N10.

*Key words and phrases*. Higher-rank graph;  $C^*$ -algebra; cohomology; crossed product.

This research was supported by the Australian Research Council and the University of Wollongong Research Committee.

a  $(k + 1)$ -graph  $\Lambda \times_{\beta} \mathbb{Z}$  such that  $C^*(\Lambda) \rtimes_{\beta} \mathbb{Z}$  is isomorphic to  $C^*(\Lambda \times_{\beta} \mathbb{Z})$ . The purpose of this paper is to examine how this situation generalizes in the setting of twisted  $k$ -graph  $C^*$ -algebras.

Recent attention has been drawn to the homological properties of a  $k$ -graph  $\Lambda$  which are nontrivial when  $k \geq 2$ . Specifically in [16, 17] two cohomology theories for a  $k$ -graph  $\Lambda$  are described: cubical and categorical. Twisted versions of  $k$ -graph  $C^*$ -algebras are introduced in both cases using 2-cocycles. Here we work with the cubical cohomology which is more tractable. If  $\varphi$  is a (cubical)  $\mathbb{T}$ -valued 2-cocycle on  $\Lambda$ , the twisted  $C^*$ -algebra is denoted by  $C_{\varphi}^*(\Lambda)$ .

In [17] it is shown that the cubical and categorical cohomologies for a  $k$ -graph agree for  $H^0, H^1$  and  $H^2$ . An isomorphism between the two twisted versions of  $k$ -graph  $C^*$ -algebras compatible with the isomorphism in  $H^2$  was also proven in [17].

If  $\beta$  is an automorphism of a  $k$ -graph  $\Lambda$ , then [16] gives a long exact sequence for the homology of the  $(k + 1)$ -graph  $\Lambda \times_{\beta} \mathbb{Z}$  in the categorical context. In this paper we describe the analogous cohomology sequence in the cubical context (see Proposition 1.8) and use it to generalise the result in [8] in three different ways.

Our main result, Theorem 2.1 shows that if we twist the  $C^*$ -algebra of  $\Lambda \times_{\beta} \mathbb{Z}$  by a 2-cocycle  $\varphi$  then the resulting  $C^*$ -algebra  $C_{\varphi}^*(\Lambda \times_{\beta} \mathbb{Z})$  is isomorphic to the crossed product of a certain twisted  $C^*$ -algebra of  $\Lambda$  (with twisting cocycle obtained by restricting  $\varphi$ ) by an automorphism associated to  $\beta$  and  $\varphi$ . Applying this result in different contexts, associated to the exact sequence outlined in Proposition 1.8 yields Corollary 2.3 which deals with the case where the class of the restriction of  $\varphi$  is trivial; Corollary 2.4 asserts that if  $\psi$  is a 2-cocycle on  $\Lambda$  whose cohomology class is left invariant by  $\beta$ , then there is an automorphism of  $C_{\psi}^*(\Lambda)$  which is compatible with  $\beta$  for which the crossed product is isomorphic to a twisted  $C^*$ -algebra of  $\Lambda \times_{\beta} \mathbb{Z}$ . The case when  $\beta$  is trivial which motivated this work is discussed in Corollary 2.5.

We conclude with a section of examples of twisted  $k$ -graph  $C^*$ -algebras arising as crossed products. In each case the twisted  $k$ -graph  $C^*$ -algebra lies in a classifiable class of  $C^*$ -algebras. In Example 3.1 we consider quasifree automorphisms on Cuntz algebras and show how they arise in the setting of Corollary 2.3. In Example 3.2 we use Theorem 2.1 to compute the cohomology of a 2-graph with infinitely many vertices which arises as a crossed product. In Example 3.3 we use other techniques to compute the cohomology of a 3-graph with one vertex which arises as a crossed product. We also consider a family of 2-cocycles on a 3-graph for which the associated  $C^*$ -algebra is isomorphic to  $\mathcal{O}_2$ .

## 1. Background

We begin by giving some background on  $k$ -graphs, their cubical cohomology, and crossed-product graphs induced by automorphisms of  $k$ -graphs. We then prove the existence of a long exact sequence involving the cohomology groups of a  $k$ -graph and a crossed product graph. We finish with recalling the twisted  $k$ -graph  $C^*$ -algebras introduced in [16].

**1.1. Higher-rank graphs.** We adopt the conventions of [14, 15, 21] for  $k$ -graphs. Given a nonnegative integer  $k$ , a  $k$ -graph is a nonempty countable small category  $\Lambda$  equipped with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$  satisfying the *factorisation property*: for all  $\lambda \in \Lambda$  and  $m, n \in \mathbb{N}^k$  such that  $d(\lambda) = m + n$  there exist unique  $\mu, \nu \in \Lambda$  such that  $d(\mu) = m$ ,  $d(\nu) = n$ , and  $\lambda = \mu\nu$ . When  $d(\lambda) = n$  we say  $\lambda$  has *degree*  $n$ . We will typically use  $d$  to denote the degree functor in any  $k$ -graph in this paper.

For  $k \geq 1$ , the standard generators of  $\mathbb{N}^k$  are denoted  $e_1, \dots, e_k$ , and for  $n \in \mathbb{N}^k$  and  $1 \leq i \leq k$  we write  $n_i$  for the  $i^{\text{th}}$  coordinate of  $n$ . For  $n = (n_1, \dots, n_k) \in \mathbb{N}^k$  let  $|n| := \sum_{i=1}^k n_i$ ; for  $\lambda \in \Lambda$  we define  $|\lambda| := |d(\lambda)|$ . For  $m, n \in \mathbb{N}^k$ , we write  $m \vee n$  for the coordinatewise maximum of the two, and write  $m \leq n$  if  $m_i \leq n_i$  for  $i = 1, \dots, k$ .

For  $n \in \mathbb{N}^k$ , we write  $\Lambda^n$  for  $d^{-1}(n)$ . The *vertices* of  $\Lambda$  are the elements of  $\Lambda^0$ . The factorisation property implies that  $o \mapsto \text{id}_o$  is a bijection from the objects of  $\Lambda$  to  $\Lambda^0$ . We will frequently and without further comment use this bijection to identify  $\text{Obj}(\Lambda)$  with  $\Lambda^0$ . The domain and codomain maps in the category  $\Lambda$  then become maps  $s, r : \Lambda \rightarrow \Lambda^0$ . More precisely, for  $\alpha \in \Lambda$ , the *source*  $s(\alpha)$  is the identity morphism associated with the object  $\text{dom}(\alpha)$  and similarly,  $r(\alpha) = \text{id}_{\text{cod}(\alpha)}$ . An *edge* is a morphism  $f$  with  $d(f) = e_i$  for some  $i = 1, \dots, k$ .

Let  $\lambda$  be an element of a  $k$ -graph  $\Lambda$  and suppose  $m, n \in \mathbb{N}^k$  satisfy  $0 \leq m \leq n \leq d(\lambda)$ . By the factorisation property there exist unique elements  $\alpha, \beta, \gamma \in \Lambda$  such that

$$\lambda = \alpha\beta\gamma, \quad d(\alpha) = m, \quad d(\beta) = n - m, \quad \text{and} \quad d(\gamma) = d(\lambda) - n.$$

We define  $\lambda(m, n) := \beta$ . In particular  $\alpha = \lambda(0, m)$  and  $\gamma = \lambda(n, d(\lambda))$ .

For  $\alpha, \beta \in \Lambda$  and  $E \subset \Lambda$ , we write  $\alpha E$  for  $\{\alpha\lambda : \lambda \in E, r(\lambda) = s(\alpha)\}$  and  $E\beta$  for  $\{\lambda\beta : \lambda \in E, s(\lambda) = r(\beta)\}$ . So for  $u, v \in \Lambda^0$ , we have  $uE = E \cap r^{-1}(u)$ ,  $E\beta = E \cap s^{-1}(v)$  and  $uE\beta = uE \cap E\beta$ .

Suppose that  $\Lambda$  is *row-finite with no sources*, that is, for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$  we have  $0 < |v\Lambda^n| < \infty$ . By [18, Remark A.3],  $\Lambda$  is *cofinal* if for all  $v, w \in \Lambda^0$  there is  $N \in \mathbb{N}^k$  such that for all  $\alpha \in v\Lambda^N$  we have  $w\Lambda s(\alpha) \neq \emptyset$ . And by [27, Lemma 3.2 (iv)],  $\Lambda$  is *aperiodic* (or satisfies the *aperiodicity condition*) if for every  $v \in \Lambda^0$  and each pair  $m \neq n \in \mathbb{N}^k$ , there is  $\lambda \in v\Lambda$  such that  $d(\lambda) \geq m \vee n$  and

$$(1) \quad \lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

A  $k$ -graph  $\Lambda$  can be visualized by its 1-*skeleton*: This is a directed graph  $E_\Lambda$  with vertices  $\Lambda^0$  and edges  $\cup_{i=1}^k \Lambda^{e_i}$  which have range and source in  $E_\Lambda$  determined by their range and source in  $\Lambda$ . Each edge in  $E_\Lambda$  with degree  $e_i$  is assigned the same colour, so  $E_\Lambda$  is a coloured graph. It is common to call edges with degree  $e_1$  in  $\Lambda$  blue edges in  $E_\Lambda$  and draw them with solid lines; edges with degree  $e_2$  in  $\Lambda$  are then called red edges and are drawn as dashed lines. In practice, along with the 1-skeleton we give a collection of commuting squares or factorisation rules which relate the edges of  $E_\Lambda$  that occur in the factorisation of morphisms of degree  $e_i + e_j$  ( $i \neq j$ ) in  $\Lambda$ . For more information about 1-skeletons we refer the reader to [25].

A functor  $\beta : \Lambda \rightarrow \Gamma$  between  $k$ -graphs is a  *$k$ -graph morphism* if it preserves degree, that is  $d_\Gamma \circ \beta = d_\Lambda$ . If  $\Gamma = \Lambda$  and  $\beta$  is invertible then  $\beta$  is an *automorphism*. The collection  $\text{Aut } \Lambda$  of automorphisms of  $\Lambda$  forms a group under composition.

Let  $T_1$  be the category  $\mathbb{N}$  regarded as a 1-graph with degree functor given by the identity map.

**1.2. Cubical cohomology of  $k$ -graphs.** For  $k \geq 0$  define

$$\mathbf{1}_k := \sum_{i=1}^k e_i \in \mathbb{N}^k.$$

By convention  $\mathbb{N}^0 = \{0\}$  and  $\mathbf{1}_0 = 0$ .

**Definition 1.1.** Let  $\Lambda$  be a  $k$ -graph. For  $r \geq 0$  let

$$Q_r(\Lambda) = \{\lambda \in \Lambda : d(\lambda) \leq \mathbf{1}_k, |\lambda| = r\}.$$

We have  $Q_0(\Lambda) = \Lambda^0$ ,  $Q_1(\Lambda) = \cup_{i=1}^k \Lambda^{e_i}$  the set of edges in  $\Lambda$  and  $Q_r(\Lambda) = \emptyset$  if  $r > k$ . For  $0 < r \leq k$  the set  $Q_r(\Lambda)$  consists of the morphisms in  $\Lambda$  which may be expressed as the composition of a sequence of  $r$  edges with distinct degrees. We regard elements of  $Q_r(\Lambda)$  as unit  $r$ -cubes in the sense that each one gives rise to a commuting diagram of edges in  $\Lambda$  shaped like an  $r$ -cube. In particular, when  $r \geq 1$ , each element of  $Q_r(\Lambda)$  has  $2r$  faces in  $Q_{r-1}(\Lambda)$  defined as follows.

