

# The choice of cofibrations of higher dimensional transition systems

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ABSTRACT. It is proved that there exists a left determined model structure of weak transition systems with respect to the class of monomorphisms and that it restricts to left determined model structures on cubical and regular transition systems. Then it is proved that, in these three model structures, for any higher dimensional transition system containing at least one transition, the fibrant replacement contains a transition between each pair of states. This means that the fibrant replacement functor does not preserve the causal structure. As a conclusion, we explain why working with star-shaped transition systems is a solution to this problem.

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## 1. Introduction

**1.1. Summary of the paper.** This work belongs to our series of papers devoted to *higher dimensional transition systems*. It is a (long) work in progress. The notion of higher dimensional transition system dates back to Cattani–Sassone’s paper [CS96]. These objects are a higher dimensional analogue of the computer-scientific notion of labelled transition system. Their purpose is to model the concurrent execution of  $n$  actions by a multiset of actions, i.e., a set with a possible repetition of some elements (e.g.,  $\{0, 0, 2, 3, 3, 3\}$ ). The higher dimensional transition  $a||b$  modeling the concurrent execution of the two actions  $a$  and  $b$ , depicted by Figure 1, consists of

Received September 1, 2015.

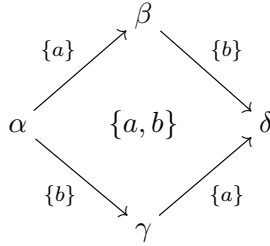
2010 *Mathematics Subject Classification*. 18C35, 18G55, 55U35, 68Q85.

*Key words and phrases*. left determined model category, combinatorial model category, higher dimensional transition system, causal structure.

the transitions  $(\alpha, \{a\}, \beta)$ ,  $(\beta, \{b\}, \delta)$ ,  $(\alpha, \{b\}, \gamma)$ ,  $(\gamma, \{a\}, \delta)$  and  $(\alpha, \{a, b\}, \delta)$ . The labelling map is the identity map. Note that with  $a = b$ , we would get the 2-dimensional transition  $(\alpha, \{a, a\}, \delta)$  which is not equal to the 1-dimensional transition  $(\alpha, \{a\}, \delta)$ . The latter actually does not exist in Figure 1. Indeed, the only 1-dimensional transitions labelled by the multiset  $\{a\}$  are  $(\alpha, \{a\}, \beta)$  and  $(\gamma, \{a\}, \delta)$ . The new formulation introduced in [Gau10] enabled us to interpret them as a small-orthogonality class of a locally finitely presentable category  $\mathcal{WTS}$  of *weak transition systems* equipped with a topological functor towards a power of the category of sets. In this new setting, the 2-dimensional transition of Figure 1 becomes the tuple  $(\alpha, a, b, \delta)$ . The set of transitions has therefore to satisfy the multiset axiom (here: if the tuple  $(\alpha, a, b, \delta)$  is a transition, then the tuple  $(\alpha, b, a, \delta)$  has to be a transition as well) and the patching axiom which is a topological version (in the sense of topological functors) of Cattani–Sassone’s interleaving axiom. We were then able to state a categorical comparison theorem between them and (labelled) symmetric precubical sets in [Gau10]. We studied in [Gau11] a homotopy theory of *cubical transition systems*  $\mathcal{CTS}$  and in [Gau15a], exhaustively, a homotopy theory of *regular transition systems*  $\mathcal{RTS}$ . The adjective cubical means that the weak transition system is the union of its subcubes. In particular this means that every higher dimensional transition has lower dimensional faces. However, a square for example may still have more than four 1-dimensional faces in the category of cubical transition systems. A cubical transition system is by definition regular if every higher dimensional transition has the expected number of faces. All known examples coming from process algebra are cubical because they are colimits of cubes, and therefore are equal to the union of their subcubes. Indeed, the associated higher dimensional transition systems are realizations in the sense of [Gau10, Theorem 9.2] (see also [Gau14, Theorem 7.4]) of a labelled precubical set obtained by following the semantics expounded in [Gau08]. It turns out that there exist colimits of cubes which are not regular (see the end of [Gau15a, Section 2]). However, it can also be proved that all process algebras for any synchronization algebra give rise to regular transition systems. The regular transition systems seem to be the only interesting ones. However, their mathematical study requires to use the whole chain of inclusion functors  $\mathcal{RTS} \subset \mathcal{CTS} \subset \mathcal{WTS}$ .

The homotopy theories studied in [Gau11] and in [Gau15a] are obtained by starting from a left determined model structure on weak transition systems with respect to the class of maps of weak transition systems which are one-to-one on the set of actions (but not necessarily one-to-one on the set of states) and then by restricting it to full subcategories (the coreflective subcategory of cubical transition systems, and then the reflective subcategory of regular ones).

In this paper, we will start from the left determined model category of weak transition systems with respect to the class of monomorphisms of weak

FIGURE 1.  $a \parallel b$  : Concurrent execution of  $a$  and  $b$ 

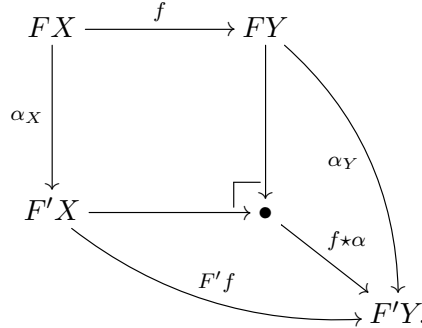
transition systems, i.e., the cofibrations are one-to-one not only on the set of actions, but also on the set of states. Indeed, it turns out that such a model structure exists: it is the first result of this paper (Theorem 2.19). And it turns out that it restricts to the full subcategories of cubical and regular transition systems as well and that it gives rise to two new left determined model structures: it is the second result of this paper (Theorem 3.3 for cubical transition systems and Theorem 3.16 for regular transition systems).

Unlike the homotopy structures studied in [Gau11] and in [Gau15a], the model structures of this paper do not have the map  $R : \{0, 1\} \rightarrow \{0\}$  identifying two states as a cofibration anymore. However, there are still cofibrations of regular transition systems which identify two different states. This is due to the fact that the set of states of a colimit of regular transition systems is in general not the colimit of the sets of states. There are identifications inside the set of states which are forced by the axioms satisfied by regular transition systems, actually CSA2. This implies that the class of cofibrations of this new left determined model structure on regular transition systems, like the one described and studied in [Gau15a], still contains cofibrations which are not monic: see an example at the very end of Section 3.

Without additional constructions, these new model structures are irrelevant for concurrency theory. Indeed, the fibrant replacement functor, in any of these model categories (the weak transition systems and also the cubical and the regular ones), destroys the causal structure of the higher dimensional transition system: this is the third result of this paper (Theorem 4.1 and Theorem 4.2).

We open this new line of research anyway because of the following discovery: by working with star-shaped transition systems, the bad behavior of the fibrant replacement just disappears. This point is discussed in the very last section of the paper. The fourth result of this paper is that left determined model structures can be constructed on star-shaped (weak or cubical or regular) transition systems (Theorem 5.10). This paper is the starting point of the study of these new homotopy theories.

Appendix A is a technical tool to relocate the map  $R : \{0, 1\} \rightarrow \{0\}$  in a transfinite composition. Even if this map is not a cofibration in this paper,

FIGURE 2. Definition of  $f \star \alpha$ .

it still plays an important role in the proofs. This map seems to play an ubiquitous role in our homotopy theories.

**1.2. Prerequisites and notations.** All categories are locally small. The set of maps in a category  $\mathcal{K}$  from  $X$  to  $Y$  is denoted by  $\mathcal{K}(X, Y)$ . The class of maps of a category  $\mathcal{K}$  is denoted by  $\text{Mor}(\mathcal{K})$ . The composite of two maps is denoted by  $fg$  instead of  $f \circ g$ . The initial (final resp.) object, if it exists, is always denoted by  $\emptyset$  ( $\mathbf{1}$  resp.). The identity of an object  $X$  is denoted by  $\text{Id}_X$ . A subcategory is always isomorphism-closed. Let  $f$  and  $g$  be two maps of a locally presentable category  $\mathcal{K}$ . Write  $f \square g$  when  $f$  satisfies the *left lifting property* (LLP) with respect to  $g$ , or equivalently  $g$  satisfies the *right lifting property* (RLP) with respect to  $f$ . Let us introduce the notations  $\mathbf{inj}_{\mathcal{K}}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f \square g\}$  and  $\mathbf{cof}_{\mathcal{K}}(\mathcal{C}) = \{f \in \mathcal{K}, \forall g \in \mathbf{inj}_{\mathcal{K}}(\mathcal{C}), f \square g\}$  where  $\mathcal{C}$  is a class of maps of  $\mathcal{K}$ . The class of morphisms of  $\mathcal{K}$  that are transfinite compositions of pushouts of elements of  $\mathcal{C}$  is denoted by  $\mathbf{cell}_{\mathcal{K}}(\mathcal{C})$ . There is the inclusion  $\mathbf{cell}_{\mathcal{K}}(\mathcal{C}) \subset \mathbf{cof}_{\mathcal{K}}(\mathcal{C})$ . Moreover, every morphism of  $\mathbf{cof}_{\mathcal{K}}(\mathcal{C})$  is a retract of a morphism of  $\mathbf{cell}_{\mathcal{K}}(\mathcal{C})$  as soon as the domains of  $\mathcal{C}$  are small relative to  $\mathbf{cell}_{\mathcal{K}}(\mathcal{C})$  [Hov99, Corollary 2.1.15], e.g., when  $\mathcal{K}$  is locally presentable. A class of maps of  $\mathcal{K}$  is cofibrantly generated if it is of the form  $\mathbf{cof}_{\mathcal{K}}(S)$  for some set  $S$  of maps of  $\mathcal{K}$ . For every map  $f : X \rightarrow Y$  and every natural transformation  $\alpha : F \rightarrow F'$  between two endofunctors of  $\mathcal{K}$ , the map  $f \star \alpha$  is defined by the diagram of Figure 2. For a set of morphisms  $\mathcal{A}$ , let  $\mathcal{A} \star \alpha = \{f \star \alpha, f \in \mathcal{A}\}$ .

We refer to [AR94] for locally presentable categories, to [Ros09] for combinatorial model categories, and to [AHS06] for topological categories, i.e., categories equipped with a topological functor towards a power of the category of sets. We refer to [Hov99] and to [Hir03] for model categories. For general facts about weak factorization systems, see also [KR05]. The reading of the first part of [Ols09a], published in [Ols09b], is recommended for any reference about good, cartesian, and very good cylinders.

We use the paper [Gau15b] as a toolbox for constructing the model structures. To keep this paper short, we refer to [Gau15b] for all notions related to Olschok model categories.

## 2. The model structure of weak transition systems

We are going first to recall a few facts about weak transition systems.

**2.1. Notation.** Let  $\Sigma$  be a fixed nonempty set of labels.

**2.2. Definition.** A weak transition system consists of a triple

$$X = \left( S, \mu : L \rightarrow \Sigma, T = \bigcup_{n \geq 1} T_n \right)$$

where  $S$  is a set of states, where  $L$  is a set of actions, where  $\mu : L \rightarrow \Sigma$  is a set map called the labelling map, and finally where  $T_n \subset S \times L^n \times S$  for  $n \geq 1$  is a set of  $n$ -transitions or  $n$ -dimensional transitions such that the two following axioms hold:

- (Multiset axiom). For every permutation  $\sigma$  of  $\{1, \dots, n\}$  with  $n \geq 2$ , if the tuple  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition, then the tuple

$$(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$$

is a transition as well.

- (Patching axiom<sup>1</sup>). For every  $(n + 2)$ -tuple  $(\alpha, u_1, \dots, u_n, \beta)$  with  $n \geq 3$ , for every  $p, q \geq 1$  with  $p + q < n$ , if the five tuples

$$(\alpha, u_1, \dots, u_n, \beta),$$

$$(\alpha, u_1, \dots, u_p, \nu_1), (\nu_1, u_{p+1}, \dots, u_n, \beta),$$

$$(\alpha, u_1, \dots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \dots, u_n, \beta)$$

are transitions, then the  $(q + 2)$ -tuple  $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$  is a transition as well.

A map of weak transition systems

$$f : (S, \mu : L \rightarrow \Sigma, (T_n)_{n \geq 1}) \rightarrow (S', \mu' : L' \rightarrow \Sigma, (T'_n)_{n \geq 1})$$

consists of a set map  $f_0 : S \rightarrow S'$ , a commutative square

$$\begin{array}{ccc} L & \xrightarrow{\mu} & \Sigma \\ \tilde{f} \downarrow & & \parallel \\ L' & \xrightarrow{\mu'} & \Sigma \end{array}$$

<sup>1</sup>This axiom is called the Coherence axiom in [Gau10] and [Gau11], and the composition axiom in [Gau15a]. I definitively adopted the terminology “patching axiom” after reading the Web page in nLab devoted to higher dimensional transition systems and written by Tim Porter.

such that if  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition, then

$$(f_0(\alpha), \tilde{f}(u_1), \dots, \tilde{f}(u_n), f_0(\beta))$$

is a transition. The corresponding category is denoted by  $\mathcal{WTS}$ . The  $n$ -transition  $(\alpha, u_1, \dots, u_n, \beta)$  is also called a transition from  $\alpha$  to  $\beta$ :  $\alpha$  is the initial state and  $\beta$  the final state of the transition. The maps  $f_0$  and  $\tilde{f}$  are sometimes denoted simply as  $f$ .

The category  $\mathcal{WTS}$  is locally finitely presentable and the functor

$$\omega : \mathcal{WTS} \longrightarrow \mathbf{Set}^{\{s\} \cup \Sigma},$$

where  $s$  is the sort of states, taking the weak higher dimensional transition system  $(S, \mu : L \rightarrow \Sigma, (T_n)_{n \geq 1})$  to the  $(\{s\} \cup \Sigma)$ -tuple of sets

$$(S, (\mu^{-1}(x))_{x \in \Sigma}) \in \mathbf{Set}^{\{s\} \cup \Sigma}$$

is topological by [Gau10, Theorem 3.4]. The terminal object of  $\mathcal{WTS}$  is the weak transition system

$$\mathbf{1} = (\{0\}, \text{Id}_\Sigma : \Sigma \rightarrow \Sigma, \bigcup_{n \geq 1} \{0\} \times \Sigma^n \times \{0\}).$$

**2.3. Notation.** For  $n \geq 1$ , let  $0_n = (0, \dots, 0)$  ( $n$  times) and  $1_n = (1, \dots, 1)$  ( $n$  times). By convention, let  $0_0 = 1_0 = ()$ .

