

# Classical theta functions from a quantum group perspective

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ABSTRACT. In this paper we show how to model classical theta functions using the quantum group of abelian Chern–Simons theory. We describe the representation of the Heisenberg and mapping class groups in this setting.

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## 1. Introduction

In 1989, Witten [31] introduced a series of knot and 3-manifold invariants based on a quantum field theory with the Chern–Simons lagrangian. Witten defined these invariants by starting with a compact simple Lie group, the gauge group of the theory, and then defined a path integral for each 3-manifold as well as for knots and links in such a 3-manifold. The most studied case is of the gauge group  $SU(2)$ , which is related to the Jones polynomial of knots [13]. Witten pointed out that the case of the gauge group  $U(1)$  (abelian Chern–Simons theory) is related to the linking number of knots and to classical theta functions. Here and below we use the term “classical” not in the complex analytical distinction between classical and canonical, but to specify the classical theta functions as opposed to the nonabelian ones.

Abelian Chern–Simons theory was studied in [2], [10], [18], [19] and [21]. Our interest was renewed by the discovery, in [10], that all constructs of abelian Chern–Simons theory can be derived from the theory of theta functions. More precisely, in [10] it was shown that the action of the Heisenberg group on theta functions discovered by A. Weil, and the action of the mapping class groups lead naturally to manifold invariants.

In this paper we show how the theory of classical theta functions can be modeled using the quantum group of abelian Chern–Simons theory. This quantum group was introduced briefly in [21], but its properties were not discussed. In particular it was not noticed that, surprisingly, this quantum group is *not* a modular Hopf algebra. Below we interpret the classical theta functions, the action of the Heisenberg group, and the action of the mapping class groups in terms of vertex models using this quantum group.

Moreover, in [17] the Heisenberg group that arises in the theory of theta functions was related to the quantum torus (also known as the noncommutative torus). More precisely, the elements of the Heisenberg group are quantized exponentials, and the multiplication rule of the Heisenberg group yields a  $*$ -product for exponentials. The quantum torus is a normed algebra in which the polynomials in quantized exponentials are dense. The question arose as to whether the quantum torus is a quantum group; a deformation of the algebra of smooth functions on the torus that is the Jacobian variety. The answer is negative, except in a very weak sense [28]. In this paper we establish the relationship between the quantum torus and quantum groups, albeit in the spirit of [1].

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**Notation and conventions.** If  $V$  and  $W$  are vector spaces, we denote the linear map that permutes  $V$  and  $W$  by

$$P : V \otimes W \rightarrow W \otimes V, \quad v \otimes w \mapsto w \otimes v.$$

Given a Hilbert space  $H$ , we denote the group of unitary transformations of  $H$  by  $U(H)$ . In particular, we denote the group of unitary transformations of  $\mathbb{C}^n$  by  $U(n)$ .

Given a ring  $R$ , we denote the ring of formal power series over  $R$  in a single variable  $h$  by  $R[[h]]$ . We denote the ring of *noncommutative* polynomials in the variables  $X_1, \dots, X_n$  by  $R\langle X_1, \dots, X_n \rangle$ . The ring of integers modulo  $N$  will be denoted by  $\mathbb{Z}_N$ .

We denote the mapping class group of a Riemann surface  $\Sigma_g$  by  $\mathcal{M}_{\Sigma_g}$ . If  $\mathfrak{g}$  is a Lie algebra, we denote its universal enveloping algebra by  $\mathcal{U}(\mathfrak{g})$ .

## 2. Classical theta functions from a topological perspective

**2.1. Theta functions and representation theory.** In this section we recall the essential facts from [10], as well as the basic (well-known) background on theta functions. Classical theta functions may be described as sections of a certain line bundle over the Jacobian variety  $\mathcal{J}(\Sigma_g)$  associated to a closed genus  $g$  Riemann surface  $\Sigma_g$ .

The Jacobian variety is constructed as follows (cf. [6]). Choose a canonical basis of  $H_1(\Sigma_g, \mathbb{Z})$  like that in Figure 1, which is given by a collection of oriented simple closed curves  $a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g$  satisfying

$$a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij}$$

with respect to the intersection form. These curves can also be interpreted as generators of the fundamental group, and in this respect they define a *marking* on the surface in the sense of [12]. The complex structure on the surface  $\Sigma_g$  together with this marking defines a point in the Teichmüller space  $\mathcal{T}_g$ , cf. [12]. There exists a unique set of holomorphic 1-forms  $\zeta_1, \zeta_2, \dots, \zeta_g$  on

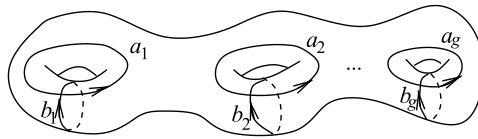


FIGURE 1. Any marked surface  $\Sigma_g$  may be canonically identified (up to isotopy) with the surface in the figure above.

$\Sigma_g$  such that

$$\int_{a_j} \zeta_k dz = \delta_{jk}.$$

The matrix  $\Pi \in M_g(\mathbb{C})$  whose entries are

$$\pi_{jk} = \int_{b_j} \zeta_k dz$$

is symmetric with positive definite imaginary part.

**Definition 2.1.** The Jacobian variety  $\mathcal{J}(\Sigma_g)$  of the marked Riemann surface  $\Sigma_g$  is the quotient

$$\mathcal{J}(\Sigma_g) := \mathbb{C}^g / L$$

of  $\mathbb{C}^g$  by the lattice subgroup

$$L := \{t_1\lambda_1 + \cdots + t_{2g}\lambda_{2g} : t_1, \dots, t_{2g} \in \mathbb{Z}\}$$

that is spanned by the columns  $\lambda_i$  of the  $g \times 2g$  matrix  $\lambda := (I_g, \Pi)$ , which we call the *period matrix*.

**Remark 2.2.** This identifies  $\mathcal{J}(\Sigma_g)$  as an abelian variety whose complex structure depends upon the matrix  $\Pi$ . Changing the marking on  $\Sigma_g$  whilst leaving the complex structure of  $\Sigma_g$  fixed gives rise to a biholomorphic Jacobian variety, albeit with a different period matrix.

The period matrix  $\lambda$  gives rise to an invertible  $\mathbb{R}$ -linear map

$$\mathbb{R}^{2g} = \mathbb{R}^g \times \mathbb{R}^g \rightarrow \mathbb{C}^g, \quad (x, y) \mapsto \lambda(x, y)^T = x + \Pi y$$

which descends to a diffeomorphism

$$(2.1) \quad \mathbb{R}^{2g} / \mathbb{Z}^{2g} \approx \mathbb{C}^g / L = \mathcal{J}(\Sigma_g).$$

Using this system of real coordinates, we define a symplectic form

$$\omega = \sum_{j=1}^g dx_j \wedge dy_j$$

on the quotient space  $\mathcal{J}(\Sigma_g)$ . This form is obviously well-defined due to the invariance of the above expression under translations. This symplectic form allows us to identify  $\mathcal{J}(\Sigma_g)$  with the phase space of a classical mechanical system, and to address questions about quantization.

Theta functions are obtained by applying the procedure of geometric quantization in a Kähler polarization, and this is explained in detail in [4]. The result is a family of functions that differs slightly from Riemann's in the sense that they depend on an integer parameter  $N$ . They are known as theta functions with rational characteristic [20] (these theta functions correspond to Mumford's in the case  $a = \frac{\mu}{N}, b = 0$ ). We point out that these theta functions with characteristic have been used to prove that abelian varieties are projective varieties (Lefschetz' Theorem).

Choose Planck's constant to be  $h = \frac{1}{N}$ , where  $N$  is a positive *even* integer. The Hilbert space of our quantum theory is obtained by applying the procedure of geometric quantization [26], [32]. It may be constructed as the space of holomorphic sections of a certain holomorphic line bundle over the Jacobian variety  $\mathcal{J}(\Sigma_g)$  which is obtained as the tensor product of a line bundle with curvature  $-\frac{2\pi i}{h}\omega = -2\pi i N\omega$  and a half-density. Such sections may be identified, in an obvious manner, with holomorphic functions on  $\mathbb{C}^g$  satisfying certain periodicity conditions determined by the period matrix  $\lambda$ , leading to the following definition.

**Definition 2.3.** The space of classical theta functions  $\Theta_N^\Pi(\Sigma_g)$  of the marked Riemann surface  $\Sigma_g$  is the vector space consisting of all holomorphic functions  $f : \mathbb{C}^g \rightarrow \mathbb{C}$  satisfying the periodicity conditions

$$f(z + \lambda_j) = f(z), \quad f(z + \lambda_{g+j}) = e^{-2\pi i N z_j - \pi i N \pi_{jj}} f(z), \quad j = 1, 2, \dots, g,$$

where  $\lambda_i$  denotes the  $i$ th column of the period matrix  $\lambda$ . The space of classical theta functions is a Hilbert space with inner product

$$(2.2) \quad \langle f, g \rangle := (2N)^{\frac{g}{2}} (\det \Pi_I)^{\frac{1}{2}} \int_{[0,1]^{2g}} f(x, y) \overline{g(x, y)} e^{-2\pi N y^T \Pi_I y} dx dy,$$

where  $\Pi_I \in M_g(\mathbb{R})$  denotes the imaginary part of  $\Pi$ .

This inner product arises from geometric quantization (see [4]), its computation uses the fact that for a holomorphic line bundle with a hermitian structure there is a unique connection compatible with both the hermitian structure and the complex structure. We point out that in [4] all computations are done in the case  $g = 1$ , but the general case is entirely parallel.

The space of theta functions depends only on the complex structure of  $\Sigma_g$  and not on the marking, in the sense that choosing a different marking on  $\Sigma_g$  yields an isomorphic Hilbert space. However, the marking on  $\Sigma_g$  specifies a particular orthonormal basis consisting of the theta series

$$\theta_\mu^\Pi(z) := \sum_{n \in \mathbb{Z}^g} e^{2\pi i N [\frac{1}{2}(\frac{\mu}{N} + n)^T \Pi (\frac{\mu}{N} + n) + (\frac{\mu}{N} + n)^T z]}, \quad \mu \in \mathbb{Z}_N^g.$$

Hence each point in  $\mathcal{T}_g$  gives rise to a space of theta functions endowed with a preferred basis.

Given  $p, q \in \mathbb{Z}^g$ , consider the exponential function

$$\mathcal{J}(\Sigma_g) \rightarrow \mathbb{C}, \quad (x, y) \mapsto e^{2\pi i (p^T x + q^T y)}$$

defined on  $\mathcal{J}(\Sigma_g)$  in the coordinates (2.1). Applying the Weyl quantization procedure in the momentum representation [2], [10] to such a function, one obtains the operator

$$(2.3) \quad O_{pq} := \text{Op} \left( e^{2\pi i (p^T x + q^T y)} \right) : \Theta_N^\Pi(\Sigma_g) \rightarrow \Theta_N^\Pi(\Sigma_g)$$

that acts on the theta series by

$$O_{pq} \theta_\mu^\Pi = e^{-\frac{\pi i}{N} p^T q - \frac{2\pi i}{N} \mu^T q} \theta_{\mu+p}^\Pi.$$

**Definition 2.4.** Given a nonnegative integer  $g$ , the *Heisenberg group*  $\mathbf{H}(\mathbb{Z}^g)$  is the group

$$\mathbf{H}(\mathbb{Z}^g) := \{(p, q, k) : p, q \in \mathbb{Z}^g, k \in \mathbb{Z}\}$$

with underlying multiplication

$$(p, q, k)(p', q', k') = \left( p + p', q + q', k + k' + \sum_{j=1}^g \begin{vmatrix} p_j & q_j \\ p'_j & q'_j \end{vmatrix} \right)$$

Given an even integer  $N$ , the *finite Heisenberg group*  $\mathbf{H}(\mathbb{Z}_N^g)$  is the quotient of the Heisenberg group  $\mathbf{H}(\mathbb{Z}^g)$  by the normal subgroup consisting of all elements of the form

$$(p, q, 2k)^N = (Np, Nq, 2Nk); \quad p, q \in \mathbb{Z}^g, k \in \mathbb{Z}.$$

**Remark 2.5.** The finite Heisenberg group is a  $\mathbb{Z}_{2N}$ -extension of  $\mathbb{Z}_N^{2g}$ , and consequently has order  $2N^{2g+1}$ . One should point out that the group  $\mathbf{H}(\mathbb{Z}^g)$  can be interpreted as the  $\mathbb{Z}$ -extension of

$$(2.4) \quad H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^g \times \mathbb{Z}^g, \quad \sum_{i=1}^g (p_i a_i + q_i b_i) \simeq (p, q)$$

by the cocycle defined by the intersection form.

