

Knot cabling and the degree of the colored Jones polynomial

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ABSTRACT. We study the behavior of the degree of the colored Jones polynomial and the boundary slopes of knots under the operation of cabling. We show that, under certain hypothesis on this degree, if a knot K satisfies the Slope Conjecture then a (p, q) -cable of K satisfies the conjecture, provided that p/q is not a Jones slope of K . As an application we prove the Slope Conjecture for iterated cables of adequate knots and for iterated torus knots. Furthermore we show that, for these knots, the degree of the colored Jones polynomial also determines the topology of a surface that satisfies the Slope Conjecture. We also state a conjecture suggesting a topological interpretation of the linear terms of the degree of the colored Jones polynomial (Conjecture 5.1), and we prove it for the following classes of knots: iterated torus knots and iterated cables of adequate knots, iterated cables of several nonalternating knots with up to nine crossings, pretzel knots of type $(-2, 3, p)$ and their cables, and two-fusion knots.

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1. Introduction

1.1. The Slope Conjecture. For a knot $K \subset S^3$, let $n(K)$ denote a tubular neighborhood of K and let

$$M_K := \overline{S^3 \setminus n(K)}$$

denote the exterior of K . Let $\langle \mu, \lambda \rangle$ be the canonical meridian-longitude basis of $H_1(\partial n(K))$. An element $a/b \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of K if there is a properly embedded essential surface $(S, \partial S) \subset (M_K, \partial n(K))$, such that ∂S represents $a\mu + b\lambda \in H_1(\partial n(K))$. Hatcher showed that every knot $K \subset S^3$ has finitely many boundary slopes [15]. We will use bs_K to denote the set of boundary slopes of K .

For a positive integer n , let $J_K(n) \in \mathbb{Z}[v^{\pm 1/2}]$ be the n -th colored Jones polynomial of K with framing 0 [18, 27], normalized so that

$$J_{\text{unknot}}(n) = \frac{v^{n/2} - v^{-n/2}}{v^{1/2} - v^{-1/2}}.$$

Here $v = A^{-4}$, where A is the variable in the Kauffman bracket [20].

For a sequence $\{x_n\}$, let $\{x_n\}'$ denote the set of its cluster points. Let $d_+[J_K(n)]$ denote the highest degree of $J_K(n)$ in v , and let $d_-[J_K(n)]$ denote the lowest degree. Elements of the sets

$$js_K := \{4n^{-2}d_+[J_K(n)]\}' \quad \text{and} \quad js_K^* := \{4n^{-2}d_-[J_K(n)]\}'$$

are called *Jones slopes* of K . Garoufalidis [9] showed that every knot has finitely many Jones slopes. Furthermore, he formulated the following conjecture and he verified it for alternating knots, nonalternating knots with up to nine crossings, torus knots, and for the family $(-2, 3, p)$ of 3-string pretzel knots [10].

Conjecture 1.1 (Slope Conjecture). *For every knot $K \subset S^3$ we have*

$$(js_K \cup js_K^*) \subset bs_K.$$

Futer, Kalfagianni and Purcell [4] verified the conjecture for *adequate knots* (see Definition 3.5 for terminology). The works of Garoufalidis and Dunfield [3] and Garoufalidis and van der Veen [11] verified the conjecture for a certain 2-parameter family of closed 3-braids, called 2-fusion knots. More recently, Lee and van der Veen [23] have proved the conjecture for several more 3-string pretzel knots.

In this paper we study the behavior of the boundary slopes and the Jones slopes of knots under the operation of cabling and prove the Slope Conjecture for cables of several classes of knots. We also formulate, and verify for several classes of knots, conjectures providing topological interpretations of the linear terms of the degree of the colored Jones polynomial (Conjectures 1.6 and 5.1). To state our results we need some preparation.

1.2. Cable knots. Suppose K is a knot with framing 0, and p, q are coprime integers. The (p, q) -cable $K_{p,q}$ of K is the 0-framed satellite of K with pattern (p, q) -torus knot (see Section 2 for more details). In the statements of results below, and throughout the paper, we will assume that our cables are nontrivial in the sense that $|q| > 1$.

Theorem 1.2. *For every knot $K \subset S^3$ and (p, q) coprime integers we have*

$$(q^2 bs_K \cup \{pq\}) \subset bs_{K_{p,q}}.$$

To continue, we recall that for any knot $K \subset S^3$ the degrees $d_+[J_K(n)]$ and $d_-[J_K(n)]$ are quadratic quasi-polynomials in n [9]. This implies that their coefficients are periodic functions $\mathbb{N} \rightarrow \mathbb{Q}$. The least common multiple of the periods of $d_+[J_K(n)]$ and $d_-[J_K(n)]$ is called *the period of K* , denoted by $\pi(K)$.

In this paper we are concerned with the Jones slopes for knots with $\pi(K) \leq 2$ and for knots where the leading coefficient of $d_+[J_K(n)]$ becomes constant for n large enough. We show that the Jones slopes of these knots behave similarly to boundary slopes under cabling operations (Propositions 3.2 and 4.4). In particular, for knots with period at most two, combining our results about Jones slopes with Theorem 1.2 we obtain the following.

Theorem 1.3. *Let K be a knot such that, for $n \gg 0$,*

$$d_+[J_K(n)] = a(n)n^2 + b(n)n + d(n)$$

and

$$d_-[J_K(n)] = a^*(n)n^2 + b^*(n)n + d^*(n)$$

are quadratic quasi-polynomials of period ≤ 2 , with $b(n) \leq 0$ and $b^*(n) \geq 0$. Suppose $\frac{p}{q} \notin js_K$. Then, we have

$$js_{K_{p,q}} \subset (q^2 js_K \cup \{pq/4\}) \quad \text{and} \quad js_{K_{p,q}}^* \subset (q^2 js_K^* \cup \{pq/4\}).$$

Furthermore, if $(js_K \cup js_K^*) \subset bs_K$ we have $(js_{K_{p,q}} \cup js_{K_{p,q}}^*) \subset bs_{K_{p,q}}$.

The proof of Theorem 1.3 reveals that the properties that $b(n) \leq 0$ and $b^*(n) \geq 0$ are preserved under cabling. We conjecture that these properties hold for all nontrivial knots, and that $b(n)$ and $b^*(n)$ detect the presence of essential annuli in the knot complement. This is stated in Conjecture 5.1, which we have verified for all the knots for which the degrees $d_+[J_K(n)]$ and $d_-[J_K(n)]$ are known.

Note that since we take $|q| > 1$, the hypothesis that p/q is not a Jones slope of K will automatically be satisfied for knots that have all of their Jones slopes integers. A large class of knots with integer Jones slopes is the class of *adequate* knots, which includes alternating knots, Montesinos knots of length at least four, pretzel knots with at least four strings and Conway sums of strongly alternating tangles. The class of *semi-adequate* knots (knots that are *A*- or *B*-adequate) is much broader including all but a handful of prime knots up to 12 crossings, all Montesinos and pretzel knots, positive knots, torus knots, and closed 3-braids. The reader is referred to Section 3 below for the precise definition (Definition 3.5) and to [7, 5, 6, 12] and references therein for more details and examples of semi-adequate knots.

Theorem 1.4. *Let K be a knot and let K' be an iterated cable knot of K .*

- (1) *If K is a B -adequate knot, then $js_{K'} \subset bs_{K'}$.*
- (2) *If K is an A -adequate knot, then $js_{K'}^* \subset bs_{K'}$.*

Hence, if K is an adequate knot then K' satisfies the Slope Conjecture.

An iterated torus knot is an iterated cable of the trivial knot. As a corollary of Theorem 1.4 we have the following.