**Definition 1.2.** Fix  $\lambda \in Q_r(\Lambda)$  and write  $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$  where  $i_1 < \cdots < i_r$ . For  $1 \leq j \leq r$ , define  $F_j^0(\lambda)$  and  $F_j^1(\lambda)$  to be the unique elements of  $Q_{r-1}(\Lambda)$  such that there exist  $\mu, \nu \in \Lambda^{e_{i_j}}$  satisfying

$$F_j^0(\lambda)\nu = \lambda = \mu F_j^1(\lambda).$$

In [16] the cubical homology of  $\Lambda$  is identified with the homology of the complex  $(\mathbb{Z}Q_*, \partial_*)$  where the boundary map  $\partial_r : \mathbb{Z}Q_r \rightarrow \mathbb{Z}Q_{r-1}$  is determined by

$$\partial_r \lambda = \sum_{j=1}^r \sum_{\ell=0}^1 (-1)^{j+\ell} F_j^\ell(\lambda).$$

**Remark 1.3.** If  $\beta \in \text{Aut}(\Lambda)$ , then it is straightforward to check that the induced action of  $\beta$  on  $\mathbb{Z}Q_r$  commutes with  $\partial_r$ . We first observe that

$$F_j^0(\beta\lambda) = \beta F_j^0(\lambda) \text{ and } F_j^1(\beta\lambda) = \beta F_j^1(\lambda);$$

and hence

$$\partial_r(\beta\lambda) = \sum_{j=1}^r \sum_{\ell=0}^1 (-1)^{j+\ell} F_j^\ell(\beta\lambda) = \sum_{j=1}^r \sum_{\ell=0}^1 (-1)^{j+\ell} \beta F_j^\ell(\lambda) = \beta \partial_r(\lambda).$$

**Notation 1.4.** Let  $\Lambda$  be a  $k$ -graph and let  $A$  be an abelian group. For  $r \geq 0$ , we write  $C^r(\Lambda, A)$  for the collection of all functions  $f : Q_r(\Lambda) \rightarrow A$ . Identify  $C^r(\Lambda, A)$  with  $\text{Hom}(\mathbb{Z}Q_r(\Lambda), A)$  in the usual way. Define maps  $\delta^r : C^r(\Lambda, A) \rightarrow C^{r+1}(\Lambda, A)$  by

$$\delta^r(f)(\lambda) := f(\partial_{r+1}(\lambda)) = \sum_{j=1}^{r+1} \sum_{\ell=0}^1 (-1)^{j+\ell} f(F_j^\ell(\lambda)).$$

Then  $(C^*(\Lambda, A), \delta^*)$  is a cochain complex.

**Definition 1.5.** We define the *cubical cohomology*  $H^*(\Lambda, A)$  of the  $k$ -graph  $\Lambda$  with coefficients in  $A$  to be the cohomology of the complex  $(C^*(\Lambda, A), \delta^*)$ ; that is  $H^r(\Lambda, A) := \ker(\delta^r) / \text{Im}(\delta^{r-1})$ . For  $r \geq 0$ , we write

$$Z^r(\Lambda, A) := \ker(\delta^r)$$

for the group of  $r$ -cocycles, and for  $r > 0$ , we write

$$B^r(\Lambda, A) = \text{Im}(\delta^{r-1})$$

for the group of  $r$ -coboundaries.

**Remark 1.6.** For each  $0 \leq r \leq k$  we define  $\beta^* : C^r(\Lambda, A) \rightarrow C^r(\Lambda, A)$  by  $\beta^*(f) = f \circ \beta$ . For each  $f \in C^r(\Lambda, A)$  and  $\lambda \in Q_r(\Lambda)$  we have

$$\delta^r \beta^*(f)(\lambda) = \sum_{j=1}^{r+1} \sum_{\ell=0}^1 f(F_j^\ell(\beta(\lambda))) = \delta^r(f)(\beta\lambda) = \beta^* \delta^r(f)(\lambda),$$

and so  $\beta^* \circ \delta^r = \delta^r \circ \beta^*$ . Hence  $\beta^*$  induces a homomorphism

$$\beta^* : H^*(\Lambda, A) \rightarrow H^*(\Lambda, A).$$

**1.3. Crossed product graphs.** Recall from [8] that if  $\Lambda$  is a row-finite  $k$ -graph with no sources and  $\beta \in \text{Aut } \Lambda$ , then there is a  $(k+1)$ -graph  $\Lambda \times_\beta \mathbb{Z}$  with morphisms  $\Lambda \times \mathbb{N}$ , range and source maps given by  $r(\lambda, n) = (r(\lambda), 0)$ ,  $s(\lambda, n) = (\beta^{-n}(s(\lambda)), 0)$ , degree map given by  $d(\lambda, n) = (d(\lambda), n)$  and composition given by

$$(\lambda, m)(\mu, n) := (\lambda \beta^m(\mu), m + n).$$

Evidently,  $\Lambda \times_\beta \mathbb{Z}$  is also row-finite with no sources and  $(\Lambda \times_\beta \mathbb{Z})^0 = \Lambda^0 \times \{0\}$ .

**Remark 1.7.** If  $\beta = \text{id}$ , note that  $\Lambda \times_\beta \mathbb{Z} = \Lambda \times T_1$ .

Recall from [16, §4] that we may describe the  $r$ -cubes of  $\Lambda \times_{\beta} \mathbb{Z}$  in terms of the cubes of  $\Lambda$ . The 0-cubes are given by  $Q_0(\Lambda \times_{\beta} \mathbb{Z}) = Q_0(\Lambda) \times \{0\}$ . For each  $0 \leq r \leq k$  the  $(r+1)$ -cubes are given by

$$Q_{r+1}(\Lambda \times_{\beta} \mathbb{Z}) = \{(\lambda, 1) : \lambda \in Q_r(\Lambda)\} \cup \{(\lambda, 0) : \lambda \in Q_{r+1}(\Lambda)\}.$$

Observe that for  $\lambda \in Q_r(\Lambda)$ ,  $F_j^{\ell}(\lambda, 0) = (F_j^{\ell}(\lambda), 0)$  and

$$(2) \quad F_j^{\ell}(\lambda, 1) = \begin{cases} (F_j^{\ell}(\lambda), 1) & \text{if } j \leq r, \\ (\lambda, 0) & \text{if } j = r+1, \ell = 0, \\ (\beta^{-1}(\lambda), 0) & \text{if } j = r+1, \ell = 1. \end{cases}$$

So for  $f \in C^r(\Lambda \times_{\beta} \mathbb{Z}, A)$ , we have

$$\delta^r(f)(\lambda, 0) = \sum_{j=1}^{r+1} \sum_{\ell=0}^1 (-1)^{j+\ell} f(F_j^{\ell}(\lambda), 0) \quad \text{for } \lambda \in Q_{r+1}(\Lambda)$$

and

$$(3) \quad \begin{aligned} \delta^r(f)(\lambda, 1) &= (-1)^{r+1} (f(\lambda, 0) - f(\beta^{-1}(\lambda), 0)) \\ &\quad + \sum_{j=1}^r \sum_{\ell=0}^1 (-1)^{j+\ell} f(F_j^{\ell}(\lambda), 1), \end{aligned}$$

for each  $\lambda \in Q_r(\Lambda)$ .

**1.4. The long exact sequence of cohomology.** Suppose  $\beta$  is an automorphism of a  $k$ -graph  $\Lambda$ . In [16, Theorem 4.13] the authors presented a long exact sequence relating the homology groups of  $\Lambda$  and  $\Lambda \times_{\beta} \mathbb{Z}$ . In the next result we present the corresponding long exact sequence of cohomology.

**Proposition 1.8.** *Suppose  $\beta$  is an automorphism of a  $k$ -graph  $\Lambda$ , and  $A$  is an abelian group. There is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(\Lambda \times_{\beta} \mathbb{Z}, A) &\xrightarrow{i^*} H^0(\Lambda, A) \xrightarrow{1-\beta^*} H^0(\Lambda, A) \\ &\xrightarrow{j^*} H^1(\Lambda \times_{\beta} \mathbb{Z}, A) \xrightarrow{i^*} \cdots \xrightarrow{1-\beta^*} H^r(\Lambda, A) \\ &\xrightarrow{j^*} H^{r+1}(\Lambda \times_{\beta} \mathbb{Z}, A) \xrightarrow{i^*} H^{r+1}(\Lambda, A) \xrightarrow{1-\beta^*} H^{r+1}(\Lambda, A) \\ &\xrightarrow{j^*} \cdots \xrightarrow{j^*} H^k(\Lambda \times_{\beta} \mathbb{Z}, A) \xrightarrow{i^*} H^k(\Lambda, A) \\ &\xrightarrow{1-\beta^*} H^k(\Lambda, A) \xrightarrow{j^*} H^{k+1}(\Lambda \times_{\beta} \mathbb{Z}, A) \longrightarrow 0, \end{aligned}$$

where

$$i^*(f)(\lambda) = f(\lambda, 0)$$

for each  $f \in Z^r(\Lambda \times_{\beta} \mathbb{Z}, A)$ , and

$$j^*(f)(\lambda, 0) = 0 \quad \text{and} \quad j^*(f)(\lambda, 1) = f(\lambda)$$

for each  $f \in Z^{r-1}(\Lambda, A)$ .