The operators (2.3) generate a subgroup of the group  $U(\Theta_N^\Pi(\Sigma_g))$  of unitary operators on  $\Theta_N^\Pi(\Sigma_g)$ , which may be identified with the finite Heisenberg group as follows.

**Proposition 2.6** (Proposition 2.3. [10]). *Given a marked Riemann surface  $\Sigma_g$  of genus  $g$  and a positive even integer  $N$ , the subgroup  $G$  of the group of unitary operators on  $\Theta_N^\Pi(\Sigma_g)$  that is generated by all operators of the form*

$$O_{pq} = \text{Op}\left(e^{2\pi i(p^T x + q^T y)}\right); \quad p, q \in \mathbb{Z}^g$$

*is isomorphic to the finite Heisenberg group  $\mathbf{H}(\mathbb{Z}_N^g)$ :*

$$(2.5) \quad \mathbf{H}(\mathbb{Z}_N^g) \cong G, \quad (p, q, k) \mapsto e^{\frac{k\pi i}{N}} O_{pq}.$$

**Remark 2.7.** Note that  $e^{\frac{\pi i}{N}} \in G$ , so (2.5) makes sense.

The multiplication in the Heisenberg group, seen as multiplication of quantized exponentials, was interpreted in [17] as a  $*$ -product for exponentials

$$\begin{aligned} \exp(p^T x + q^T y) * \exp(p'^T x + q'^T y) \\ = e^{\pi i h (p^T q' - q^T p')} \exp((p + p')^T x + (q + q')^T y), \end{aligned}$$

with  $h = 1/N$ . Exponentials generated an algebra that is norm dense in the quantum torus, and hence we have an action of the quantum torus on theta functions.

The representation of  $\mathbf{H}(\mathbb{Z}_N^g)$  on the space of theta functions  $\Theta_N^\Pi(\Sigma_g)$  defined by (2.5), which we refer to as the *Schrödinger representation*, was first discovered by A. Weil [30] by examining translations in the line bundle over the Jacobian. Like the Schrödinger representation of the Heisenberg group with real entries, it satisfies a Stone–von Neumann theorem (for a proof see [10]).

**Theorem 2.8.** *Any irreducible unitary representation of the finite Heisenberg group in which the element  $(0, 0, 1) \in \mathbf{H}(\mathbb{Z}_N^g)$  acts as multiplication by the scalar  $e^{\frac{\pi i}{N}}$  is unitarily equivalent to the Schrödinger representation (2.5).*

This Stone–von Neumann theorem provides a reason for the existence of the action of the mapping class group on theta functions, whose discovery can be traced back to Jacobi. Let us denote by  $\mathcal{M}_{\Sigma_g}$  the mapping class group of the surface  $\Sigma_g$ . An element  $h \in \mathcal{M}_{\Sigma_g}$  induces a linear endomorphism  $h_*$  of  $H_1(\Sigma_g, \mathbb{Z})$  preserving the intersection form. By Remark 2.5, this endomorphism gives rise to an automorphism

$$(2.6) \quad \mathbf{H}(\mathbb{Z}^g) \rightarrow \mathbf{H}(\mathbb{Z}^g), \quad ((p, q), k) \mapsto (h_*(p, q), k)$$

of the Heisenberg group. This automorphism descends to an automorphism of the finite Heisenberg group  $\mathbf{H}(\mathbb{Z}_N^g)$ , yielding an action of the mapping class group of  $\Sigma_g$  on  $\mathbf{H}(\mathbb{Z}_N^g)$ ;

$$(2.7) \quad \mathcal{M}_{\Sigma_g} \rightarrow \text{Aut}(\mathbf{H}(\mathbb{Z}_N^g)) \quad h \mapsto \tilde{h} : ((p, q), k) \mapsto (h_*(p, q), k).$$

Pulling the Schrödinger representation (2.5) back via the automorphism  $\tilde{h}$  yields another representation

$$(p, q, k) \mapsto e^{\frac{k\pi i}{N}} O_{h_*(p, q)}$$

of the finite Heisenberg group  $\mathbf{H}(\mathbb{Z}_N^g)$ . By Theorem 2.8, this representation is unitarily equivalent to the Schrödinger representation, and therefore there exists a unitary map

$$\rho(h) : \Theta_N^{\text{II}}(\Sigma_g) \rightarrow \Theta_N^{\text{II}}(\Sigma_g),$$

satisfying what is called the *exact Egorov identity*

$$e^{\frac{k\pi i}{N}} O_{h_*(p, q)} = \rho(h) \circ \left( e^{\frac{k\pi i}{N}} O_{pq} \right) \circ \rho(h)^{-1}; \quad p, q \in \mathbb{Z}^g, k \in \mathbb{Z}.$$

By Schur’s Lemma,  $\rho(h)$  is well-defined up to multiplication by a complex scalar of unit modulus. Consequently, this construction yields a projective unitary representation

$$\mathcal{M}_{\Sigma_g} \rightarrow U(\Theta_N^{\text{II}}(\Sigma_g)) / U(1), \quad h \mapsto \rho(h)$$

of the mapping class group on the space of theta functions. The maps  $\rho(h)$  can be described as *discrete Fourier transforms* as we will explain below.

The Schrödinger representation can be obtained as an induced representation, and this allows us to relate theta functions to knots without the use of Witten’s quantum field theoretic insights.

Consider the submodule

$$\mathbf{L} := \{(0, q) : q \in \mathbb{Z}^g\}$$

of  $\mathbb{Z}^g \times \mathbb{Z}^g$  that is isotropic with respect to the intersection pairing induced by (2.4) and let

$$\tilde{\mathbf{L}} := \{(p, q, k) \in \mathbf{H}(\mathbb{Z}^g) : (p, q) \in \mathbf{L}\}$$

be the corresponding maximal abelian subgroup of  $\mathbf{H}(\mathbb{Z}^g)$ . Denote the image of  $\tilde{\mathbf{L}}$  in  $\mathbf{H}(\mathbb{Z}_N^g)$  under the natural projection by  $\tilde{\mathbf{L}}_N$ . Being abelian, this group has only one-dimensional irreducible representations, which are therefore characters. In view of the Stone–von Neumann theorem, choose the character

$$\chi_{\mathbf{L}} : \tilde{\mathbf{L}}_N \rightarrow \mathbb{C}, \quad \chi_{\mathbf{L}}(p, q, k) := e^{\frac{k\pi i}{N}}.$$

Consider the group algebras  $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$  and  $\mathbb{C}[\tilde{\mathbf{L}}_N]$ . Note that the latter acts on  $\mathbb{C}$  by the character  $\chi_{\mathbf{L}}$ . The induced representation is

$$\text{Ind}_{\tilde{\mathbf{L}}_N}^{\mathbf{H}(\mathbb{Z}_N^g)} = \mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)] \otimes_{\mathbb{C}[\tilde{\mathbf{L}}_N]} \mathbb{C},$$

with  $\mathbf{H}(\mathbb{Z}_N^g)$  acting on the left in the first factor of the tensor product.

We denote  $\mathcal{H}_{N,g}(\mathbf{L}) := \text{Ind}_{\tilde{\mathbf{L}}_N}^{\mathbf{H}(\mathbb{Z}_N^g)}$ . This space can be described as the quotient of  $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$  by the relations

$$(2.8) \quad \chi_{\mathbf{L}}(u_1)u = uu_1; \quad u \in \mathbf{H}(\mathbb{Z}_N^g), u_1 \in \tilde{\mathbf{L}}_N.$$

Denote the quotient map by

$$\pi_{\mathbf{L}} : \mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)] \rightarrow \mathcal{H}_{N,g}(\mathbf{L}).$$

The left regular action of  $\mathbf{H}(\mathbb{Z}_N^g)$  on  $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$  descends to an action of  $\mathbf{H}(\mathbb{Z}_N^g)$  on the quotient  $\mathcal{H}_{N,g}(\mathbf{L})$  that gives rise to the induced representation.

**Proposition 2.9** ([10]). *The map*

$$(2.9) \quad \Theta_N^{\Pi}(\Sigma_g) \rightarrow \mathcal{H}_{N,g}(\mathbf{L}) \quad \theta_{\mu}^{\Pi} \mapsto \pi_{\mathbf{L}}[(\mu, 0, 0)]$$

defines an  $\mathbf{H}(\mathbb{Z}_N^g)$ -equivariant  $\mathbb{C}$ -linear isomorphism between the space of theta functions  $\Theta_N^{\Pi}(\Sigma_g)$  equipped with the Schrödinger representation and the space  $\mathcal{H}_{N,g}(\mathbf{L})$  equipped with the left regular action of  $\mathbf{H}(\mathbb{Z}_N^g)$ .

**Remark 2.10.** Let  $h$  be an element of the mapping class group of  $\Sigma_g$  and set  $\mathbf{L}' := h_*(\mathbf{L})$ . As above, we may construct the vector space  $\mathcal{H}_{N,g}(\mathbf{L}')$  as the quotient of  $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$  by the same relations (2.8), where  $\mathbf{L}$  is replaced by  $\mathbf{L}'$  and the formula for the character  $\chi_{\mathbf{L}'}$  is the same as that for  $\chi_{\mathbf{L}}$ .

Following [22] (see also [10]) we consider the map  $\mathcal{H}_{N,g}(\mathbf{L}) \rightarrow \mathcal{H}_{N,g}(\mathbf{L}')$  given by

$$(2.10) \quad \pi_{\mathbf{L}}(\mathbf{u}) \mapsto \frac{1}{\left[ \tilde{\mathbf{L}}_N : (\tilde{\mathbf{L}}_N \cap \tilde{\mathbf{L}}_N') \right]^{\frac{1}{2}}} \sum_{u_1 \in \tilde{\mathbf{L}}_N / (\tilde{\mathbf{L}}_N \cap \tilde{\mathbf{L}}_N')} \chi_{\mathbf{L}}(u_1)^{-1} \pi_{\mathbf{L}'}(\mathbf{u}u_1).$$

where  $\mathbf{u} \in \mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ . On the other hand, the automorphism  $\tilde{h}$  of  $\mathbf{H}(\mathbb{Z}_N^g)$  defined by (2.7) induces a canonical identification

$$\mathcal{H}_{N,g}(\mathbf{L}) \cong \mathcal{H}_{N,g}(\mathbf{L}'), \quad \pi_{\mathbf{L}}(\mathbf{u}) \mapsto \pi_{\mathbf{L}'}(\tilde{h}(\mathbf{u}))$$

Composing (2.10) with the inverse of this map yields an endomorphism of  $\mathcal{H}_{N,g}(\mathbf{L})$  and consequently, by Proposition 2.9, an endomorphism of  $\Theta_N^{\Pi}(\Sigma_g)$ . This endomorphism is (a unitary representative for)  $\rho(h)^{-1}$ . There is a



general philosophy with identifies the intetwiners of the representations of Heisenberg groups as Fourier transforms, which is explained for the case of the metaplectic representation in [16] and for our situation in [22]. In that sense,  $\rho(h)$  is a discrete Fourier transform, and formula (2.10) is useful in establishing this correspondence.

Denote by  $L(\Theta_N^{\text{II}}(\Sigma_g))$  the algebra of  $\mathbb{C}$ -linear endomorphisms of  $\Theta_N^{\text{II}}(\Sigma_g)$ . The Schrödinger representation (2.5) provides a way to describe this space of linear operators in terms of the finite Heisenberg group  $\mathbf{H}(\mathbb{Z}_N^g)$ .