Here are some important examples of weak transition systems:

- (1) Every set  $S$  can be identified with the weak transition system having the set of states  $S$ , with no actions and no transitions. For all weak transition system  $X$ , the set  $\mathcal{WTS}(\{0\}, X)$  is the set of states of  $X$ . The empty set is the initial object of  $\mathcal{WTS}$ .
- (2) The weak transition system  $\underline{x} = (\emptyset, \{x\} \subset \Sigma, \emptyset)$  for  $x \in \Sigma$ . For all weak transition system  $X$ , the set  $\mathcal{WTS}(\underline{x}, X)$  is the set of actions of  $X$  labelled by  $x$  and  $\bigsqcup_{x \in \Sigma} \mathcal{WTS}(\underline{x}, X)$  is the set of actions of  $X$ .
- (3) Let  $n \geq 0$ . Let  $x_1, \dots, x_n \in \Sigma$ . The *pure  $n$ -transition*

$$C_n[x_1, \dots, x_n]^{\text{ext}}$$

is the weak transition system with the set of states  $\{0_n, 1_n\}$ , with the set of actions

$$\{(x_1, 1), \dots, (x_n, n)\}$$

and with the transitions all  $(n + 2)$ -tuples

$$(0_n, (x_{\sigma(1)}, \sigma(1)), \dots, (x_{\sigma(n)}, \sigma(n)), 1_n)$$

for  $\sigma$  running over the set of permutations of the set  $\{1, \dots, n\}$ . Intuitively, the pure transition is a cube without faces of lower dimension. For all weak transition system  $X$ , the set  $\mathcal{WTS}(C_n[x_1, \dots, x_n]^{\text{ext}}, X)$

is the set of transitions  $(\alpha, u_1, \dots, u_n, \beta)$  of  $X$  such that for all  $1 \leq i \leq n$ ,  $\mu(u_i) = x_i$  and

$$\bigsqcup_{x_1, \dots, x_n \in \Sigma} WTS(C_n[x_1, \dots, x_n]^{\text{ext}}, X)$$

is the set of transitions of  $X$ .

The purpose of this section is to prove the existence of a left determined combinatorial model structure on the category of weak transition systems with respect to the class of monomorphisms.

We first have to check that the class of monomorphisms of weak transition systems is generated by a set. The set of generating cofibrations is obtained by removing the map  $R : \{0, 1\} \rightarrow \{0\}$  from the set of generating cofibrations of the model structure studied in [Gau11] and in [Gau15a].

**2.4. Notation** (Compare with [Gau11, Notation 5.3]). Let  $\mathcal{I}$  be the set of maps  $C : \emptyset \rightarrow \{0\}$ ,  $\emptyset \subset \underline{x}$  for  $x \in \Sigma$  and

$$\{0_n, 1_n\} \sqcup \underline{x}_1 \sqcup \dots \sqcup \underline{x}_n \subset C_n[x_1, \dots, x_n]^{\text{ext}}$$

for  $n \geq 1$  and  $x_1, \dots, x_n \in \Sigma$ .

**2.5. Lemma.** *The forgetful functor mapping a weak transition system to its set of states is colimit-preserving. The forgetful functor mapping a weak transition system to its set of actions is colimit-preserving.*

**Proof.** The lemma is a consequence of the fact that the forgetful functor  $\omega : WTS \rightarrow \mathbf{Set}^{\{s\} \cup \Sigma}$  taking the weak higher dimensional transition system  $(S, \mu : L \rightarrow \Sigma, (T_n)_{n \geq 1})$  to the  $(\{s\} \cup \Sigma)$ -tuple of sets  $(S, (\mu^{-1}(x))_{x \in \Sigma}) \in \mathbf{Set}^{\{s\} \cup \Sigma}$  is topological.  $\square$

**2.6. Lemma.** *All maps of  $\mathbf{cell}_{WTS}(\{R\})$  are epic.*

**Proof.** Let  $f, g, h$  be three maps of  $WTS$  with  $f \in \mathbf{cell}_{WTS}(\{R\})$  such that  $gf = hf$ . By functoriality, we obtain the equality  $\omega(g)\omega(f) = \omega(h)\omega(f)$ . All maps of  $\mathbf{cell}_{WTS}(\{R\})$  are onto on states and the identity on actions by Lemma 2.5. Therefore  $\omega(f)$  is epic and we obtain  $\omega(g) = \omega(h)$ . Since the forgetful functor  $\omega : WTS \rightarrow \mathbf{Set}^{\{s\} \cup \Sigma}$  is topological, it is faithful by [AHS06, Theorem 21.3]. Thus, we obtain  $g = h$ .  $\square$

**2.7. Proposition.** *There is the equality  $\mathbf{cell}_{WTS}(\mathcal{I}) = \mathbf{cof}_{WTS}(\mathcal{I})$  and this class of maps is the class of monomorphisms of weak transition systems.*

**Proof.** By [Gau11, Proposition 3.1], a map of weak transition systems is a monomorphism if and only if it induces a one-to-one set map on states and on actions. Consequently, by [Gau11, Proposition 5.4], a cofibration of weak transition systems  $f$  belongs to  $\mathbf{cell}_{WTS}(\mathcal{I} \cup \{R\})$ . All maps of  $\mathcal{I}$  belong to  $\mathbf{inj}_{WTS}(\{R\})$  because they are one-to-one on states. Using Lemma 2.6, we apply Theorem A.2:  $f$  factors uniquely, up to isomorphism, as a composite  $f = f^+f^-$  with  $f^+ \in \mathbf{cell}_{WTS}(\mathcal{I})$  and  $f^- \in \mathbf{cell}_{WTS}(\{R\})$ . The map  $f^-$

is one-to-one on states because  $f$  is one-to-one on states. We obtain the equalities  $f^- = \text{Id}$  and  $f = f^+$ . Therefore  $f$  belongs to  $\mathbf{cell}_{\mathcal{WTS}}(\mathcal{I})$ . Conversely, every map of  $\mathbf{cell}_{\mathcal{WTS}}(\mathcal{I})$  is one-to-one on states and on actions by Lemma 2.5. Thus, the class of cofibrations is  $\mathbf{cell}_{\mathcal{WTS}}(\mathcal{I})$ . Since the underlying category  $\mathcal{WTS}$  is locally presentable, every map of  $\mathbf{cof}_{\mathcal{WTS}}(\mathcal{I})$  is a retract of a map of  $\mathbf{cell}_{\mathcal{WTS}}(\mathcal{I})$ . This implies that every map of  $\mathbf{cof}_{\mathcal{WTS}}(\mathcal{I})$  is one-to-one on states and actions. Thus, we obtain  $\mathbf{cof}_{\mathcal{WTS}}(\mathcal{I}) \subset \mathbf{cell}_{\mathcal{CTS}}(\mathcal{I})$ . Hence we have obtained  $\mathbf{cof}_{\mathcal{WTS}}(\mathcal{I}) = \mathbf{cell}_{\mathcal{WTS}}(\mathcal{I})$  and the proof is complete.  $\square$

Let us now introduce the interval object of this model structure.

**2.8. Definition.** Let  $V$  be the weak transition system defined as follows:

- The set of states is  $\{0, 1\}$ .
- The set of actions is  $\Sigma \times \{0, 1\}$ .
- The labelling map is the projection  $\Sigma \times \{0, 1\} \rightarrow \Sigma$ .
- The transitions are the tuples

$$(\epsilon_0, (x_1, \epsilon_1), \dots, (x_n, \epsilon_n), \epsilon_{n+1})$$

for all  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$  and all  $x_1, \dots, x_n \in \Sigma$ .

**2.9. Notation.** Denote by  $\text{Cyl} : \mathcal{WTS} \rightarrow \mathcal{WTS}$  the functor  $- \times V$ .

**2.10. Proposition.** Let  $X = (S, \mu : L \rightarrow \Sigma, T)$  be a weak transition system. The weak transition system  $\text{Cyl}(X)$  has the set of states  $S \times \{0, 1\}$ , the set of actions  $L \times \{0, 1\}$ , the labelling map the composite map

$$\mu : L \times \{0, 1\} \rightarrow L \rightarrow \Sigma,$$

and a tuple

$$((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$$

is a transition of  $\text{Cyl}(X)$  if and only if the tuple  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition of  $X$ . There exists a unique map of weak transition systems  $\gamma_X^\epsilon : X \rightarrow \text{Cyl}(X)$  for  $\epsilon = 0, 1$  defined on states by  $s \mapsto (s, \epsilon)$  and on actions by  $u \mapsto (u, \epsilon)$ . There exists a unique map of weak transition systems  $\sigma_X : \text{Cyl}(X) \rightarrow X$  defined on states by  $(s, \epsilon) \mapsto s$  and on actions by  $(u, \epsilon) \mapsto u$ . There is the equality  $\sigma_X \gamma_X^\epsilon = \text{Id}_X$ . The composite map  $\sigma_X \gamma_X$  with  $\gamma_X = \gamma_X^0 \sqcup \gamma_X^1$  is the codiagonal of  $X$ .

Note that if  $T_n$  denotes the set of  $n$ -transitions of  $X$ , then the set of  $n$ -transitions of  $\text{Cyl}(X)$  is  $T_n \times \{0, 1\}^{n+2}$ .

**Proof.** The binary product in  $\mathcal{WTS}$  is described in [Gau11, Proposition 5.5]. The set of states of  $\text{Cyl}(X)$  is  $S \times \{0, 1\}$ . The set of actions of  $\text{Cyl}(X)$  is the product  $L \times_\Sigma (\Sigma \times \{0, 1\}) \cong L \times \{0, 1\}$  and the transitions of  $\text{Cyl}(X)$  are the tuples of the form  $((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$  such that  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition of  $X$  and such that the tuple

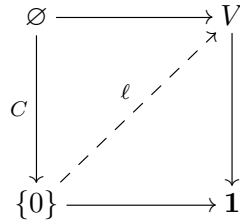
$$(\epsilon_0, (\mu(u_1), \epsilon_1), \dots, (\mu(u_n), \epsilon_n), \epsilon_{n+1})$$



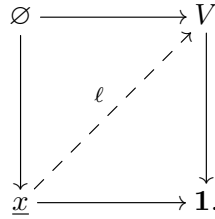
is a transition of  $V$ . The latter holds for any choice of  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$  by definition of  $V$ .  $\square$

**2.11. Proposition.** *Let  $X$  be a weak transition system. Then the map  $\gamma_X : X \sqcup X \rightarrow \text{Cyl}(X)$  is a monomorphism of weak transition systems and the map  $\sigma_X : \text{Cyl}(X) \rightarrow X$  satisfies the right lifting property (RLP) with respect to the monomorphisms of weak transition systems.*

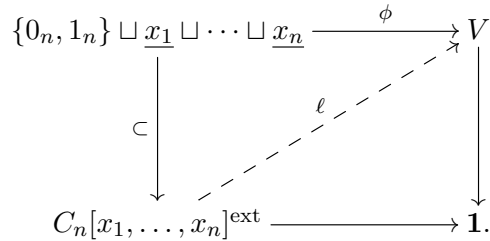
**Proof.** By [Gau11, Proposition 3.1], the map  $\gamma_X : X \sqcup X \rightarrow \text{Cyl}(X)$  is a monomorphism of  $\mathcal{WTS}$  since it is bijective on states and on actions. The lift  $\ell$  exists in the following diagram:



where  $\mathbf{1} = (\{0\}, \text{Id}_\Sigma : \Sigma \rightarrow \Sigma, \bigcup_{n \geq 1} \{0\} \times \Sigma^n \times \{0\})$  is the terminal object of  $\mathcal{WTS}$ : take  $\ell(0) = 0$ . The lift  $\ell$  exists in the following diagram:



Indeed,  $\ell(x) = x$  is a solution. Finally, consider a commutative diagram of the form:



Then let

$$\ell(0_n, (x_{\sigma(1)}, 1), \dots, (x_{\sigma(n)}, n), 1_n) = (\phi(0_n), (x_{\sigma(1)}, 0), \dots, (x_{\sigma(n)}, 0), \phi(1_n))$$

for any permutation  $\sigma$ : it is a solution. Therefore by Proposition 2.7, the map  $V \rightarrow \mathbf{1}$  satisfies the RLP with respect to all monomorphisms. Finally,

consider the commutative diagram of solid arrows:

$$\begin{array}{ccc}
 A & \longrightarrow & \text{Cyl}(X) \\
 \downarrow f & \nearrow \ell & \downarrow \sigma_X \\
 B & \longrightarrow & X
 \end{array}$$

where  $f$  is a monomorphism. Then the lift  $\ell$  exists because there are the isomorphisms  $\text{Cyl}(X) \cong X \times V$  and  $X \cong X \times \mathbf{1}$  and because the map  $\sigma_X$  is equal to the product  $\text{Id}_X \times (V \rightarrow \mathbf{1})$ .  $\square$

**2.12. Corollary.** *The functor  $\text{Cyl} : \mathcal{WTS} \rightarrow \mathcal{WTS}$  together with the natural transformations  $\gamma : \text{Id} \Rightarrow \text{Cyl}$  and  $\text{Cyl} \Rightarrow \text{Id}$  gives rise to a very good cylinder with respect to  $\mathcal{I}$ .*

**2.13. Proposition.** *The functor  $\text{Cyl} : \mathcal{WTS} \rightarrow \mathcal{WTS}$  is colimit-preserving.*

We will use the following notation: let  $I$  be a small category. For any diagram  $D$  of weak transition systems over  $I$ , the canonical map  $D_i \rightarrow \varinjlim D_i$  is denoted by  $\phi_{D,i}$ .