Corollary 1.5. *Iterated torus knots satisfy the Slope Conjecture.*

Theorem 1.3 also applies to several nonalternating prime knots with up to nine crossings (see Corollary 3.10). We should mention that Motegi–Takata [26] used Theorem 1.3 to generalize Corollary 1.5 to all knots of zero Gromov norm (graph knots).

The proofs of Theorems 1.2 and 1.3 reveal that there is a remarkable similarity in the behaviors, under cabling, of the linear terms of $d_+[J_K(n)]$ and $d_-[J_K(n)]$ and the Euler characteristic of the essential surfaces “selected” by the Slope Conjecture. We conjecture that the cluster points of the sets $\{2b_K(n)\}$ and $\{2b_K^*(n)\}$ contain information about the topology of essential surfaces that satisfy the Slope Conjecture for K . To state the conjecture, let $ld_+[J_K(n)]$ denote the linear term of $d_+[J_K(n)]$ and let

$$jx_K := \{2n^{-1}ld_+[J_K(n)]\}' = \{2b_K(n)\}'.$$

Conjecture 1.6 (Strong Slope Conjecture). *Let K be a knot and let $a/b \in js_K$, with $b > 0$ and $\gcd(a, b) = 1$, be a Jones slope of K . Then there is an essential surface $S \subset M_K$, with $|\partial S|$ boundary components, and such that each component of ∂S has slope a/b and*

$$\frac{\chi(S)}{|\partial S|b} \in jx_K.$$

Conjecture 1.6 implies a similar statement for $\{2b_K^*(n)\}'$ since it is known that $d_-[J_K(n)] = -d_+[J_{K^*}(n)]$, where K^* denotes the mirror image of K . An immediate corollary of Theorem 3.9 is the following:

Corollary 1.7. *Iterated cables of adequate knots satisfy the Strong Slope Conjecture. In particular, iterated torus knots satisfy the Strong Slope Conjecture.*

We also prove Conjecture 1.6 for pretzel knots of type $(-2, 3, p)$ and all Montesinos knots with up to nine crossings (see Section 5).

1.3. Organization. The paper is organized as follows. In Section 2 we study boundary slopes of cable knots and we prove Theorem 1.2. In Sections 3 and 4 we study the behavior of the degree of the colored Jones polynomial under knot cabling. In particular, we discuss cables of knots of period at most two and we prove Theorem 1.3. In fact, the proof of this theorem allows us to describe explicitly how the Jones slopes of the cable knot $K_{p,q}$ are related to those of the original knot K . We apply our results on knots of period at most two to adequate knots to prove Theorem 1.4. In Section 5 we state, and partially verify, some conjectures about the degree of the colored Jones polynomial. Finally in Section 6 we verify Conjecture 5.1 for two-fusion knots.

2. Boundary slopes of cable knots

In this section we study how the boundary slopes of knots in S^3 affect the boundary slopes of their cables. The main result is Theorem 2.2 that implies in particular Theorem 1.2 stated in the Introduction. Theorem 2.2 and Corollary 2.8 are key ingredients in the proofs of the results of the paper concerning relations of the colored Jones polynomial to essential surfaces.

2.1. Preliminaries and statement of main result. Let V be a standardly embedded solid torus in S^3 and let $V' \subset V$ be a second standard solid torus that is concentric to V . On $\partial V'$ we choose a pair of meridian and canonical longitude (which also determines such a pair on ∂V). For coprime integers p, q , let $T_{p,q} \subset \partial V'$ be a simple closed curve of slope p/q ; that is a (p, q) -torus knot.

Recall that for a knot K , $n(K)$ denotes a neighborhood of K . Embed V in S^3 by a homeomorphism $f : V \rightarrow n(K)$ that preserves the homology classes of the canonical longitudes. The (p, q) -cable of K is the image $K_{p,q} := f(T_{p,q})$. The space

$$C_{p,q} := f(\overline{V \setminus n(T_{p,q})})$$

is called a (p, q) -cable space. The complement of $K_{p,q}$, denoted by $M_{K_{p,q}}$, is obtained from the complement of K by attaching $C_{p,q}$. The space $C_{p,q}$ has two boundary components; the inner one $T_- = f(\partial n(T_{p,q})) = \partial M_{K_{p,q}}$ and the outer one $T_+ = f(\partial V) = \partial M_K$.

Definition 2.1. For a knot $K \subset S^3$, let $\langle \mu, \lambda \rangle$ be the canonical meridian-longitude basis of $H_1(\partial n(K))$. For a pair of integers (a, b) , the ratio $a/b \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of K if there is a properly embedded essential surface $(S, \partial S) \subset (M_K, \partial n(K))$, such that ∂S represents $a\mu + b\lambda \in H_1(\partial n(K))$.

In Definition 2.1, a, b do not need to be coprime. In fact if $d = \gcd(a, b)$ then we have a surface S as above with d boundary components. To stress this point sometimes we will say that the *total slope* of ∂S is a/b . Recall that every knot K has finitely many boundary slopes and that bs_K denotes the set of boundary slopes of K . The rest of the section is devoted to the proof of the following theorem.

Theorem 2.2.

- (a) Let $K \subset S^3$ be a nontrivial knot and (p, q) coprime integers. If a/b is a boundary slope of K , then $q^2 a/b$ is a boundary slope of $K_{p,q}$.
- (b) For every knot $K \subset S^3$ and (p, q) coprime integers, we have

$$(q^2 bs_K \cup \{pq\}) \subset bs_{K_{p,q}}.$$

The reader is referred to [14, 16] for basic definitions and terminology. Let (p, q) be coprime integers. The cable space $C_{p,q}$ is a Seifert fibered manifold over an annulus B , with one singular fiber of multiplicity q . In $C_{p,q}$ there is an essential annulus A , that is vertical with respect to the fibration, with $\partial A \subset T_-$ and with boundary slope equal to pq ; this annulus is the *cabling annulus*. There are two additional essential annuli in $C_{p,q}$. One with both boundary components on T_- , each with slope p/q . The other annulus A' has one component of $\partial A'$ on T_- with slope pq , and the second component of $\partial A'$ on T_+ with slope p/q . See [13]. In particular, we have the following.

Lemma 2.3. For any cable knot $K_{p,q}$, $s = pq$ is a boundary slope in $M_{K_{p,q}}$.

Recall that a properly embedded surface S in a 3-manifold M with boundary, is *essential* if the map on π_1 induced by inclusion is injective. If S is orientable this is equivalent to saying that S is *incompressible* and ∂ -*incompressible* in M . If S is nonorientable, then S being essential is equivalent to saying that the surface $\tilde{S} := \overline{\partial(S \times I) \setminus \partial M}$ is incompressible and ∂ -incompressible in M .

We need the following lemma, a proof of which is given, for example, in [21, Proposition 1.1].

Lemma 2.4. Let M be a knot complement in S^3 and let Σ be a properly embedded essential surface in M . Suppose that a path $\alpha \subset \Sigma$ that has its endpoints on $\partial \Sigma$ is homotopic relative endpoints in M to a path in ∂M . Then α is homotopic relative endpoints in Σ to a path in $\partial \Sigma$.

The complement $M_{K_{p,q}}$ of the cable knot $K_{p,q}$ is obtained by gluing $C_{p,q}$ and the complement of K along the torus T_+ . If K is a nontrivial knot, then the torus T_+ is essential in $M_{K_{p,q}}$; that is a companion of $K_{p,q}$.

We will use $M_{K_{p,q}} \setminus \setminus T_+$ to denote the 3-manifold obtained by splitting $M_{K_{p,q}}$ along T_+ . Also given a properly embedded surface S in $M_{K_{p,q}}$ we will use $S \setminus \setminus T_+$ to denote the image of S in $M_{K_{p,q}} \setminus \setminus T_+$.