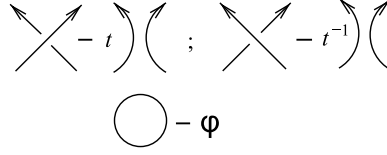
**Proposition 2.11** (Proposition 2.5 in [10]). *The quotient of the algebra  $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$  by the ideal  $I$  generated by  $(0, 0, 1) - e^{\frac{\pi i}{N}}(0, 0, 0)$  is isomorphic, via the Schrödinger representation, to the algebra of linear operators  $L(\Theta_N^{\text{II}}(\Sigma_g))$  with isomorphism defined by  $(p, q, k) \mapsto e^{\frac{k\pi i}{N}} O_{pq}$ .*

**2.2. Theta functions modeled using skein modules.** By Proposition 2.9, the map (2.9) intertwines the Schrödinger representation and the left action of the finite Heisenberg group. As explained in [10], this abstract version of the Schrödinger representation has topological flavor, which allows us to model the space of theta functions, the Schrödinger representation, and the action of the mapping class group using the skein modules corresponding to the linking number introduced by Przytycki in [23]. The key idea is that the group algebra of the Heisenberg group can be interpreted as a skein algebra of curves on the surface, while the factorization relation that gives rise to the induced representation is the result of filling in the surface with a handlebody. Here is the topological model.

For an oriented 3-manifold  $M$ , consider the free  $\mathbb{C}[t, t^{-1}]$ -module with basis the set of isotopy classes of framed oriented links contained in the interior of  $M$ , including the empty link  $\emptyset$ . Factor it by the  $\mathbb{C}[t, t^{-1}]$ -submodule spanned by the elements from Figure 2, where the two terms depict framed links that are identical, except in an embedded ball, in which they look as shown and have the blackboard framing; the orientation in Figure 2 that is induced by the orientation of  $M$  must coincide with the canonical orientation of  $\mathbb{R}^3$ . In other words, we are allowed to smooth each crossing, provided that we multiply with the appropriate power of  $t$ , and we are allowed to delete trivial link components. The result of this factorization is called the *linking number skein module* of  $M$  and is denoted by  $\mathcal{L}(M)$ . Its elements are called skeins. The name is due to the fact that the skein relations (Figure 2) are used in computing the Gaussian linking number of two curves.

From the even positive integer  $N$  we define the *reduced linking number skein module* of  $M$ , denoted by  $\mathcal{L}_N(M)$ , as follows. Consider the  $\mathbb{C}$ -module

$$\mathcal{L}(M) \otimes_{\mathbb{C}[t, t^{-1}]} \mathbb{C},$$

FIGURE 2. The skein relations defining  $\mathcal{L}(M)$ .

where  $\mathbb{C}[t, t^{-1}]$  acts on  $\mathbb{C}$  by setting  $t = e^{\frac{\pi i}{N}}$ . The reduced linking number skein module  $\mathcal{L}_N(M)$  is then the quotient of this  $\mathbb{C}$ -module by the relation,

$$\gamma^{\parallel N} \cup L = L;$$

where this relation holds for all framed oriented knots  $\gamma$  and framed oriented links  $L$  disjoint from  $\gamma$ , and  $\gamma^{\parallel N}$  is the multicurve<sup>1</sup> in a regular neighborhood of  $\gamma$  disjoint from  $L$  obtained by replacing  $\gamma$  by  $N$  parallel copies of it (where “parallel” is defined using the framing of  $\gamma$ ).

If  $M = \Sigma_g \times [0, 1]$ , then the identification

$$(2.11) \quad (\Sigma_g \times [0, 1]) \bigsqcup_{\Sigma_g \times \{0\} = \Sigma_g \times \{1\}} (\Sigma_g \times [0, 1]) \approx \Sigma_g \times [0, 1]$$

induces a multiplication of skeins in  $\mathcal{L}(\Sigma_g \times [0, 1])$  and  $\mathcal{L}_N(\Sigma_g \times [0, 1])$  which transforms them into algebras. Moreover, the identification

$$(2.12) \quad (\partial M \times [0, 1]) \bigsqcup_{\partial M \times \{0\} = \partial M} M \approx M,$$

canonically defined up to isotopy, makes  $\mathcal{L}(M)$  into a  $\mathcal{L}(\partial M \times [0, 1])$ -module and  $\mathcal{L}_N(M)$  into a  $\mathcal{L}_N(\partial M \times [0, 1])$ -module.

We are only interested in the situation where the cylinder  $\Sigma_g \times [0, 1]$  is glued to the boundary of the standard handlebody  $H_g$  of genus  $g$ , which we take to be the 3-manifold enclosed by the surface depicted in Figure 1. The marking on  $\Sigma_g$  leads to a canonical (up to isotopy) identification  $\partial H_g = \Sigma_g$  making  $\mathcal{L}(H_g)$  into a  $\mathcal{L}(\Sigma_g \times [0, 1])$ -module and  $\mathcal{L}_N(H_g)$  into a  $\mathcal{L}_N(\Sigma_g \times [0, 1])$ -module. Note that any link in a cylinder over a surface is skein-equivalent to a link in which every link component is endowed with a framing that is parallel to the surface. Consider the curves  $a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g$  from Figure 1 that define the marking on  $\Sigma_g$ , equipped with this parallel framing.

The surface  $\Sigma_g$  can be decomposed into the union of a sphere with  $g$  punctures and  $g$  punctured tori in such a way that for each  $j$ ,  $a_j, b_j$  form a basis of the first homology group of the  $j$ th punctured torus. Using the canonical basis we identify  $H_1(\Sigma_g, \mathbb{Z})$  with  $\mathbb{Z}^g \times \mathbb{Z}^g$ . Then an element  $(p, q) \in \mathbb{Z}^g \times \mathbb{Z}^g$  can be identified with a multicurve which is the union of the multicurves  $(p_j, q_j)$  on the  $j$ th torus,  $j = 1, 2, \dots, g$ , where  $(p_j, q_j)$  consists of  $\gcd(p_j, q_j)$  copies of the curve of slope  $p_j/q_j$  on the  $j$ th torus oriented from the origin

<sup>1</sup>We use this notation to distinguish this from an  $N$ -fold product, which will appear later.

towards the point with coordinates  $(p_j, q_j)$  (when viewing this curve as a segment in the universal covering space  $\mathbb{R}^2$  of the torus). Turn the multicurve  $(p, q)$  into a skein in  $\Sigma_g \times [0, 1]$  by placing it on  $\Sigma_g \times \{1/2\}$  and endowing each curve with the blackboard framing of the surface.

**Theorem 2.12** (Theorem 4.5 and 4.7 in [10]). *The algebras  $\mathbb{C}[\mathbf{H}(\mathbb{Z}^g)]$  and  $\mathcal{L}(\Sigma_g \times [0, 1])$  are isomorphic, with the isomorphism defined by the map*

$$(2.13) \quad \mathbb{C}[\mathbf{H}(\mathbb{Z}^g)] \cong \mathcal{L}(\Sigma_g \times [0, 1]), \quad (p, q, k) \mapsto t^k(p, q).$$

*This isomorphism is equivariant with respect to the action of the mapping class group of  $\Sigma_g$ , where  $\mathcal{M}_{\Sigma_g}$  acts on the left by (2.6) and on the right in an obvious fashion.*

*Furthermore, this isomorphism descends, using Proposition 2.11, to an isomorphism of the algebras*

$$(2.14) \quad L(\Theta_N^\Pi(\Sigma_g)) \cong \mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)] / \langle (0, 0, 1) - e^{\frac{\pi i}{N}}(0, 0, 0) \rangle \cong \mathcal{L}_N(\Sigma_g \times [0, 1]).$$

**Remark 2.13.** We should point out that in terms of the canonical basis depicted in Figure 1 this isomorphism is,

$$(p, q, k) \mapsto t^{(k-p^T q)} a_1^{p_1} \dots a_g^{p_g} b_1^{q_1} \dots b_g^{q_g};$$

where we consider the curves  $a_i$  and  $b_i$  as oriented framed curves in the cylinder  $\Sigma_g \times [0, 1]$  whose framing is parallel to the surface. We remind the reader that according to the multiplication defined by (2.11), the curves to the left are placed on top of the curves to the right.

Consider now the oriented framed curves  $a_1, a_2, \dots, a_g$  on  $\Sigma_g = \partial H_g$ . These curves give rise to oriented framed curves in the handlebody  $H_g$  which, by an abuse of notation, we denote in the same way. By (2.14),  $\mathcal{L}_N(H_g)$  is a  $L(\Theta_N^\Pi(\Sigma_g))$ -module.

**Theorem 2.14** (Theorem 4.7 in [10]). *The map*

$$(2.15) \quad \Theta_N^\Pi(\Sigma_g) \rightarrow \mathcal{L}_N(H_g), \quad \theta_\mu^\Pi \mapsto a_1^{\mu_1} \dots a_g^{\mu_g}$$

*is an isomorphism of  $L(\Theta_N^\Pi(\Sigma_g))$ -modules.*

Consider the orientation reversing diffeomorphism

$$(2.16) \quad \partial H_g \approx \partial H_g, \quad a_i \mapsto b_i, \quad b_i \mapsto a_i$$

of the boundary of the standard handlebody  $H_g$  that is canonically determined (up to isotopy) by the above action on the marking depicted in Figure 1. This identification gives rise to a Heegaard splitting

$$H_g \bigsqcup_{\partial H_g \approx \partial H_g} H_g = S^3$$

of  $S^3$  in which the leftmost handlebody corresponds to the interior of Figure 1 and the rightmost handlebody corresponds to the exterior. In turn, this Heegaard splitting defines a pairing

$$(2.17) \quad [\cdot, \cdot] : \Theta_N^\Pi(\Sigma_g) \otimes \Theta_N^\Pi(\Sigma_g) \cong \mathcal{L}_N(H_g) \otimes \mathcal{L}_N(H_g) \rightarrow \mathcal{L}_N(S^3) = \mathbb{C},$$

where we make use of Theorem 2.14 and the fact<sup>2</sup> that  $\mathcal{L}_N(S^3)$  is generated by the empty link. A direct computation illustrated in Figure 3 yields

$$[\theta_\mu^\Pi, \theta_{\mu'}^\Pi] = t^{-2\mu^T \mu'}; \quad \mu, \mu' \in \mathbb{Z}_N^g.$$

Since  $t = e^{\frac{\pi i}{N}}$  is a primitive  $2N$ th root of unity, it follows from this formula that this pairing is nondegenerate (the inverse matrix is  $\frac{1}{N^g} t^{2\mu^T \mu'}$ ). It is important to point out that this pairing is not the inner product (2.2).

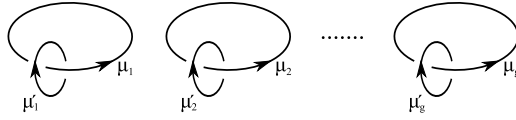


FIGURE 3. The pairing  $[\theta_\mu^\Pi, \theta_{\mu'}^\Pi]$  yields the above skein in  $S^3$ . The labels  $\mu_i, \mu'_i$  indicate the number of curves.

Let us now turn to the action of the mapping class group. Recall that any element  $h$  of the mapping class group of  $\Sigma_g$  can be represented as surgery on a framed link in  $\Sigma_g \times [0, 1]$ . Suppose that  $L$  is a framed link in  $\Sigma_g \times [0, 1]$  such that the 3-manifold  $K$  that is obtained from  $\Sigma_g \times [0, 1]$  by surgery along  $L$  is diffeomorphic to  $\Sigma_g \times [0, 1]$  by a diffeomorphism (where  $K$  is the domain and  $\Sigma_g \times [0, 1]$  is the codomain) that is the identity on  $\Sigma_g \times \{1\}$  and  $h \in \mathcal{M}_{\Sigma_g}$  on  $\Sigma_g \times \{0\}$ . Of course, not all links have this property, but for those that do, the diffeomorphism  $h_L := h$  is well-defined up to isotopy.

In particular, if  $T$  is a simple closed curve on  $\Sigma_g$ , then a Dehn twist along  $T$  is represented by the curve  $T^+$  that is obtained from  $T$  by endowing  $T$  with the framing that is parallel to the surface and placing a single positive twist in  $T$  (here, our convention for a Dehn Twist is such that a Dehn twist along the curve  $b_1$  in Figure 1 maps  $a_1 \in H_1(\Sigma_g)$  to  $a_1 + b_1$ ). The inverse twist is represented by the curve  $T^-$  obtained by placing a negative twist in  $T$ .

By Theorem 2.12, any linear operator on  $\Theta_N^\Pi(\Sigma_g)$  may be uniquely represented by a skein in  $\mathcal{L}_N(\Sigma_g \times [0, 1])$ . To describe the skein associated to the discrete Fourier transform  $\rho(h)$ , we need the following definition.