**Proof.** Let  $I$  be a small category. Let  $X : i \mapsto X_i$  be a small diagram of weak transition systems over  $I$ . By Lemma 2.5, for all objects  $i$  of  $I$ , the map  $\phi_{X,i} : X_i \rightarrow \varinjlim X_i$  is the inclusion  $S_i \subset \varinjlim S_i$  on states and the inclusion  $L_i \subset \varinjlim L_i$  on actions if  $S_i$  ( $L_i$  resp.) is the set of states (of actions resp.) of  $X_i$ . By definition of the functor  $\text{Cyl}$ , for all objects  $i$  of  $I$ , the map  $\text{Cyl}(\phi_{X,i}) : \text{Cyl}(X_i) \rightarrow \text{Cyl}(\varinjlim X_i)$  is then the inclusion

$$S_i \times \{0, 1\} \subset (\varinjlim S_i) \times \{0, 1\}$$

on states and the inclusion

$$L_i \times \{0, 1\} \subset (\varinjlim L_i) \times \{0, 1\}$$

on actions. Thus, the map  $\varinjlim \text{Cyl}(\phi_{X,i}) : \varinjlim \text{Cyl}(X_i) \rightarrow \text{Cyl}(\varinjlim X_i)$  induces a bijection on states and on actions since the category of sets is cartesian-closed (for the sequel, we will suppose that  $\varinjlim \text{Cyl}(\phi_{X,i})$  is the identity on states and on actions by abuse of notation). Consequently, by [Gau14, Proposition 4.4], the map

$$\varinjlim \text{Cyl}(\phi_{X,i}) : \varinjlim \text{Cyl}(X_i) \rightarrow \text{Cyl}(\varinjlim X_i)$$

is one-to-one on transitions. Let  $((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$  be a transition of  $\text{Cyl}(\varinjlim X_i)$ . By definition of  $\text{Cyl}$ , the tuple  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition of  $\varinjlim X_i$ . Let  $T_i$  be the image by the map

$$\phi_{X,i} : X_i \rightarrow \varinjlim X_i$$

of the set of transitions of  $X_i$ . Let  $G_0(\bigcup T_i) = \bigcup T_i$ . Let us define  $G_\lambda(\bigcup_i T_i)$  by induction on the transfinite ordinal  $\lambda \geq 0$  by  $G_\lambda(\bigcup_i T_i) = \bigcup_{\kappa < \lambda} G_\kappa(\bigcup_i T_i)$  for every limit ordinal  $\lambda$  and  $G_{\lambda+1}(\bigcup_i T_i)$  is obtained from  $G_\lambda(\bigcup_i T_i)$  by adding to  $G_\lambda(\bigcup_i T_i)$  all tuples obtained by applying the patching axiom to tuples of  $G_\lambda(\bigcup_i T_i)$  in  $\varinjlim_i X_i$ . Hence we have the inclusions

$$G_\lambda \left( \bigcup_i T_i \right) \subset G_{\lambda+1} \left( \bigcup_i T_i \right)$$

for all  $\lambda \geq 0$ . For cardinality reason, there exists an ordinal  $\lambda_0$  such that for every  $\lambda \geq \lambda_0$ , there is the equality  $G_\lambda(\bigcup_i T_i) = G_{\lambda_0}(\bigcup_i T_i)$ . The set  $G_{\lambda_0}(\bigcup_i T_i)$  is the set of transitions of  $\varinjlim_i X_i$  by [Gau10, Proposition 3.5]. We are going to prove by transfinite induction on  $\lambda \geq 0$  the assertion:

$\mathcal{A}_\lambda$ : If  $(\alpha, u_1, \dots, u_n, \beta) \in G_\lambda(\bigcup_i T_i)$ , then the tuple

$$((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$$

is a transition of  $\varinjlim_i \text{Cyl}(X_i)$  for any choice of  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ .

Assume that  $\lambda = 0$ . This implies that there exists a transition

$$(\alpha^{i_0}, u_1^{i_0}, \dots, u_n^{i_0}, \beta^{i_0})$$

of some  $X_{i_0}$  such that  $\phi_{X_{i_0}}(\alpha^{i_0}, u_1^{i_0}, \dots, u_n^{i_0}, \beta^{i_0}) = (\alpha, u_1, \dots, u_n, \beta)$ . In particular, this means that  $\phi_{X_{i_0}}(\alpha^{i_0}) = \alpha$ ,  $\phi_{X_{i_0}}(\beta^{i_0}) = \beta$  and for all  $1 \leq i \leq n$ ,  $\phi_{X_{i_0}}(u_i^{i_0}) = u_i$ . By definition of the functor  $\text{Cyl}$ , we obtain  $\text{Cyl}(\phi_{X_{i_0}})(\alpha^{i_0}, \epsilon_0) = (\alpha, \epsilon_0)$ ,  $\text{Cyl}(\phi_{X_{i_0}})(\beta^{i_0}, \epsilon_{n+1}) = (\beta, \epsilon_{n+1})$  and for all  $1 \leq i \leq n$ ,  $\text{Cyl}(\phi_{X_{i_0}})(u_i^{i_0}, \epsilon_i) = (u_i, \epsilon_i)$ . Since we have

$$\left( \varinjlim_i \text{Cyl}(\phi_{X_i}) \right) \phi_{\text{Cyl} X_{i_0}} = \text{Cyl}(\phi_{X_{i_0}})$$

by the universal property of the colimit, we obtain  $\phi_{\text{Cyl} X_{i_0}}(\alpha^{i_0}, \epsilon_0) = (\alpha, \epsilon_0)$ ,  $\phi_{\text{Cyl} X_{i_0}}(\beta^{i_0}, \epsilon_{n+1}) = (\beta, \epsilon_{n+1})$  and for all  $1 \leq i \leq n$ ,  $\phi_{\text{Cyl} X_{i_0}}(u_i^{i_0}, \epsilon_i) = (u_i, \epsilon_i)$ . However, the tuple  $((\alpha^{i_0}, \epsilon_0), (u_1^{i_0}, \epsilon_1), \dots, (u_n^{i_0}, \epsilon_n), (\beta^{i_0}, \epsilon_{n+1}))$  is a transition of  $\text{Cyl}(X_{i_0})$  by definition of the functor  $\text{Cyl}$ . This implies that

$$\begin{aligned} \phi_{\text{Cyl} X_{i_0}}((\alpha^{i_0}, \epsilon_0), (u_1^{i_0}, \epsilon_1), \dots, (u_n^{i_0}, \epsilon_n), (\beta^{i_0}, \epsilon_{n+1})) \\ = ((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1})) \end{aligned}$$

is a transition of  $\varinjlim_i \text{Cyl}(X_i)$ . We have proved  $\mathcal{A}_0$ . Assume  $\mathcal{A}_\kappa$  proved for all  $\kappa < \lambda$  for some limit ordinal  $\lambda$ . If  $(\alpha, u_1, \dots, u_n, \beta) \in G_\lambda(\bigcup_i T_i)$ , then  $(\alpha, u_1, \dots, u_n, \beta) \in G_\kappa(\bigcup_i T_i)$  for some  $\kappa < \lambda$ , and therefore the tuple

$$((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$$

is a transition of  $\varinjlim_i \text{Cyl}(X_i)$  as well by induction hypothesis. We have proved  $\mathcal{A}_\lambda$ . Assume  $\mathcal{A}_\lambda$  proved for  $\lambda \geq 0$  and assume that  $(\alpha, u_1, \dots, u_n, \beta)$

belongs to  $G_{\lambda+1}(\bigcup_i T_i) \setminus G_\lambda(\bigcup_i T_i)$ . Then there exist five tuples

$$\begin{aligned} &(\alpha', u'_1, \dots, u'_{n'}, \beta') \\ &(\alpha', u'_1, \dots, u'_p, \nu'_1) \\ &(\nu'_1, u'_{p+1}, \dots, u'_{n'}, \beta') \\ &(\alpha', u'_1, \dots, u'_{p+q}, \nu'_2) \\ &(\nu'_2, u'_{p+q+1}, \dots, u'_{n'}, \beta') \end{aligned}$$

of  $G_\lambda(\bigcup_i T_i)$  such that  $(\nu'_1, u'_{p+1}, \dots, u'_{p+q}, \nu'_2) = (\alpha, u_1, \dots, u_n, \beta)$ . By induction hypothesis, the five tuples

$$\begin{aligned} &((\alpha', 0), (u'_1, \epsilon'_1), \dots, (u'_{n'}, \epsilon_{n'}), (\beta', 0)) \\ &((\alpha', 0), (u'_1, \epsilon'_1), \dots, (u'_p, \epsilon'_p), (\nu'_1, \epsilon_0)) \\ &((\nu'_1, \epsilon_0), (u'_{p+1}, \epsilon'_{p+1}), \dots, (u'_{n'}, \epsilon'_{n'}), (\beta', 0)) \\ &((\alpha', 0), (u'_1, \epsilon'_1), \dots, (u'_{p+q}, \epsilon'_{p+q}), (\nu'_2, \epsilon_{n+1})) \\ &((\nu'_2, \epsilon_{n+1}), (u'_{p+q+1}, \epsilon'_{p+q+1}), \dots, (u'_{n'}, \epsilon'_{n'}), (\beta', 0)) \end{aligned}$$

are transitions of  $\varinjlim_i \text{Cyl}(X_i)$  for any choice of  $\epsilon'_i \in \{0, 1\}$ . Therefore the tuple

$$((\nu'_1, \epsilon_0), (u'_{p+1}, \epsilon'_{p+1}), \dots, (u'_{p+q}, \epsilon'_{p+q}), (\nu'_2, \epsilon_{n+1}))$$

is a transition of  $\varinjlim_i \text{Cyl}(X_i)$  by applying the patching axiom in  $\varinjlim_i \text{Cyl}(X_i)$ . Let  $\epsilon'_i = \epsilon_{i-p}$  for  $p+1 \leq i \leq p+n$  and  $\epsilon'_i = 0$  otherwise. Since there is the equality

$$\begin{aligned} &((\nu'_1, \epsilon_0), (u'_{p+1}, \epsilon'_{p+1}), \dots, (u'_{p+q}, \epsilon'_{p+q}), (\nu'_2, \epsilon_{n+1})) \\ &= ((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1})), \end{aligned}$$

we deduce that  $\mathcal{A}_{\lambda+1}$  holds. The transfinite induction is complete. We have proved that  $\varinjlim_i \text{Cyl}(\phi_{X,i}) : \varinjlim_i \text{Cyl}(X_i) \rightarrow \text{Cyl}(\varinjlim_i X_i)$  is onto on transitions. The latter map is bijective on states, bijective on actions and bijective on transitions: it is an isomorphism of weak transition systems and the proof is complete.  $\square$

**2.14. Proposition.** *Let  $X = (S, \mu : L \rightarrow \Sigma, T)$  be a weak transition system. There exists a well-defined weak transition system  $\text{Path}(X)$  such that:*

- *The set of states is the set  $S \times S$ .*
- *The set of actions is the set  $L \times_\Sigma L$  and the labelling map is the canonical map  $L \times_\Sigma L \rightarrow \Sigma$ .*
- *The transitions are the tuples*

$$((\alpha^0, \alpha^1), (u_1^0, u_1^1), \dots, (u_n^0, u_n^1), (\beta^0, \beta^1))$$

*such that for any  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ , the tuple*

$$(\alpha^{\epsilon_0}, u_1^{\epsilon_1}, \dots, u_n^{\epsilon_n}, \beta^{\epsilon_{n+1}})$$

*is a transition of  $X$ .*

Let  $f : X \rightarrow Y$  be a map of weak transition systems. There exists a map of weak transition systems  $\text{Path}(f) : \text{Path}(X) \rightarrow \text{Path}(Y)$  defined on states by the mapping  $(\alpha^0, \alpha^1) \mapsto (f(\alpha^0), f(\alpha^1))$  and on actions by the mapping  $(u^0, u^1) \mapsto (f(u^0), f(u^1))$ .

**Proof.** Let

$$((\alpha^0, \alpha^1), (u_1^0, u_1^1), \dots, (u_n^0, u_n^1), (\beta^0, \beta^1))$$

be a transition of  $\text{Path}(X)$ . Let  $\sigma$  be a permutation of  $\{1, \dots, n\}$  with  $n \geq 2$ . Then for any  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ , the tuple  $(\alpha^{\epsilon_0}, u_{\sigma(1)}^{\epsilon_1}, \dots, u_{\sigma(n)}^{\epsilon_n}, \beta^{\epsilon_{n+1}})$  is a transition of  $X$  by the multiset axiom. Thus, the tuple

$$((\alpha^0, \alpha^1), (u_{\sigma(1)}^0, u_{\sigma(1)}^1), \dots, (u_{\sigma(n)}^0, u_{\sigma(n)}^1), (\beta^0, \beta^1))$$

is a transition of  $\text{Path}(X)$ . Let  $n \geq 3$ . Let  $p, q \geq 1$  with  $p + q < n$ . Suppose that the five tuples

$$\begin{aligned} &((\alpha^0, \alpha^1), (u_1^0, u_1^1), \dots, (u_n^0, u_n^1), (\beta^0, \beta^1)) \\ &((\alpha^0, \alpha^1), (u_1^0, u_1^1), \dots, (u_p^0, u_p^1), (\nu_1^0, \nu_1^1)) \\ &((\nu_1^0, \nu_1^1), (u_{p+1}^0, u_{p+1}^1), \dots, (u_n^0, u_n^1), (\beta^0, \beta^1)) \\ &((\alpha^0, \alpha^1), (u_1^0, u_1^1), \dots, (u_{p+q}^0, u_{p+q}^1), (\nu_2^0, \nu_2^1)) \\ &((\nu_2^0, \nu_2^1), (u_{p+q+1}^0, u_{p+q+1}^1), \dots, (u_n^0, u_n^1), (\beta^0, \beta^1)) \end{aligned}$$

are transitions of  $\text{Path}(X)$ . Then for any  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ , the tuple

$$(\nu_1^{\epsilon_0}, u_{p+1}^{\epsilon_{p+1}}, \dots, u_{p+q}^{\epsilon_{p+q}}, \nu_2^{\epsilon_{n+1}})$$

is a transition of  $X$  by the patching axiom. Thus, the tuple

$$((\nu_1^0, \nu_1^1), (u_{p+1}^0, u_{p+1}^1), \dots, (u_{p+q}^0, u_{p+q}^1), (\nu_2^0, \nu_2^1))$$

is a transition of  $\text{Path}(X)$ . Hence  $\text{Path}(X)$  is well-defined as a weak transition system. Let  $f : X \rightarrow Y$  be a map of weak transition systems. For any state  $(\alpha^0, \alpha^1)$  of  $\text{Path}(X)$ , the pair  $(f(\alpha^0), f(\alpha^1))$  is a state of  $\text{Path}(Y)$  by definition of the functor  $\text{Path}$ . For any state  $(u^0, u^1)$  of  $\text{Path}(X)$ , we have  $\mu(u^0) = \mu(u^1)$  by definition of the functor  $\text{Path}$ . We deduce that  $\mu(f(u^0)) = \mu(u^0) = \mu(u^1) = \mu(f(u^1))$ . Hence the pair  $(f(u^0), f(u^1))$  is an action of  $\text{Path}(Y)$  by definition of the functor  $\text{Path}$ . Let

$$((\alpha^0, \alpha^1), (u_1^0, u_1^1), \dots, (u_n^0, u_n^1), (\beta^0, \beta^1))$$

be a transition of  $\text{Path}(X)$ . By definition of the functor  $\text{Path}$ , the tuple

$$(\alpha^{\epsilon_0}, u_1^{\epsilon_1}, \dots, u_n^{\epsilon_n}, \beta^{\epsilon_{n+1}})$$

is a transition of  $X$  for any choice of  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ . Consequently, the tuple

$$(f(\alpha^{\epsilon_0}), f(u_1^{\epsilon_1}), \dots, f(u_n^{\epsilon_n}), f(\beta^{\epsilon_{n+1}}))$$

is a transition of  $Y$  for any choice of  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ . By definition of the functor  $\text{Path}$ , we deduce that the tuple

$$((f(\alpha^0), f(\alpha^1)), (f(u_1^0), f(u_1^1)), \dots, (f(u_n^0), f(u_n^1)), (f(\beta^0), f(\beta^1)))$$

is a transition of  $\text{Path}(Y)$ . We have proved the last part of the statement.  $\square$

We obtain a well-defined functor  $\text{Path} : \mathcal{WTS} \rightarrow \mathcal{WTS}$ . For  $\epsilon \in \{0, 1\}$ , there exists a unique map of weak transition systems  $\pi_X^\epsilon : \text{Path}(X) \rightarrow X$  induced by the mappings  $(\alpha^0, \alpha^1) \mapsto \alpha^\epsilon$  on states and  $(u^0, u^1) \mapsto u^\epsilon$  on actions. Let  $\pi_X = (\pi_X^0, \pi_X^1)$ . This defines a natural transformation

$$\pi : \text{Path} \Rightarrow \text{Id} \times \text{Id}.$$

Since  $\mathcal{WTS}$  is locally presentable, and since the functor  $\text{Cyl} : \mathcal{WTS} \rightarrow \mathcal{WTS}$  is colimit-preserving by Proposition 2.13, we can deduce that it is a left adjoint by applying the opposite of the Special Adjoint Functor Theorem. The right adjoint is calculated in the following proposition.