Lemma 2.5. *Let $K \subset S^3$ be a nontrivial knot and (p, q) coprime integers. Let S be a properly embedded surface in $M_{K_{p,q}}$. Suppose that each component of $S \setminus \setminus T_+$ is essential in the component of $M_{K_{p,q}} \setminus \setminus T_+$ it lies in. Then S is essential in $M_{K_{p,q}}$.*

Proof. Since S may be nonorientable we will work with the orientable double \tilde{S} . By way of contradiction, suppose that \tilde{S} is not π_1 -injective. This means that \tilde{S} is either compressible or ∂ -compressible. Since $\partial M_{K_{p,q}}$ consists of tori, incompressibility implies ∂ -incompressibility [14]. Thus we may assume that there is a compression disk $(E, \partial E) \subset (M_{K_{p,q}}, \tilde{S})$. Since each component of $\tilde{S} \setminus \setminus T_+$ is incompressible, the intersection $E \cap T_+$ must be nonempty. Since T_+ is essential (and thus incompressible in $M_{K_{p,q}}$) we may eliminate the closed components of $E \cap T_+$. Thus we may assume that each component of $E \cap T_+$ is an arc properly embedded in E . By further isotopy of E , during which ∂E moves on \tilde{S} , we may assume that the intersection $E \cap T_+$ is minimal. Now let α be a component of $E \cap T_+$ that is outermost on E : It cuts off a disc $E' \subset E$ whose interior contains no further intersections with T_+ . Now $\partial E'$ consists of α and an arc β that is properly embedded on a component, say Σ , of $\tilde{S} \setminus \setminus T_+$. We can use E' to isotope β , relatively $\partial\beta$, on T_+ ; this isotopy takes place in $M_{K_{p,q}} \setminus \setminus T_+$. Since Σ is essential in the component of $M_{K_{p,q}} \setminus \setminus T_+$ it lies in, we conclude that the arc β may be isotoped on Σ , relatively $\partial\beta$, to an arc on $\partial\Sigma$. This follows from Lemma 2.4. But this isotopy will reduce the components of the intersection $E \cap T_+$, contradicting our assumption of minimality. Thus \tilde{S} must be incompressible and therefore, by above discussion, essential in $M_{K_{p,q}}$. \square

2.2. Boundary slopes and homology of cable spaces. A slope s on a torus T is the isotopy class of a simple closed curve on T . Let $\mathcal{S}(T)$ denote the set of slopes of T . The elements in $\mathcal{S}(T)$ are represented by elements of $\mathbb{Q} \cup \{1/0\}$. With this in mind we will often refer to a slope by its corresponding numerical value. A homology class in $H_1(T, \mathbb{Z})$ is called *primitive* if it is not a nontrivial integer multiple of another element in $H_1(T, \mathbb{Z})$. There is a 2-1 correspondence between primitive classes in $H_1(T, \mathbb{Z})$ and elements in $\mathcal{S}(T)$, where $\alpha, \beta \in H_1(T, \mathbb{Z})$ give the same slope if and only if $\alpha = \pm\beta$.

Next we need the following lemma.

Lemma 2.6. *Consider a cable knot complement $M_{K_{p,q}} = C_{p,q} \cup M_K$ as above. Let $s \in \mathcal{S}(T_-)$ be a slope on T_- of $C_{p,q}$, corresponding to $a/b \in \mathbb{Q} \cup \{1/0\}$. Suppose that in $C_{p,q}$ we have a properly embedded, connected, essential surface F such that:*

- (1) *The boundary ∂F intersects both of T_- and T_+ .*
- (2) *Each component of ∂F on T_- has slope s .*

(3) *The total slope of $\partial F \cap T_+$ is a boundary slope in M_K .*

Then a/b is a boundary slope in $M_{K_{p,q}}$.

Proof. Suppose that the total slope of $\partial F \cap T_+$ corresponds to $c/d \in \mathbb{Q} \cup \{1/0\}$. Let E be a essential surface in M_K with ∂E having total slope c/d . By passing to the doubles if necessary, we may assume that E and F are orientable. Let x and y denote the number of components of ∂F and ∂E on T_+ , respectively. In $C_{p,q}$ consider a surface F' that is y copies of F and in M_K consider a surface E' that is x copies of E . Each of $\partial E'$ and $\partial F'$ has xy components on T_+ . After isotopy on T_+ we may assume that $\partial E' = \partial F'$. Now $S = E' \cup F'$ is a properly embedded surface in $M_{K_{p,q}}$. By assumption, each component of $S \setminus T_+$ is essential in the component of $M_{K_{p,q}} \setminus T_+$ it lies in. Thus by Lemma 2.5, S is essential in $M_{K_{p,q}}$. \square

The following lemma should be compared with [21, Lemma 2.3].

Lemma 2.7. *Let $C_{p,q}$ be a cable space with $\partial C = T_- \cup T_+$ as above. Let \mathcal{S}_- and \mathcal{S}_+ denote the set of slopes on T_- and T_+ respectively. There is a bijection*

$$\phi : \mathcal{S}_- \longrightarrow \mathcal{S}_+,$$

such that for every $s \in \mathcal{S}_-$ there is a connected, essential properly embedded surface $F \subset C_{p,q}$, intersecting both components of $\partial C_{p,q}$, and such that each component $\partial F \cap T_-$ has slope s while each component of $\partial F \cap T_+$ has slope $\phi(s)$.

Proof. To simplify our notation, throughout this proof, we will use $X := C_{p,q}$. Identify $H_1(\partial X; \mathbb{Q})$ with $H_1(T_-; \mathbb{Q}) \oplus H_1(T_+; \mathbb{Q})$.

We claim that the maps $i_{\pm} : H_1(T_{\pm}; \mathbb{Q}) \longrightarrow H_1(X; \mathbb{Q})$, induced by the inclusions of T_{\pm} in X are isomorphisms. To see that i_- is an isomorphism consider the solid torus $X \cup n(T_{p,q})$, and apply the Mayer–Vietoris long exact sequence to this decomposition. To see that i_+ is an isomorphism decompose X into a fibered solid torus and an I -bundle $T_+ \times I$ along a vertical annulus and again apply the Mayer–Vietoris long exact sequence.

The fact that i_{\pm} are isomorphisms implies the following: Given a primitive class $\alpha_- \in H_1(T_-; \mathbb{Z}) \subset H_1(T_-; \mathbb{Q})$ there is a unique primitive class $\alpha_+ \in H_1(T_+; \mathbb{Z}) \subset H_1(T_+; \mathbb{Q})$ so that

$$(2.1) \quad \alpha_+ = r i_+^{-1} \circ i_-(\alpha_-), \text{ for some } r \in \mathbb{Q}.$$

As discussed above, a slope $s \in \mathcal{S}_{\pm}$ determines a primitive class $\alpha_{\pm} \in H_1(T_{\pm}; \mathbb{Z})$ up to sign. Given $s \in \mathcal{S}_-$, determining a primitive element $\alpha_- \in H_1(T_-, \mathbb{Z})$ up to sign, define $\phi(s)$ to be the slope in \mathcal{S}_+ that describes the class $\alpha_+ \in H_1(T_+, \mathbb{Z})$ defined in Equation (2.1). This clearly defines a bijection.

By above discussion, there are relatively prime integers m, n such that

$$m i_-(\alpha_-) + n i_+(\alpha_+) = 0.$$

Thus the element $m\alpha_- + n\alpha_+$ is in the kernel of the map

$$i_- \oplus i_+ : H_1(\partial X; \mathbb{Z}) \longrightarrow H_1(X; \mathbb{Z}).$$

Looking at the homology long exact sequence for the pair $(X, \partial X)$, we conclude that there is a class $A \in H_1(X, \partial X; \mathbb{Z})$ such that $\partial(A) = m\alpha_- + n\alpha_+$ under the boundary map $\partial : H_1(X, \partial X; \mathbb{Z}) \longrightarrow H_1(\partial X; \mathbb{Z})$. Now 3-manifold theory assures that there is a 2-sided, embedded, essential surface S that represents A [16, Lemma 6.6]. That is $[S] = A$. By construction, each component of $\partial S \cap T_-$ has slope s while each component of $\partial S \cap T_+$ has slope $\phi(s)$.