**Definition 2.15.** Denote by  $\Omega$  the skein in the solid torus depicted in Figure 4 multiplied by  $N^{-\frac{1}{2}}$ .

<sup>2</sup>Note that in  $S^3$ , the relation  $\gamma^{\parallel N} \cup L = L$  is redundant, as using the skein relations of Figure 2, any  $N$ -fold multicurve may be disentangled from any link and transformed into a union of unknots, which may then be deleted.

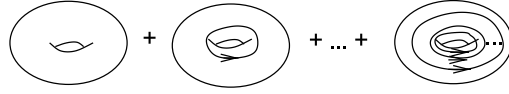


FIGURE 4. The skein  $\Omega$ . The sum has  $N$  terms.

For a framed link  $L$  in a 3-manifold  $M$ , denote by  $\Omega(L) \in \mathcal{L}_N(M)$  the skein that is obtained by replacing each component of  $L$  by  $\Omega$  using the framing on  $L$ . Specifically, if  $L = L_1 \cup L_2 \cup \dots \cup L_m$  then

$$\Omega(L) := \frac{1}{N^{\frac{m}{2}}} \sum_{i_1, \dots, i_m=0}^{N-1} L_1^{\parallel i_1} \cup L_2^{\parallel i_2} \cup \dots \cup L_m^{\parallel i_m}.$$

Note that  $\Omega(L)$  is independent of the orientation of  $L$ .

**Remark 2.16.** Given an oriented 3-manifold  $M$ , let  $\widehat{\mathcal{L}}(M)$  denote the free  $\mathbb{C}[t, t^{-1}]$ -module generated by isotopy classes of framed oriented links. This is distinct from  $\mathcal{L}(M)$  in that we do not quotient out by the skein relations of Figure 2. The definition of  $\Omega$  extends to give rise to a linear endomorphism,

$$L \mapsto \Omega(L),$$

of  $\widehat{\mathcal{L}}(M)$  (but not of  $\mathcal{L}(M)$ ). As before, the identification (2.11) makes  $\widehat{\mathcal{L}}(\Sigma_g \times [0, 1])$  into an algebra. In this case, the endomorphism defined by  $\Omega$  is multiplicative.

**Theorem 2.17** (Theorem 5.3 in [10]). *Let  $h_L \in \mathcal{M}_{\Sigma_g}$  be a diffeomorphism that is represented by surgery on a framed link  $L$ . Then the skein associated to the discrete Fourier transform  $\rho(h_L)$  by (2.14) is  $\Omega(L)$ . Consequently, if we consider  $\rho(h_L)$  as an endomorphism of  $\mathcal{L}_N(H_g)$  using Theorem 2.14, then*

$$(2.18) \quad \rho(h_L)(\beta) = \Omega(L) \cdot \beta, \quad \beta \in \mathcal{L}_N(H_g).$$

**Remark 2.18.** Of course, since  $\rho(h_L)$  is a projective unitary representation, what is meant by (2.18) is that left multiplication by the skein  $\Omega(L)$  is a unitary representative for the equivalence class represented by  $\rho(h_L)$ .

**Remark 2.19.** Since the mapping class group is generated by Dehn twists, Theorem 2.17 is sufficient to describe the action of the mapping class group. Consider the subalgebra  $E$  of  $\widehat{\mathcal{L}}(\Sigma_g \times [0, 1])$  that is linearly generated by isotopy classes of links for which surgery along that link does not change the diffeomorphism type of  $\Sigma_g \times [0, 1]$ . We may consider the diffeomorphism represented by surgery on a framed link as a multiplicative map

$$E \rightarrow \mathbb{C}[\mathcal{M}_{\Sigma_g}], \quad L \mapsto h_L.$$

If

$$h = h_{T_1^\pm} \circ h_{T_2^\pm} \circ \dots \circ h_{T_n^\pm}$$

is a product of Dehn twists, it follows that  $h$  is represented by surgery on the framed link  $T_1^\pm \cdots T_n^\pm$ . Consequently,

$$\rho(h) = \rho(h_{T_1^\pm \cdots T_n^\pm}) = \Omega(T_1^\pm \cdots T_n^\pm).$$

### 3. The quantum group of abelian Chern–Simons theory

Any isotopy of knots can be decomposed into a sequence of Reidemeister moves. Of them, the third Reidemeister move, an instance of which is depicted in Figure 5, was interpreted by Drinfeld [5] as a symmetry which leads to the existence of quantum groups. It follows that the linking number skein modules, and hence the theory of classical theta functions, should have an associated quantum group. This is the quantum group of abelian Chern–Simons theory. In what follows we explain how this quantum group is constructed and how the theory of classical theta functions is modeled using it.

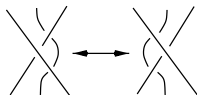


FIGURE 5. The third Reidemeister move leads to the study of quantum groups.

**3.1. The quantum group.** From now on we set  $t = e^{\frac{i\pi}{N}}$ , where  $N$  is the even integer from §2. Note that  $t$  is a primitive  $2N$ th root of unity. It is worth mentioning that, for the purposes of §3.2 and §3.3 alone, everything still holds true if  $N$  is an odd integer. The quantum group associated to  $U(1)$  is very simple; it is nothing more than the group algebra of the cyclic group  $\mathbb{Z}_{2N}$  of order  $2N$ . However, in order to explain how one arrives at this as the correct definition of the quantum group for  $U(1)$ , we need to revisit the construction of the quantum group for  $SL_2(\mathbb{C})$ , from which it is deduced. The reader unconcerned with the derivation of this quantum group may skip ahead to Definition 3.1.

The Lie algebra  $\mathfrak{u}(1)$  is not semisimple, so the Drinfeld–Jimbo construction does not apply directly. Nevertheless, the construction of the corresponding quantum group can be performed by following closely the constructions from [5], [14]. Recall the Drinfeld–Jimbo construction of  $\mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C}))$ . This is the quotient of the algebra  $\mathbb{C}\langle X, Y, H \rangle[[\hbar]]$  by the closure of the ideal generated by the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = \frac{e^{\frac{\hbar H}{2}} - e^{-\frac{\hbar H}{2}}}{e^{\frac{\hbar}{2}} - e^{-\frac{\hbar}{2}}}$$

in the  $h$ -adic topology. It carries the structure of a Hopf algebra;

$$\begin{aligned} \Delta(H) &:= H \otimes 1 + 1 \otimes H, & \Delta(X) &:= X \otimes e^{\frac{hH}{4}} + e^{-\frac{hH}{4}} \otimes X, \\ \Delta(Y) &:= Y \otimes e^{\frac{hH}{4}} + e^{-\frac{hH}{4}} \otimes Y, & S(H) &:= -H, \\ S(X) &:= -X, & S(Y) &:= -Y. \end{aligned}$$

There is a surjective map of Hopf algebras  $\mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C})) \rightarrow \mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$  defined by its action on the generators as follows;

$$h \mapsto 0, \quad X \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now recall that according to [25, §8], the quantum group  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$  is defined as the quotient of  $\mathbb{C}\langle K, K^{-1}, X, Y \rangle[q, q^{-1}]$  by the relations

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K, & (q - q^{-1})[X, Y] &= K^2 - K^{-2}, \\ KXK^{-1} &= qX, & KYK^{-1} &= q^{-1}Y. \end{aligned}$$

It carries the Hopf algebra structure

$$\begin{aligned} \Delta(K) &= K \otimes K, & \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ S(K) &= K^{-1}, & S(K^{-1}) &= K, \\ \Delta(X) &= X \otimes K + K^{-1} \otimes X, & \Delta(Y) &= Y \otimes K + K^{-1} \otimes Y, \\ S(X) &= -X, & S(Y) &= -Y. \end{aligned}$$

There is a map of Hopf algebras  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C})) \rightarrow \mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C}))$  defined by its action on the generators as follows;

$$K \mapsto e^{\frac{hH}{4}}, \quad K^{-1} \mapsto e^{-\frac{hH}{4}}, \quad X \mapsto X, \quad Y \mapsto Y, \quad q \mapsto e^{\frac{h}{2}}.$$

Finally, Reshetikhin and Turaev [25] define the quantum group  $\mathcal{U}_t(\mathfrak{sl}_2(\mathbb{C}))$  as the quotient of  $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$  by the relations

$$K^{2N} = 1, \quad X^{\frac{N}{2}} = Y^{\frac{N}{2}} = 0, \quad q = t^2.$$

Having recalled the construction of the quantum group for  $SL_2(\mathbb{C})$ , we may now deduce from it the construction of the quantum group for  $U(1)$ . There is an inclusion of groups

$$U(1) \rightarrow SL_2(\mathbb{C}), \quad z \mapsto \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$$

giving rise to an inclusion of (real) Lie algebras

$$(3.1) \quad \mathfrak{u}(1) = \mathbb{R} \rightarrow \mathfrak{sl}_2(\mathbb{C}), \quad x \mapsto \begin{bmatrix} ix & 0 \\ 0 & -ix \end{bmatrix}.$$

If we denote by

$$\mathfrak{u}_{\mathbb{C}}(1) := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(1) = \mathbb{C}$$

the complexification of  $\mathfrak{u}(1)$ , then (3.1) extends to a  $\mathbb{C}$ -linear map

$$\mathfrak{u}_{\mathbb{C}}(1) \rightarrow \mathfrak{sl}_2(\mathbb{C})$$

of complex Lie algebras.

We wish to define analogues  $\mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1))$ ,  $\mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1))$  and  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  of the above quantum groups for  $SL_2(\mathbb{C})$  in such a way that they fit naturally into a commutative diagram

$$(3.2) \quad \begin{array}{ccc} \mathcal{U}(\mathfrak{u}_{\mathbb{C}}(1)) & \longrightarrow & \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \\ \uparrow \text{---} & & \uparrow \\ \mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1)) & \dashrightarrow & \mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C})) \\ \uparrow \text{---} & & \uparrow \\ \mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1)) & \dashrightarrow & \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C})) \\ \downarrow \text{---} & & \downarrow \\ \mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1)) & \dashrightarrow & \mathcal{U}_t(\mathfrak{sl}_2(\mathbb{C})). \end{array}$$

Since the Lie algebra  $\mathfrak{u}_{\mathbb{C}}(1)$  is not semi-simple, we cannot use the Drinfeld–Jimbo construction to define its quantized enveloping algebra  $\mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1))$ . However, by the definition of quantum enveloping algebra, we must have

$$\mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1)) = \mathcal{U}(\mathfrak{u}_{\mathbb{C}}(1))[[h]]$$

as a  $\mathbb{C}[[h]]$ -module. The only reasonable way to place a Hopf algebra structure on  $\mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1))$  so that it fits into the commutative diagram (3.2) appears to be to choose the trivial deformation of  $\mathcal{U}(\mathfrak{u}_{\mathbb{C}}(1))$ . Consequently, since  $\mathcal{U}(\mathfrak{u}_{\mathbb{C}}(1))$  is a polynomial algebra in a single variable, we define

$$\mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1)) := \mathbb{C}\langle H \rangle[[h]]$$

and define

$$\begin{aligned} \mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1)) &\rightarrow \mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C})); & h &\mapsto h, & H &\mapsto H \\ \mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1)) &\rightarrow \mathcal{U}(\mathfrak{u}_{\mathbb{C}}(1)); & h &\mapsto 0, & H &\mapsto -i \in \mathfrak{u}_{\mathbb{C}}(1). \end{aligned}$$

Similarly, one now realizes the only reasonable way to define  $\mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1))$  is to define it as the quotient

$$\mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1)) := \mathbb{C}\langle K, K^{-1} \rangle[q, q^{-1}] / (KK^{-1} = 1 = K^{-1}K)$$

and choose the maps to be

$$\begin{aligned} \mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1)) &\rightarrow \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C})); & K &\mapsto K, & q &\mapsto q \\ \mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1)) &\rightarrow \mathcal{U}_h(\mathfrak{u}_{\mathbb{C}}(1)); & K &\mapsto e^{\frac{hH}{4}}, & q &\mapsto e^{\frac{h}{2}}. \end{aligned}$$

Finally, one arrives at the only sensible choice for  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$ .