**2.15. Proposition.** *There is a natural bijection of sets*

$$\Phi : \mathcal{WTS}(\text{Cyl}(X), X') \xrightarrow{\cong} \mathcal{WTS}(X, \text{Path}(X'))$$

for any weak transition systems  $X$  and  $X'$ .

**Proof.** The proof is in seven parts.

(1) *Construction of  $\Phi$ .* Let

$$X = (S, \mu : L \rightarrow \Sigma, T) \quad \text{and} \quad X' = (S', \mu : L' \rightarrow \Sigma, T')$$

be two weak transition systems. let  $f \in \mathcal{WTS}(\text{Cyl}(X), X')$ . Let

$$g^0 : S \rightarrow S' \times S'$$

be the set map defined by  $g^0(\alpha) = (f^0(\alpha, 0), f^0(\alpha, 1))$ . Let  $\tilde{g} : L \rightarrow L' \times_\Sigma L'$  be the set map defined by  $\tilde{g}(u) = (f(u, 0), f(u, 1))$ . Let  $(\alpha, u_1, \dots, u_n, \beta)$  be a transition of  $X$ . Then for any  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$ , the tuple

$$((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$$

is a transition of  $\text{Cyl}(X)$  by definition of the functor  $\text{Cyl}$ . Thus, the tuple

$$(f(\alpha, \epsilon_0), f(u_1, \epsilon_1), \dots, f(u_n, \epsilon_n), f(\beta, \epsilon_{n+1}))$$

is a transition of  $X'$  since  $f$  is a map of weak transition systems. We deduce that the tuple

$$\begin{aligned} ((f(\alpha, 0), f(\alpha, 1)), (f(u_1, 0), f(u_1, 1)), \\ \dots, (f(u_n, 0), f(u_n, 1)), (f(\beta, 0), f(\beta, 1))) \end{aligned}$$

is a transition of  $\text{Path}(X')$  by definition of  $\text{Path}$ . We have obtained a natural set map

$$g = \Phi(f) : \mathcal{WTS}(\text{Cyl}(X), X') \longrightarrow \mathcal{WTS}(X, \text{Path}(X')).$$

(2) *The case  $X = \emptyset$ .* There is the equality  $\text{Cyl}(\emptyset) = \emptyset$ . We obtain the bijection  $\mathcal{WTS}(\text{Cyl}(\emptyset), X') \cong \mathcal{WTS}(\emptyset, \text{Path}(X'))$ . We have proved that  $\Phi$  induces a bijection for  $X = \emptyset$ .

(3) *The case  $X = \{0\}$ .* There is the equality

$$\mathcal{WTS}(\text{Cyl}(\{0\}), X') \cong \mathcal{WTS}(\{(0, 0), (0, 1)\}, X')$$

by definition of Cyl. There is the equality

$$\mathcal{WTS}(\{(0,0), (0,1)\}, X') \cong \mathcal{WTS}(\{(0,0)\} \sqcup \{(0,1)\}, X')$$

by [Gau11, Proposition 5.6]. Hence we obtain the bijection

$$\mathcal{WTS}(\text{Cyl}(\{0\}), X') \cong \mathcal{WTS}(\{(0,0)\}, X') \times \mathcal{WTS}(\{(0,1)\}, X').$$

The right-hand term is equal to  $S' \times S'$ , which is precisely

$$\mathcal{WTS}(\{0\}, \text{Path}(X'))$$

by definition of Path. We have proved that  $\Phi$  induces a bijection for  $X = \{0\}$ .

(4) *The case  $X = \underline{x}$  for  $x \in \Sigma$ .* There is the equality  $\text{Cyl}(\underline{x}) = \underline{x} \sqcup \underline{x}$ . Therefore we obtain the bijections

$$\mathcal{WTS}(\text{Cyl}(\underline{x}), X') \cong \mathcal{WTS}(\underline{x} \sqcup \underline{x}, X') \cong \mathcal{WTS}(\underline{x}, X') \times \mathcal{WTS}(\underline{x}, X').$$

The set  $\mathcal{WTS}(\text{Cyl}(\underline{x}), X')$  is then equal to  $\mu^{-1}(x) \times \mu^{-1}(x)$ . And the set  $\mathcal{WTS}(\underline{x}, \text{Path}(X'))$  is the set of actions of  $\text{Path}(X')$  labelled by  $x$ , i.e.,  $\mu^{-1}(x) \times \mu^{-1}(x)$ . We have proved that  $\Phi$  induces a bijection for  $X = \underline{x}$  for all  $x \in \Sigma$ .

(5) *The case  $X = C_n^{\text{ext}}[x_1, \dots, x_n]$ .* The set of transitions of

$$\text{Cyl}(C_n^{\text{ext}}[x_1, \dots, x_n])$$

is the set of tuples

$$((0_n, \epsilon_0), ((x_{\sigma(1)}, \sigma(1)), \epsilon_1), \dots, ((x_{\sigma(n)}, \sigma(n)), \epsilon_n), (1_n, \epsilon_{n+1}))$$

for  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$  and all permutation  $\sigma$  of  $\{1, \dots, n\}$ . A map

$$f : \text{Cyl}(C_n^{\text{ext}}[x_1, \dots, x_n]) \longrightarrow X'$$

is then determined by the choice of four states  $f(0_n, 0)$ ,  $f(0_n, 1)$ ,  $f(1_n, 0)$ ,  $f(1_n, 1)$  of  $X'$  and for every  $1 \leq i \leq n$  by the choice of two actions  $f((x_i, i), 0)$  and  $f((x_i, i), 1)$  of  $X'$  such that the tuples

$$(f(0_n, \epsilon_0), f((x_{\sigma(1)}, \sigma(1)), \epsilon_1), \dots, f((x_{\sigma(n)}, \sigma(n)), \epsilon_n), f(1_n, \epsilon_{n+1}))$$

are transitions of  $X'$  for all  $\epsilon_0, \dots, \epsilon_{n+1} \in \{0, 1\}$  and all permutation  $\sigma$  of  $\{1, \dots, n\}$ . By definition of the functor Path, the latter assertion is equivalent to saying that the tuple

$$((f(0_n, 0), f(0_n, 1)), (f((x_1, 1), 0), f((x_1, 1), 1)), \dots, (f((x_n, n), 0), f((x_n, n), 1)), (f(1_n, 0), f(1_n, 1)))$$

is a transition of  $\text{Path}(X')$ . Choosing a map  $f$  from  $\text{Cyl}(C_n^{\text{ext}}[x_1, \dots, x_n])$  to  $X'$  is therefore equivalent to choosing a map of

$$\mathcal{WTS}(C_n^{\text{ext}}[x_1, \dots, x_n], \text{Path}(X')).$$

We have proved that  $\Phi$  induces a bijection for  $X = C_n^{\text{ext}}[x_1, \dots, x_n]$  for  $n \geq 1$  and for all  $x_1, \dots, x_n \in \Sigma$ .

(6) *The case*  $X = X_1 \sqcup X_2$ . If  $\Phi$  induces the bijections of sets

$$\mathcal{WTS}(\text{Cyl}(X_i), X') \cong \mathcal{WTS}(X_i, \text{Path}(X')) \quad \text{for } i = 1, 2,$$

then we obtain the sequence of bijections

$$\begin{aligned} & \mathcal{WTS}(\text{Cyl}(X), X') \\ & \cong \mathcal{WTS}(\text{Cyl}(X_1 \sqcup X_2), X') && \text{by definition of } X \\ & \cong \mathcal{WTS}(\text{Cyl}(X_1) \sqcup \text{Cyl}(X_2), X') && \text{by Proposition 2.13} \\ & \cong \mathcal{WTS}(\text{Cyl}(X_1), X') \times \mathcal{WTS}(\text{Cyl}(X_2), X') && \text{since } \mathcal{WTS}(-, X') \text{ is limit-preserving} \\ & \cong \mathcal{WTS}(X_1, \text{Path}(X')) \times \mathcal{WTS}(X_2, \text{Path}(X')) && \text{by hypothesis} \\ & \cong \mathcal{WTS}(X_1 \sqcup X_2, \text{Path}(X')) && \text{since } \mathcal{WTS}(-, \text{Path}(X')) \text{ is limit-preserving} \\ & \cong \mathcal{WTS}(X, \text{Path}(X')) && \text{by definition of } X. \end{aligned}$$

We have proved that  $\Phi$  induces a bijection for  $X = X_1 \sqcup X_2$ .

(7) *End of the proof.* The functor  $X \mapsto \mathcal{WTS}(\text{Cyl}(X), X')$  from the opposite of the category  $\mathcal{WTS}$  to the category of sets is limit-preserving by Proposition 2.13. The functor  $X \mapsto \mathcal{WTS}(X, \text{Path}(X'))$  from the opposite of the category  $\mathcal{WTS}$  to the category of sets is limit-preserving as well since the functor  $\mathcal{WTS}(-, Z)$  is limit-preserving as well for any weak transition system  $Z$ . The proof is complete by observing that the canonical map  $\emptyset \rightarrow X$  belongs to  $\mathbf{cell}_{\mathcal{WTS}}(\mathcal{I})$  by Proposition 2.7.  $\square$

**2.16. Corollary.** *The weak transition system  $V$  is exponential.*

**2.17. Proposition.** *Let  $f : X \rightarrow X'$  be a monomorphism of  $\mathcal{WTS}$ . Then the maps  $f \star \gamma^0$ ,  $f \star \gamma^1$  and  $f \star \gamma$  are monomorphisms of  $\mathcal{WTS}$ .*

**Proof.** Let  $X = (S, \mu : L \rightarrow \Sigma, T)$  and  $X' = (S', \mu' : L' \rightarrow \Sigma, T')$ . The map  $f \star \gamma^\epsilon$  induces on states the set map  $S' \sqcup_{S \times \{0,1\}} (S \times \{0,1\}) \rightarrow S' \times \{0,1\}$  which is one-to-one since the map  $S \rightarrow S'$  is one-to-one. And it induces on actions the set map  $L' \sqcup_{L \times \{0,1\}} (L \times \{0,1\}) \rightarrow L' \times \{0,1\}$  which is one-to-one since the map  $L \rightarrow L'$  is one-to-one. So by [Gau11, Proposition 3.1], the map  $f \star \gamma^\epsilon : X' \sqcup_X \text{Cyl}(X) \rightarrow \text{Cyl}(X')$  is a monomorphism of  $\mathcal{WTS}$ . The map  $f \star \gamma$  induces on states the set map  $(S' \sqcup S') \sqcup_{S \sqcup S} (S \times \{0,1\}) \rightarrow S' \times \{0,1\}$  which is the identity of  $S' \sqcup S'$ . And it induces on actions the identity of  $L' \rightarrow L'$ . So by [Gau11, Proposition 3.1], the map  $f \star \gamma : (X' \sqcup X') \sqcup_{X \sqcup X} \text{Cyl}(X) \rightarrow \text{Cyl}(X')$  is a monomorphism of  $\mathcal{WTS}$ .  $\square$

**2.18. Corollary.** *The cylinder  $\text{Cyl} : \mathcal{WTS} \rightarrow \mathcal{WTS}$  is cartesian with respect to the class of monomorphisms of weak transition systems.*

We have all the ingredients leading to an Olschok model structure (see [Gau15b, Definition 2.7] for the definition of an Olschok model structure):

**2.19. Theorem.** *There exists a unique left determined model category on  $\mathcal{WTS}$  such that the cofibrations are the monomorphisms. This model structure is an Olschok model structure, with the very good cylinder  $\text{Cyl}$  above defined.*

**Proof.** This a consequence of Olschok's theorems.  $\square$



### 3. Restricting the model structure of weak transition systems

We start this section by restricting the model structure to the full subcategory of cubical transition systems.

By definition, a *cubical transition system* satisfies all axioms of weak transition systems and the following two additional axioms (with the notations of Definition 2.2):

- (All actions are used). For every  $u \in L$ , there is a 1-transition  $(\alpha, u, \beta)$ .
- (Intermediate state axiom). For every  $n \geq 2$ , every  $p$  with  $1 \leq p < n$  and every transition  $(\alpha, u_1, \dots, u_n, \beta)$  of  $X$ , there exists a state  $\nu$  such that both  $(\alpha, u_1, \dots, u_p, \nu)$  and  $(\nu, u_{p+1}, \dots, u_n, \beta)$  are transitions.

By definition, a cubical transition system is *regular* if it satisfies the Unique intermediate state axiom, also called CSA2:

- (Unique intermediate state axiom or CSA2). For every  $n \geq 2$ , every  $p$  with  $1 \leq p < n$  and every transition  $(\alpha, u_1, \dots, u_n, \beta)$  of  $X$ , there exists a unique state  $\nu$  such that both  $(\alpha, u_1, \dots, u_p, \nu)$  and  $(\nu, u_{p+1}, \dots, u_n, \beta)$  are transitions.

Here is an important example of regular transition systems:

- For every  $x \in \Sigma$ , let us denote by  $\uparrow x \uparrow$  the cubical transition system with four states  $\{1, 2, 3, 4\}$ , one action  $x$  and two transitions  $(1, x, 2)$  and  $(3, x, 4)$ . The cubical transition system  $\uparrow x \uparrow$  is called the *double transition (labelled by  $x$ )* where  $x \in \Sigma$ .