Now S may not be connected. However since S represents A , we have $\partial(A) = m\alpha_- + n\alpha_+ \neq 0$. There must be a component $F \subset S$ such that $\partial([F]) \neq 0$. We claim that F must intersect both components of ∂X . For, suppose that it doesn't intersect one component of ∂X ; say $F \cap T_+ = \emptyset$. Then the class $\partial([F]) \neq 0$ would be a nonzero multiple of $\alpha_- \in H_1(\partial X; \mathbb{Z})$. But this is impossible, since $i_-(\alpha_-)$ has infinite order in $H_1(X; \mathbb{Z})$. Thus $F \cap T_{\pm} \neq \emptyset$. To finish the proof of the lemma, note that since $F \subset S$ and S is essential, each component $\partial F \cap T_-$ has slope s while each component of $\partial F \cap T_+$ has slope $\phi(s)$. \square

2.3. The proof of Theorem 2.2. We are now ready to prove Theorem 2.2. For part (a) let K be a nontrivial knot and let (p, q) coprime integers. Suppose that a/b is a boundary slope of K . Let $K_{p,q}$ be the cable knot of K . As above we will consider the complement $M_{K_{p,q}}$ obtained by gluing a cable space $C_{p,q}$ to the complement M_K . We must show that there is an essential surface S in $M_{K_{p,q}}$ such that the total slope of $S \cap \partial M_{K_{p,q}}$ is $q^2 a/b$.

On the boundary component $T_- = \partial M_{K_{p,q}}$ consider a pair (μ, λ) of meridian and canonical longitude whose homology classes generate $H_1(T_-; \mathbb{Z})$. Let $r := \gcd(aq^2, b)$ and let $x = aq^2/r$ and $y = b/r$. Consider a simple closed curve γ whose numerical slope is x/y . That is in $H_1(T_-; \mathbb{Z})$ we have

$$[\gamma] = x\mu + y\lambda.$$

By Lemma 2.7, applied for $s = x/y$ we have a slope $\phi(s)$ on the other component T_+ of $C_{p,q}$ such that the following is true: There is an essential, connected, properly embedded surface $F \subset C_{p,q}$, with ∂F intersecting both components of $\partial C_{p,q}$, and such that each component $\partial F \cap T_-$ has slope s while each component of $\partial F \cap T_+$ has slope $\phi(s)$.

Since $C_{p,q}$ is a Seifert fibered space, up to isotopy, essential surfaces are either vertical or horizontal with respect to the Seifert fibration [14]. Since the base space of $C_{p,q}$ is an annulus, the only vertical surfaces in $C_{p,q}$ are annuli. In fact, essential surfaces in cable spaces are classified by Gordon and Litherland in [13, Lemma 3.1]. From that lemma and its proof, it follows that if F is a vertical annulus in $C_{p,q}$, then every component of $\partial F \cap T_-$ has slope pq , while every component of $\partial F \cap T_+$ has slope p/q . In particular, we

have $a/b = p/q$ and our hypothesis implies that for this to happen p/q must be a boundary slope of K . On the other hand $x/y = pq$ which, by Lemma 2.3, is always a boundary slope of $M_{K_{p,q}}$. Thus the desired conclusion holds in this case.

Now suppose that the surface F is horizontal with respect to the Seifert fibration of $C_{p,q}$. On T_- a regular fiber of the fibration can be identified with the knot $H := K_{p,q}$. The only singular fiber of the fibration (that has multiplicity $|q|$) may be identified with the knot K on T_+ . If N is the number of times that F intersects the regular fibers of $C_{p,q}$, then since the base of the fibration is an annulus we have

$$0 - \chi(F)/N = 1 - 1/|q|.$$

It follows that $\chi(F) = n'(1 - |q|)$, where $N = n'q$. Since the intersection number of μ and H is 1 we conclude that on ∂T_- , the meridian curve μ is a cross section of the fibration and in $H_1(\partial M_{K_{p,q}})$ we have $[\gamma] = nq\mu + mH$, where $n = n'/r, m \in \mathbb{Z}$ and we have $(nq, m) = 1$. On the other hand we must have $H = pq\mu + \lambda$. Hence we obtain

$$[\partial F] = r(nq + mq)\mu + rm\lambda, \quad \text{thus} \quad s = x/y = (nq + mpq)/m.$$

In $C_{p,q}$ we have a horizontal planar surface that intersects T_+ in a single meridian, call it μ' , and it intersects T_- in q copies of μ . This surface may be taken to be the image of a meridian disk of a neighborhood of K in $C_{p,q}$. By choosing appropriate orientations of μ, μ' , and by the proof of Lemma 2.7, we may assume that $\phi(\mu) = \mu'$ and that in $H_1(C_{p,q}; \mathbb{Z})$ we have $\mu' = qm$.

In $H_1(T_+; \mathbb{Z})$, the fiber H corresponds to the slope p/q and each component of $T_+ \cap \partial F$ has the form $n\mu' + mH$. Thus, as also stated in [13, Lemma 3.1], $T_+ \cap \partial F$ has total slope $(n + mp)/(mq)$. The number of components $T_+ \cap \partial F$ is $t = \gcd(n + mp, mq)$. Since in \mathbb{Q} we have that $(n + mp)/(mq) = a/b$ we may take $a' := a/w = n + mp/t$ and $b' := b/w = mq/t$, where $w = \gcd(a, b)$. Thus we may assume that $\phi(s) = a'/b'$. By assumption a/b is a boundary slope in M_K . Thus, Lemma 2.6 applies to conclude that $s = q^2 a/b$ is a boundary slope in $M_{K_{p,q}}$. This finishes the proof of part (a) of Theorem 2.2.

For a nontrivial knot K , part (b) follows at once from part (a). To finish the proof of part (b) assume that K is the trivial knot and let (p, q) coprime integers. Now $K_{p,q}$ is the (p, q) torus knot. The only boundary slope of K is $a/b = 0/1$ and the boundary slopes of $K_{p,q}$ are 0 and pq . Thus $q^2 bs_K \cup \{pq\} = \{0, pq\} = bs_{K_{p,q}}$. \square

We close the section with the following corollary that will be useful to us in subsequent sections.

Corollary 2.8. *Let $M_{K_{p,q}} = C_{p,q} \cup M_K$ be the complement of a cable knot, where $|q| > 1$. Let $F \subset C_{p,q}$ be a properly embedded essential surface, that is not an annulus, and such that each component of $\partial F \cap T_+$ has integral*

slope a . Suppose that there is a connected essential surface $S' \subset M_K$ such that each component of $\partial S'$ has slope a . Then the following hold:

- (1) $\partial F \cap T_+$ has $|q|$ components and $\partial F \cap T_-$ has a single component of slope $q^2a \in \mathbb{Z}$.
- (2) There is a connected essential surface $S \subset M_{K_{p,q}}$, such that each component of ∂S has slope q^2a and

$$\chi(S) = |q|\chi(S') + |\partial S'| |(1 - |q|)p - aq|,$$

where $|\partial S'|$ denotes the number of boundary components of S' . Furthermore, we have $|\partial S| = |\partial S'|$.