**Proof.** For each  $0 \leq r \leq k$  the maps  $i : \mathbb{Z}Q_r(\Lambda) \rightarrow \mathbb{Z}Q_r(\Lambda \times_\beta \mathbb{Z})$  and  $j : \mathbb{Z}Q_{r+1}(\Lambda \times_\beta \mathbb{Z}) \rightarrow \mathbb{Z}Q_r(\Lambda)$  determined by

$$i(\lambda) = (\lambda, 0) \quad \text{for } \lambda \in Q_r(\Lambda)$$

$$j(\lambda, \ell) = \begin{cases} 0 & \text{if } \ell = 0 \\ \lambda & \text{if } \ell = 1 \end{cases} \quad \text{for } (\lambda, \ell) \in Q_{r+1}(\Lambda \times_\beta \mathbb{Z})$$

induce maps

$$i^* : C^r(\Lambda \times_\beta \mathbb{Z}, A) \rightarrow C^r(\Lambda, A) \quad \text{and} \quad j^* : C^r(\Lambda, A) \rightarrow C^{r+1}(\Lambda \times_\beta \mathbb{Z}, A)$$

given by

$$i^*(f)(\lambda) = f(\lambda, 0) \quad \text{for } f \in C^r(\Lambda \times_\beta \mathbb{Z}, A)$$

$$j^*(f)(\lambda, \ell) = \begin{cases} 0 & \text{if } \ell = 0 \\ f(\lambda) & \text{if } \ell = 1 \end{cases} \quad \text{for } f \in C^r(\Lambda, A).$$

Using the description of the cubes in  $\Lambda \times_\beta \mathbb{Z}$  we obtain a short exact sequence of complexes  $E$  where

$$(4) \quad E_r : 0 \rightarrow C^{r-1}(\Lambda, A) \xrightarrow{j^*} C^r(\Lambda \times_\beta \mathbb{Z}, A) \xrightarrow{i^*} C^r(\Lambda, A) \rightarrow 0,$$

since  $i^*$  and  $j^*$  commute with the coboundary maps. Indeed,

$$(\delta^r i^*)f(\lambda) = \sum_{j=1, \ell=0}^{r+1, 1} (i^* f)(F_j^\ell(\lambda)) = \sum_{j=1, \ell=0}^{r+1, 1} f(F_j^\ell(\lambda), 0) = \delta^r f(\lambda, 0) = (i^* \delta^r)f(\lambda)$$

for  $f \in C^r(\Lambda \times_\beta \mathbb{Z}, A)$ , and

$$(\delta^r j^*)f(\lambda, 1) = (-1)^{r+1}((j^* f)(\lambda, 0) - (j^* f)(\beta^{-1}(\lambda), 0))$$

$$+ \sum_{j=1, \ell=0}^{r, 1} (-1)^{j+\ell} (j^* f)(F_j^\ell(\lambda), 1)$$

$$= \sum_{j=1, \ell=0}^{r, 1} (-1)^{j+\ell} f(F_j^\ell(\lambda)) = \delta^{r-1} f(\lambda) = (j^* \delta^{r-1})f(\lambda, 1)$$

and

$$(\delta^r j^*)f(\lambda, 0) = \sum_{j=1, \ell=0}^{r+1, 1} (-1)^{j+\ell} (j^* f)(F_j^\ell(\lambda), 0) = 0 = (j^* \delta^{r-1})f(\lambda, 0)$$

for  $f \in C^{r-1}(\Lambda, A)$ .

Using the long exact sequence associated to a short exact sequence of homology complexes (see [19, Theorem II.4.1]) applied to a short exact sequence of cohomology complexes with the appropriate reindexing, we obtain

the long exact sequence

$$\begin{aligned} \cdots \xrightarrow{\delta_E^r} H^r(\Lambda, A) \xrightarrow{j^*} H^{r+1}(\Lambda \times_\beta \mathbb{Z}, A) \\ \xrightarrow{i^*} H^{r+1}(\Lambda, A) \xrightarrow{\delta_E^{r+1}} H^{r+1}(\Lambda, A) \rightarrow \cdots \end{aligned}$$

Indeed, the boundary map  $\delta_E$  (see [19, II.4]) is defined as follows. Start with  $m \in Z^r(\Lambda, A)$  and take the lift  $n \in C^r(\Lambda \times_\beta \mathbb{Z}, A)$  given by  $n(\lambda, 0) = m(\lambda)$  and  $n(\lambda, 1) = 0$ . Then  $\delta^r n(\lambda, 0) = 0$  for all  $\lambda$ , so there is  $c \in Z^r(\Lambda, A)$  such that  $\delta^r n = j^*c$ , that is  $\delta^r n(\lambda, 1) = c(\lambda)$ . Then  $\delta_E$  takes the class of  $m$  into the class of  $c$ . Using (2) and (3) we get

$$\begin{aligned} c(\lambda) &= j^*(c)(\lambda, 1) = \delta^r n(\lambda, 1) = (-1)^{r+1}(n(\lambda, 0) - n(\beta^{-1}(\lambda), 0)) \\ &= (-1)^{r+1}(m - m \circ \beta^{-1})(\lambda). \end{aligned}$$

Hence we see that  $\delta_E^r = (-1)^{r+1}(1 - (\beta^{-1})^*)$ . By using a similar argument as in [16, Theorem 4.13], the sequence remains exact after replacing  $\delta_E$  with  $1 - \beta^*$ .  $\square$

If  $\beta = \text{id}$ , then as noted in Remark 1.7  $\Lambda \times_\beta \mathbb{Z}$  may be identified with  $\Lambda \times T_1$  and  $1 - \beta^* = 0$ . Hence, for all  $r$ , we have the short exact sequence

$$0 \rightarrow H^r(\Lambda, A) \xrightarrow{j^*} H^{r+1}(\Lambda \times T_1, A) \xrightarrow{i^*} H^{r+1}(\Lambda, A) \rightarrow 0.$$

Moreover there is a map  $\sigma : C^r(\Lambda, A) \rightarrow C^r(\Lambda \times T_1, A)$  such that  $\sigma(f)(\lambda, 0) = f(\lambda)$  for  $\lambda \in Q_r(\Lambda)$  and, if  $r \geq 1$ ,  $\sigma(f)(\lambda, 1) = 0$  for  $\lambda \in Q_{r-1}(\Lambda)$ . It is straightforward to check that  $\sigma$  intertwines boundary maps and that  $i^*\sigma(f) = f$  for all  $f \in C^r(\Lambda, A)$ . Hence, the map

$$(5) \quad \Xi : (f, g) \in Z^r(\Lambda, A) \oplus Z^{r+1}(\Lambda, A) \mapsto j^*(f) + \sigma(g) \in Z^{r+1}(\Lambda \times T_1, A)$$

is an isomorphism which intertwines the boundary maps. We thereby obtain the following result:

**Corollary 1.9.** *If  $\beta = \text{id}$ , then  $H^0(\Lambda \times T_1, A) \cong H^0(\Lambda, A)$  and for  $r \geq 0$  the map on cohomology induced by  $\Xi$  is an isomorphism*

$$\Xi : H^r(\Lambda, A) \oplus H^{r+1}(\Lambda, A) \cong H^{r+1}(\Lambda \times_\beta \mathbb{Z}, A).$$

### 1.5. Twisted $k$ -graph $C^*$ -algebras.

**Definition 1.10.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and fix  $\varphi \in Z^2(\Lambda, \mathbb{T})$ . A Cuntz–Krieger  $\varphi$ -representation of  $\Lambda$  in a  $C^*$ -algebra  $A$  is a set  $\{p_v : v \in \Lambda^0\} \subseteq A$  of mutually orthogonal projections and a set  $\{s_\lambda : \lambda \in Q_1(\Lambda)\} \subseteq A$  satisfying:

(TG1) for all  $1 \leq i \leq k$  and  $\lambda \in \Lambda^{e_i}$ ,  $s_\lambda^* s_\lambda = p_{s(\lambda)}$ ;

(TG2) for all  $1 \leq i < j \leq k$  and  $\mu, \mu' \in \Lambda^{e_i}$ ,  $\nu, \nu' \in \Lambda^{e_j}$  such that  $\mu\nu = \nu'\mu'$ ,

$$s_{\nu'} s_{\mu'} = \varphi(\mu\nu) s_\mu s_\nu; \text{ and}$$



(TG3) for all  $v \in \Lambda^0$  and all  $i = 1, \dots, k$  such that  $v\Lambda^{e_i} \neq \emptyset$ ,

$$p_v = \sum_{\lambda \in v\Lambda^{e_i}} s_\lambda s_\lambda^*.$$

**Definition 1.11.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and let  $\varphi \in Z^2(\Lambda, \mathbb{T})$ . We define  $C_\varphi^*(\Lambda)$  to be the universal  $C^*$ -algebra generated by a Cuntz–Krieger  $\varphi$ -representation of  $\Lambda$ .

**Remark 1.12.** If  $\varphi \in Z^2(\Lambda, \mathbb{T})$  is the trivial cocycle, then  $\varphi \in B^2(\Lambda, \mathbb{T})$  and so by [17, Proposition 5.3] and [17, Proposition 5.6] we have  $C_\varphi^*(\Lambda) \cong C^*(\Lambda)$ .

**Remark 1.13.** In the context of twisted  $k$ -graph  $C^*$ -algebras, we shall be particularly interested in the following part of the exact sequence for  $A = \mathbb{T}$

$$\begin{aligned} \dots \rightarrow H^1(\Lambda, \mathbb{T}) \xrightarrow{1-\beta^*} H^1(\Lambda, \mathbb{T}) \xrightarrow{j^*} H^2(\Lambda \times_\beta \mathbb{Z}, \mathbb{T}) \\ \xrightarrow{i^*} H^2(\Lambda, \mathbb{T}) \xrightarrow{1-\beta^*} H^2(\Lambda, \mathbb{T}) \rightarrow \dots \end{aligned}$$

## 2. Main results

In this section we present our  $C^*$ -algebraic results. In our main result we generalise the isomorphism  $C^*(\Lambda \times_\beta \mathbb{Z}) \cong C^*(\Lambda) \rtimes_{\tilde{\beta}} \mathbb{Z}$  from [8, Theorem 3.4] (in the case  $l = 1$ ) to the twisted setting. Note that for  $A = \mathbb{T}$  we use multiplicative notation; inverses are given by conjugation, and the identity element is  $1 \in \mathbb{T}$ .

**Theorem 2.1.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, let  $\beta \in \text{Aut}(\Lambda)$  and let  $\varphi \in Z^2(\Lambda \times_\beta \mathbb{Z}, \mathbb{T})$ . Then*

- (i) *There is an automorphism  $\beta_\varphi$  of  $C_{i^*(\varphi)}^*(\Lambda)$  such that*
- (6) 
$$\beta_\varphi(p_v) = p_{\beta v} \quad \text{and} \quad \beta_\varphi(s_e) = \varphi(\beta e, 1)s_{\beta e},$$

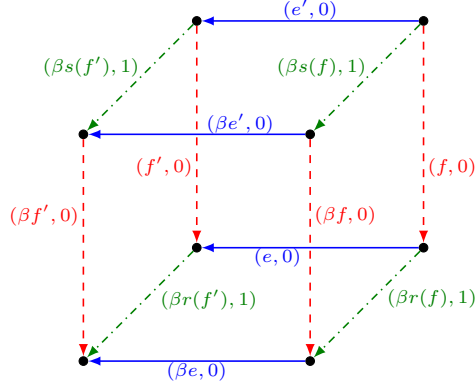
*for all  $v \in Q_0(\Lambda)$  and  $e \in Q_1(\Lambda)$ .*
- (ii) *Let  $(F_n)_{n \in \mathbb{N}}$  be an increasing family of finite subsets of  $\Lambda^0$  such that  $\cup_{n \in \mathbb{N}} F_n = \Lambda^0$ . The sequence  $(\sum_{v \in F_n} s_{(v,1)})_{n \in \mathbb{N}}$  converges strictly to a unitary  $U \in \mathcal{M}C_\varphi^*(\Lambda \times_\beta \mathbb{Z})$  satisfying*
- (7) 
$$Up_{(v,0)}U^* = p_{(\beta v,0)} \quad \text{and} \quad Us_{(e,0)}U^* = \varphi(\beta e, 1)s_{(\beta e,0)},$$

*for all  $v \in Q_0(\Lambda)$  and  $e \in Q_1(\Lambda)$ .*
- (iii) *There is a homomorphism  $\pi : C_{i^*(\varphi)}^*(\Lambda) \rightarrow C_\varphi^*(\Lambda \times_\beta \mathbb{Z})$  which forms a covariant pair  $(\pi, U)$  whose integrated form*

$$\pi \times U : C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_\varphi} \mathbb{Z} \rightarrow C_\varphi^*(\Lambda \times_\beta \mathbb{Z})$$