**3.1. Notation.** The full subcategory of  $\mathcal{WTS}$  of cubical transition systems is denoted by  $\mathcal{CTS}$ . The full subcategory of  $\mathcal{CTS}$  of regular transition systems is denoted by  $\mathcal{RTS}$ .

The category  $\mathcal{RTS}$  of regular transition systems is a reflective subcategory of the category  $\mathcal{CTS}$  of cubical transition systems by [Gau15a, Proposition 4.4]. The reflection is denoted by  $\text{CSA}_2 : \mathcal{CTS} \rightarrow \mathcal{RTS}$ . The unit of the adjunction  $\text{Id} \Rightarrow \text{CSA}_2$  forces CSA2 to be true by identifying the states provided by a same application of the intermediate state axiom (see [Gau15a, Proposition 4.2]).

Let us introduce now the weak transition system corresponding to the labelled  $n$ -cube.

**3.2. Proposition.** [Gau10, Proposition 5.2] *Let  $n \geq 0$  and  $x_1, \dots, x_n \in \Sigma$ . Let  $T_d \subset \{0, 1\}^n \times \{(x_1, 1), \dots, (x_n, n)\}^d \times \{0, 1\}^n$  (with  $d \geq 1$ ) be the subset of  $(d+2)$ -tuples*

$$((\epsilon_1, \dots, \epsilon_n), (x_{i_1}, i_1), \dots, (x_{i_d}, i_d), (\epsilon'_1, \dots, \epsilon'_n))$$

*such that:*

- $i_m = i_n$  implies  $m = n$ , i.e., there are no repetitions in the list  $(x_{i_1}, i_1), \dots, (x_{i_d}, i_d)$ .

- for all  $i$ ,  $\epsilon_i \leq \epsilon'_i$ .
- $\epsilon_i \neq \epsilon'_i$  if and only if  $i \in \{i_1, \dots, i_d\}$ .

Let  $\mu : \{(x_1, 1), \dots, (x_n, n)\} \rightarrow \Sigma$  be the set map defined by  $\mu(x_i, i) = x_i$ . Then

$$C_n[x_1, \dots, x_n] = (\{0, 1\}^n, \mu : \{(x_1, 1), \dots, (x_n, n)\} \rightarrow \Sigma, (T_d)_{d \geq 1})$$

is a well-defined regular transition system called the  $n$ -cube.

The  $n$ -cubes  $C_n[x_1, \dots, x_n]$  for all  $n \geq 0$  and all  $x_1, \dots, x_n \in \Sigma$  are regular by [Gau10, Proposition 5.2] and [Gau10, Proposition 4.6]. For  $n = 0$ ,  $C_0[\ ]$ , also denoted by  $C_0$ , is nothing else but the one-state higher dimensional transition system  $(\{\()\}, \mu : \emptyset \rightarrow \Sigma, \emptyset)$ .

Since the tuple  $(0, (x, 0), 0)$  is a transition of  $V$  for all  $x \in \Sigma$ , all actions are used. The intermediate state axiom is satisfied since both the states 0 or 1 can always divide a transition in two parts. Therefore the weak transition system  $V$  is cubical. Note that the cubical transition system  $V$  is not regular.

**3.3. Theorem.** *There exists a unique left determined model category on  $CTS$  such that the cofibrations are the monomorphisms of weak transition systems between cubical transition systems. This model structure is an Olschok model structure, with the very good cylinder  $Cyl$  above defined.*

**Proof.** The category  $CTS$  is a full coreflective locally finitely presentable subcategory of  $WTS$  by [Gau11, Corollary 3.15]. The full subcategory of cubical transition systems is a small injectivity class by [Gau11, Theorem 3.6]: more precisely being cubical is equivalent to being injective with respect to the set of inclusions  $C_n[x_1, \dots, x_n]^{ext} \subset C_n[x_1, \dots, x_n]$  and  $\underline{x_1} \subset C_1[x_1]$  for all  $n \geq 0$  and all  $x_1, \dots, x_n \in \Sigma$ . Therefore, by [AR94, Proposition 4.3], it is closed under binary products. Hence we obtain the inclusion  $Cyl(CTS) \subset CTS$  since  $V$  is cubical. Then [Gau15b, Theorem 4.3] can be applied because all maps  $C_n[x_1, \dots, x_n]^{ext} \subset C_n[x_1, \dots, x_n]$  and  $\underline{x_1} \subset C_1[x_1]$  for all  $n \geq 0$  and all  $x_1, \dots, x_n \in \Sigma$  are cofibrations.  $\square$

The right adjoint  $Path^{CTS} : CTS \rightarrow CTS$  of the restriction of  $Cyl$  to the full subcategory of cubical transition systems is the composite map

$$Path^{CTS} : CTS \subset WTS \xrightarrow{Path} WTS \rightarrow CTS$$

where the right-hand map is the coreflection, obtained by taking the canonical colimit over all cubes and all double transitions [Gau11, Theorem 3.11]:

$$Path^{CTS}(X) = \lim_{\substack{\longrightarrow \\ f : C_n[x_1, \dots, x_n] \rightarrow Path(X) \\ \text{or } f : \uparrow x \uparrow \rightarrow Path(X)}} \text{dom}(f).$$

Therefore, we obtain:

**3.4. Proposition.** *The counit map  $\text{Path}^{\mathcal{CTS}}(X) \rightarrow \text{Path}(X)$  is bijective on states and one-to-one on actions and transitions.*

**Proof.** This is a consequence of the first part of [Gau11, Theorem 3.11].  $\square$

**3.5. Lemma.** *The forgetful functor mapping a cubical transition system to its set of states is colimit-preserving. The forgetful functor mapping a cubical transition system to its set of actions is colimit-preserving.*

**Proof.** Since the category of cubical transition systems is a coreflective subcategory of the category of weak transition systems by [Gau11, Corollary 3.15], this lemma is a consequence of Lemma 2.5.  $\square$

Theorem 3.3 proves the existence of a set of generating cofibrations for the model structure. It does not give any way to find it.

**3.6. Lemma.** *All maps of  $\text{cell}_{\mathcal{CTS}}(\{R\})$  are epic.*

**Proof.** Let  $f, g, h$  be three maps of  $\mathcal{CTS}$  with  $f \in \text{cell}_{\mathcal{CTS}}(\{R\})$  such that  $gf = hf$ . Since  $\mathcal{CTS}$  is coreflective in  $\mathcal{WTS}$ , we obtain  $f \in \text{cell}_{\mathcal{WTS}}(\{R\})$ . Since  $\mathcal{CTS}$  is a full subcategory of  $\mathcal{WTS}$ , we obtain  $gf = hf$  in  $\mathcal{WTS}$ . By Lemma 2.6, we obtain  $g = h$ .  $\square$

**3.7. Theorem** (Compare with [Gau14, Notation 4.5] and [Gau14, Theorem 4.6]). *The set of maps*

$$\begin{aligned} \mathcal{I}^{\mathcal{CTS}} = \{ & C : \emptyset \rightarrow \{0\} \} \\ & \cup \{ \partial C_n[x_1, \dots, x_n] \rightarrow C_n[x_1, \dots, x_n] \mid n \geq 1 \text{ and } x_1, \dots, x_n \in \Sigma \} \\ & \cup \{ C_1[x] \rightarrow \uparrow x \uparrow \mid x \in \Sigma \}. \end{aligned}$$

*generates the class of cofibrations of the model structure of  $\mathcal{CTS}$ .*

**Proof.** By [Gau14, Theorem 4.6], a cofibration between cubical transition systems belongs to  $\text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}} \cup \{R\})$  where  $R : \{0, 1\} \rightarrow \{0\}$  is the map identifying two states since it is one-to-one on actions. Every map of  $\mathcal{I}^{\mathcal{CTS}}$  is one-to-one on states. Therefore, there is the inclusion  $\mathcal{I}^{\mathcal{CTS}} \subset \text{inj}_{\mathcal{CTS}}(\{R\})$ . Every map of  $\text{cell}_{\mathcal{CTS}}(\{R\})$  is epic by Lemma 3.6. By Theorem A.2, every cofibration  $f$  then factors as a composite  $f = f^+ f^-$  such that  $f^- \in \text{cell}_{\mathcal{CTS}}(\{R\})$  and  $f^+ \in \text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$ , i.e.,  $R$  can be relocated at the beginning of the cellular decomposition. Since the cofibration  $f$  is also one-to-one on states by definition of a cofibration, the map  $f^- \in \text{cell}_{\mathcal{CTS}}(\{R\})$  is one-to-one on states as well. Therefore  $f^-$  is trivial and there is the equality  $f = f^+$ . We deduce that  $f$  belongs to  $\text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$ . Conversely, by Lemma 3.5, every map of  $\text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$  is one-to-one on states and on actions. Consequently, the class of cofibrations of  $\mathcal{CTS}$  is  $\text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$ . Since the underlying category  $\mathcal{CTS}$  is locally presentable, every map of  $\text{cof}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$  is a retract of a map of  $\text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$ . Therefore every map of  $\text{cof}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$  is one-to-one on states and on actions. We obtain  $\text{cof}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}}) \subset \text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$ . Hence we have obtained  $\text{cof}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}}) = \text{cell}_{\mathcal{CTS}}(\mathcal{I}^{\mathcal{CTS}})$  and the proof is complete.  $\square$

**3.8. Definition.** Let  $X$  be a weak transition system. A state  $\alpha$  of  $X$  is internal if there exists three transitions

$$(\gamma, u_1, \dots, u_n, \delta), (\gamma, u_1, \dots, u_p, \alpha), (\alpha, u_{p+1}, \dots, u_n, \delta)$$

with  $n \geq 2$  and  $p \geq 1$ . A state  $\alpha$  is external if it is not internal.

**3.9. Notation.** The set of internal states of a weak transition system  $X$  is denoted by  $X_{\text{int}}^0$ . The complement is denoted by  $X_{\text{ext}}^0 = X^0 \setminus X_{\text{int}}^0$ .

An internal state cannot be initial or final. The converse is false. Consider the amalgamated sum  $C_1[x] * C_1[y]$  with  $x, y \in \Sigma$  where the final state of  $C_1[x]$  is identified with the initial state of  $C_1[y]$ : the intermediate state is not internal because  $C_1[x] * C_1[y]$  does not contain any 2-transition.

**3.10. Proposition.** Let  $X$  be a regular transition system. Then the cubical transition system  $\text{Path}^{\text{CTS}}(X)$  is regular.

**Proof.** Let  $(\gamma^-, \gamma^+)$  and  $(\delta^-, \delta^+)$  be two states of  $\text{Path}^{\text{CTS}}(X)$  such that the four tuples

$$\begin{aligned} &((\alpha^-, \alpha^+), (u_1^-, u_1^+), \dots, (u_p^-, u_p^+), (\gamma^-, \gamma^+)) \\ &((\gamma^-, \gamma^+), (u_{p+1}^-, u_{p+1}^+), \dots, (u_n^-, u_n^+), (\beta^-, \beta^+)) \\ &((\alpha^-, \alpha^+), (u_1^-, u_1^+), \dots, (u_p^-, u_p^+), (\delta^-, \delta^+)) \\ &((\delta^-, \delta^+), (u_{p+1}^-, u_{p+1}^+), \dots, (u_n^-, u_n^+), (\beta^-, \beta^+)) \end{aligned}$$

are transitions of  $\text{Path}^{\text{CTS}}(X)$ , and therefore of  $\text{Path}(X)$  by Proposition 3.4. By definition of  $\text{Path}(X)$ , the tuples

$$\begin{aligned} &(\alpha^\pm, u_1^\pm, \dots, u_p^\pm, \gamma^\pm) \\ &(\gamma^\pm, u_{p+1}^\pm, \dots, u_n^\pm, \beta^\pm) \\ &(\alpha^\pm, u_1^\pm, \dots, u_p^\pm, \delta^\pm) \\ &(\delta^\pm, u_{p+1}^\pm, \dots, u_n^\pm, \beta^\pm) \end{aligned}$$

are transitions of  $X$ . Since  $X$  is regular, we obtain  $\gamma^- = \gamma^+ = \delta^- = \delta^+$ . In particular, this implies that  $(\gamma^-, \gamma^+) = (\delta^-, \delta^+)$ . Hence the cubical transition system  $\text{Path}^{\text{CTS}}(X)$  is regular as well.  $\square$

**3.11. Lemma.** Let  $X = (S, \mu : L \rightarrow \Sigma, T)$  be a weak transition system. Let  $S' \subset S$ . Let  $T \upharpoonright S'$  be the subset of tuples of  $T$  such that both the initial and the final states belong to  $S'$ . Then the triple  $(S', \mu : L \rightarrow \Sigma, T \upharpoonright S')$  yields a well-defined weak transition system denoted by  $X \upharpoonright S'$ .

**Proof.** For every permutation  $\sigma$  of  $\{1, \dots, n\}$  with  $n \geq 2$ , if the tuple  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition such that  $\alpha, \beta \in S'$ , then

$$(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta) \in T \upharpoonright S'$$

Therefore the set of tuples  $T \upharpoonright S'$  satisfies the multiset axiom. For every  $(n+2)$ -tuple  $(\alpha, u_1, \dots, u_n, \beta)$  with  $n \geq 3$ , for every  $p, q \geq 1$  with  $p+q < n$ ,

if the five tuples  $(\alpha, u_1, \dots, u_n, \beta)$ ,  $(\alpha, u_1, \dots, u_p, \nu_1)$ ,  $(\nu_1, u_{p+1}, \dots, u_n, \beta)$ ,  $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$  and  $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$  are transitions of  $T \upharpoonright S'$ , then  $\nu_1, \nu_2 \in S'$ . Therefore the transition  $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$  belongs to  $T \upharpoonright S'$ . Thus, the set of tuples  $T \upharpoonright S'$  satisfies the patching axiom.  $\square$

**3.12. Lemma.** *Let  $X$  be a weak transition system. Let  $Z \subset X^0$  be a subset of the set  $X^0$  of states of  $X$ . Consider a map  $f : \text{Cyl}(X) \rightarrow Y$  of WTS such that  $f$  belongs to  $\mathbf{cell}_{\text{WTS}}(\{R\})$  where  $R : \{0, 1\} \rightarrow \{0\}$  is the set map identifying two states. Suppose that every cell of  $f$  is of the form*

$$\begin{array}{ccc}
 \{0, 1\} & \xrightarrow{m_\alpha: \epsilon \mapsto (\alpha, \epsilon)} & \bullet \\
 \downarrow R & & \downarrow \\
 \{0\} & \xrightarrow{\quad} & \bullet
 \end{array}$$

*i.e., the identifications of states made by  $f$  are of the form  $(\alpha, 0) = (\alpha, 1)$  for  $\alpha \in Z$ . Then  $f$  is onto on maps, bijective on actions, onto on transitions and split epic. There is an inclusion  $Y \subset \text{Cyl}(X)$  which is a section. Moreover, if  $X$  is cubical, then  $Y$  is cubical as well.*

In general, identifications of states may generate new transitions by the patching axiom. The point is that it is not the case for this particular situation.