Proof. Let F be as in the statement above. By the proof of Theorem 2.2, since F is not an annulus, F may be isotoped to be horizontal with respect to the Seifert fibration of $C_{p,q}$. Furthermore, $\partial F \cap T_+$ has total slope

$$(n + mp)/(mq),$$

while $\partial F \cap T_-$ has total slope $(nq + mpq)/m$, for some coprime integers m, n . The number of components of $\partial F \cap T_+$ is $t = \gcd(n + mp, mq)$, and each has slope b/c where $b = (n + mp)/t$ and $c = mq/t$. Since $b/c \in \mathbb{Z}$ and $\gcd(m, n) = 1$, it follows that $m = \pm 1$ and $t = |q|$. Hence $n = m(qa - p)$, where we will have $m = 1$ or $m = -1$ according to whether $qa \geq p$ or $qa \leq p$. Furthermore $n > 0$, $\chi(F) = n(1 - |q|)$. The rest of the claims in part (1) follow.

Now we prove part (2). By the proof of Theorem 2.2, an essential surface S realizing the boundary slope q^2a for $K_{p,q}$ is as in the proof of Lemma 2.6. Since $\partial F \cap T_+$ has $|q|$ components, S is constructed by gluing $|q|$ copies of S' with $|\partial S'|$ copies of F . Hence we have

$$\chi(S) = \chi(M_K \cap S) + \chi(C_{p,q} \cap S) = |q|\chi(S') + |\partial S'| |aq - p|(1 - |q|).$$

The last equation follows from the fact that $\chi(C_{p,q} \cap S) = |\partial S'| \chi(F)$ and the above discussion on $\chi(F)$. □

3. Cables of knots with period at most two

In this section we study the behavior of the Jones slopes of knots under the operation of cabling. The main result is Theorem 3.4 that relates the Jones slopes of knots of period at most two to the Jones slopes of their cables. We apply this theorem to prove the Strong Slope Conjecture for iterated cables of adequate knots and iterated torus knots and the Slope Conjecture for cables of all the nonalternating knots up to nine crossings that have period two.

3.1. The colored Jones polynomial. To define the colored Jones polynomial, we first recall the definition of the Chebyshev polynomials of the

second kind. For $n \geq 0$, the polynomial $S_n(x)$ is defined recursively as follows:

$$(3.1) \quad S_{n+2}(x) = xS_{n+1}(x) - S_n(x), \quad S_1(x) = x, \quad S_0(x) = 1.$$

Let D be a diagram of a knot K . For an integer $m > 0$, let D^m denote the diagram obtained from D by taking m parallels copies of K . This is the m -cable of D using the blackboard framing; if $m = 1$ then $D^1 = D$. Let $\langle D^m \rangle$ denote the Kauffman bracket of D^m : this is a Laurent polynomial over the integers in a variable $v^{-1/4}$ normalized so that $\langle \text{unknot} \rangle = -(v^{1/2} + v^{-1/2})$. Let $c = c(D) = c_+ + c_-$ denote the crossing number and $w = w(D) = c_+ - c_-$ denote the writhe of D .

For $n > 0$, we define

$$J_K(n) := ((-1)^{n-1}v^{(n^2-1)/4})^w (-1)^{n-1} \langle S_{n-1}(D) \rangle$$

where $S_{n-1}(D)$ is a linear combination of blackboard cablings of D , obtained via Equation (3.1), and the notation $\langle S_{n-1}(D) \rangle$ means extend the Kauffman bracket linearly. That is, for diagrams D_1 and D_2 and scalars a_1 and a_2 , $\langle a_1D_1 + a_2D_2 \rangle = a_1 \langle D_1 \rangle + a_2 \langle D_2 \rangle$.

For a Laurent polynomial $f(v) \in \mathbb{C}[v^{\pm 1/4}]$, let $d_+[f]$ and $d_-[f]$ be respectively the maximal and minimal degree of f in v .

Definition 3.1. A *quasi-polynomial* is a function

$$f : \mathbb{N} \rightarrow \mathbb{C}, \quad f(n) = \sum_{i=0}^d c_i(n)n^i$$

for some $d \in \mathbb{N}$, where $c_i(n)$ is a periodic function with integral period for $i = 1, \dots, d$. If $c_d(n)$ is not identically zero, then the degree of $f(n)$ is d .

The *period* π of a quasi-polynomial $f(n)$ as above is the least common multiple of the periods of the $c_i(n)$.

Garoufalidis [9] showed that for any knot $K \subset S^3$ the degrees $d_+[J_K(n)]$ and $d_-[J_K(n)]$ are quadratic quasi-polynomials. The least common multiple of the periods of $d_+[J_K(n)]$ and $d_-[J_K(n)]$ is called *the period of K* , denoted by $\pi(K)$.

3.2. Cables of knots of period at most 2. In this subsection we will study knots with period at most two. Examples of such knots include all the adequate knots and the torus knots. We show that, under a mild hypothesis satisfied by all the known examples, the property of having period at most two is preserved under cabling (Proposition 3.2). As a result, if a knot $K \subset S^3$ satisfies the Slope Conjecture and $\pi(K) \leq 2$, then all but at most two cables of K also satisfy the conjecture.

Proposition 3.2. *Let K be a knot such that for $n \gg 0$ we have*

$$d_+[J_K(n)] = a(n)n^2 + b(n)n + d(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $b(n) \leq 0$.

Suppose $\frac{p}{q} \notin \{4a(n)\}$. Then for $n \gg 0$ we have

$$d_+[J_{K_{p,q}}(n)] = A(n)n^2 + B(n)n + D(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with

$$\{A(n)\} \subset (\{q^2a(n)\} \cup \{pq/4\}) \quad \text{and} \quad B(n) \leq 0.$$

Proof. Since $d_+[J_K(n)]$ is a quadratic quasi-polynomial of period ≤ 2 , for $n \gg 0$ we can write

$$d_+[J_K(n)] = \begin{cases} a_0n^2 + b_0n + d_0 & \text{if } n \text{ is even,} \\ a_1n^2 + b_1n + d_1 & \text{if } n \text{ is odd.} \end{cases}$$

Recall that $K_{p,q}$ is the (p, q) -cable of a knot K , where p, q are coprime integers and $|q| > 1$. It is known that $K_{-p,-q} = rK_{p,q}$, where $rK_{p,q}$ denotes $K_{p,q}$ with the opposite orientation, and that the colored Jones polynomial of a knot is independent of the orientation of the knot. Hence, without loss of generality, we will assume that $q > 1$.

For $n > 0$, let \mathcal{S}_n be the set of all k such that

$$|k| \leq (n - 1)/2 \quad \text{and} \quad k \in \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z} + \frac{1}{2} & \text{if } n \text{ is even.} \end{cases}$$

By [33], for $n > 0$ we have

$$(3.2) \quad J_{K_{p,q}}(n) = v^{pq(n^2-1)/4} \sum_{k \in \mathcal{S}_n} v^{-pk(qk+1)} J_K(2qk + 1),$$

where it is understood that $J_K(-m) = -J_K(m)$.

In the above formula, there is a sum. Under the assumption of the proposition, we will show that there is a unique term of the sum whose highest degree is strictly greater than those of the other terms. This implies that the highest degree of the sum is exactly equal to the highest degree of that unique term.