*is an isomorphism.*

**Remark 2.2.** In the proof of this theorem we need to calculate the faces of a cube  $(\beta\lambda, 1) \in Q_3(\Lambda \times_\beta \mathbb{Z})$ , where  $\lambda \in Q_2(\Lambda)$ . Suppose  $\lambda = ef = f'e'$ , where  $e, e' \in \Lambda^{e_i}$  and  $f, f' \in \Lambda^{e_j}$  such that  $1 \leq i < j \leq k$ . We can factorise  $(\beta\lambda, 1)$  according to the following diagram, and then calculate its faces.



$$\begin{aligned}
F_1^0(\beta\lambda, 1) &= (\beta f', 1) \\
F_1^1(\beta\lambda, 1) &= (\beta f, 1) \\
F_2^0(\beta\lambda, 1) &= (\beta e, 1) \\
F_2^1(\beta\lambda, 1) &= (\beta e', 1) \\
F_3^0(\beta\lambda, 1) &= (\beta\lambda, 0) \\
F_3^1(\beta\lambda, 1) &= (\lambda, 0)
\end{aligned}$$

**Proof of Theorem 2.1.** Let  $\lambda \in Q_2(\Lambda)$ . Write  $\lambda = ef = f'e'$ , where  $e, e' \in \Lambda^{e_i}$  and  $f, f' \in \Lambda^{e_j}$  such that  $1 \leq i < j \leq k$ . Using (3) and the identities in Remark 2.2 we get

$$(8) \quad 1 = \delta^2(\varphi)(\beta\lambda, 1) = \overline{\varphi(\beta\lambda, 0)}\varphi(\lambda, 0)\overline{\varphi(\beta f', 1)}\varphi(\beta e, 1)\overline{\varphi(\beta e', 1)}\varphi(\beta f, 1).$$

For each  $v \in Q_0(\Lambda)$  let  $P_v := p_{\beta v}$  and for each  $e \in Q_1(\Lambda)$  let

$$S_e := \varphi(\beta e, 1)s_{\beta e}.$$

We claim that  $\{P, S\}$  defines a  $i^*(\varphi)$ -representation of  $\Lambda$  in  $C_{i^*(\varphi)}^*(\Lambda)$ . We check condition (TG2) using (8):

$$\begin{aligned}
S_{f'}S_{e'} &= \varphi(\beta f', 1)\varphi(\beta e', 1)s_{\beta f'}s_{\beta e'} \\
&= \varphi(\beta f', 1)\varphi(\beta e', 1)i^*(\varphi)(\beta e\beta f)s_{\beta e}s_{\beta f} \\
&= \varphi(\beta f', 1)\varphi(\beta e', 1)\varphi(\beta(ef), 0)s_{\beta e}s_{\beta f} \\
&= \varphi(\beta f', 1)\varphi(\beta e', 1)\varphi(\beta(ef), 0)\overline{\varphi(\beta e, 1)}\varphi(\beta f, 1)S_eS_f \\
&= \varphi(ef, 0)S_eS_f = i^*(\varphi)(ef)S_eS_f.
\end{aligned}$$

Conditions (TG1) and (TG3) follow easily. By the universal property of  $C_{i^*(\varphi)}^*(\Lambda)$  we obtain a homomorphism  $\beta_\varphi$  satisfying

$$\beta_\varphi(p_v) = p_{\beta v} \quad \text{and} \quad \beta_\varphi(s_e) = \varphi(\beta e, 1)s_{\beta e}.$$

Similar calculations show that the collection  $\{p_{\beta^{-1}v}, \overline{\varphi(e, 1)}s_{\beta^{-1}e}\}$  is also an  $i^*(\varphi)$ -representation of  $\Lambda$  in  $C_{i^*(\varphi)}^*(\Lambda)$ , and the corresponding homomorphism coming from the universal property of  $C_{i^*(\varphi)}^*(\Lambda)$  is the inverse of  $\beta_\varphi$ . So  $\beta_\varphi$  is an automorphism, and (i) holds.

For any finite subset  $F \subseteq \Lambda^0$  we denote by

$$P(F) := \sum_{v \in F} p_{(v, 0)} \in C^*(\Lambda \times_\beta \mathbb{Z}).$$

To see that (ii) holds, first let  $(F_n)_{n \in \mathbb{N}}$  be an increasing sequence of finite subsets of  $\Lambda^0$  such that  $\cup_{n \in \mathbb{N}} F_n = \Lambda^0$ . Then a standard argument shows that  $P(F_n) \rightarrow 1$  strictly in  $\mathcal{MC}_\varphi^*(\Lambda \times_\beta \mathbb{Z})$ . For  $n \geq 1$  let  $U_n := \sum_{v \in F_n} s_{(v, 1)}$ .

Since the elements in the sum defining  $U_n$  have the same degree, by (TG1) we have

$$(9) \quad U_n^* U_n = \sum_{v,w \in F_n} s_{(v,1)}^* s_{(w,1)} = \sum_{v \in F_n} s_{(v,1)}^* s_{(v,1)} = \sum_{v \in F_n} p_{(\beta^{-1}v,0)} = P(\beta^{-1}(F_n)),$$

and by (TG3) we have

$$(10) \quad U_n U_n^* = \sum_{v,w \in F_n} s_{(v,1)} s_{(w,1)}^* = \sum_{v \in F_n} s_{(v,1)} s_{(v,1)}^* = \sum_{v \in F_n} p_{(v,0)} = P(F_n).$$

Hence  $U_n$  is a partial isometry, with initial projection  $P(\beta^{-1}(F_n))$  and final projection  $P(F_n)$ .

For  $(w, 0) \in (\Lambda \times_{\beta} \mathbb{Z})^0$  and  $(e, 0) \in (\Lambda \times_{\beta} \mathbb{Z})^{e_j}$ ,  $1 \leq j \leq k$  we have

$$(11) \quad U_n p_{(w,0)} = s_{(\beta w,1)} \text{ if } \beta w \in F_n, \text{ and zero otherwise, and} \\ U_n s_{(e,0)} = s_{(\beta r(e),1)} s_{(e,0)} \text{ if } \beta r(e) \in F_n, \text{ and zero otherwise;}$$

and

$$(12) \quad p_{(w,0)} U_n = s_{(w,1)} \text{ if } w \in F_n \text{ and zero otherwise, and} \\ s_{(e,0)} U_n = s_{(e,0)} s_{(s(e),1)} \text{ if } s(e) \in F_n, \text{ and zero otherwise.}$$

Hence  $U_n$  multiplied on the left or right of any product of generators of  $C_{\varphi}^*(\Lambda \times_{\beta} \mathbb{Z})$  is eventually constant as  $n \rightarrow \infty$ . A standard argument shows that  $U_n$  converges strictly to an element  $U \in \mathcal{MC}^*(\Lambda \times_{\beta} \mathbb{Z})$ . Moreover,  $U$  is independent of the choice of  $F_n$ , and from (10) and (9) we see that  $UU^* = 1 = UU^*$ . Finally from (11) and (12) it follows that for  $w, \beta w \in F_n$  we have  $U_n p_{(w,0)} = p_{(\beta w,0)} U_n$ , and for  $\beta r(e), \beta s(e) \in F_n$  we have

$$U_n s_{(e,0)} = s_{(e,0)} s_{(\beta r(e),1)} = \varphi(\beta e, 1) s_{(\beta e,0)} s_{(\beta s(e),1)} = \varphi(\beta e, 1) s_{(\beta e,0)} U_n$$

We see that the identities (7) hold by taking  $n \rightarrow \infty$ . This completes the proof of (ii).

For (iii) we first claim that  $\{p_{(v,0)}, s_{(e,0)}\}$  is an  $i^*(\varphi)$ -representation of  $\Lambda$  in  $C_{\varphi}^*(\Lambda \times_{\beta} \mathbb{Z})$ . We check (TG2): for  $e, e' \in \Lambda^{e_i}$  and  $f, f' \in \Lambda^{e_j}$  such that  $ef = f'e'$ , where  $1 \leq i < j \leq k$ , we have

$$s_{(f',0)} s_{(e',0)} = \varphi(ef, 0) s_{(e,0)} s_{(f,0)}.$$

Checking conditions (TG1) and (TG3) is straightforward. The universal property of  $C_{i^*(\varphi)}^*(\Lambda)$  now gives a homomorphism

$$\pi : C_{i^*(\varphi)}^*(\Lambda) \rightarrow C_{\varphi}^*(\Lambda \times_{\beta} \mathbb{Z})$$

satisfying  $\pi(p_v) = p_{(v,0)}$  and  $\pi(s_e) = s_{(e,0)}$ .

The homomorphism  $\pi$  and the unitary  $U$  from (ii) satisfy

$$U \pi(s_e) U^* = U s_{(e,0)} U^* = \varphi(\beta e, 1) S'_{\beta e} = \pi(\varphi(\beta e, 1) s_{\beta e}) = \pi(\beta_{\varphi}(s_e)),$$

for each  $e \in Q_1(\Lambda)$ . It follows that  $(\pi, U)$  is a covariant representation of  $(C_{i^*(\varphi)}^*(\Lambda), \beta_\varphi)$ , and hence by the universal property of the full crossed product  $C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_\varphi} \mathbb{Z}$  we get a homomorphism

$$\pi \times U : C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_\varphi} \mathbb{Z} \rightarrow C_\varphi^*(\Lambda \times_\beta \mathbb{Z}).$$

If we denote the universal covariant pair by  $(i_\Lambda, i_{\mathbb{Z}}(1))$ , then we know that  $(\pi \times U) \circ i_\Lambda = \pi$  and  $\overline{\pi \times U}(i_{\mathbb{Z}}(1)) = U$ , where  $\overline{\pi \times U}$  is the extension of  $\pi \times U$  to the multiplier algebra  $\mathcal{M}C_\varphi^*(\Lambda \times_\beta \mathbb{Z})$ .