**Definition 3.1.** The quantum group  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  is defined to be the quotient of the quantum group  $\mathcal{U}_q(\mathfrak{u}_{\mathbb{C}}(1))$  by the ideal generated by the relations

$$K^{2N} = 1, \quad q = t^2.$$



The quantum group  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  may be identified with the group algebra of  $\mathbb{Z}_{2N}$  by identifying  $K$  with the generator of  $\mathbb{Z}_{2N}$ . The remaining two maps at the bottom of (3.2) are defined in the obvious manner.

The irreducible representations of  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1)) = \mathbb{C}[\mathbb{Z}_{2N}]$  are

$$V^k; \quad k = 0, 1, \dots, 2N - 1$$

where  $V^k \cong \mathbb{C}$  and  $K$  acts by  $K \cdot v = t^k v$ . We denote by  $e_k$  the canonical basis element of  $V^k$ .

Because of the Hopf algebra structure, finite-dimensional representations form a ring (with the underlying abelian group defined á la Grothendieck from the monoid whose addition is the direct sum of representations) in which the product is provided by the tensor product of representations. Since  $(V^1)^{\otimes k} \cong V^k$ , the representation ring is

$$\mathbb{C}[V^1] / ((V^1)^{2N} - 1) = \mathbb{C}[\mathbb{Z}_{2N}].$$

The fact that, in this case, the representation ring coincides with the quantum group, is purely coincidental. Since  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  is a Hopf algebra, this implies that the dual space of each representation is itself a representation. It is easy to see that there are natural isomorphisms

$$D : (V^k)^* \rightarrow V^{2N-k}, \quad De^k = e_k, \quad k = 1, \dots, 2N - 1,$$

where the functional  $e^k$  is defined by  $e^k(e_k) = 1$ .

In what follows, we will explain how  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  may be given the structure of a ribbon Hopf algebra. Everything will be phrased using the terminology from [29]. Most of our computations are based on the simple fact that if  $z \neq 1$  is a  $k$ th root of unity, then

$$(3.3) \quad 1 + z + z^2 + \dots + z^{k-1} = 0.$$

**3.2. The universal  $R$ -matrix.** Just as in the Reshetikhin–Turaev theory, we want to model the braiding of strands in Section 2 by  $R$ -matrices. The operator  $\check{R} = P \circ R : V^m \otimes V^n \rightarrow V^n \otimes V^m$  (where  $P$  is the map that transposes the factors) should then come from the crossing of  $m$  strands by  $n$  strands, as in Figure 6, and hence should equal multiplication by  $t^{mn}$  for all  $m, n \in \{0, 1, \dots, 2N - 1\}$ .

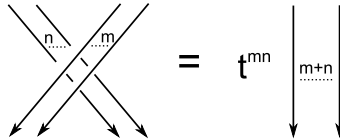


FIGURE 6. The skein relation for the crossing of  $n$  strands by  $m$  strands.

We claim that, as for the quantum group of  $SU(2)$  [25], these  $R$ -matrices are induced by a universal  $R$ -matrix in  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$ . The universal  $R$ -matrix

should be of the form

$$R := \sum_{j,k=0}^{2N-1} c_{jk} K^j \otimes K^k.$$

Let us compute the coefficients  $c_{jk}$ . Because  $K^j = t^{jn}$  id on  $V^n$  for all  $j$  and  $n$ , we obtain the system of equations

$$(3.4) \quad \sum_{j,k=0}^{2N-1} c_{jk} t^{mj} t^{nk} = t^{mn}; \quad m, n \in \{0, 1, \dots, 2N-1\}.$$

If  $\mathcal{T}$  is the matrix whose  $mn$ th entry is  $t^{mn}$  and  $C = (c_{jk})$ , then this equation becomes

$$\mathcal{T}C\mathcal{T} = \mathcal{T}$$

and hence we find that  $C = \mathcal{T}^{-1}$ . Since  $t$  is a primitive  $2N$ th root of unity, it follows from equation (3.3) that

$$(3.5) \quad c_{jk} = \frac{1}{2N} t^{-jk}.$$

Hence, we arrive at the following formula for the  $R$ -matrix,

$$(3.6) \quad R = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-jk} K^j \otimes K^k.$$

Note that this formula for  $R$  implies that the  $R$ -matrix is symmetric in the sense that  $P(R) = R$ .

**Theorem 3.2.** *( $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1)), R$ ) is a quasi-triangular Hopf algebra.*

**Proof.** We must show the following:

- (1)  $R$  is invertible.
- (2) For all  $a \in A$ ,  $\Delta_{\text{op}}(a) = R\Delta(a)R^{-1}$ ; where  $\Delta_{\text{op}} := P \circ \Delta$ .
- (3) The identities
  - (a)  $R_{13}R_{12} = (\text{id} \otimes \Delta)(R)$ ,
  - (b)  $R_{13}R_{23} = (\Delta \otimes \text{id})(R)$ .

If  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  is to be a quasi-triangular Hopf algebra, then a formula for  $R^{-1}$  should be given by

$$(3.7) \quad \begin{aligned} R^{-1} &= (S \otimes \text{id})[R] = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-jk} K^{-j} \otimes K^k \\ &= \frac{1}{2N} \sum_{i,j \in \mathbb{Z}_{2N}} t^{jk} K^{-j} \otimes K^{-k}. \end{aligned}$$

We may check that this element is inverse to  $R$  as follows;

$$RR^{-1} = \frac{1}{4N^2} \sum_{j,j',k,k' \in \mathbb{Z}_{2N}} t^{j'k'-jk} K^{j-j'} \otimes K^{k-k'}$$

$$= \frac{1}{4N^2} \sum_{m,n \in \mathbb{Z}_{2N}} \left( \sum_{j,k \in \mathbb{Z}_{2N}} t^{(j-m)(k-n)-jk} \right) K^m \otimes K^n.$$

But

$$\sum_{j,k \in \mathbb{Z}_{2N}} t^{(j-m)(k-n)-jk} = t^{mn} \left( \sum_{j \in \mathbb{Z}_{2N}} t^{-jn} \right) \left( \sum_{k \in \mathbb{Z}_{2N}} t^{-km} \right).$$

By (3.3), this is zero unless  $m = n = 0$ , in which case it is equal to  $4N^2$ . Hence  $RR^{-1} = 1 \otimes 1$ .

Having proven (1), (2) follows trivially from the fact that

$$\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1)) = \mathbb{C}[\mathbb{Z}_{2N}]$$

is both commutative and co-commutative.

Finally, to establish (3), one may compute directly from equation (3.6) that

$$\begin{aligned} R_{13}R_{12} &= \frac{1}{4N^2} \sum_{j,j',k,k' \in \mathbb{Z}_{2N}} t^{-jk-j'k'} K^{j+j'} \otimes K^{k'} \otimes K^k \\ &= \frac{1}{4N^2} \sum_{m \in \mathbb{Z}_{2N}} \sum_{j,k,k' \in \mathbb{Z}_{2N}} t^{-jk-(m-j)k'} K^m \otimes K^{k'} \otimes K^k \\ &= \frac{1}{4N^2} \sum_{m,k,k' \in \mathbb{Z}_{2N}} t^{-mk'} \left( \sum_{j \in \mathbb{Z}_{2N}} t^{j(k'-k)} \right) K^m \otimes K^{k'} \otimes K^k. \end{aligned}$$

Now by (3.3),  $\sum_{j \in \mathbb{Z}_{2N}} t^{j(k'-k)}$  is equal to zero, unless  $k' = k$ , in which case it is equal to  $2N$ . Hence,

$$R_{13}R_{12} = \frac{1}{2N} \sum_{m,k \in \mathbb{Z}_{2N}} t^{-mk} K^m \otimes K^k \otimes K^k = (\text{id} \otimes \Delta)[R]$$

which establishes (3a). Item (3b) follows from a similar argument.  $\square$

For further use we introduce the element  $u$ . In general, if  $R = \sum_j \alpha_j \otimes \beta_j$ , then  $u = \sum_j S(\beta_j)\alpha_j$ . In our particular situation

$$(3.8) \quad u = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-jk} K^{j-k}.$$

**3.3. The universal twist.** We want to prove the existence of a universal twist  $v$  that defines on  $(\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1)), R)$  a ribbon Hopf algebra structure.

Following [25], the maps

$$\phi^+ : V^k \rightarrow V^k, \quad \phi^+(x) := v^{-1}x$$

and

$$\phi^- : V^k \rightarrow V^k, \quad \phi^-(x) := vx$$

are intended to model a positive twist respectively a negative twist, as in Figure 7.

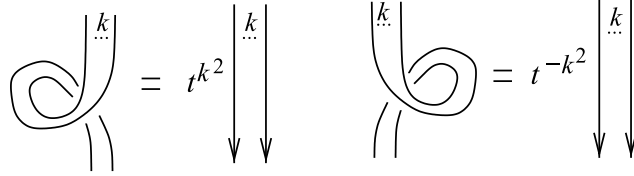


FIGURE 7. The skein relation for a positive twist of  $k$  strands

If we write  $v = \sum_{j \in \mathbb{Z}_{2N}} c_j K^j$ , then the above requirement leads us to the system of equations

$$(3.9) \quad \sum_{j \in \mathbb{Z}_{2N}} c_j t^{jk} = t^{-k^2}, \quad k \in \mathbb{Z}_{2N}$$

for the coefficients  $c_j$ . From the formula (3.5) for the matrix  $\mathcal{T}^{-1}$  introduced in the previous section, we find

$$c_j = \frac{1}{2N} \sum_{k \in \mathbb{Z}_{2N}} t^{-jk} t^{-k^2},$$

which yields

$$(3.10) \quad v = \frac{1}{2N} \sum_{j, k \in \mathbb{Z}_{2N}} t^{-k(k+j)} K^j.$$

Before proceeding further, we determine a simpler expression for  $v$ . Note that

$$\begin{aligned} \sum_{k \in \mathbb{Z}_{2N}} t^{-(k+j)k} &= \sum_{k=0}^{N-1} \left( t^{-(k+j)k} + t^{-(k+N+j)(k+N)} \right) \\ &= \sum_{k=0}^{N-1} t^{-(k+j)k} (1 + t^{-2kN - jN - N^2}) \\ &= (1 + (-1)^{j+N}) \sum_{k=0}^{N-1} t^{-(k+j)k}. \end{aligned}$$

Hence

$$\begin{aligned} v &= \frac{1}{2N} \sum_{j \in \mathbb{Z}_{2N}} \left[ (1 + (-1)^{j+N}) \sum_{k=0}^{N-1} t^{-(k+j)k} K^j \right] \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} t^{-(k+2j+N)k} K^{2j+N} = \frac{1}{N} \sum_{j, k \in \mathbb{Z}_N} (-1)^k t^{-(k+2j)k} K^{2j+N} \end{aligned}$$

$$= \frac{1}{N} \sum_{j,k \in \mathbb{Z}_N} (-1)^{k-j} t^{-(k+j)(k-j)} K^{2j+N}$$

yielding the formula

$$(3.11) \quad v = \frac{1}{N} \left( \sum_{k \in \mathbb{Z}_N} (-1)^k t^{-k^2} \right) \left( \sum_{j \in \mathbb{Z}_N} (-1)^j t^{j^2} K^{2j+N} \right).$$

A similar computation starting with the negative twist yields

$$v^{-1} = \frac{1}{N} \left( \sum_{k \in \mathbb{Z}_N} (-1)^k t^{k^2} \right) \left( \sum_{j \in \mathbb{Z}_N} (-1)^j t^{-j^2} K^{2j+N} \right).$$

Let us check that

$$v^{-1}v = vv^{-1} = 1.$$

Firstly

$$\begin{aligned} \left( \sum_{k \in \mathbb{Z}_N} (-1)^k t^{k^2} \right) \left( \sum_{k' \in \mathbb{Z}_N} (-1)^{k'} t^{-k'^2} \right) &= \sum_{k,k' \in \mathbb{Z}_N} (-1)^{k+k'} t^{(k-k')(k+k')} \\ &= \sum_{k,k' \in \mathbb{Z}_N} (-1)^k t^{k(k+2k')} \\ &= \sum_{k \in \mathbb{Z}_N} \left[ (-1)^k t^{k^2} \sum_{k' \in \mathbb{Z}_N} t^{2kk'} \right] = N \end{aligned}$$

where the last equality follows from (3.3). Hence

$$\begin{aligned} v^{-1}v &= \frac{1}{N} \left( \sum_{j \in \mathbb{Z}_N} (-1)^j t^{-j^2} K^{2j+N} \right) \left( \sum_{j' \in \mathbb{Z}_N} (-1)^{j'} t^{j'^2} K^{2j'+N} \right) \\ &= \frac{1}{N} \sum_{j,j' \in \mathbb{Z}_N} (-1)^{j+j'} t^{j'^2-j^2} K^{2j+2j'} \\ &= \frac{1}{N} \sum_{m \in \mathbb{Z}_N} \sum_{j \in \mathbb{Z}_N} (-1)^m t^{(m-j)^2-j^2} K^{2m} \\ &= \frac{1}{N} \sum_{m \in \mathbb{Z}_N} (-1)^m t^{m^2} \left( \sum_{j \in \mathbb{Z}_N} t^{-2mj} \right) K^{2m} = 1, \end{aligned}$$

where the last line follows from (3.3), proving that  $v$  and  $v^{-1}$  are the inverse of each other.