**3.13. Notation.** With the notations of Lemma 3.12, let  $Y = \text{Cyl}(X)//Z$ .

**Proof of Lemma 3.12.** The map  $R : \{0, 1\} \rightarrow \{0\}$  is onto on states and bijective on actions. Therefore the map  $f$  is onto on states and bijective on actions by Lemma 2.5. Consider the cocone of  $\mathbf{Set}^{\{s\} \cup \Sigma}$  consisting of the unique map

$$\omega(\text{Cyl}(X)) \longrightarrow (Z \times \{0\} \sqcup ((X^0 \setminus Z) \times \{0, 1\}), L_X \times \{0, 1\})$$

where  $L_X$  is the set of actions of  $X$ . The  $\omega$ -final lift gives rise to the map of weak transition systems  $f : \text{Cyl}(X) \rightarrow \text{Cyl}(X)//Z$ . The final structure contains all tuples of the form  $((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$  such that  $(\alpha, u_1, \dots, u_n, \beta)$  is a transition of  $X$ . The weak transition system  $X \upharpoonright (Z \times \{0\} \sqcup ((X^0 \setminus Z) \times \{0, 1\}))$  has exactly this set of transitions. Hence this set of transitions is the final structure and  $X \upharpoonright (Z \times \{0\} \sqcup ((X^0 \setminus Z) \times \{0, 1\})) = \text{Cyl}(X)//Z$ . The identity on states and the identity on actions induce a section of  $f$ , actually the inclusion  $\text{Cyl}(X)//Z \subset \text{Cyl}(X)$ . Suppose moreover that  $X$  is cubical. Then the weak transition  $\text{Cyl}(X)$  is cubical by Theorem 3.3. Since the cocone above induces the identity on actions,  $\text{Cyl}(X)//Z$  is then cubical by [Gau15a, Theorem 3.3].  $\square$

**3.14. Lemma.** *Let  $X$  be a regular transition system. Then there is the natural isomorphism*

$$\text{CSA}_2(\text{Cyl}(X)) \cong \text{Cyl}(X) // X_{\text{int}}^0.$$

**Proof.** By Lemma 3.12, the weak transition system  $\text{Cyl}(X) // X_{\text{int}}^0$  is cubical. Let  $(\mu_1, \zeta_1)$  and  $(\mu_2, \zeta_2)$  be two states of  $\text{Cyl}(X) // X_{\text{int}}^0$  such that there exists a transition

$$((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1}))$$

of  $\text{Cyl}(X) // X_{\text{int}}^0$  such that the four tuples

$$\begin{aligned} &((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_p, \epsilon_p), (\mu_1, \zeta_1)) \\ &((\alpha, \epsilon_0), (u_1, \epsilon_1), \dots, (u_p, \epsilon_p), (\mu_2, \zeta_2)) \\ &((\mu_1, \zeta_1), (u_{p+1}, \epsilon_{p+1}), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1})) \\ &((\mu_2, \zeta_2), (u_{p+1}, \epsilon_{p+1}), \dots, (u_n, \epsilon_n), (\beta, \epsilon_{n+1})) \end{aligned}$$

are transitions of  $\text{Cyl}(X) // X_{\text{int}}^0$  as well. Then the five tuples

$$\begin{aligned} &(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \mu_1), (\alpha, u_1, \dots, u_p, \mu_2) \\ &(\mu_1, u_{p+1}, \dots, u_n, \beta), (\mu_2, u_{p+1}, \dots, u_n, \beta) \end{aligned}$$

are transitions of  $X$  by definition of  $\text{Cyl}(X)$ . Since  $X$  is regular, there is the equality  $\mu_1 = \mu_2$ . Moreover, the state  $\mu_1 = \mu_2$  belongs to  $X_{\text{int}}^0$ . Therefore  $\zeta_1 = \zeta_2 = 0$  and  $(\mu_1, \zeta_1) = (\mu_2, \zeta_2)$ . We deduce that  $\text{Cyl}(X) // X_{\text{int}}^0$  is regular. Consider a map  $\text{Cyl}(X) \rightarrow Z$  with  $Z$  regular. The map  $\omega(\text{Cyl}(X)) \rightarrow \omega(Z)$  makes the identifications  $(u, 0) = (u, 1)$  for all  $u \in X_{\text{int}}^0$  by CSA2. Therefore it factors uniquely as a composite

$$\omega(\text{Cyl}(X)) \rightarrow (X_{\text{int}}^0 \times \{0\} \sqcup (X_{\text{ext}}^0 \times \{0, 1\}), L_X \times \{0, 1\}) \rightarrow \omega(Z).$$

Hence the map  $\text{Cyl}(X) \rightarrow Z$  factors uniquely as a composite

$$\text{Cyl}(X) \longrightarrow \text{Cyl}(X) // X_{\text{int}}^0 \longrightarrow Z.$$

However,  $\text{Cyl}(X) // X_{\text{int}}^0$  is regular. Hence we obtain the isomorphism

$$\text{Cyl}(X) // X_{\text{int}}^0 \cong \text{CSA}_2(\text{Cyl}(X)). \quad \square$$

**3.15. Proposition.** *Let  $X$  be a regular transition system. Then the map*

$$\eta_{\text{Cyl}(X)} : \text{Cyl}(X) \rightarrow \text{CSA}_2(\text{Cyl}(X))$$

*has a section  $s_X : \text{CSA}_2(\text{Cyl}(X)) \rightarrow \text{Cyl}(X)$ .*

**Proof.** By Lemma 3.14, the inclusion  $s_X : \text{CSA}_2(\text{Cyl}(X)) \subset \text{Cyl}(X)$  is a section of the natural map  $\eta_{\text{Cyl}(X)} : \text{Cyl}(X) \rightarrow \text{CSA}_2(\text{Cyl}(X))$ .  $\square$

We can now prove:

**3.16. Theorem.** *There exists a unique left determined model category on  $\mathcal{RTS}$  such that the set of generating cofibrations is  $\text{CSA}_2(\mathcal{I}^{\text{CTS}}) = \mathcal{I}^{\text{CTS}}$ . This model structure is an Olschok model structure with the very good cylinder  $\text{CSA}_2 \text{Cyl}$ .*

**Proof.** Thanks to Proposition 3.10 and Proposition 3.15, the theorem is a consequence of [Gau15b, Theorem 3.1].  $\square$

For any regular transition system  $X$ , the map

$$\gamma_X : X \sqcup X \rightarrow \text{CSA}_2(\text{Cyl}(X))$$

is a cofibration of the left determined model structure of  $\mathcal{RTS}$ . If  $\alpha$  is an internal state of  $X$ , then the two states  $(\alpha, 0)$  and  $(\alpha, 1)$  of  $X \sqcup X$  are identified in  $\text{CSA}_2(\text{Cyl}(X))$  by  $\gamma_X$  by Lemma 3.14. Consequently, as soon as  $X$  contains internal states, the cofibration  $\gamma_X : X \sqcup X \rightarrow \text{CSA}_2(\text{Cyl}(X))$  is not one-to-one on states.

#### 4. The fibrant replacement functor destroys the causal structure

We are going to prove in this section that the model structure of weak transition systems as well as all its restrictions interact extremely badly with the causal structure of the higher dimensional transition systems. More precisely, the fibrant replacement functor destroys the causal structure.

For the three model structures (on  $\mathcal{WTS}$ , on  $\mathcal{CTS}$  and  $\mathcal{RTS}$ ), we start from a weak transition system  $X$  containing at least one transition. We then consider the fibrant replacement  $X^{\text{fib}}$  of  $X$  in  $\mathcal{WTS}$  (in  $\mathcal{CTS}$  or in  $\mathcal{RTS}$  resp.) by factoring the canonical map  $X \rightarrow \mathbf{1}$  as a composite

$$X \xleftarrow[i_X]{\cong} X^{\text{fib}} \longrightarrow \mathbf{1}$$

in  $\mathcal{WTS}$  (in  $\mathcal{CTS}$  or in  $\mathcal{RTS}$  resp.). The canonical map  $X^{\text{fib}} \rightarrow \mathbf{1}$  is a fibration: therefore it satisfied the RLP with respect to any trivial cofibration, in particular with respect to any cofibration of the form  $f \star \gamma^0$  where  $f : A \rightarrow B$  is a cofibration. By adjunction, the lift  $\ell$  in any commutative diagram of solid arrows of the form

$$\begin{array}{ccc} A & \xrightarrow{\phi} & P(X^{\text{fib}}) \\ f \downarrow & \nearrow \ell & \downarrow \pi^0 \\ B & \xrightarrow{\psi} & X^{\text{fib}} \end{array}$$

exists, where  $P(X^{\text{fib}})$  is the path space of  $X^{\text{fib}}$  in  $\mathcal{WTS}$  (in  $\mathcal{CTS}$  or in  $\mathcal{RTS}$  resp.). Since  $X$  contains at least one transition, the image by  $i_X$  gives rise to a transition  $(\alpha, u_1, \dots, u_n, \beta)$  of  $X^{\text{fib}}$ . Let us now treat first the case of  $\mathcal{WTS}$ ,

and then together the case of  $\mathcal{CTS}$  and  $\mathcal{RTS}$ . The key point in what follows is that, if  $S$  denotes the set of states of  $X^{\text{fib}}$ , then the cartesian product  $S \times S$  is the set of states of  $P(X^{\text{fib}})$ : for  $\mathcal{WTS}$ , this is due to Proposition 2.14, for  $\mathcal{CTS}$ , this is a consequence of Proposition 3.4, and finally for  $\mathcal{RTS}$ , this is a consequence of Proposition 3.10. The crucial fact is that the coordinates in a cartesian product are independent from each other.

**4.1. Theorem.** *With the notations above, for any pair of states  $(\gamma, \delta)$  of the fibrant replacement  $X^{\text{fib}}$  of  $X$  in  $\mathcal{WTS}$ , the tuple  $(\gamma, u_1, \dots, u_n, \delta)$  is a transition of  $X^{\text{fib}}$ .*

**Proof.** The transition  $(\alpha, u_1, \dots, u_n, \beta)$  of  $X^{\text{fib}}$  gives rise to a map

$$\psi : C_n^{\text{ext}}[\mu(u_1), \dots, \mu(u_n)] \longrightarrow X^{\text{fib}}.$$

We then consider the diagram above with the cofibration

$$f : \{0_n, 1_n\} \subset C_n^{\text{ext}}[\mu(u_1), \dots, \mu(u_n)]$$

and with  $\phi(0_n) = (\alpha, \gamma)$  and  $\phi(1_n) = (\beta, \delta)$ . The existence of the lift  $\ell$  yields a transition  $((\alpha, \gamma), \ell(\mu(u_1), 1), \dots, \ell(\mu(u_n), n), (\beta, \delta))$  of  $P(X^{\text{fib}})$  with  $\ell(\mu(u_i), i) = (u_i, u'_i)$  for some  $u'_i$  for  $1 \leq i \leq n$ . By Proposition 2.14, we deduce that the tuple  $(\gamma, u_1, \dots, u_n, \delta)$  is a transition of  $X^{\text{fib}}$ .  $\square$

Since the path functor in the category of cubical transition systems is a subobject of the path functor in the category of weak transition systems by Proposition 3.4, and since the path space in  $\mathcal{CTS}$  of a regular transition system is regular by Proposition 3.10, we can conclude in the same way:

**4.2. Theorem.** *With the notations above. For any pair of states  $(\gamma, \delta)$  of the fibrant replacement  $X^{\text{fib}}$  of  $X$  in  $\mathcal{CTS}$  (in  $\mathcal{RTS}$  resp.), the tuple*

$$(\gamma, u_1, \dots, u_n, \delta)$$

*is a transition of  $X^{\text{fib}}$ .*

**Sketch of proof.** Since we work now in  $\mathcal{CTS}$  (in  $\mathcal{RTS}$  resp.), we must start from the cofibration  $f : \{0_n, 1_n\} \subset C_n[\mu(u_1), \dots, \mu(u_n)]$  (the weak transition  $C_n^{\text{ext}}[\mu(u_1), \dots, \mu(u_n)]$  is not cubical nor regular since it does not satisfy the intermediate state axiom). The transition  $(\alpha, u_1, \dots, u_n, \beta)$  of  $X^{\text{fib}}$  gives rise to a map

$$\psi : C_n^{\text{ext}}[\mu(u_1), \dots, \mu(u_n)] \longrightarrow X^{\text{fib}}.$$

By [Gau11, Theorem 3.6], the map  $\psi$  factors as a composite

$$\psi : C_n^{\text{ext}}[\mu(u_1), \dots, \mu(u_n)] \longrightarrow C_n[\mu(u_1), \dots, \mu(u_n)] \xrightarrow{\bar{\psi}} X^{\text{fib}}.$$

The rest of the proof is mutatis mutandis the proof of Theorem 4.1.  $\square$



### 5. The homotopy theory of star-shaped transition systems

We need first to introduce some definitions and notations. In this section,  $\mathcal{K}$  is one of the three model categories  $\mathcal{WTS}$ ,  $\mathcal{CTS}$  or  $\mathcal{RTS}$  equipped with the left determined model structure constructed in this paper. Consider the one-state weak cubical regular transition system  $\{\iota\}$ . The forgetful functor  $\omega^{\{\iota\}} : \{\iota\}\downarrow\mathcal{K} \rightarrow \mathcal{K}$  defined on objects by  $\omega^{\{\iota\}}(\{\iota\} \rightarrow X) = X$  and on maps by  $\omega^{\{\iota\}}(\{\iota\} \rightarrow f) = f$  is a right adjoint. The left adjoint  $\rho^{\{\iota\}} : \mathcal{K} \rightarrow \{\iota\}\downarrow\mathcal{K}$  is defined on objects by  $\rho^{\{\iota\}}(X) = (\{\iota\} \rightarrow \{\iota\} \sqcup X)$  and on morphisms by  $\rho^{\{\iota\}}(f) = \text{Id}_{\{\iota\}} \sqcup f$ . The locally presentable category  $\{\iota\}\downarrow\mathcal{K}$  is equipped with the model structure described in [Hir15, Theorem 2.7]: a map  $f$  is a cofibration (fibration, weak equivalence resp.) of  $i\downarrow\mathcal{K}$  if and only if  $\omega^{\{\iota\}}(f)$  is a cofibration (fibration, weak equivalence resp.) of  $\mathcal{K}$ . For the sequel, it is important to keep in mind that the forgetful functor  $\omega^{\{\iota\}} : \{\iota\}\downarrow\mathcal{K} \rightarrow \mathcal{K}$  preserves colimits of connected diagrams, in particular pushouts and transfinite compositions.