We claim that  $\pi \times U$  is an isomorphism. To find the inverse we construct a  $\varphi$ -representation of  $\Lambda \times_\beta \mathbb{Z}$  in  $C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_\varphi} \mathbb{Z}$ . For each

$$(v, 0) \in Q_0(\Lambda \times_\beta \mathbb{Z})$$

let  $P_{(v,0)} := i_\Lambda(p_v)$ , for each  $(e, 0) \in Q_1(\Lambda \times_\beta \mathbb{Z})$  let  $S_{(e,0)} := i_\Lambda(s_e)$ , and for each  $(v, 1) \in Q_1(\Lambda \times_\beta \mathbb{Z})$  let  $S_{(v,1)} := i_\Lambda(p_v)i_{\mathbb{Z}}(1)$ . We claim that  $\{P, S\}$  is a Cuntz–Krieger  $\varphi$ -representation of  $\Lambda \times_\beta \mathbb{Z}$  in  $C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_\varphi} \mathbb{Z}$ . To check (TG2) we have two cases to consider. For

$$(\beta e, 1) = (\beta r(e), 1)(e, 0) = (\beta e, 0)(\beta s(e), 1) \in (\Lambda \times_\beta \mathbb{Z})^{e_i + e_{k+1}}$$

we have

$$\begin{aligned} S_{(\beta r(e), 1)} S_{(e, 0)} &= i_\Lambda(p_{\beta r(e)}) i_{\mathbb{Z}}(1) i_\Lambda(s_e) \\ &= i_\Lambda(p_{\beta r(e)}) i_\Lambda(\beta_\varphi(s_e)) i_{\mathbb{Z}}(1) \\ &= \varphi(\beta e, 1) i_\Lambda(p_{\beta r(e)}) i_\Lambda(s_{\beta e}) i_{\mathbb{Z}}(1) \\ &= \varphi(\beta e, 1) i_\Lambda(s_{\beta e}) i_\Lambda(p_{\beta s(e)}) i_{\mathbb{Z}}(1) \\ &= \varphi(\beta e, 1) S_{(\beta e, 0)} S_{(\beta s(e), 1)}. \end{aligned}$$

The other case is when  $ef = f'e'$ , where

$$e, e' \in (\Lambda \times_\beta \mathbb{Z})^{e_i} \quad \text{and} \quad f, f' \in (\Lambda \times_\beta \mathbb{Z})^{e_j}$$

and  $1 \leq i < j \leq k$ . Then

$$(ef, 0) = (e, 0)(f, 0) = (f', 0)(e', 0) = (f'e', 0) \in \Lambda \times_\beta \mathbb{Z},$$

and

$$S_{(f', 0)} S_{(e', 0)} = i_\Lambda(s_{f'} s_{e'}) = i^*(\varphi)(ef) i_\Lambda(s_e s_f) = \varphi(ef, 0) S_{(e, 0)} S_{(f, 0)}.$$

Properties (TG1) and (TG3) follow more easily. The universal property of  $C_\varphi^*(\Lambda \times_\beta \mathbb{Z})$  now gives a homomorphism  $\rho_{P,S} : C_\varphi^*(\Lambda \times_\beta \mathbb{Z}) \rightarrow C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_\varphi} \mathbb{Z}$  such that  $\rho_{P,S}(p_{(v,0)}) = P_{(v,0)}$ ,  $\rho_{P,S}(s_{(e,0)}) = S_{(e,0)}$ , and  $\rho_{P,S}(s_{(v,1)}) = S_{(v,1)}$ . One checks on generators that  $\rho_{P,S}$  is the inverse of  $\pi \times U$ .  $\square$

**Corollary 2.3.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, let  $\beta \in \text{Aut}(\Lambda)$  and let  $c \in Z^1(\Lambda, \mathbb{T})$ . There is an automorphism  $\beta^c$  of  $C^*(\Lambda)$  satisfying*

$$(13) \quad \beta^c(p_v) = p_{\beta v} \quad \text{and} \quad \beta^c(s_e) = c(\beta e) s_{\beta e},$$

for all  $v \in Q_0(\Lambda)$  and  $e \in Q_1(\Lambda)$ , and an isomorphism

$$C^*(\Lambda) \rtimes_{\beta^c} \mathbb{Z} \cong C_{j^*(c)}^*(\Lambda \times_{\beta} \mathbb{Z}).$$

**Proof.** We apply Theorem 2.1 with  $\varphi := j^*(c) \in Z^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T})$ . Then  $\beta^c$  is just  $\beta_{j^*(c)}$ , and (13) follows because  $\varphi(e, 1) = j^*(c)(e, 1) = c(e)$  for all  $e \in Q_1(\Lambda)$ . The isomorphism  $C^*(\Lambda) \rtimes_{\beta^c} \mathbb{Z} \cong C_{j^*(c)}^*(\Lambda \times_{\beta} \mathbb{Z})$  follows by Theorem 2.1 and realising that  $i^*(\varphi) = i^*(j^*(c)) = 1$ .  $\square$

**Corollary 2.4.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and let  $\beta \in \text{Aut}(\Lambda)$ . Suppose that  $\psi \in Z^2(\Lambda, \mathbb{T})$  such that  $[\psi] \in \ker(1 - \beta^*)$ . Then there is  $\varphi \in Z^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T})$  such that  $\psi = i^*(\varphi)$  and so Theorem 2.1 applies. In particular there is an automorphism  $\beta_{\varphi}$  of  $C_{\psi}^*(\Lambda)$  such that*

$$C_{i^*(\varphi)}^*(\Lambda) \rtimes_{\beta_{\varphi}} \mathbb{Z} \cong C_{\varphi}^*(\Lambda \times_{\beta} \mathbb{Z}).$$

**Proof.** Since  $[\psi] \in \ker(1 - \beta^*)$ ,  $[\psi] = [\beta^*\psi]$  and so there is a map  $b : Q_1(\Lambda) \rightarrow \mathbb{T}$  such that  $(\beta^*\psi)\overline{\psi} = \delta^1(b)$ . So for  $\lambda = ef = f'e'$  where  $e, e' \in \Lambda^{e_i}$  and  $f, f' \in \Lambda^{e_j}$  such that  $1 \leq i < j \leq k$ , we get

$$\psi(\beta\lambda)\overline{\psi(\lambda)} = b(e)b(f)\overline{b(f')b(e')}.$$

We define  $\varphi \in Z^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T})$  by

$$\varphi(\lambda, \ell) = \begin{cases} \psi(\lambda) & \text{if } \lambda \in Q_2(\Lambda), \ell = 0, \\ b(\beta^{-1}\lambda) & \text{if } \lambda \in Q_1(\Lambda), \ell = 1. \end{cases}$$

Then by a computation as in Equation (8) we have  $\delta^2(\varphi)(\beta\lambda, 1) = 1$  for all  $\lambda \in Q_2(\Lambda)$ ; moreover, for  $\lambda \in Q_3(\Lambda)$ , we have  $\delta^2(\varphi)(\lambda, 0) = (\delta^2(\psi)(\lambda), 0) = 1$  since  $\psi \in Z^2(\Lambda, \mathbb{T})$ . Hence,  $\varphi \in Z^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T})$ . By construction  $\psi = i^*(\varphi)$  and so Theorem 2.1 gives the result.  $\square$

Recall that if  $\beta$  is the identity automorphism of  $\Lambda$  then  $\Lambda \times_{\beta} \mathbb{Z} \cong \Lambda \times T_1$  by Remark 1.7 and  $\Xi : Z^1(\Lambda, \mathbb{T}) \oplus Z^2(\Lambda, \mathbb{T}) \cong Z^2(\Lambda \times T_1, \mathbb{T})$  by Equation (5). In this case Theorem 2.1 reduces to the following result:

**Corollary 2.5.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources and let  $\varphi \in Z^2(\Lambda \times T_1, \mathbb{T})$ . Then  $\varphi = \Xi(\varphi_1, \varphi_2)$  where  $(\varphi_1, \varphi_2) \in Z^1(\Lambda, \mathbb{T}) \oplus Z^2(\Lambda, \mathbb{T})$ . Moreover,*

- (i) *We have  $\varphi(e, 1) = \varphi_1(e)$  for all  $e \in Q_1(\Lambda)$  and  $\varphi(\lambda, 0) = \varphi_2(\lambda)$  for all  $\lambda \in Q_2(\Lambda)$ .*
- (ii) *There is an automorphism  $\alpha = \alpha_{\varphi}$  of  $C_{\varphi_2}^*(\Lambda)$  such that  $\alpha(p_v) = p_v$  and  $\alpha(s_e) = \varphi_1(e)s_e$  for all  $v \in Q_0(\Lambda)$  and  $e \in Q_1(\Lambda)$ .*
- (iii) *There is an isomorphism  $C_{\varphi_2}^*(\Lambda) \rtimes_{\alpha} \mathbb{Z} \rightarrow C_{\varphi}^*(\Lambda \times T_1)$ .*

### 3. Examples

We consider some examples of automorphisms and the associated crossed products. In Example 3.1 we consider quasifree automorphisms on Cuntz algebras. In Examples 3.2 and 3.3 we compute cohomology of the crossed product; in Example 3.3 we adduce conditions under which the twisted crossed product  $C^*$ -algebra is simple and purely infinite and use classification results to show that it is isomorphic to  $\mathcal{O}_2$ .

**Example 3.1.** For  $n > 1$  let  $B_n$  denote the 1-graph which is the path category of the directed graph with a single vertex  $v$  and edges  $f_1, \dots, f_n$ . It is well-known that  $C^*(B_n) \cong \mathcal{O}_n$ . It is straightforward to see that  $Z^1(B_n, \mathbb{T}) \cong \mathbb{T}^n$ , since we may label each edge  $f_i$  with an independent element of  $\mathbb{T}$ . It is also straightforward to see that  $\text{Aut } B_n$  is isomorphic to  $S_n$ , the symmetric group of order  $n$ , which acts by permuting the edges  $f_i$ .

Following [7], an automorphism  $\alpha$  of  $\mathcal{O}_n$  is said to be *quasifree* if it is determined by a unitary matrix  $u \in U(n)$  in the following sense

$$\alpha(s_{f_i}) = \sum_{j=1}^n u_{i,j} s_{f_j} \text{ for } i = 1, \dots, n.$$

We write  $\alpha = \alpha_u$ . Given  $u, u' \in U(n)$ , we have  $\alpha_{uu'} = \alpha_u \circ \alpha_{u'}$ . Moreover, if  $u, u' \in U(n)$  are conjugate, the corresponding automorphisms  $\alpha_u, \alpha_{u'}$  are conjugate.

Evans notes on [7, Page 917] (citing an argument of Archbold from [1]) that  $\alpha_u$  is outer if and only if  $u \neq 1$ . Hence by [13, Lemma 10] the crossed product  $\mathcal{O}_n \rtimes_{\alpha_u} \mathbb{Z}$  is simple and purely infinite if and only if  $u^m \neq 1$  for all  $m \neq 0$ . By the Pimsner–Voiculescu six-term exact sequence we have  $K_i(\mathcal{O}_n \rtimes_{\alpha_u} \mathbb{Z}) = \mathbb{Z}/(n-1)\mathbb{Z}$  for  $i = 0, 1$ . Hence if  $u^m \neq 1$  for all  $m \neq 0$ , the Kirchberg–Phillips Theorem [11, 24] yields that the isomorphism class of  $\mathcal{O}_n \rtimes_{\alpha_u} \mathbb{Z}$  is independent of  $u$ .

We consider the situation covered in Corollary 2.3 in the case  $\Lambda = B_n$ . If the action  $\beta$  on  $B_n$  is induced by the identity permutation and  $c = (c_1, \dots, c_n) \in Z^1(B_n, \mathbb{T})$ , then the automorphism  $\beta^c$  of Corollary 2.3 is the quasifree automorphism  $\alpha_u$  of  $\mathcal{O}_n$  arising from the  $n \times n$  diagonal matrix  $u$  with entries determined by  $c$  (see [10, 7, 12]). By Remark 1.7 we have that  $B_n \rtimes_{\beta} \mathbb{Z} \cong B_n \times T_1$  and so by Corollary 2.3 we have

$$\mathcal{O}_n \rtimes_{\beta^c} \mathbb{Z} \cong C_{j^*(c)}^*(B_n \times T_1).$$

Moreover,  $u^m \neq 1$  for all  $m \neq 0$  if and only if  $c_i$  is not a root of unity for some  $1 \leq i \leq n$ , and hence in this case by the above paragraph we have  $C_{j^*(c)}^*(B_n \times T_1)$  simple and purely infinite with

$$K_i(C_{j^*(c)}^*(B_n \times T_1)) = \mathbb{Z}/(n-1)\mathbb{Z}$$

for  $i = 0, 1$ .