**Theorem 3.3.**  $(\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1)), R, v)$  is a ribbon Hopf algebra.

**Proof.** The fact that  $v$  is central is obvious since the Hopf algebra is commutative. We must check the following identities:

- (1)  $v^2 = S(u)u$ , where  $u$  is given by (3.8);
- (2)  $\Delta(v) = (P(R)R)^{-1}(v \otimes v)$ ;
- (3)  $S(v) = v$ ;
- (4)  $\epsilon(v) = 1$ .

Using (3.10) we have

$$v^2 = \frac{1}{4N^2} \sum_{j,k,l,m \in \mathbb{Z}_{2N}} t^{-k(j+k)-l(m+l)} K^j K^m$$

which after the change  $j \mapsto j - k$ ,  $m \mapsto m - l$  becomes

$$v^2 = \frac{1}{4N^2} \sum_{j,k,l,m \in \mathbb{Z}_{2N}} t^{-kj-lm} K^{j-k+l-m}.$$

On the other hand, using (3.8) we have

$$\begin{aligned} uS(u) &= \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-jk} K^{j-k} \frac{1}{2N} \sum_{m,l \in \mathbb{Z}_{2N}} t^{-ml} S(K^{m-l}) \\ &= \frac{1}{4N^2} \sum_{j,k,l,m \in \mathbb{Z}_{2N}} t^{-jk} t^{-ml} K^{j-k} K^{l-m} \\ &= \frac{1}{4N^2} \sum_{j,k,l,m \in \mathbb{Z}_{2N}} t^{-jk-ml} K^{j-k+l-m}. \end{aligned}$$

This proves (1).

Because  $P(R) = R$ ,  $(P(R)R)^{-1} = R^{-2}$ . To show (2) we need to compute  $R^{-2}$ . From (3.7) we calculate

$$\begin{aligned} R^{-2} &= \frac{1}{4N^2} \sum_{j,j',k,k' \in \mathbb{Z}_{2N}} t^{jk+j'k'} K^{-j-j'} \otimes K^{-k-k'} \\ &= \frac{1}{4N^2} \sum_{m,n \in \mathbb{Z}_{2N}} \sum_{j,k \in \mathbb{Z}_{2N}} t^{jk+(m-j)(n-k)} K^{-m} \otimes K^{-n} \\ &= \frac{1}{4N^2} \sum_{m,n,k \in \mathbb{Z}_{2N}} t^{m(n-k)} \left( \sum_{j \in \mathbb{Z}_{2N}} t^{(2k-n)j} \right) K^{-m} \otimes K^{-n} \\ &= \frac{1}{2N} \sum_{m,k \in \mathbb{Z}_{2N}} t^{mk} K^{-m} \otimes K^{-2k} = \frac{1}{2N} \sum_{m,k \in \mathbb{Z}_{2N}} t^{mk} K^m \otimes K^{2k} \\ &= \frac{1}{2N} \sum_{m \in \mathbb{Z}_{2N}} \sum_{k=0}^{N-1} (t^{mk} + t^{m(k+N)}) K^m \otimes K^{2k} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2N} \sum_{m \in \mathbb{Z}_{2N}} \left[ (1 + (-1)^m) \sum_{k=0}^{N-1} t^{mk} K^m \otimes K^{2k} \right] \\
 &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} t^{2mk} K^{2m} \otimes K^{2k}.
 \end{aligned}$$

where line 4 follows from (3.3). This yields the following formula for  $R^{-2}$ ,

$$(3.12) \quad R^{-2} = \frac{1}{N} \sum_{j,k \in \mathbb{Z}_N} t^{2jk} K^{2j} \otimes K^{2k}.$$

Next, using (3.11), we may write

$$v \otimes v = \frac{1}{N^2} \left( \sum_{r \in \mathbb{Z}_N} (-1)^r t^{-r^2} \right)^2 \left( \sum_{j,k \in \mathbb{Z}_N} (-1)^{j+k} t^{j^2+k^2} K^{2j+N} \otimes K^{2k+N} \right).$$

From this and (3.12), we obtain that  $R^{-2}(v \otimes v)$  is equal to

$$\begin{aligned}
 &\frac{1}{N^3} \left( \sum_{r \in \mathbb{Z}_N} (-1)^r t^{-r^2} \right)^2 \sum_{m,n,j,k \in \mathbb{Z}_N} (-1)^{j+k} t^{2mn+j^2+k^2} K^{2m+2j+N} \otimes K^{2n+2k+N} \\
 &= \frac{1}{N^3} \left( \sum_{r \in \mathbb{Z}_N} (-1)^r t^{-r^2} \right)^2 \sum_{s,s' \in \mathbb{Z}_N} \left( \sum_{j,k \in \mathbb{Z}_N} (-1)^{j+k} t^{2(s-j)(s'-k)+j^2+k^2} \right) \\
 &\quad \times K^{2s+N} \otimes K^{2s'+N} \\
 &= \frac{1}{N^3} \left( \sum_{r \in \mathbb{Z}_N} (-1)^r t^{-r^2} \right)^2 \sum_{s,s' \in \mathbb{Z}_N} t^{2ss'} \left( \sum_{j,k \in \mathbb{Z}_N} (-1)^{j+k} t^{(j+k)^2-2sk-2s'j} \right) \\
 &\quad \times K^{2s+N} \otimes K^{2s'+N}.
 \end{aligned}$$

Now consider the coefficient

$$\begin{aligned}
 \sum_{j,k \in \mathbb{Z}_N} (-1)^{j+k} t^{(j+k)^2-2sk-2s'j} &= \sum_{j,k \in \mathbb{Z}_N} (-1)^j t^{j^2-2sk-2s'(j-k)} \\
 &= \left( \sum_{j \in \mathbb{Z}_N} (-1)^j t^{j^2-2s'j} \right) \left( \sum_{k \in \mathbb{Z}_N} t^{2(s'-s)k} \right).
 \end{aligned}$$

By (3.3), the right-hand factor is zero, unless  $s = s'$ , in which case it is equal to  $N$ . The left-hand factor is

$$\sum_{j \in \mathbb{Z}_N} (-1)^j t^{j^2-2s'j} = t^{-s'^2} \sum_{j \in \mathbb{Z}_N} (-1)^j t^{(j-s')^2} = (-1)^{s'} t^{-s'^2} \sum_{j \in \mathbb{Z}_N} (-1)^j t^{j^2}.$$

Consequently,  $R^{-2}(v \otimes v)$  equals

$$\begin{aligned} & \frac{1}{N^2} \left( \sum_{r \in \mathbb{Z}_N} (-1)^r t^{-r^2} \right)^2 \left( \sum_{j \in \mathbb{Z}_N} (-1)^j t^{j^2} \right) \left( \sum_{s \in \mathbb{Z}_N} (-1)^s t^{s^2} K^{2s+N} \otimes K^{2s+N} \right) \\ &= \frac{1}{N} \left( \sum_{r \in \mathbb{Z}_N} (-1)^r t^{-r^2} \right) \left( \sum_{s \in \mathbb{Z}_N} (-1)^s t^{s^2} K^{2s+N} \otimes K^{2s+N} \right) = \Delta(v), \end{aligned}$$

where on the last line we have used the fact that  $\Delta(K^j) = K^j \otimes K^j$ . This proves (2).

Using (3.10), we have

$$\begin{aligned} S(v) &= \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-k(k+j)} S(K^j) = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-k(k+j)} K^{-j} \\ &= \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-(-k)(-k-j)} K^j = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-k(k+j)} K^j = v. \end{aligned}$$

This proves (3).

Finally, for (4), we have

$$\begin{aligned} \epsilon(v) &= \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-k(k+j)} \epsilon(K^j) = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-k(k+j)} \\ &= \frac{1}{2N} \sum_{k \in \mathbb{Z}_{2N}} t^{-k^2} \sum_{j \in \mathbb{Z}_{2N}} t^{-kj}. \end{aligned}$$

By (3.3) the second sum is 0 unless  $k = 0$ , in which case it is equal to  $2N$ . Hence the result of the computation is  $\frac{1}{2N} \cdot 2N = 1$ . The theorem is proved.  $\square$

It is important to point out that  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  is not a modular Hopf algebra. We will explain this in the next section, and return to this matter in [8].

## 4. Modeling classical theta functions using the quantum group

**4.1. Quantum link invariants.** Theorem 3.3 implies that the quantum group, which we shall denote

$$A_t := \mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$$

can be used to define invariants of oriented framed knots and links in  $S^3$ , the 3-dimensional sphere. Let us recall the construction, which is described in detail in the general setting in [25] and [29].

View  $S^3$  as  $\mathbb{R}^3$  compactified with the point at infinity, and in it fix a plane and a direction in the plane called the vertical direction. Given an oriented framed link  $L$  in  $S^3$ , deform it through an isotopy to a link whose framing is parallel to the plane, and whose projection onto the plane is a link diagram that can be sliced by finitely many horizontal lines into pieces, each of which consists of several vertical strands and exactly one of the events



from Figure 8. The lines in these figures will be oriented, inheriting their orientation from the orientation of the link.

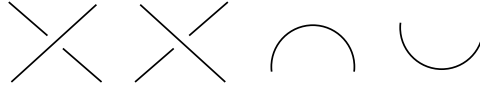


FIGURE 8. Any link diagram may be decomposed into a sequence involving the above events.

Now consider a coloring  $\mathbf{V}$  of the components of  $L$  by irreducible representations of  $A_t$ . This means that if  $L$  has components  $L_1, L_2, \dots, L_m$ , then  $\mathbf{V}$  is a map

$$\mathbf{V} : \{L_1, L_2, \dots, L_m\} \rightarrow \{V^0, V^1, \dots, V^{2N-1}\}.$$

We shall denote the data consisting of  $L$  together with its coloring  $\mathbf{V}$  by  $\mathbf{V}(L)$ . Each of the horizontal lines considered is crossed by finitely many strands of  $L$ . To a strand that crosses downwards, associate to that crossing point the corresponding irreducible representation decorating that strand; to a strand that crosses upwards, associate the dual of that representation. Then take the tensor product of all these irreducible representations for that given horizontal line, from left to right. This defines a representation of  $A_t$ . If the horizontal line does not intersect the link, the representation is automatically  $V^0$ .

The link diagram defines a composition of homomorphisms between the representations on each horizontal line when traveling in the vertical direction from below the link diagram to above the link diagram. To the events from Figure 8 we associate operators as follows:

- To a crossing like the first event in Figure 8 we associate  $\check{R} := P \circ R$ .
- To a crossing like the second event in Figure 8 we associate

$$\check{R}^{-1} := (P \circ R)^{-1}.$$

- If the event is like the third in Figure 8, then there are two possibilities depending on the orientation of the strand:
  - If the strand is oriented left to right, so that the homomorphism should be  $V^* \otimes V \rightarrow \mathbb{C}$ , then the homomorphism is

$$E(f \otimes x) := f(x).$$

- If the strand is oriented right to left, so that the homomorphism should be  $V \otimes V^* \rightarrow \mathbb{C}$ , then the homomorphism is

$$E_{op}(x \otimes f) := f(v^{-1}ux).$$

- If the event is like the fourth in Figure 8, then there are two possibilities depending on the orientation of the strand:

- If the strand is oriented right to left, so that the homomorphism should be  $\mathbb{C} \rightarrow V^* \otimes V$ , then the homomorphism is defined by

$$N_{op}(1) := e^k \otimes u^{-1}ve_k.$$

- If the strand is oriented left to right, so that the homomorphism should be  $\mathbb{C} \rightarrow V \otimes V^*$ , then the homomorphism is defined by

$$N(1) := e_k \otimes e^k.$$

Here  $e_k$  denotes the basis vector of  $V$  and  $e^k$  is the dual basis vector in  $V^*$ , meaning that  $e^k(e_m) = \delta_{km}$ . These operators are  $A_t$ -linear. Vertical strands away from the events define the identity operators, which are then tensored with the operators of the events to form a homomorphism between the representations associated to the horizontal lines.