Let $\mathcal{S}_n^+ = \{k \in \mathcal{S}_n \mid k \geq 0\}$ and $\mathcal{S}_n^- = \{k \in \mathcal{S}_n \mid k \leq -\frac{1}{2}\}$. For $k \in \mathcal{S}_n$ let

$$f(k) = d_+[v^{-pk(qk+1)} J_K(2qk + 1)].$$

Let $g_i^\pm(x)$, for $i \in \{0, 1\}$, be the quadratic real polynomials defined by

$$g_i^\pm(x) = (-pq + 4q^2a_i)x^2 + (-p + 4qa_i \pm 2qb_i)x + a_i \pm b_i + d_i.$$

For $k \in \mathcal{S}_n$ we have $f(k) = -pk(qk + 1) + d_+[J_K(|2qk + 1|)]$, which gives

$$f(k) = \begin{cases} g_0^+(k) & \text{if } k \in \mathcal{S}_n^+ \text{ and } 2qk + 1 \text{ is even,} \\ g_0^-(k) & \text{if } k \in \mathcal{S}_n^- \text{ and } 2qk + 1 \text{ is even,} \\ g_1^+(k) & \text{if } k \in \mathcal{S}_n^+ \text{ and } 2qk + 1 \text{ is odd,} \\ g_1^-(k) & \text{if } k \in \mathcal{S}_n^- \text{ and } 2qk + 1 \text{ is odd.} \end{cases}$$

Case 1. Suppose that q is even. For $k \in \mathcal{S}_n$, since $2qk + 1$ is odd we have

$$f(k) = \begin{cases} g_1^+(k) & \text{if } k \in \mathcal{S}_n^+, \\ g_1^-(k) & \text{if } k \in \mathcal{S}_n^-. \end{cases}$$

Subcase 1.1. Assume that $p/q < 4a_1$.

Since $-pq + 4q^2a_1 > 0$, the quadratic polynomial $g_1^+(x)$ is concave up. Hence, for $n \gg 0$, $g_1^+(k)$ is maximized on \mathcal{S}_n^+ at $k = (n-1)/2$. Similarly, $g_1^-(k)$ is maximized on \mathcal{S}_n^- at $k = (1-n)/2$. Note that

$$g_1^+((n-1)/2) - g_1^-((1-n)/2) = (-p + 4qa_1)(n-1) + 2b_1 > 0$$

for $n \gg 0$. Hence $f(k)$ is maximized on the set \mathcal{S}_n at $k = (n-1)/2$. Since $f((n-1)/2) = g_1^+((n-1)/2)$, Equation (3.2) then implies that

$$\begin{aligned} d_+[J_{K_{p,q}}(n)] &= pq(n^2 - 1)/4 + g_1^+((n-1)/2) \\ &= q^2a_1n^2 + (qb_1 + (q-1)(p-4qa_1)/2)n \\ &\quad + a_1(q-1)^2 - (b_1 + p/2)(q-1) + d_1 \end{aligned}$$

for $n \gg 0$. Since we assumed that $q > 1$, we have that

$$B(n) = qb_1 + (q-1)(p-4qa_1)/2 < 0,$$

and the conclusion follows in this case.

Subcase 1.2. Assume that $p/q > 4a_1$.

Since $-pq + 4q^2a_1 < 0$, the quadratic polynomial $g_1^+(x)$ is concave down and attains its maximum at

$$x = x_0 := -\left(\frac{1}{2q} + \frac{b_1}{-p + 4qa_1}\right).$$

Since $b_1 \leq 0$, we have $x_0 < 0$. This implies that $g_1^+(x)$ is a strictly decreasing function on $[0, \infty)$. Similarly, $g_1^-(x)$ is a strictly increasing function on $(-\infty, -\frac{1}{2}]$.

First suppose n is even. Then $k \in \mathbb{Z} + \frac{1}{2}$. In this subcase, $g_1^+(k)$ is maximized on \mathcal{S}_n^+ at $k = \frac{1}{2}$ and $g_1^-(k)$ is maximized on \mathcal{S}_n^- at $k = -\frac{1}{2}$. Note that $g_1^+(\frac{1}{2}) - g_1^-(-\frac{1}{2}) = (-p + 4qa_1) + 2b_1 < 0$. Hence $f(k)$ is maximized on \mathcal{S}_n at $k = -1/2$. Since $f(-\frac{1}{2}) = g_1^-(-\frac{1}{2})$, Equation (3.2) then implies that

$$d_+[J_{K_{p,q}}(n)] = pq(n^2 - 1)/4 + g_1^-(-1/2)$$

for even $n \gg 0$. Similarly, for odd $n \gg 0$ we obtain

$$d_+[J_{K_{p,q}}(n)] = pq(n^2 - 1)/4 + g_1^+(0).$$

Note that $B(n) = 0$ in this case.

Case 2. Suppose that q is odd. As in Case 1 we have the following.

Subcase 2.1. Suppose that n is even.

For $k \in \mathcal{S}_n$, we have $k \in \mathbb{Z} + \frac{1}{2}$ and $2qk + 1$ is even. Hence

$$f(k) = \begin{cases} g_0^+(k) & \text{if } k \in \mathcal{S}_n^+, \\ g_0^-(k) & \text{if } k \in \mathcal{S}_n^-. \end{cases}$$

If $p/q < 4a_0$ then $f(k)$ is maximized on \mathcal{S}_n at $k = (n - 1)/2$. Hence

$$\begin{aligned} d_+[J_{K_{p,q}}(n)] &= pq(n^2 - 1)/4 + g_0^+((n - 1)/2) \\ &= q^2 a_0 n^2 + (qb_0 + (q - 1)(p - 4qa_0)/2)n \\ &\quad + a_0(q - 1)^2 - (b_0 + p/2)(q - 1) + d_0. \end{aligned}$$

In this case we have $B(n) = qb_0 + (q - 1)(p - 4qa_0)/2 < 0$.

If $p/q > 4a_0$ then $f(k)$ is maximized on \mathcal{S}_n at $k = -1/2$. Hence

$$d_+[J_{K_{p,q}}(n)] = pq(n^2 - 1)/4 + g_0^-(-1/2).$$

Note that $B(n) = 0$ in this case.

Subcase 2.2. Suppose that n is odd.

For $k \in \mathcal{S}_n$, we have $k \in \mathbb{Z}$ and $2qk + 1$ is odd. Hence

$$f(k) = \begin{cases} g_1^+(k) & \text{if } k \in \mathcal{S}_n^+, \\ g_1^-(k) & \text{if } k \in \mathcal{S}_n^-. \end{cases}$$

If $p/q < 4a_1$ then $f(k)$ is maximized on \mathcal{S}_n at $k = (n - 1)/2$. Hence

$$\begin{aligned} d_+[J_{K_{p,q}}(n)] &= pq(n^2 - 1)/4 + g_1^+((n - 1)/2) \\ &= q^2 a_1 n^2 + (qb_1 + (q - 1)(p - 4qa_1)/2)n \\ &\quad + a_1(q - 1)^2 - (b_1 + p/2)(q - 1) + d_1. \end{aligned}$$

In this case we have $B(n) = qb_1 + (q - 1)(p - 4qa_1)/2 < 0$.

If $p/q > 4a_1$ then $f(k)$ is maximized on \mathcal{S}_n at $k = 0$. Hence

$$d_+[J_{K_{p,q}}(n)] = pq(n^2 - 1)/4 + g_1^+(0).$$

Note that $B(n) = 0$ in this case.

This completes the proof of Proposition 3.2. □

Remark 3.3.

- (1) Proposition 3.2 generalizes [29, Lemma 2.2], [28, Lemma 2.2], [31, Lemma 3.1] and [32, Lemma 3.2].
- (2) When $\pi(K)$ is greater than 2 then determining the highest degree of $J_{K_{p,q}}(n)$ in Equation (3.2) becomes harder as there might be more opportunities for cancellation between terms. This case will be discussed in Section 4.

Proposition 3.2 and Theorem 2.2 imply the following.

Theorem 3.4. *Let K be a knot such that for $n \gg 0$ we have*

$$d_+[J_K(n)] = a(n)n^2 + b(n)n + d(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $b(n) \leq 0$.

Suppose $\frac{p}{q} \notin js_K$. Then for $n \gg 0$ we have

$$d_+[J_{K_{p,q}}(n)] = A(n)n^2 + B(n)n + D(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $B(n) \leq 0$. Moreover, if

$js_K \subset bs_K$ we have $js_{K_{p,q}} \subset bs_{K_{p,q}}$.