Now let  $\alpha$  be a quasifree automorphism of  $\mathcal{O}_n$ ; then  $\alpha = \alpha_u$  for some unitary matrix  $u \in U(n)$ . Then since every unitary is conjugate to a diagonal unitary,  $\alpha$  is conjugate to  $\beta^c$  for some  $c \in Z^1(B_n, \mathbb{T})$  (where  $\beta$  is the identity permutation). Hence,

$$\mathcal{O}_n \rtimes_{\alpha} \mathbb{Z} \cong \mathcal{O}_n \rtimes_{\beta^c} \mathbb{Z} \cong C_{j^*(c)}^*(B_n \times T_1).$$

Let  $\gamma$  be an arbitrary permutation of the edges of  $B_n$ . Then the induced automorphism  $\gamma$  of  $\mathcal{O}_n$  is the quasifree automorphism associated to the unitary corresponding to the permutation. So if  $c \in Z^1(B_n, \mathbb{T})$ , then  $\gamma^c = \beta^c \circ \gamma$  is also a quasifree automorphism.

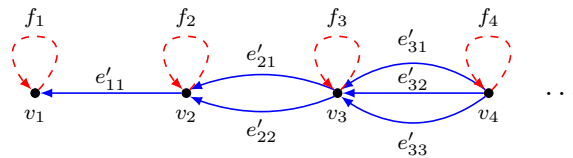
**Example 3.2.** Consider the infinite 1-graph  $\Lambda$  with vertices  $v_n$  for  $n \geq 1$  and edges  $e_{nj}$ ,  $n \geq 1, j = 1, \dots, n$ , where  $s(e_{nj}) = v_{n+1}$  and  $r(e_{nj}) = v_n$  for  $j = 1, \dots, n$ . Then  $C^*(\Lambda)$  is strongly Morita equivalent to the UHF-algebra with  $K_0$ -group isomorphic to  $\mathbb{Q}$ . Let  $\beta$  be the automorphism of  $\Lambda$  which fixes the vertices and cyclically permutes the edges between two successive vertices, i.e.

$$\beta(e_{nj}) = e_{n,j+1}, j = 1, \dots, n - 1, \beta(e_{nn}) = e_{n1}.$$

The crossed product graph  $\Lambda \times_{\beta} \mathbb{Z}$  has degree  $(1, 0)$  edges  $e'_{nj} = (e_{nj}, 0)$  for  $n \geq 1, j = 1, \dots, n$  and it has a degree  $(0, 1)$  loop  $f_n = (v_n, 1)$  at each vertex  $v_n$  (we identify  $(v_n, 0)$  with  $v_n$ ), with commuting squares

$$(14) \quad e'_{nj}f_{n+1} = f_n e'_{n,j+1}, \text{ for } j = 1, \dots, n - 1, \quad e'_{nn}f_{n+1} = f_n e'_{n1}.$$

It is a rank-2 Bratteli diagram (see [22, Definition 4.1]) with 1-skeleton shown below



By [22, Corollary 3.12, Theorem 4.3 and Lemma 4.8] the  $C^*$ -algebra

$$C^*(\Lambda \times_{\beta} \mathbb{Z})$$

is strongly Morita equivalent to an AT-algebra, the inductive limit of

$$C(\mathbb{T}) \rightarrow C(\mathbb{T}) \otimes M_{2!} \rightarrow C(\mathbb{T}) \otimes M_{3!} \rightarrow \dots$$

with  $K$ -theory groups isomorphic to  $\mathbb{Q}$ . Fix  $m \leq n$ . Then for all  $\alpha \in v_n(\Lambda \times_{\beta} \mathbb{Z})^{(m,0)}$  we have  $v_m(\Lambda \times_{\beta} \mathbb{Z})s(\alpha) \neq \emptyset$  and so  $\Lambda \times_{\beta} \mathbb{Z}$  is cofinal. Fix  $\alpha \in v_n(\Lambda \times_{\beta} \mathbb{Z})^{(N,0)}$ , then by repeated use of (14) it follows that  $o(\alpha) = n(n+1) \cdots (n+N)$  where  $o(\alpha)$  is the order of  $\alpha$  (see [22, §5]). It is then easy to see that  $\Lambda \times_{\beta} \mathbb{Z}$  has large-permutation factorisations (see [22, Definition 5.6]) and so  $C^*(\Lambda \times_{\beta} \mathbb{Z})$  is simple and has real-rank zero by [22, Theorem 5.7].

We now turn our attention to computing  $H^1(\Lambda \times_\beta \mathbb{Z}, A)$  for  $A$  an abelian group; first we compute  $H^1(\Lambda, A)$ . Observe that

$$\delta^0(f)(\lambda) = f(s(\lambda)) - f(r(\lambda))$$

where  $f \in C^0(\Lambda, A)$ ,  $\lambda \in \Lambda$  and  $\delta^1 : C^1(\Lambda, A) \rightarrow C^2(\Lambda, A)$  is the zero map. Hence  $H^1(\Lambda, A) = C^1(\Lambda, A)/\text{im } \delta^0$  and, identifying elements of  $C^1(\Lambda, A)$  with “double sequences”  $a = (a_j^n)_{n \geq 1, 1 \leq j \leq n}$  with  $a_j^n \in A$ , then  $H^1(\Lambda, A)$  may be identified with the set of equivalence classes of such elements where  $a \sim b$  if there are  $c_n \in A$  with  $b_j^n = a_j^n + c_n, n \geq 1, j = 1, \dots, n$ . It follows that  $H^1(\Lambda, A) \cong (\prod_{n \geq 1} A^n)/\sim$ .

To compute the cohomology  $H^1(\Lambda \times_\beta \mathbb{Z}, A)$ , observe that

$$\delta^1(\varphi)(e'_{nj} f_{n+1} - f_n e'_{n,j+1}) = \varphi(f_{n+1}) + \varphi(e'_{nj}) - \varphi(e'_{n,j+1}) - \varphi(f_n),$$

so for  $\varphi \in Z^1(\Lambda \times_\beta \mathbb{Z}, A)$  we have

$$\varphi(e'_{n,j+1}) - \varphi(e'_{nj}) = \varphi(f_{n+1}) - \varphi(f_n) = \varphi(e'_{n1}) - \varphi(e'_{nn}),$$

for  $n \geq 1, j = 1, \dots, n-1$ . Summing over  $j$  we obtain  $n(\varphi(f_{n+1}) - \varphi(f_n)) = 0$  for  $n \geq 1$ . If  $A$  is torsion free, then  $\varphi(f_n)$  is constant and therefore  $\varphi(e'_{nj})$  is constant. In this case,  $H^1(\Lambda \times_\beta \mathbb{Z}, A) \cong A$ . If  $A$  has torsion, then

$$H^1(\Lambda \times_\beta \mathbb{Z}, A) \cong A \times T_2(A) \times T_3(A) \times \dots,$$

where  $T_n(A)$  denotes the  $n$ -torsion subgroup of  $A$  for  $n \geq 2$ .

The last part of the long exact sequence

$$\dots \rightarrow H^1(\Lambda, A) \xrightarrow{1-\beta^*} H^1(\Lambda, A) \xrightarrow{j^*} H^2(\Lambda \times_\beta \mathbb{Z}, A) \rightarrow 0,$$

implies that

$$H^2(\Lambda \times_\beta \mathbb{Z}, A) \cong \text{coker}(1 - \beta^*).$$

Since  $\beta$  cyclically permutes the edges at each stage, the map

$$\beta^* : C^1(\Lambda, A) \rightarrow C^1(\Lambda, A)$$

is  $\prod \beta_n : \prod_{n \geq 1} A^n \rightarrow \prod_{n \geq 1} A^n$ , where

$$\beta_n : A^n \rightarrow A^n, \beta_n(a_1, \dots, a_n) = (a_n, a_1, \dots, a_{n-1}),$$

an automorphism of order  $n$ . Therefore, the map

$$1 - \beta^* : C^1(\Lambda, A) \rightarrow C^1(\Lambda, A)$$

is determined by

$$(1 - \beta_n)(a_1, \dots, a_n) = (a_1 - a_n, a_2 - a_1, \dots, a_n - a_{n-1}).$$

Since  $H^1(\Lambda, A) = C^1(\Lambda, A)/\text{im } \delta^0$ , we conclude that

$$\text{coker}(1 - \beta^*) \cong \left( \prod_{n \geq 1} A^n \right) / (\text{im}(1 - \beta^*) + \text{im } \delta^0).$$



First observe that

$$\left( \prod_{n \geq 1} A^n \right) / \text{im}(1 - \beta^*) \cong \prod_{n \geq 1} A$$

by the map  $(a_j^n) \mapsto (\sum_{j=1}^n a_j^n)$ . Since  $(a_j^n) \in \text{im } \delta^0$  iff there are  $c_n \in A$  with  $a_j^n = c_n$  for all  $j = 1, \dots, n$ , we conclude that  $\text{im } \delta^0$  in  $\prod_{n \geq 1} A$  is  $\prod_{n \geq 1} nA$ . It follows that

$$H^2(\Lambda \times_{\beta} \mathbb{Z}, A) \cong \left( \prod_{n \geq 1} A \right) / \text{im } \delta^1 \cong \prod_{n \geq 1} A/nA.$$

If  $A$  is divisible, in particular if  $A = \mathbb{T}$ , then  $H^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T}) \cong 0$ .

Given  $\varphi \in Z^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T})$ , both  $[\varphi]$  and  $[i^*(\varphi)]$  are trivial since both  $H^2(\Lambda \times_{\beta} \mathbb{Z}, \mathbb{T})$  and  $H^2(\Lambda, \mathbb{T})$  are trivial. It follows by Theorem 2.1 that

$$C^*(\Lambda) \rtimes_{\beta, \varphi} \mathbb{Z} \cong C^*_{\varphi}(\Lambda \times_{\beta} \mathbb{Z}) \cong C^*(\Lambda \times_{\beta} \mathbb{Z}).$$

**Example 3.3.** For  $n \geq 1$  let  $\underline{n} = \{0, \dots, n - 1\}$ . Define a bijection  $\theta : \underline{2} \times \underline{3} \rightarrow \underline{2} \times \underline{3}$  by  $\theta(i, j) = (i+1 \pmod{2}, j+1 \pmod{3})$ . Consider the 2-graph  $\mathbb{F}_{\theta}^+$  defined in [2]:  $\mathbb{F}_{\theta}^+$  is the unital semigroup generated by  $f_0, f_1, g_0, g_1, g_2$  subject to the relations  $f_i g_j = g_{j'} f_{i'}$  where  $\theta(i, j) = (i', j')$ ; that is,

$$(15) \quad \begin{aligned} f_0 g_0 &= g_1 f_1, & f_1 g_0 &= g_1 f_0, & f_0 g_1 &= g_2 f_1, \\ f_1 g_1 &= g_2 f_0, & f_0 g_2 &= g_0 f_1, & f_1 g_2 &= g_0 f_0, \end{aligned}$$

where the degree of  $f_0, f_1$  is  $e_1$  and the degree of  $g_0, g_1, g_2$  is  $e_2$ .