The colored link diagram defines an endomorphism of  $V^0 = \mathbb{C}$ , which is given by multiplication by a complex number  $\langle \mathbf{V}(L) \rangle$ . Because  $A_t$  has the structure of a ribbon Hopf algebra, it follows from [24] (see also [29, §I.2, XI.3]) that  $\langle \mathbf{V}(L) \rangle$  is an invariant of colored framed oriented links, meaning that it depends only on the isotopy class of the link and not on its projection onto the plane.

In fact, according to [29], we may color these links using any representations of  $A_t$ , and the above algorithm still leads to an isotopy invariant. There is even a simple “cabling formula”, which states that if a link component is colored by the representation  $U \otimes W$ , then we may replace that link component by two parallel copies, one colored by  $U$  and the other by  $W$ . In particular, if our link component is colored by  $V^k$ , then we may replace it with  $k$  parallel copies which are each colored by  $V^1$ .

Furthermore, since this invariant is distributive with respect to direct sums of representations, its definition extends to links that are colored by elements of the representation ring of our quantum group  $A_t$ . We may describe this extension explicitly as follows. If  $L$  is an oriented framed link whose components  $L_1, \dots, L_m$  are colored by elements of the representation ring of  $A_t$ ,

$$(4.1) \quad \mathbf{V} : L_j \mapsto \sum_{k=0}^{2N-1} c_{jk} V^k, \quad 1 \leq j \leq m;$$

then

$$\langle \mathbf{V}(L) \rangle = \sum_{k_1=0}^{2N-1} \sum_{k_2=0}^{2N-1} \cdots \sum_{k_m=0}^{2N-1} c_{1k_1} c_{2k_2} \cdots c_{mk_m} \langle \mathbf{V}_{k_1, k_2, \dots, k_m}(L) \rangle;$$

where  $\mathbf{V}_{k_1, k_2, \dots, k_m}$  is the coloring of  $L$  that decorates the component  $L_j$  with the color  $V^{k_j}$ .

We wish to connect these colored links and their invariants to the skein theory of §2. We need the following definition.

**Definition 4.1.** Given an oriented 3-manifold  $M$ , we define  $\mathcal{V}_{A_t}(M)$  to be the vector space whose basis consists of isotopy classes of oriented framed links, whose components are colored by *irreducible* representations of  $A_t$ .

There is a map

$$(4.2) \quad \mathcal{V}_{A_t}(M) \rightarrow \mathcal{L}_N(M)$$

which is defined by the cabling formula which replaces any link component colored by  $V^j$  with  $j$  copies of that component that are parallel in the framing of the link component.

**Theorem 4.2.** *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{V}_{A_t}(S^3) & & \\ \text{cabling} \downarrow & \searrow \text{invariant} & \\ \mathcal{L}_N(S^3) & \xlongequal{\quad} & \mathbb{C} \end{array}$$

where the map on the right assigns to a colored link  $\mathbf{V}(L)$  its invariant  $\langle \mathbf{V}(L) \rangle$ , and that on the left is the cabling map (4.2).

**Proof.** This is a basic consequence of our construction of the quantum group  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  which we carried out in §3. Present the colored link diagram as a composition of horizontal slices, each containing exactly one event from Figure 8. Using the canonical basis elements  $e_j$  of  $V^j$ , the representation associated to each horizontal line may be identified with  $\mathbb{C}$ . Consequently, the homomorphism assigned to each horizontal slice is just multiplication by some complex number.

For the first event in Figure 8 with the “over” strand decorated by  $V^m$  and the “under” strand decorated by  $V^n$ , we have the following possibilities:

- If both strands are oriented downwards or both are oriented upwards, then the complex number is  $t^{mn}$ .
- If both strands are oriented to the right or to the left, then the number is  $t^{-mn}$ .

This is precisely what we get when we unlink  $n$  strands crossed by  $m$  strands, which should be so because we constructed the  $R$  matrix to satisfy this condition. A similar analysis for the second event with the same orientations yields respectively the numbers  $t^{-mn}, t^{mn}, t^{-mn}, t^{mn}$ , in agreement with what the linking number skein relations yield.

For the third event, if the strand is oriented from left to right and decorated by  $V^l$ , then  $E(e^l \otimes e_l) = e^l(e_l) = 1$ , so if we identify  $(V^l)^* \otimes V^l$  with  $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$ , then  $E$  is just multiplication by 1. We recall that here  $e_l$  is the basis element of  $V^l$ .

If the strand is oriented from right to left, then the operator is  $E_{op}$ , for which the situation is slightly more complicated. We compute

$$ue_l = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-jk} K^{j-k} e_l = \frac{1}{2N} \sum_{j,k \in \mathbb{Z}_{2N}} t^{-jk} t^{(j-k)l} e_l$$

$$= \frac{1}{2N} \sum_{k \in \mathbb{Z}_{2N}} t^{-kl} \sum_{j \in \mathbb{Z}_{2N}} (t^{l-k})^j e_l.$$

The inner sum is zero unless  $k \equiv l$ . So  $ue_l = t^{-l^2} e_l$ . But  $v$  was constructed so that  $ve_l = t^{-l^2} e_l$  and  $v^{-1}e_l = t^{l^2} e_l$ . Hence

$$E_{op}(e_l \otimes e^l) = e^l(v^{-1}ue_l) = 1$$

Similar computations show that the complex numbers corresponding to  $N$  and  $N_{op}$  are 1. So the homomorphisms  $\mathbb{C} \rightarrow \mathbb{C}$  associated to maxima and minima are trivial. In particular trivial knot components have the quantum invariant equal to 1, as required by the skein module picture. The theorem is proved.  $\square$

**Corollary 4.3.** *Let  $\mathbf{V}$  be a coloring of an oriented framed link  $L$  in  $S^3$  by irreducible representations, and suppose that some link component of  $L$  is colored by  $V^n$  with  $0 \leq n \leq N-1$ . If  $\mathbf{V}'$  denotes the coloring of  $L$  obtained by replacing the color of that link component by  $V^{n+N}$ , then*

$$\langle \mathbf{V}(L) \rangle = \langle \mathbf{V}'(L) \rangle.$$

It follows that we may factor the representation ring by the ideal generated by the single relation  $V^N = 1$ , without any affect on the invariants  $\langle \mathbf{V}(L) \rangle$ . Let

$$R(A_t) := \mathbb{C}[V^1] / ((V^1)^N - 1)$$

denote this quotient. Note that in  $R(A_t)$ ,

$$(V^k)^* = V^{N-k} \quad \text{and} \quad V^m \otimes V^n = V^{m+n \pmod{N}},$$

where these equalities should be interpreted formally (i.e., inside  $R(A_t)$ ). Thus, we can think of the link invariant defined above as an invariant of oriented framed links colored by elements of  $R(A_t)$ .

**4.2. Theta functions as colored links in a handlebody.** Consider the Heegaard decomposition

$$H_g \bigsqcup_{\partial H_g \approx \partial H_g} H_g = S^3$$

of  $S^3$  given by (2.16). This decomposition gives rise to a bilinear pairing

$$(4.3) \quad \begin{array}{ccccc} [\cdot, \cdot]_{\text{qgr}} : \mathcal{V}_{A_t}(H_g) \otimes \mathcal{V}_{A_t}(H_g) & \rightarrow & \mathcal{V}_{A_t}(S^3) & \rightarrow & \mathbb{C} \\ & & \mathbf{V}(L) \otimes \mathbf{V}'(L') & \mapsto & \mathbf{V}(L) \cup \mathbf{V}'(L') \end{array}$$

on  $\mathcal{V}_{A_t}(H_g)$ . Due to the obvious diffeomorphism of  $S^3$  that swaps the component handlebodies, this pairing is symmetric. However, it is far from being nondegenerate, which leads us to the following definition.

**Definition 4.4.** The vector space  $\tilde{\mathcal{L}}_{A_t}(H_g)$  is defined to be the quotient of the vector space  $\mathcal{V}_{A_t}(H_g)$  by the annihilator

$$\text{Ann}(\mathcal{V}_{A_t}(H_g)) := \{x \in \mathcal{V}_{A_t}(H_g) : [x, y]_{\text{qgr}} = 0, \text{ for all } y \in \mathcal{V}_{A_t}(H_g)\}$$

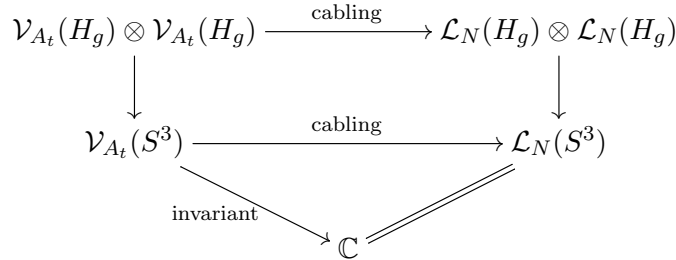
of the form (4.3).

The pairing induced on  $\tilde{\mathcal{L}}_{A_t}(H_g)$  by (4.3) is nondegenerate.

**Proposition 4.5.** *Consider the cabling map from  $\mathcal{V}_{A_t}(H_g)$  to  $\mathcal{L}_N(H_g)$  described in Definition 4.1. This map factors to an isomorphism,*

$$\tilde{\mathcal{L}}_{A_t}(H_g) \cong \mathcal{L}_N(H_g).$$

**Proof.** By Theorem 4.2, the following diagram commutes;



Since the cabling map from  $\mathcal{V}_{A_t}(H_g)$  to  $\mathcal{L}_N(H_g)$  is obviously surjective, the result follows from the fact that the pairing (2.17) appearing on the right of the above diagram is nondegenerate.  $\square$

**Corollary 4.6.** *The space  $\tilde{\mathcal{L}}_{A_t}(H_g)$  is isomorphic to the space of theta functions  $\Theta_N^{\text{II}}(\Sigma_g)$ .*

**Proof.** This is a consequence of Theorem 2.14.  $\square$

Consequently, we may represent the theta series  $\theta_k^{\text{II}}$  as colored links in the handlebody  $H_g$ . More precisely, the coloring represented by Figure 9 of  $a_1, \dots, a_g$  by  $V^{k_1}, \dots, V^{k_g}$ ,  $k_1, k_2, \dots, k_g \in \{0, 1, \dots, N - 1\}$  respectively, corresponds to the theta series  $\theta_{k_1, \dots, k_g}^{\text{II}}$ .

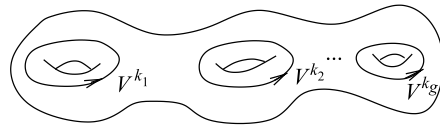


FIGURE 9. The presentation of  $\theta_{k_1, \dots, k_g}^{\text{II}}$  as a colored link in a handlebody.

**Remark 4.7.** At this point we should remark that theta functions are modeled using only half of the irreducible representations of the quantum group. This should be compared to the case of  $SU(2)$ , where there were 4 families of irreducible representations of the quantum group and only one was used for constructing the  $SU(2)$  Chern–Simons theory [25] and [15] and for modeling the nonabelian theta functions [11]. However, there is a major difference. In the present situation, the representations that model theta functions do not form a ring. This has an interesting consequence:

The quantum group  $\mathcal{U}_t(\mathfrak{u}_{\mathbb{C}}(1))$  is *not* a modular Hopf algebra when equipped with the irreducible representations  $V^1, V^2, \dots, V^{2N}$ .