Similarly, let K be a knot such that for $n \gg 0$ we have

$$d_-[J_K(n)] = a^*(n)n^2 + b^*(n)n + d^*(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $b^(n) \geq 0$. Suppose*

$\frac{p}{q} \notin js_K^$. Then for $n \gg 0$ we have*

$$d_-[J_{K_{p,q}}(n)] = A^*(n)n^2 + B^*(n)n + D^*(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $B^(n) \geq 0$. Moreover, if*

$js_K^ \subset bs_K$ we have $js_{K_{p,q}}^* \subset bs_{K_{p,q}}$.*

Proof. The first part of the theorem follows immediately by Proposition 3.2 and Theorem 2.2. To obtain the second part recall that if K^* denotes the mirror image of K then $J_{K^*}(n)$ is obtained from $J_K(n)$ by replacing the variable v with v^{-1} . Now the result will follow by applying Proposition 3.2 and Theorem 2.2 to K^* . □

3.3. Strong Slope Conjecture for cables of adequate knots. Let D be a link diagram, and x a crossing of D . Associated to D and x are two link diagrams, each with one fewer crossing than D , called the *A-resolution* and *B-resolution* of the crossing. See Figure 1.

A Kauffman state σ is a choice of *A-resolution* or *B-resolution* at each crossing of D . Corresponding to every state σ is a crossing-free diagram s_σ : this is a collection of circles in the projection plane. We can encode the choices that lead to the state σ in a graph G_σ , as follows. The vertices of G_σ are in 1–1 correspondence with the state circles of s_σ . Every crossing x of D corresponds to a pair of arcs that belong to circles of s_σ ; this crossing gives rise to an edge in G_σ whose endpoints are the state circles containing those arcs.

Every Kauffman state σ also gives rise to a surface S_σ , as follows. Each state circle of σ bounds a disk in S^3 . This collection of disks can be disjointly embedded in the ball below the projection plane. At each crossing of D , we connect the pair of neighboring disks by a half-twisted band to construct a surface $S_\sigma \subset S^3$ whose boundary is K . See Figure 2 for an example where σ is the all-*B* state.

Definition 3.5. A link diagram D is called *A-adequate* if the state graph \mathbb{G}_A corresponding to the all-*A* state contains no 1-edge loops. Similarly, D

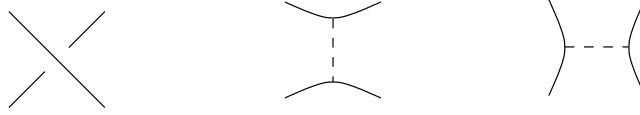


FIGURE 1. From left to right: A crossing, the A -resolution and the the B -resolution.

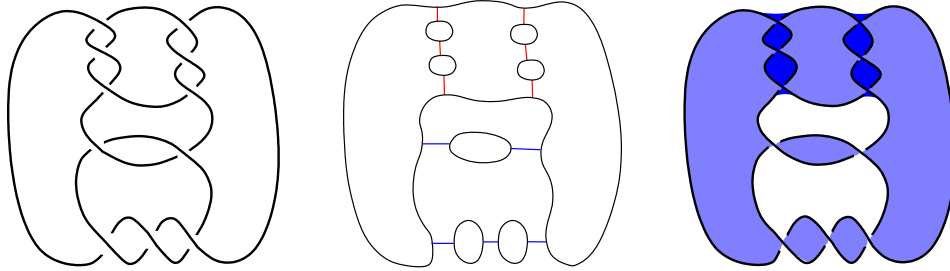


FIGURE 2. From left to right: An example link diagram, the graph \mathbb{G}_σ corresponding to state surface S_σ . Figure borrowed from [8].

is called B -adequate if the all- B graph \mathbb{G}_B contains no 1-edge loops. A link diagram is *adequate* if it is both A - and B -adequate. A link that admits an adequate diagram is also called *adequate*.

The number of negative crossings c_- of an A -adequate knot diagram is a knot invariant. Similarly, the number of positive crossings c_+ of a B -adequate knot diagram is a knot invariant. Let v_A (resp. v_B) be the number of state circles in the all- A (resp. all- B) state of the knot diagram D .

The following summarizes [24, Lemma 5.4], [22, Proposition 2.1] and [19, Theorem 3.1].

Lemma 3.6. *Let D be a diagram of a knot K .*

(1) *We have*

$$2d_-[J_K(n)] \geq -c_-n^2 + (c - v_A)n + v_A - c_+.$$

Equality holds for all $n \geq 1$ if D is A -adequate. Moreover, if equality holds for some $n \geq 3$ then D is A -adequate.

(2) *We have*

$$2d_+[J_K(n)] \leq c_+n^2 + (v_B - c)n + c_- - v_B.$$

Equality holds for all $n \geq 1$ if D is B -adequate. Moreover, if equality holds for some $n \geq 3$ then D is B -adequate.

The following theorem, which implies that the Slope Conjecture is true for adequate knots, summarizes results proved in [4, 5].

Theorem 3.7. *Let D be an A -adequate diagram of a knot K . Then the state surface S_A is essential in the knot complement M_K , and it has boundary slope $-2c_-$. Furthermore, we have*

$$-2c_- = \lim_{n \rightarrow \infty} 4n^{-2}d_-[J_K(n)].$$

Similarly, if D is a B -adequate diagram of a knot K , then S_B is essential in the knot complement M_K , and it has boundary slope $-2c_+$. Furthermore, we have

$$2c_+ = \lim_{n \rightarrow \infty} 4n^{-2}d_+[J_K(n)].$$

By Lemma 3.6, the highest degree of the colored Jones polynomial of a B -adequate knot is an actual quadratic polynomial in n . That is the period is one. The following lemma shows that the term $b \leq 0$ and thus nontrivial B -adequate knots satisfy the hypothesis of Proposition 3.2.

Lemma 3.8. *Suppose D is a diagram of a nontrivial knot K . Then $v_B \leq c$. Furthermore, if $v_B = c$ then K is a torus knot.*

Proof. Let D be a diagram of a nontrivial knot and let S_B be the all- B state surface obtained from D . Recall that S_B is a surface with a single boundary component obtained by starting with v_B disks and attaching a half-twisted band for each crossing of D . Thus the Euler characteristic of S_B is $\chi(S_B) = v_B - c$. Since D represents a nontrivial knot, we have $\chi(S_B) \leq 0$ and thus $v_B \leq c$.

If $\chi(S_B) = v_B - c = 0$, then (since ∂S_B has one component) S_B must be a Mobius band. This implies that D is the standard closed 2-braid diagram of a $(2, q)$ -torus knot. □

The above discussion shows that the first part of Theorem 3.4 applies to nontrivial B -adequate knots. Similarly the second part of the theorem applies to nontrivial A -adequate knots. We are now ready to prove the following theorem that implies Theorem 1.4 stated in the introduction.

Theorem 3.9. *Let K be a B -adequate knot and $K' := K_{(p_1, q_1), (p_2, q_2), \dots, (p_r, q_r)}$ an iterated cable knot of K . Then we have $js_{K'} \subset bs_{K'}$. Furthermore, for $n \gg 0$,*

$$d_+[J_{K'}(n)] = An^2 + Bn + D(n),$$

is a quadratic quasi-polynomial of period ≤ 2 , with $4A \in \mathbb{Z}$, $2B \in \mathbb{Z}$ and $B \leq 0$, and there is an essential surface S' in the complement of K' with boundary slope $4A$ and such that $\chi(S') = 2B$. In particular K' satisfies the Strong Slope Conjecture.

Proof. Suppose K is a B -adequate knot. First, we prove that the conclusion of Theorem 3.9 holds true for the cable knot K_{p_1, q_1} . We will distinguish two cases according to whether K is a nontrivial knot or not.