If  $\beta_1 = (01) \in S_2$  and  $\beta_2 = (012) \in S_3$ , then it is easy to check that  $\theta \circ (\beta_1 \times \beta_2) = (\beta_1 \times \beta_2) \circ \theta$  and so  $\beta = \beta_1 \times \beta_2$  induces an automorphism of  $\mathbb{F}_{\theta}^+$ .

Since  $\mathbb{F}_{\theta}^+$  has only one vertex  $v$ , it follows that  $v\mathbb{F}_{\theta}^+s(\alpha) \neq \emptyset$  for all  $\alpha \in \mathbb{F}_{\theta}^+$  and so  $\mathbb{F}_{\theta}^+$  is cofinal. Furthermore  $\mathbb{F}_{\theta}^+$  is aperiodic by [3, Corollary 3.2] since  $\log 3$  and  $\log 2$  are rationally independent. Hence  $C^*(\mathbb{F}_{\theta}^+)$  is simple by [27, Theorem 3.1]. Moreover, the loop  $f_0$  has an entrance  $f_1 g_0$  and since there is only one vertex, it follows by [28, Proposition 8.8] that  $C^*(\mathbb{F}_{\theta}^+)$  is purely infinite. By [6, Proposition 3.16] it follows that

$$K_0(C^*(\mathbb{F}_{\theta}^+)) = K_1(C^*(\mathbb{F}_{\theta}^+)) = 0$$

since  $M_1 = [2]$  and  $M_2 = [3]$ . Since  $C^*(\mathbb{F}_{\theta}^+)$  is a Kirchberg algebra,  $C^*(\mathbb{F}_{\theta}^+) \cong \mathcal{O}_2$  by the Kirchberg–Phillips Theorem [11, 24].

To compute the cohomology of  $\Lambda = \mathbb{F}_{\theta}^+$  and of  $\Lambda \times_{\beta} \mathbb{Z}$ , we first compute the homology  $H_n(\Lambda)$ . Next we determine the maps  $\beta_* : H_n(\Lambda) \rightarrow H_n(\Lambda)$  induced by the automorphism  $\beta$  and then use the exact sequence (see [16, Theorem 4.13]) to compute  $H_n(\Lambda \times_{\beta} \mathbb{Z})$ . Thereafter, we apply the Universal

Coefficient Theorem (see [16, Theorem 7.3])

$$0 \rightarrow \text{Ext}(H_{n-1}(\Lambda \times_{\beta} \mathbb{Z}), A) \rightarrow H^n(\Lambda \times_{\beta} \mathbb{Z}, A) \\ \rightarrow \text{Hom}(H_n(\Lambda \times_{\beta} \mathbb{Z}), A) \rightarrow 0$$

to compute  $H^n(\Lambda \times_{\beta} \mathbb{Z}, A)$ . We have

$$Q_0(\Lambda) = \{v\}, \\ Q_1(\Lambda) = \{f_0, f_1, g_0, g_1, g_2\}, \\ Q_2(\Lambda) = \{f_0g_0, f_0g_1, f_0g_2, f_1g_0, f_1g_1, f_1g_2\},$$

from (15). Moreover, we have  $\partial_0 = \partial_1 = 0$  and

$$\partial_2(f_0g_0) = f_0 + g_0 - g_1 - f_1, \quad \partial_2(f_1g_0) = f_1 + g_0 - g_1 - f_0, \\ \partial_2(f_0g_1) = f_0 + g_1 - g_2 - f_1, \quad \partial_2(f_1g_1) = f_1 + g_1 - g_2 - f_0, \\ \partial_2(f_0g_2) = f_0 + g_2 - g_0 - f_1, \quad \partial_2(f_1g_2) = f_1 + g_2 - g_0 - f_0.$$

Using the Smith normal form of  $\partial_2$  we see that  $\ker \partial_2$  has generators

$$b_1 = f_0g_0 + f_1g_0 + 2f_1g_1 + 2f_0g_2, \\ b_2 = -2f_0g_0 + f_0g_1 - 3f_1g_1 - 2f_0g_2, \\ b_3 = 2f_0g_0 + 2f_1g_1 + f_0g_2 + f_1g_2$$

and  $\text{im } \partial_2$  has generators

$$f_0 - f_1 + g_0 - g_1, \quad g_0 - g_2, \quad g_1 - g_2.$$

It follows that

$$H_0(\Lambda) = \mathbb{Z}, \quad H_1(\Lambda) = \mathbb{Z}^5 / \text{im } \partial_2 \cong \mathbb{Z}^2, \quad H_2(\Lambda) = \ker \partial_2 \cong \mathbb{Z}^3.$$

Since  $\beta(f_0) = f_1, \beta(f_1) = f_0, \beta(g_0) = g_1, \beta(g_1) = g_2, \beta(g_2) = g_0$  and  $H_1(\Lambda)$  is generated by  $[f_0], [g_0]$ , we see that  $\beta_* : H_1(\Lambda) \rightarrow H_1(\Lambda)$  is the identity map.

Since  $\beta(b_1) = 2b_1 + b_2, \beta(b_2) = -2b_1 + b_3, \beta(b_3) = b_1$ , it follows that  $\beta_* : H_2(\Lambda) \rightarrow H_2(\Lambda)$  has matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and  $\ker(1 - \beta_*) \cong \mathbb{Z}, \text{im}(1 - \beta_*) \cong \mathbb{Z}^2$ .

Then by the long exact sequence of homology (see [16, Theorem 4.13]) we obtain

$$H_1(\Lambda \times_{\beta} \mathbb{Z}) \cong \mathbb{Z}^3, \quad H_2(\Lambda \times_{\beta} \mathbb{Z}) \cong \mathbb{Z}^3, \quad H_3(\Lambda \times_{\beta} \mathbb{Z}) \cong \mathbb{Z}.$$

It follows that

$$H^0(\Lambda, A) \cong A, \quad H^1(\Lambda, A) \cong A^2, \quad H^2(\Lambda, A) \cong A^3.$$

and

$$H^1(\Lambda \times_{\beta} \mathbb{Z}, A) \cong A^3, \quad H^2(\Lambda \times_{\beta} \mathbb{Z}, A) \cong A^3, \quad H^3(\Lambda \times_{\beta} \mathbb{Z}, A) \cong A.$$

We construct a representation of  $C^*(\mathbb{F}_\theta^+)$  on the Hilbert space  $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$  with basis  $\{e_{i,j,k}\}_{i,k \geq 1, j=0,1}$  by taking

$$\begin{aligned} S_{f_0}(e_{i,0,k}) &= e_{2i-1,0,k}, & S_{f_0}(e_{i,1,k}) &= e_{2i,1,k}, \\ S_{f_1}(e_{i,0,k}) &= e_{2i,0,k}, & S_{f_1}(e_{i,1,k}) &= e_{2i-1,1,k}, \end{aligned}$$

and

$$T_{g_0}(e_{i,j,k}) = e_{i,1-j,3k-2}, \quad T_{g_1}(e_{i,j,k}) = e_{i,1-j,3k-1}, \quad T_{g_2}(e_{i,j,k}) = e_{i,1-j,3k}.$$

Then  $S_{f_0}, S_{f_1}, T_{g_0}, T_{g_1}, T_{g_2}$  are isometries and

$$S_{f_0}S_{f_0}^* + S_{f_1}S_{f_1}^* = T_{g_0}T_{g_0}^* + T_{g_1}T_{g_1}^* + T_{g_2}T_{g_2}^* = I.$$

The commutation relations (15) are also satisfied, for example

$$(S_{f_0}T_{g_2})(e_{i,0,k}) = S_{f_0}(e_{i,1,3k}) = e_{2i,1,3k}$$

and

$$(S_{f_0}T_{g_2})(e_{i,1,k}) = S_{f_0}(e_{i,0,3k}) = e_{2i-1,0,3k}.$$

Fix  $z, w \in \mathbb{T}$  and let  $c = c(z, w) \in Z^1(\mathbb{F}_\theta^+, \mathbb{T})$  be such that

$$c(f_0) = c(f_1) = z, \quad c(g_i) = w,$$

for  $i = 0, 1, 2$ . Suppose that  $z^n, w^n \neq 1$  for all  $n$ . As in Corollary 2.3 let  $\kappa = \beta^c$  be the automorphism of  $C^*(\mathbb{F}_\theta^+)$  such that

$$\kappa(S_{f_i}) = zS_{f_{\beta_1(i)}} \text{ for } i = 0, 1 \quad \text{and} \quad \kappa(T_{g_i}) = wT_{g_{\beta_2(i)}} \text{ for } i = 0, 1, 2.$$

We claim that  $\kappa^n$  is outer for all  $n$ . To prove this we follow the technique of [1, 5, 4].

Fix  $n$ . For contradiction assume there is a unitary  $V \in C^*(\mathbb{F}_\theta^+)$  such that  $\kappa^n = AdV$ . We have  $S_{f_0}e_{1,0,1} = e_{1,0,1}$  and

$$\kappa^n(S_{f_0})Ve_{1,0,1} = VS_{f_0}V^*Ve_{1,0,1} = Ve_{1,0,1} = \sum c_{i,j,k}e_{i,j,k} \neq 0.$$

On the other hand if  $n$  is odd then

$$\begin{aligned} \kappa^n(S_{f_0})Ve_{1,0,1} &= z^n S_{f_1} \sum c_{i,j,k}e_{i,j,k} \\ &= z^n \sum_{i,k=1}^{\infty} c_{i,0,k}e_{2i,0,k} + z^n \sum_{i,k=1}^{\infty} c_{i,1,k}e_{2i-1,1,k}. \end{aligned}$$

Identifying coefficients, we get  $c_{2i-1,0,k} = c_{2i,1,k} = 0$ ,  $c_{2i,0,k} = z^n c_{i,0,k}$  and  $c_{2i-1,1,k} = z^n c_{i,1,k}$ ; since  $z^n \neq 1$  for all  $n$  it follows by induction that  $c_{i,j,k} = 0$  for all  $i, j, k$ . If  $n$  is even a similar argument applies.