This can be checked easily as follows. In the framework of [29], the representations of a modular Hopf algebra give rise to a modular category. The  $S$  matrix of this theory (§ 2.1.4 in [29]) has dimension  $2N$  and its  $j, k$  entry is equal to the quantum invariant associated to the Hopf link with components colored by  $V^j$  and  $V^k$  (Figure 10). This is equal to  $t^{jk}$ . Consequently

$$S = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix}$$

where  $A$  is  $N^{-1/2}$  times the standard discrete Fourier transform whose  $jk$  entry is  $e^{\frac{2\pi ijk}{N}}$ . Unfortunately this matrix is singular, violating the nondegeneracy axiom (1.4.4 in [29]).

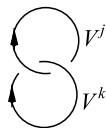


FIGURE 10. The  $S$  matrix.

This problem cannot be solved by identifying  $V^N$  with  $V^0$ , since this identification can only be done at the level of link invariants and not at the level of tangles. There is an approach in [27] that attempts to resolve this problem by twisting the standard associators for the tensor product of irreducible representations by roots of unity.

**4.3. The Schrödinger representation and the action of the mapping class groups via quantum group representations.** Consider the spaces  $\mathcal{V}_{A_t}(\Sigma_g \times [0, 1])$  and  $\mathcal{V}_{A_t}(H_g)$ . As before, by (2.11), we see that  $\mathcal{V}_{A_t}(\Sigma_g \times [0, 1])$  is an algebra and, by (2.12), that  $\mathcal{V}_{A_t}(H_g)$  is a module over this algebra. Since the cabling maps to the corresponding reduced skein modules are equivariant with respect to this action, it follows from Proposition 4.5 that this action descends to  $\tilde{\mathcal{L}}_{A_t}(H_g)$ .

We wish to cast the action of the finite Heisenberg group on the space of theta functions within this setting. Consider the surface  $\Sigma_g$  endowed with the canonical basis  $a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g$ . Then each element

$$(p, q) = (p_1, p_2, \dots, p_g, q_1, q_2, \dots, q_g)$$

in  $H_1(\Sigma_g, \mathbb{Z})$  can be represented as a multicurve on  $\Sigma_g$  as explained in Section 2.2, and by endowing it with the blackboard framing of  $\Sigma_g$  it can be turned into a skein in  $\mathcal{L}(\Sigma_g \times [0, 1])$ . Next, for an element of the finite Heisenberg group  $\mathbf{H}(\mathbb{Z}_N^g)$  of the form

$$(p, q, k), \quad p, q \in \{0, 1, \dots, N-1\}^g, k \in \{0, 1, \dots, 2N-1\}$$

consider the skein  $(p, q) \in \mathcal{L}(\Sigma_g \times [0, 1])$ . Add to it a trivial framed link component, with framing twisted  $k$  times. Color all link components by the representation  $V^1$  of  $A_t$ . Denote the resulting colored link by  $(p, q, k)_{\text{qgr}}$ . This is an element of  $\mathcal{V}_{A_t}(\Sigma_g \times [0, 1])$ .

**Theorem 4.8.** *Let  $\Phi : \Theta_N^{\Pi}(\Sigma_g) \rightarrow \tilde{\mathcal{L}}_{A_t}(H_g)$  denote the isomorphism of Corollary 4.6. Then*

$$\Phi[(p, q, k) \cdot f] = (p, q, k)_{\text{qgr}} \Phi(f); \quad (p, q, k) \in \mathbf{H}(\mathbb{Z}_N^g), f \in \Theta_N^{\Pi}(\Sigma_g).$$

**Proof.** Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{V}_{A_t}(\Sigma_g \times [0, 1]) \otimes \tilde{\mathcal{L}}_{A_t}(H_g) & \xrightarrow{\text{cabling}} & \mathcal{L}_N(\Sigma_g \times [0, 1]) \otimes \mathcal{L}_N(H_g) & \longleftarrow & \mathbb{C}(\mathbf{H}(\mathbb{Z}_N^g)) \otimes \Theta_N^{\Pi}(\Sigma_g) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{L}}_{A_t}(H_g) & \xrightarrow[\cong]{\text{cabling}} & \mathcal{L}_N(H_g) & \longleftarrow[\cong]{} & \Theta_N^{\Pi}(\Sigma_g) \end{array}$$

where we have used Proposition 4.5 and Theorems 2.12 and 2.14. Since the image of  $(p, q, k)$  in  $\mathcal{L}_N(\Sigma_g \times [0, 1])$  under (2.13) coincides with the image of  $(p, q, k)_{\text{qgr}}$  under the cabling map, this proves the result.  $\square$

The following corollary is the abelian analogue of the main result in [9].

**Corollary 4.9.** *The Weyl quantization and the quantum group quantization of the moduli space of flat  $\mathfrak{u}(1)$ -connections on a closed surface are unitary equivalent.*

We can describe the reduced linking number skein algebra  $\mathcal{L}_N(\Sigma_g)$  as a quotient of  $\mathcal{V}_{A_t}(\Sigma_g \times [0, 1])$  and within it we can find a quantum group model of the finite Heisenberg group  $\mathbf{H}(\mathbb{Z}_N^g)$ .

Embed the cylinder  $\Sigma_g \times [0, 1]$  in the standard way in  $S^3$ , so that on each side lies one handlebody. We then have a decomposition of the 3-dimensional sphere as

$$(4.4) \quad S^3 = H_g \bigsqcup (\Sigma_g \times [0, 1]) \bigsqcup H_g.$$

Let  $L$  be an oriented framed link in  $\Sigma_g \times [0, 1]$  endowed with a coloring  $\mathbf{V}$  by representations of  $A_t$ . Insert the colored link in (4.4). Then  $\mathbf{V}(L)$  defines a bilinear pairing

$$\mathcal{V}_{A_t}(H_g) \otimes \mathcal{V}_{A_t}(H_g) \rightarrow \mathbb{C}$$

via the Reshetikhin–Turaev invariant in  $S^3$ . This pairing descends to a pairing

$$(4.5) \quad [\cdot, \cdot]_{\mathbf{V}(L)} : \mathcal{L}_{A_t}(H_g) \otimes \mathcal{L}_{A_t}(H_g) \rightarrow \mathbb{C}.$$

Because  $[\cdot, \cdot]_{\text{qgr}}$  is nondegenerate, the bilinear map (4.5) defines a linear map

$$\text{Op}(\mathbf{V}(L)) : \mathcal{L}_{A_t}(H_g) \rightarrow \mathcal{L}_{A_t}(H_g),$$

by

$$[\text{Op}(\mathbf{V}(L))x, y]_{\text{qgr}} = [x, y]_{\mathbf{V}(L)}.$$

Using the identification of  $\mathcal{L}_{A_t}(H_g)$  with  $\Theta_N^{\Pi}(\Sigma_g)$ , we deduce that

$$\text{Op}(\mathbf{V}(L)) \in L(\Theta_N^{\Pi}(\Sigma_g)).$$

**Example 4.10.** Let us consider the operator obtained by decorating the  $(1, 1)$  curve on the torus by the irreducible representation  $V^m$ . In this situation  $\Pi$  is just a complex number in the upper half-plane, and it is customary to denote it by  $\tau$ . The theta series  $\theta_j^{\tau}(z)$  is represented in the solid torus by the curve that is the core of the solid torus, decorated by  $V^j$ .

$$[\theta_j^{\tau}(z), \theta_k^{\tau}(z)]_{\text{qgr}} = t^{-2jk}.$$

The operator  $\text{Op}(V^m(1, 1))$  defined by coloring the  $(1, 1)$  curve by  $V^m$  is determined by requiring that for every  $j, k = 0, 1, 2, \dots, N - 1$ ,

$$[\text{Op}(V^m(1, 1))\theta_j^{\tau}(z), \theta_k^{\tau}(z)]_{\text{qgr}}$$

is equal to the Reshetikhin–Turaev invariant of the link from Figure 11.

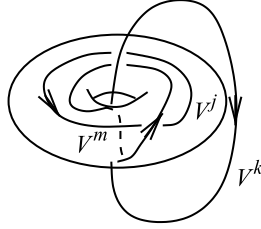


FIGURE 11. The  $jk$  entry of  $\text{Op}(\mathbf{V}(1, 1))$

This operator coincides with the one defined by the skein

$$\langle (m, m) \rangle \in \mathcal{L}_N(\Sigma_g \times [0, 1])$$

acting on  $\mathcal{L}_N(H_g)$ .

The cabling map

$$\mathcal{V}_{A_t}(\Sigma_g \times [0, 1]) \rightarrow \mathcal{L}_N(\Sigma_g \times [0, 1]),$$

is onto. Let  $\mathcal{L}_{A_t}(\Sigma_g \times [0, 1])$  be the quotient of  $\mathcal{V}_{A_t}(\Sigma_g \times [0, 1])$  by the kernel of this map. Then  $\mathcal{L}_{A_t}(\Sigma_g \times [0, 1])$  is an algebra isomorphic to  $\mathcal{L}_N(\Sigma_g \times [0, 1])$ , which is therefore isomorphic to the algebra of linear operators on the space of theta functions.

Intuitively, the algebra  $L(\Theta_N^{\Pi}(\Sigma_g))$  is an algebra of oriented simple closed curves on  $\Sigma_g$  colored by irreducible representations of  $A_t$ .

In particular,  $\mathbf{H}(\mathbb{Z}_N^g)$  lies inside  $\mathcal{L}_{A_t}(\Sigma_g \times [0, 1])$  consisting of the equivalence classes of framed oriented links colored by irreducible representations. Its elements are  $(p, q, k)_{\text{qgr}}$ ,  $p, q \in \{0, 1, \dots, N - 1\}^g$ ,  $k \in \{0, 1, \dots, 2N - 1\}$ .

This construction allows us to define a star product for exponential functions using quantum groups. This is a particular case of the general construction from [1]. Hence we have a quantum group model for the  $*$ -product



of the quantum torus. We point out that the key ingredient in defining the  $*$ -product is the  $R$ -matrix, since the  $R$ -matrix determines the skein relation by which we smooth the crossings, and hence defines the multiplication rule in  $\mathcal{L}_N(\Sigma_g \times [0, 1]) = L(\Theta_N^{\Pi}(\Sigma_g))$ .

Next, denote by  $\mathcal{V}'_{A_t}(\Sigma_g \times [0, 1])$  the vector space that is freely generated by isotopy classes of oriented framed links colored by elements of the *representation ring* of  $A_t$ . There is a multiplicative map from this space onto  $\mathcal{V}_{A_t}(\Sigma_g \times [0, 1])$ ; if  $L$  is a link with  $m$  components whose coloring  $\mathbf{V}$  by the representation ring is written as in (4.1), then this map is given by

$$\mathbf{V}(L) \mapsto \sum_{k_1, \dots, k_m=0}^{2N-1} c_{1k_1} \cdots c_{mk_m} \mathbf{V}_{k_1, \dots, k_m}(L);$$

where  $\mathbf{V}_{k_1, \dots, k_m}$  is the coloring of  $L$  that decorates the component  $L_j$  with the color  $V^{k_j}$ .

Consequently,  $\tilde{\mathcal{L}}_{A_t}(H_g)$  is a module over  $\mathcal{V}'_{A_t}(\Sigma_g \times [0, 1])$ . In fact, in view of Corollary 4.3, we may assume that our links are colored by elements of the quotient ring  $R(A)$ . If  $L$  is an oriented framed link in  $\Sigma_g \times [0, 1]$ , we denote by  $\Omega_{A_t}(L)$  the element of  $\mathcal{V}'_{A_t}(\Sigma_g \times [0, 1])$  that is obtained by coloring each component of  $L$  by  $N^{-\frac{1}{2}} \sum_{k=0}^{N-1} V^k$ .

**Proposition 4.11.** *Let  $h_L \in \mathcal{M}_{\Sigma_g}$  be a diffeomorphism that is represented by surgery on a framed link  $L$ . By Corollary 4.6 we may consider the discrete Fourier transform  $\rho(h_L)$  as an endomorphism of  $\tilde{\mathcal{L}}_{A_t}(H_g)$ . This endomorphism is given (projectively) by:*

$$\rho(h_L)[\beta] = \Omega_{A_t}(L) \cdot \beta, \quad \beta \in \tilde{\mathcal{L}}_{A_t}(H_g).$$

**Proof.** Since the image of  $\Omega_{A_t}(L)$  in  $\mathcal{L}_N(\Sigma_g \times [0, 1])$  under the cabling map coincides with  $\Omega(L)$ , this is a consequence of Theorem 2.17 and the fact that the cabling map is equivariant.  $\square$

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