**5.1. Theorem.** *Let  $\mathcal{K}$  be  $\mathcal{WTS}$  or  $\mathcal{CTS}$  or  $\mathcal{RTS}$ . Then the model category  $\{\iota\}\downarrow\mathcal{K}$  is an Olschok model category and is left determined.*

**Proof.** The map  $\gamma_{\{\iota\}} : \{\iota\} \sqcup \{\iota\} \rightarrow \text{Cyl}(\{\iota\})$  is an isomorphism by Proposition 2.10. Therefore it is epic. We can then apply [Gau15b, Theorem 5.8] to obtain an Olschok model structure. Let  $\underline{C} : \mathcal{K} \rightarrow \mathcal{K}$  be the cylinder functor. By [Gau15b, Lemma 5.7], there is the pushout diagram of  $\mathcal{K}$ :

$$\begin{array}{ccc}
 \{\iota\} \sqcup \{\iota\} & \xrightarrow{\quad} & \{\iota\} \\
 \downarrow & & \downarrow \underline{C}_{\{\iota\}}(\iota \rightarrow X) \\
 \underline{C}(X) & \xrightarrow{\quad} & \omega^{\{\iota\}}(\underline{C}_{\{\iota\}}(\iota \rightarrow X))
 \end{array}$$

where  $\underline{C}_{\{\iota\}} : \{\iota\}\downarrow\mathcal{K} \rightarrow \{\iota\}\downarrow\mathcal{K}$  is the cylinder functor of the comma category  $\{\iota\}\downarrow\mathcal{K}$ . The map

$$\underline{C}(X) \rightarrow \omega^{\{\iota\}}(\underline{C}_{\{\iota\}}(\{\iota\} \rightarrow X))$$

consists of the identification  $(\iota, 0) = (\iota, 1)$ . In  $\mathcal{WTS}$ , and in  $\mathcal{CTS}$  which is coreflective in  $\mathcal{WTS}$ , the latter map is the map

$$\text{Cyl}(X) \rightarrow \text{Cyl}(X) // \{\iota\}$$

which has a section by Lemma 3.12. Colimits in  $\mathcal{RTS}$  are calculated first by taking the colimit in  $\mathcal{CTS}$  and then by applying the reflection  $\text{CSA}_2 : \mathcal{CTS} \rightarrow \mathcal{RTS}$ . Let  $X \in \mathcal{RTS}$ . The map

$$\underline{C}(X) \rightarrow \omega^{\{\iota\}}(\underline{C}_{\{\iota\}}(\{\iota\} \rightarrow X))$$

consisting of the identification  $(\iota, 0) = (\iota, 1)$  in  $\mathcal{RTS}$  is then equal to the composite map

$$\begin{aligned} \text{CSA}_2(\text{Cyl}(X)) &\cong \text{Cyl}(X)//X_{\text{int}}^0 \rightarrow \text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\}) \\ &\rightarrow \text{CSA}_2(\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\})). \end{aligned}$$

By Lemma 3.12, the weak transition system  $\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\})$  is cubical and the map  $\text{Cyl}(X)//X_{\text{int}}^0 \rightarrow \text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\})$  has a section: the inclusion

$$\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\}) \subset \text{Cyl}(X)//X_{\text{int}}^0.$$

Therefore, there exists a map

$$\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\}) \rightarrow \text{CSA}_2(\text{Cyl}(X))$$

which is one-to-one on states. Since  $\text{CSA}_2(\text{Cyl}(X))$  is regular, the cubical transition system  $\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\})$  is regular as well by [Gau15a, Proposition 4.1]. Hence there is an isomorphism

$$\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\}) \cong \text{CSA}_2(\text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\})).$$

We have proved that the map

$$\underline{C}(X) \rightarrow \omega^{\{\iota\}}(\underline{C}_{\{\iota\}}(\{\iota\} \rightarrow X))$$

consisting of the identification  $(\iota, 0) = (\iota, 1)$  in  $\mathcal{RTS}$  is the map

$$\text{Cyl}(X)//X_{\text{int}}^0 \rightarrow \text{Cyl}(X)//(X_{\text{int}}^0 \cup \{\iota\})$$

which has a section by Lemma 3.12.

Thanks to [Gau15b, Corollary 5.9], we deduce that the cylinder  $\underline{C}_{\{\iota\}}$  is very good and that the Olschok model structure is left determined for the three cases  $\mathcal{K} = \mathcal{WTS}$ ,  $\mathcal{K} = \mathcal{CTS}$  and  $\mathcal{K} = \mathcal{RTS}$ .  $\square$

It is usual in computer science to work in the comma category  $\{\iota\} \downarrow \mathcal{K}$  where the image of the state  $\iota$  represents the initial state of the process which is modeled. It then makes sense to restrict to the states which are reachable from this initial state by a path of transitions. Hence we introduce the following definitions:

**5.2. Definition.** Let  $X$  be a weak transition system and let  $\iota$  be a state of  $X$ . A state  $\alpha$  of  $X$  is reachable from  $\iota$  if it is equal to  $\iota$  or if there exists a finite sequence of transitions  $t_i$  of  $X$  from  $\alpha_i$  to  $\alpha_{i+1}$  for  $0 \leq i \leq n$  with  $n \geq 0$ ,  $\alpha_0 = \iota$  and  $\alpha_{n+1} = \alpha$ .

**5.3. Definition.** A star-shaped weak (cubical regular resp.) transition system is an object  $\{\iota\} \rightarrow X$  of the comma category  $\{\iota\} \downarrow \mathcal{K}$  such that every state of the underlying weak transition system  $X$  is reachable from  $\iota$ . The full subcategory of  $\{\iota\} \downarrow \mathcal{K}$  of star-shaped weak (cubical regular resp.) transition systems is denoted by  $\mathcal{K}_\bullet$ .

**5.4. Proposition.** Let  $\mathcal{K}$  be  $\mathcal{WTS}$  or  $\mathcal{CTS}$ .

- (1) Every map of  $\mathbf{cell}_{\{\iota\}\downarrow\mathcal{K}}(\{\rho^{\{\iota\}}(R)\})$  is onto on states and the identity on actions.
- (2) Every map of  $\mathbf{cell}_{\{\iota\}\downarrow\mathcal{K}}(\{\rho^{\{\iota\}}(R)\})$  is epic.

Proposition 5.4 also holds for  $\mathcal{K} = \mathcal{RTS}$  with a slightly different proof.

**Proof.** By adjunction, if  $f \in \mathbf{cell}_{\{\iota\}\downarrow\mathcal{K}}(\{\rho^{\{\iota\}}(R)\})$ , then  $\omega^{\{\iota\}}(f) \in \mathbf{cell}_{\mathcal{K}}(R)$ . Hence the first assertion of the proposition is a consequence of Lemma 2.5 for  $\mathcal{K} = \mathcal{WTS}$  and of Lemma 3.5 for  $\mathcal{K} = \mathcal{CTS}$ . Let  $f, g, h$  be three maps of  $\{\iota\}\downarrow\mathcal{K}$  such that  $f \in \mathbf{cell}_{\{\iota\}\downarrow\mathcal{K}}(\{\rho^{\{\iota\}}(R)\})$  and  $gf = hf$ . Then we have by functoriality  $\omega^{\{\iota\}}(g)\omega^{\{\iota\}}(f) = \omega^{\{\iota\}}(h)\omega^{\{\iota\}}(f)$ . By Lemma 2.6 if  $\mathcal{K} = \mathcal{WTS}$  and by Lemma 3.6 if  $\mathcal{K} = \mathcal{CTS}$ , we obtain  $\omega^{\{\iota\}}(g) = \omega^{\{\iota\}}(h)$ . Thus, there is the equality  $g = h$  and the proof is complete.  $\square$

**5.5. Proposition.** *Let  $\mathcal{K}$  be  $\mathcal{WTS}$  or  $\mathcal{CTS}$  or  $\mathcal{RTS}$ . The category  $\mathcal{K}_\bullet$  is a coreflective full subcategory of  $\{\iota\}\downarrow\mathcal{K}$ .*

**Sketch of proof.** The coreflection is described in [Gau15b, Proposition 6.5] for  $\mathcal{K} = \mathcal{WTS}$ . For  $\mathcal{K} = \mathcal{CTS}$  or  $\mathcal{K} = \mathcal{RTS}$ , the coreflection  $\{\iota\}\downarrow\mathcal{K} \rightarrow \mathcal{K}_\bullet$  removes all states which are not reachable from  $\iota$ , it removes all transitions not starting from a reachable state and it removes all actions which are not used by reachable transitions. This functor clearly takes a cubical (regular resp.) transition system to a cubical (regular resp.) one since the state(s) dividing a transition of the image is (are) reachable from  $\iota$ .  $\square$

**5.6. Proposition.** *Let  $\mathcal{K}$  be  $\mathcal{WTS}$  or  $\mathcal{CTS}$  or  $\mathcal{RTS}$ . The category  $\mathcal{K}_\bullet$  is a small-cone injectivity class of  $\{\iota\}\downarrow\mathcal{K}$  such that the top of the cone is  $\rho^{\{\iota\}}(\alpha) = \{\iota, \alpha\}$ , such that the cone contains only cofibrations and also the map  $\{\iota, \alpha\} \rightarrow \iota$ .*

**Proof.** For  $\mathcal{K} = \mathcal{WTS}$ , this is [Gau15b, Proposition 6.6]. For  $\mathcal{K} = \mathcal{CTS}$  or  $\mathcal{K} = \mathcal{RTS}$ , and since a cubical (regular resp.) transition system satisfies the intermediate state axiom, a state is reachable from  $\iota$  if and only if it is reachable from  $\iota$  by a path of 1-dimensional transitions. The cone consists of the map  $\{\iota, \alpha\} \rightarrow \iota$  and of the inclusions of  $\{\iota, \alpha\}$  into the cubical transition systems

$$\iota \xrightarrow{t_1} \bullet \longrightarrow \dots \longrightarrow \bullet \xrightarrow{t_n} \alpha$$

for all  $n \geq 1$  and all 1-transitions  $t_1, \dots, t_n$  with the labelling map  $\text{Id}_\Sigma$ .  $\square$

**5.7. Corollary.** *Let  $\mathcal{K}$  be  $\mathcal{WTS}$  or  $\mathcal{CTS}$  or  $\mathcal{RTS}$ . The category  $\mathcal{K}_\bullet$  is a small-cone injectivity class of  $\{\iota\}\downarrow\mathcal{K}$  such that the cone contains only maps which are one-to-one on actions.*

**5.8. Corollary.** *Let  $\mathcal{K}$  be  $\mathcal{WTS}$  or  $\mathcal{CTS}$  or  $\mathcal{RTS}$ . The category  $\mathcal{K}_\bullet$  is locally presentable.*

**Proof.** Since the category  $\mathcal{K}_\bullet$  is a small cone-injectivity class by Proposition 5.6, it is accessible by [AR94, Proposition 4.16]. Therefore it is locally

presentable because it is a full coreflective subcategory of a cocomplete category.  $\square$

**5.9. Theorem.** *Let  $\mathcal{K}$  be  $WTS$  or  $CTS$  or  $RTS$ . The class of cofibrations of  $\{\iota\}\downarrow\mathcal{K}$  between objects of  $\mathcal{K}_\bullet$  is cofibrantly generated.*

**Proof.** By Proposition 5.6, there exists a set of cofibrations  $\{g : \{\iota, \alpha\} \rightarrow P\}$  of  $\{\iota\}\downarrow\mathcal{K}$  such that  $\mathcal{K}_\bullet$  is the subcategory of  $\{\iota\}\downarrow\mathcal{K}$  of objects which are injective with respect to  $\{g : \{\iota, \alpha\} \rightarrow P\} \cup \{\{\iota, \alpha\} \rightarrow \{\iota\}\}$ . Consider a commutative square of  $\{\iota\}\downarrow\mathcal{K}$  of the form

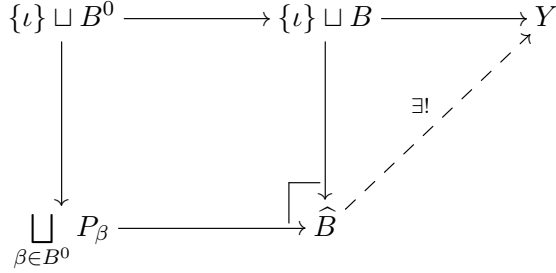
$$\begin{array}{ccc} \{\iota\} \sqcup A & \xrightarrow{\phi} & X \\ \rho^{\{\iota\}}(f) \downarrow & & \downarrow g \\ \{\iota\} \sqcup B & \xrightarrow{\psi} & Y \end{array}$$

where  $f$  is a generating cofibration of  $\mathcal{K}$  and where the map  $g : X \rightarrow Y$  is a map between star-shaped objects. For every state  $\alpha$  of  $A$ , and since  $X$  is star-shaped, the composite map  $\{\iota, \alpha\} \rightarrow \{\iota\} \sqcup A \rightarrow X$  factors as a composite  $\{\iota, \alpha\} \xrightarrow{g_\alpha} P_\alpha \rightarrow X$  with  $g_\alpha \in \{g : \{\iota, \alpha\} \rightarrow P\} \cup \{\{\iota, \alpha\} \rightarrow \{\iota\}\}$ . We obtain the commutative diagram of  $\{\iota\}\downarrow\mathcal{K}$