Similarly, consider  $x = \frac{1}{2}(e_{1,0,1} + e_{1,1,1})$ . Then  $T_{g_0}x = x$  and

$$\kappa^n(T_{g_0})Vx = VT_{g_0}V^*Vx = Vx = \sum_{i,k=1}^{\infty} x_{i,j,k}e_{i,j,k} \neq 0.$$

On the other hand if  $n$  is congruent to 1 mod 3 then we have

$$\begin{aligned}\beta(T_{g_0})Vx &= w^n T_{g_1} \sum_{i,k=1}^{\infty} x_{i,j,k} e_{i,j,k} \\ &= w^n \sum_{i,k=1}^{\infty} x_{i,0,k} e_{i,1,3k-1} + w^n \sum_{i,k=1}^{\infty} x_{i,1,k} e_{i,0,3k-1}.\end{aligned}$$

Identifying coefficients,

$$x_{i,0,3k} = x_{i,0,3k-2} = x_{i,1,3k} = x_{i,1,3k-2} = 0,$$

$x_{i,0,3k-1} = w^n x_{i,1,k}$ ,  $x_{i,1,3k-1} = w^n x_{i,0,k}$  and  $x_{i,0,2} = w^n x_{i,1,1} = 0$ . Similarly  $x_{i,1,2} = w^n x_{i,0,1} = 0$ ; since  $w^n \neq 1$  for all  $n$  it follows by induction that  $c_{i,j,k} = 0$  for all  $i, j, k$ . Other cases for  $n$  follow in a similar manner. Together they give a contradiction, so  $\kappa^n$  is outer for all  $n$ . Hence by [13, Lemma 10] and Corollary 2.3 it follows that  $C^*(\mathbb{F}_\theta^+) \rtimes_{\beta^c} \mathbb{Z} \cong C_{j^*(c)}(\mathbb{F}_\theta^+ \times_\beta \mathbb{Z})$  is simple and purely infinite for all  $c = c(z, w) \in Z^1(\mathbb{F}_\theta^+, \mathbb{T})$  such that  $z^n, w^n \neq 1$  for all  $n$ . For such  $c$  the Pimsner–Voiculescu six-term exact sequence and classification results show that  $C^*(\mathbb{F}_\theta^+) \rtimes_{\beta^c} \mathbb{Z} \cong \mathcal{O}_2$ .

## References

- [1] ARCHBOLD, ROBERT J. On the “flip-flop” automorphism of  $C^*(S_1, S_2)$ . *Quart. J. Math. Oxford Ser. (2)* **30** (1979), no. 118, 129–132. MR0534827, Zbl 0405.46051, doi:10.1093/qmath/30.2.129.
- [2] DAVIDSON, KENNETH R.; POWER, STEPHEN C.; YANG, DILIAN. Atomic representations of rank 2 graph algebras. *J. Funct. Anal.* **255** (2008), no. 4, 819–853. MR2433954, Zbl 1154.46030, arXiv:0705.4498, doi:10.1016/j.jfa.2008.05.008.
- [3] DAVIDSON, KENNETH R.; YANG, DILIAN. Periodicity in rank 2 graph algebras. *Canad. J. Math.* **61** (2009), no. 6, 1239–1261. MR2588421, Zbl 1177.47087, arXiv:0705.4499, doi:10.4153/CJM-2009-058-0.
- [4] ENOMOTO, MASATOSHI; FUJII, MASATOSHI; TAKEHANA, HIROAKI; WATATANI, YASUO. Automorphisms on Cuntz algebras. II. *Math. Japon.* **24** (1979/80), no. 4, 463–468. MR0557480, Zbl 0419.46043.
- [5] ENOMOTO, MASATOSHI; TAKEHANA, HIROAKI; WATATANI, YASUO. Automorphisms on Cuntz algebras. *Math. Japon.* **24** (1979/80), no. 2, 231–234. MR0550001, Zbl 0418.46053.
- [6] EVANS, D. GWION. On the  $K$ -theory of higher-rank graph  $C^*$ -algebras. *New York J. Math.* **14** (2008), 1–31. MR2383584, Zbl 1146.46048, arXiv:math/0406458.
- [7] EVANS, DAVID E. On  $O_n$ . *Publ. Res. Inst. Math. Sci.* **16** (1980), 915–927. MR0602475, Zbl 0461.46042, doi:10.2977/prims/1195186936.
- [8] FARTHING, CYNTHIA; PASK, DAVID; SIMS, AIDAN. Crossed products of  $k$ -graph  $C^*$ -algebras by  $\mathbb{Z}^l$ . *Houston J. Math.* **35** (2009), no. 3, 903–933. MR2534288, Zbl 1186.46056, arXiv:0706.3547.
- [9] HATCHER, ALLEN. Algebraic topology. *Cambridge University Press, Cambridge*, 2002. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0. MR1867354, Zbl 1044.55001.
- [10] KATSURA, TAKESHI. The ideal structures of crossed products of Cuntz algebras by quasi-free actions of abelian groups. *Canad. J. Math.* **55** (2003), no. 6, 1302–1338. MR2016248, Zbl 1068.46041, arXiv:math/0108047, doi:10.4153/CJM-2003-050-6.

- [11] KIRCHBERG, EBERHARD. The classification of purely infinite  $C^*$ -algebras using Kasparov's theory. Preprint, 1994.
- [12] KISHIMOTO, AKITAKA. Simple crossed products of  $C^*$ -algebras by locally compact abelian groups. *Yokohama Math. J.* **28** (1980), no. 1–2, 69–85. MR0623751, Zbl 0467.46042.
- [13] KISHIMOTO, AKITAKA; KUMJIAN, ALEXANDER. Crossed products of Cuntz algebras by quasi-free automorphisms. *Operator algebras and their applications* (Waterloo, ON, 1994/1995), 173–192, Fields Inst. Commun., **13**. Amer. Math. Soc., Providence, RI, 1997. MR1424962, Zbl 0951.46039.
- [14] KUMJIAN, ALEX; PASK, DAVID. Higher rank graph  $C^*$ -algebras. *New York J. Math.* **6** (2000), 1–20. MR1745529, Zbl 0946.46044, arXiv:math/0007029.
- [15] KUMJIAN, ALEX; PASK, DAVID; SIMS, AIDAN. Generalised morphisms of  $k$ -graphs:  $k$ -morphs. *Trans. Amer. Math. Soc.* **363** (2011), 2599–2626. MR2763728, Zbl 1222.46041, arXiv:0712.1072, doi: 10.1090/S0002-9947-2010-05152-9.
- [16] KUMJIAN, ALEX; PASK, DAVID; SIMS, AIDAN. Homology for higher-rank graphs and twisted  $C^*$ -algebras. *J. Funct. Anal.* **263** (2012), no. 6, 1539–1574. MR2948223, Zbl 1253.55006, arXiv:1110.1433, doi: 10.1016/j.jfa.2012.05.023.
- [17] KUMJIAN, ALEX; PASK, DAVID; SIMS, AIDAN. On twisted higher-rank graph  $C^*$ -algebras. *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 5177–5216. MR3335414, Zbl 06429159, arXiv:1112.6233v1, doi: 10.1090/S0002-9947-2015-06209-6.
- [18] LEWIN, PETER; SIMS, AIDAN. Aperiodicity and cofinality for finitely aligned higher-rank graphs. *Math. Proc. Cambridge Philosophical Soc.* **149** (2010), no. 2, 333–350. MR2670219, Zbl 1213.46047, arXiv:0905.0735, doi: 10.1017/S0305004110000034.
- [19] MACLANE, SAUNDERS. Homology. Die Grundlehren der mathematischen Wissenschaften, 114. Springer-Verlag, Berlin-Heidelberg-New York, 1967. x+422 pp. MR0349792, Zbl 0328.18009.
- [20] MALONEY, BEN; PASK, DAVID; RAEBURN, IAIN. Skew-products of higher-rank graphs and crossed products by semigroups. *Semigroup Forum* **88** (2014), no. 1, 162–176. MR3164157, Zbl 1296.46055, arXiv:1203.4860, doi: 10.1007/s00233-013-9525-4.
- [21] PASK, DAVID; QUIGG, JOHN; RAEBURN, IAIN. Fundamental groupoids of  $k$ -graphs. *New York J. Math.* **10** (2004), 195–207. MR2114786, Zbl 1064.05073, arXiv:math/0401016.
- [22] PASK, DAVID; RAEBURN, IAIN; RØRDAM, MIKAEL; SIMS, AIDAN. Rank-two graphs whose  $C^*$ -algebras are direct limits of circle algebras. *J. Funct. Anal.* **239** (2006), no. 1, 137–178. MR2258220, Zbl 1112.46042, arXiv:math/0512254, doi: 10.1016/j.jfa.2006.04.003.
- [23] PASK, DAVID; RAEBURN, IAIN; WEAVER, NATASHA A. Periodic 2-graphs arising from subshifts. *Bull. Aust. Math. Soc.* **82** (2010), no. 1, 120–138. MR2669150, Zbl 1200.46060, arXiv:0911.0730, doi: 10.1017/S0004972709001269.
- [24] PHILLIPS, N. CHRISTOPHER. A classification theorem for nuclear purely infinite simple  $C^*$ -algebras. *Doc. Math.* **5** (2000), 49–114. MR1745197, Zbl 0943.46037, arXiv:funct-an/9506010.
- [25] RAEBURN, IAIN; SIMS, AIDAN; YEEND, TRENT. Higher-rank graphs and their  $C^*$ -algebras. *Proc. Edinb. Math. Soc.* (2) **46** (2003), no. 1, 99–115. MR1961175, Zbl 1031.46061, arXiv:math/0107222, doi: 10.1017/S0013091501000645.
- [26] RENAULT, JEAN. A groupoid approach to  $C^*$ -algebras. Lecture Notes in Mathematics, 793. Springer, Berlin, 1980. ii+160 pp. ISBN: 3-540-09977-8. MR0584266, Zbl 0433.46049.
- [27] ROBERTSON, DAVID I.; SIMS, AIDAN. Simplicity of  $C^*$ -algebras associated to higher-rank graphs. *Bull. London Math. Soc.* **39** (2007), no. 2, 337–344. MR2323468, Zbl 1125.46045, arXiv:math/0602120, doi: 10.1112/blms/bdm006.

- [28] SIMS, AIDAN. Gauge-invariant ideals in the  $C^*$ -algebras of finitely aligned higher-rank graphs. *Canad. J. Math.* **58** (2006), no. 6, 1268–1290. MR2270926, Zbl 1115.46050, arXiv:math/0406592, doi: 10.4153/CJM-2006-045-2.
- [29] WESTMAN, JOEL J. Cohomology for ergodic groupoids. *Trans. Amer. Math. Soc.* **146** (1969), 465–471. MR0255771, Zbl 0271.18013, doi: 10.2307/1995186.

(Nathan Brownlowe) SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, THE UNIVERSITY OF WOLLONGONG, NSW 2522, AUSTRALIA  
[nathanb@uow.edu.au](mailto:nathanb@uow.edu.au)

(Valentin Deaconu) DEPARTMENT OF MATHEMATICS (084), UNIVERSITY OF NEVADA, RENO NV 89557-0084, USA  
[vdeaconu@unr.edu](mailto:vdeaconu@unr.edu)

(Alex Kumjian) DEPARTMENT OF MATHEMATICS (084), UNIVERSITY OF NEVADA, RENO NV 89557-0084, USA  
[alex@unr.edu](mailto:alex@unr.edu)

(David Pask) SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, THE UNIVERSITY OF WOLLONGONG, NSW 2522, AUSTRALIA  
[dpask@uow.edu.au](mailto:dpask@uow.edu.au)

This paper is available via <http://nyjm.albany.edu/j/2016/22-10.html>.