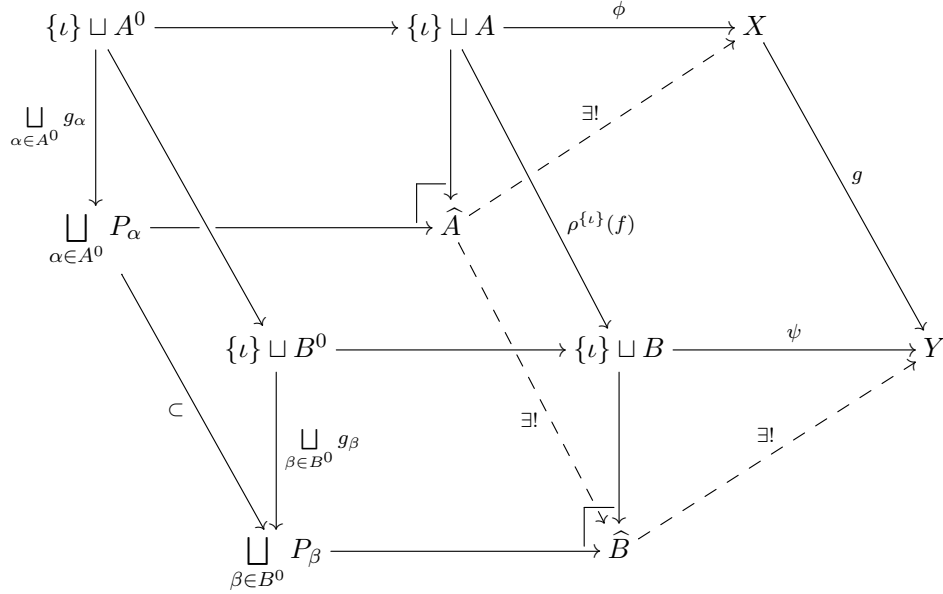
$$\begin{array}{ccccccc} \{\iota\} \sqcup A^0 & \longrightarrow & \{\iota\} \sqcup A & \longrightarrow & X & \longrightarrow & Y \\ \downarrow g_\alpha & & \downarrow & & \nearrow \exists! & & \\ \bigsqcup_{\alpha \in A^0} P_\alpha & \longrightarrow & \widehat{A} & & & & \end{array}$$

where  $A^0$  is the set of states of  $A$  and where the sum  $\bigsqcup_{\alpha \in A^0}$  is taken in the category  $\{\iota\}\downarrow\mathcal{K}$ . The lift  $\widehat{A} \rightarrow X$  exists and is unique by the universal property of the pushout. All generating cofibrations of  $WTS$  described in Notation 2.4 and all generating cofibrations of  $CTS$  and  $RTS$  described in Theorem 3.7 and in Theorem 3.16 respectively are one-to-one on states and on actions. Thus, we can identify the states of  $A$  with states of  $B$  and the

same argument leads us to the commutative diagram of  $\{\iota\}\downarrow\mathcal{K}$



where  $B^0$  is the set of states of  $B$  and where the sum  $\bigsqcup_{\beta \in B^0}$  is taken in the category  $\{\iota\}\downarrow\mathcal{K}$  (it is understood that we choose for  $\beta \in A^0$  the same  $P_\alpha$  as above). We obtain the commutative diagram of  $\{\iota\}\downarrow\mathcal{K}$ :



The map  $\widehat{A} \rightarrow \widehat{B}$  making the diagram commutative exists by the universal property of the pushout and it is one-to-one on states and on actions, i.e.,

it is a cofibration. We obtain the factorization

$$\begin{array}{ccccc}
 & & \phi & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \{\iota\} \sqcup A & \xrightarrow{\subset} & \widehat{A} & \xrightarrow{\quad} & X \\
 \downarrow \rho^{\{\iota\}}(f) & & \downarrow & & \downarrow g \\
 \{\iota\} \sqcup B & \xrightarrow{\subset} & \widehat{B} & \xrightarrow{\quad} & Y \\
 & \curvearrowleft & & \curvearrowright & \\
 & \psi & & & 
 \end{array}$$

By construction, the transition systems  $\widehat{A}$  and  $\widehat{B}$  are star-shaped. The map of star-shaped transition systems  $\widehat{A} \rightarrow \widehat{B}$  is obtained by choosing for each state of  $B$  a map of the set  $\{g : \{\iota, \alpha\} \rightarrow P\} \cup \{\{\iota, \alpha\} \rightarrow \{\iota\}\}$ . We have therefore constructed a solution set of cofibrations for the set of generating cofibrations of  $\{\iota\} \downarrow \mathcal{K}$  with respect to  $\mathcal{K}_\bullet$ , i.e., there exists a set  $J$  of cofibrations of  $\{\iota\} \downarrow \mathcal{K}$  between star-shaped objects such that every map  $i \rightarrow w$  from a generating cofibration  $i$  of  $\{\iota\} \downarrow \mathcal{K}$  to a map  $w$  of  $\mathcal{K}_\bullet$  factors as a composite  $i \rightarrow j \rightarrow w$  with  $j \in J$ . The proof is complete thanks to [Gau11, Lemma A.3].  $\square$

**5.10. Theorem.** *Let  $\mathcal{K}$  be WTS or CTS or RTS. There exists a left determined Olschok model structure on the category  $\mathcal{K}_\bullet$  of star-shaped weak (cubical, regular resp.) transition systems with respect to the class of maps such that the underlying map is a cofibration of  $\mathcal{K}$ .*

Note that unlike in the proof of [Gau15b, Theorem 6.8], we cannot use [Gau15b, Theorem 4.3]. Indeed,  $\mathcal{K}_\bullet$  is still a small cone-injectivity class by Proposition 5.6. However, the cone contains the map  $\{\iota, \alpha\} \rightarrow \iota$  which is not a cofibration in this paper.

**Proof.** Thanks to Theorem 5.1, Proposition 5.5, Corollary 5.8 and Theorem 5.9, we can apply [Gau15b, Theorem 4.1] if we can prove that the cylinder functor  $C_{\{\iota\}} : \{\iota\} \downarrow \mathcal{K} \rightarrow \{\iota\} \downarrow \mathcal{K}$  of  $\{\iota\} \downarrow \mathcal{K}$  takes a star-shaped (weak, cubical, regular resp.) transition system to a star-shaped one. Let  $\{\iota\} \rightarrow X$  be an object of  $\mathcal{K}_\bullet$ . We have the pushout diagram of  $\mathcal{K}$ :

$$\begin{array}{ccc}
 \{\iota\} \sqcup \{\iota\} & \xrightarrow{\quad} & \{\iota\} \\
 \downarrow & & \downarrow \\
 C(X) & \xrightarrow{\quad} & \omega^{\{\iota\}}(C_{\{\iota\}}(\{\iota\} \rightarrow X))
 \end{array}$$

where  $C$  denotes the cylinder of  $\mathcal{K}$ . Therefore if a state  $\alpha$  is reachable from  $\iota$ , then the state  $(\alpha, \epsilon)$  with  $\epsilon = 0, 1$  is reachable from  $(\iota, \epsilon) = (\iota, 0) = (\iota, 1)$  in  $C_{\{\iota\}}(\iota \rightarrow X)$ .  $\square$

Let us now reconsider the argument of Section 4. We obtain what follows (the functor  $P_{\{\iota\}} : \{\iota\} \downarrow \mathcal{K} \rightarrow \{\iota\} \downarrow \mathcal{K}$  denoting the right adjoint to the functor  $C_{\{\iota\}}$ ). Let  $\{\iota\} \rightarrow X$  be a star-shaped transition system of  $\mathcal{K}_\bullet$  which contains at least one transition. Let  $(\alpha, u_1, \dots, u_n, \beta)$  be a transition of a fibrant replacement  $(\{\iota\} \rightarrow X)^{\text{fib}}$  of  $\{\iota\} \rightarrow X$  in  $\mathcal{K}_\bullet$ . Let  $(\gamma, \delta)$  be a pair of states of  $(\{\iota\} \rightarrow X)^{\text{fib}}$ . If  $(\alpha, \gamma)$  and  $(\beta, \delta)$  are two reachable states of  $P_{\{\iota\}}(\{\iota\} \rightarrow X)$ , then the triple  $(\gamma, u_1, \dots, u_n, \delta)$  is a transition of  $(\{\iota\} \rightarrow X)^{\text{fib}}$ . The crucial difference with Section 4 is that  $(\alpha, \gamma)$  and  $(\beta, \delta)$  must now be reachable states of  $P_{\{\iota\}}((\{\iota\} \rightarrow X)^{\text{fib}})$ , and *not* any pair of states of  $X^{\text{fib}}$ . We have to understand now the intuitive meaning of a reachable state of

$$P_{\{\iota\}}((\{\iota\} \rightarrow X)^{\text{fib}}).$$

Let  $(\kappa, \lambda)$  be a reachable state of  $P_{\{\iota\}}((\{\iota\} \rightarrow X)^{\text{fib}})$ . That means that there exists a finite sequence of transitions  $t_i$  of  $P(X)$  (the path space of  $X$  in  $\mathcal{K}$ ) from  $(\alpha_i, \alpha'_i)$  to  $(\alpha_{i+1}, \alpha'_{i+1})$  for  $0 \leq i \leq n$  with  $n \geq 0$ , with  $(\alpha_0, \alpha'_0) = (\iota, \iota)$  and  $(\alpha_{n+1}, \alpha'_{n+1}) = (\kappa, \lambda)$ . By definition of a transition in  $P(X^{\text{fib}})$ , that means not only that the states  $\kappa$  and  $\lambda$  are reachable, but also that the transitions relating  $\iota$  to  $\kappa$  have the same labels as the transitions relating  $\iota$  to  $\lambda$ . Indeed, by Proposition 2.14 for  $\mathcal{WTS}$ , by Proposition 3.4 for  $\mathcal{CTS}$  and by Proposition 3.10 for  $\mathcal{RTS}$ , the set of actions of the path space of  $X^{\text{fib}}$  is a subset of  $L \times_\Sigma L$  where  $L$  is the set of actions of  $X^{\text{fib}}$ . Roughly speaking, *the states  $\kappa$  and  $\lambda$  have the same past*.

The interaction between the fibrant replacement in  $\mathcal{K}_\bullet$  and the causal structure can now be reformulated in plain English as follows:

*Let  $(\alpha, u_1, \dots, u_n, \beta)$  be a transition of a fibrant replacement  $(\{\iota\} \rightarrow X)^{\text{fib}}$  of  $\{\iota\} \rightarrow X$  in  $\mathcal{K}_\bullet$ . Let  $(\gamma, \delta)$  be a pair of states of  $(\{\iota\} \rightarrow X)^{\text{fib}}$  such that  $\alpha$  and  $\gamma$  ( $\beta$  and  $\delta$  resp.) have the same past. Then the triple  $(\gamma, u_1, \dots, u_n, \delta)$  is a transition of  $(\{\iota\} \rightarrow X)^{\text{fib}}$ .*

## Appendix A. Relocating maps in a transfinite composition

For this section,  $\mathcal{K}$  is a locally presentable category and  $R$  is a map such that all maps of  $\mathbf{cell}_{\mathcal{K}}(\{R\})$  are epic. Proposition A.1 and Theorem A.2 are used as follows:

- In the proof of Proposition 2.7 with the category  $\mathcal{WTS}$  and with the map  $R : \{0, 1\} \rightarrow \{0\}$ .
- In the proof of Theorem 3.7 with the category  $\mathcal{CTS}$  and with the map  $R : \{0, 1\} \rightarrow \{0\}$ .

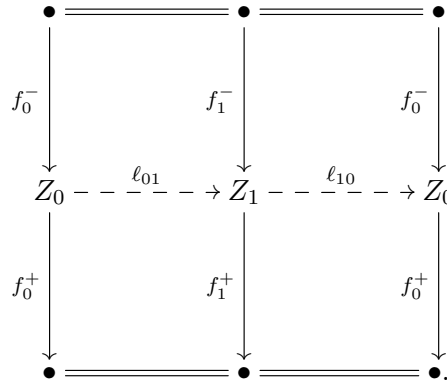


FIGURE 3. Unique factorization  $f = f^+ f^-$ .

**A.1. Proposition.** *Every map  $f$  of  $\mathcal{K}$  factors functorially as a composite  $f = f^+ f^-$  with  $f^- \in \mathbf{cell}_{\mathcal{K}}(\{R\})$  and  $f^+ \in \mathbf{inj}_{\mathcal{K}}(\{R\})$ . This factorization is unique up to isomorphism.*

**Proof.** The existence of the functorial factorization is a consequence of [Bek00, Proposition 1.3]. Consider the commutative diagram of solid arrows of  $\mathcal{K}$  of Figure 3 with  $f_0^+ f_0^- = f_1^+ f_1^-$ . The lift  $\ell_{01}$  exists since  $f_0^- \in \mathbf{cell}_{\mathcal{K}}(\{R\})$  and  $f_1^+ \in \mathbf{inj}_{\mathcal{K}}(\{R\})$ . If  $\ell'_{01}$  is another lift, then there is the equality  $\ell_{01} f_0^- = f_1^- = \ell'_{01} f_0^-$ . By hypothesis, the map  $f_0^-$  is epic. Therefore  $\ell_{01} = \ell'_{01}$ , which means that the lift  $\ell_{01}$  is unique. By switching the two columns, we obtain another lift  $\ell_{10}$ . By uniqueness of the lift, the composite  $\ell_{10} \ell_{01}$  is equal to the lift  $\ell_{00} = \text{Id}_{Z_0}$  and the composite  $\ell_{01} \ell_{10}$  is equal to the lift  $\ell_{11} = \text{Id}_{Z_1}$ .  $\square$

**A.2. Theorem.** *With the notations of Proposition A.1. Let  $A$  be a set of maps of  $\mathcal{K}$  such that  $A \subset \mathbf{inj}_{\mathcal{K}}(\{R\})$ . Then every map  $f \in \mathbf{cell}_{\mathcal{K}}(A \cup \{R\})$  factors uniquely, up to isomorphism, as a composite  $f = f^+ f^-$  with  $f^- \in \mathbf{cell}_{\mathcal{K}}(\{R\})$  and  $f^+ \in \mathbf{cell}_{\mathcal{K}}(A)$ .*

Theorem A.2 means that the cells  $R$  of a cellular complex  $\mathbf{cell}_{\mathcal{K}}(A \cup \{R\})$  can be relocated at the beginning of the cellular decomposition.

**Proof.** Let  $(q_\alpha : X_\alpha \rightarrow X_{\alpha+1})_{\alpha \geq 0}$  be a transfinite tower of pushouts of maps of  $A \cup \{R\}$ . Consider the commutative diagram of solid arrows of Figure 4. It represents the  $\alpha$ -th stage of the tower  $(q_\alpha : X_\alpha \rightarrow X_{\alpha+1})_{\alpha \geq 0}$  which is supposed to be a pushout of a map  $f$  of  $A \cup \{R\}$ . Since the factorizations  $f = f^+ f^-$  and  $p_\alpha = p_\alpha^+ p_\alpha^-$  are functorial, there exists a map  $\ell : \bullet \rightarrow \bar{X}_\alpha$  such that  $\ell f^- = p_\alpha^- \phi$ . We obtain  $\bar{q}_\alpha \ell f^- = \bar{q}_\alpha p_\alpha^- \phi = \bar{\psi} \psi f = \bar{\psi} \psi f^+ f^-$ . By hypothesis, the map  $f^-$  is epic. We obtain  $\bar{q}_\alpha \ell = \bar{\psi} \psi f^+$ . Finally, an immediate application of the universal property of a pushout square shows





and the proof is complete.  $\square$

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