Case 1. Suppose that K is nontrivial. Then, by Lemmas 3.6 and 3.8 we have

$$d_+[J_K(n)] = an^2 + bn + d$$

for $n > 0$, where $a = c_+/2$, $b = (v_B - c)/2 \leq 0$ and $d = (c_- - v_B)/2$. By Theorem 3.7, $4a = 2c_+$ is a boundary slope of K . Furthermore, an essential surface that realizes this boundary slope is the state surface S_B . Since S_B is constructed by joining v_B disks with c bands we have $|\partial S_B| = 1$ and $\chi(S_B) = v_B - c = 2b$. Thus the conclusion is true in this case.

Now we consider a cable K_{p_1, q_1} of K . Since $|q_1| > 1$, and the Jones slopes of K are integers, we have $\frac{p_1}{q_1} \notin js_K$. Theorem 3.4 and the proof of Proposition 3.2 then imply that

$$d_+[J_{K_{p_1, q_1}}(n)] = A_1n^2 + B_1n + D_1(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $4A_1 \in \mathbb{Z}, 2B_1 \in \mathbb{Z}$ and $B_1 \leq 0$. Moreover, since $js_K \subset bs_K$ we have $js_{K_{p_1, q_1}} \subset bs_{K_{p_1, q_1}}$. Furthermore, the proof of Proposition 3.2 shows that one of the following is true:

- (1) We have $4A_1 = pq$ and $B_1 = 0$.
- (2) We have $4A_1 = 4q^2a$ and $2B_1 = 2|q|b + (1 - |q|)|4aq - p|$.

In case (1), the surface S with boundary slope pq is the cabling annulus; thus $\chi(S) = 0 = 2B_1$. In case (2), an essential surface S realizing the boundary slope $4A_1 = 4q^2a = 2q^2c_+$ is obtained by Theorem 2.2. By Corollary 2.8 we have $|\partial S| = |\partial S_B| = 1$ and $\chi(S) = |q|\chi(S_B) + (1 - |q|)|4aq - p| = 2B_1$. Thus the conclusion follows for K_{p_1, q_1} .

Case 2. Suppose that K is the trivial knot. Then K_{p_1, q_1} is the (p_1, q_1) -torus knot. Note that 0 and p_1q_1 are boundary slopes of K_{p_1, q_1} ; realized by a Seifert surface and an annulus respectively.

For $n > 0$, by [25] (or by Equation (3.2)) we have

$$J_{K_{p_1, q_1}}(n) = v^{p_1q_1(n^2-1)/4} \sum_{k \in \mathcal{S}_n} v^{-p_1k(q_1k+1)} \frac{v^{(2qk+1)/2} - v^{-(2qk+1)/2}}{v^{1/2} - v^{-1/2}}.$$

By [10, Section 4.8], [30, Lemma 1.4] and [29, Lemma 2.1], we have the following. If $p_1 > 0$ and $q_1 > 0$ then

$$d_+[J_{K_{p_1, q_1}}(n)] = (p_1q_1n^2 + d(n))/4$$

where $d(n) = -p_1q_1 - \frac{1}{2}(1 + (-1)^n)(p_1 - 2)(q_1 - 2)$ is a periodic sequence of period ≤ 2 . In this case we have $A_1(n) = p_1q_1/4$ and $B_1(n) = 0$.

If $p_1 < 0 < q_1$ then

$$d_+[J_{K_{p_1, q_1}}(n)] = ((p_1q_1 - p_1 + q_1)n - (p_1q_1 - p_1 + q_1))/2.$$

In this case we have $A_1(n) = 0$ and $B_1(n) = (p_1q_1 - p_1 + q_1)/2$. Note that $p_1q_1 - p_1 + q_1 = 1 + (p_1 + 1)(q_1 - 1) \leq 0$.

K	$js_K \cup js_K^*$	$b(n)$	$b^*(n)$
8 ₁₉	{12, 0}	0	5/2
8 ₂₁	{1, -12}	-1	3/2
9 ₄₂	{6, -8}	-1/2	5/2
9 ₄₅	{1, -14}	-1	2
9 ₄₆	{2, -12}	-1/2	5/2
9 ₄₇	{9, -6}	-1	2
9 ₄₈	{11, -4}	-3/2	3/2
9 ₄₉	{15, 0}	-3/2	3/2

TABLE 1. The knots up to nine crossings of period two.

In both cases we have that

$$d_+[J_{K_{p_1, q_1}}(n)] = A_1 n^2 + B_1 n + D_1(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $4A_1 \in \mathbb{Z}$, $2B_1 \in \mathbb{Z}$ and $B_1 \leq 0$. Moreover, we have

$$js_{K_{p_1, q_1}} \subset bs_{K_{p_1, q_1}}.$$

For $p_1 > 0$ and $q_1 > 0$, an essential surface with slope $p_1 q_1$ for K_{p_1, q_1} is the cabling annulus A ; thus $\chi(A_1) = 0 = 2B_1$. For $p_1 < 0 < q_1$, the genus of K_{p_1, q_1} is $g = -(q_1 - 1)(p_1 + 1)/2$. A Seifert surface S of minimal genus for K_{p_1, q_1} has boundary slope 0 and $\chi(S) = 1 - 2g = p_1 q_1 - p_1 + q_1 = 2B_1$. This proves the desired conclusion for the torus knot K_{p_1, q_1} .

We have proved that the conclusion of Theorem 3.9 holds true for the cable knot K_{p_1, q_1} . Now, applying the arguments in Case 1 repeatedly will finish the proof of the theorem for iterated cables. \square

3.4. Proof of Theorem 1.4. Part (1) is immediate from Theorem 3.9. Suppose now that K is A -adequate. Then the mirror image K^* is B -adequate. Furthermore, $J_{K^*}(n)$ is obtained from $J_K(n)$ by replacing v with v^{-1} . Thus in this case the result will follow by applying Theorem 3.9 to K^* .

3.5. Low crossing knots with period two. Theorem 3.4 applies to several nonalternating knots with 8 and 9 crossings. The Jones slopes of all nonalternating prime knots with up to nine crossings were calculated by Garoufalidis in Section 4 of [10]. According to [10] the knots of period two are the ones shown in Table 1. The remaining knots, which are 8₂₀, 9₄₃, 9₄₄, have periods 3 and they will be treated in Section 4.

Corollary 3.10. *Let K be any knot of Table 1 and let K' be an iterated cable of K . Then K' satisfies the Slope Conjecture.*

Proof. Due to different conventions and normalizations of the colored Jones polynomial, $d_+[J_K(n)]$ (resp. $d_-[J_K(n)]$) in our paper is different from $\delta_K(n)$

(resp. $\delta_K^*(n)$) in [10]. For $n > 0$ we have

$$\begin{aligned} d_+[J_K(n)] &= \delta_K(n-1) + (n-1)/2, \\ d_-[J_K(n)] &= \delta_K^*(n-1) + (1-n)/2. \end{aligned}$$

Consider the nonalternating knot $K = 8_{19}$. By [10, Section 4] we have

$$\begin{aligned} d_+[J_K(n)] &= 3n^2 - (13 + (-1)^n)/4, \\ d_-[J_K(n)] &= 5(n-1)/2. \end{aligned}$$

Moreover, $js_K = \{12\} \subset bs_K$ and $js_K^* = \{0\} \subset bs_K$. In particular, the Jones slopes are integers and we have $b(n) \leq 0$ and $b^*(n) \geq 0$. Similar analysis applies to the knots of Table 1.

Now given a cable $K_{p,q}$ of K , since $|q| > 1$, p/q is not a Jones slope of K . Theorem 3.4 implies that $js_{K_{p,q}} \subset bs_{K_{p,q}}$ and that $js_{K_{p,q}}^* \subset bs_{K_{p,q}}$. In particular, $K_{p,q}$ satisfies the Slope Conjecture. Applying this argument repeatedly we obtain the result for iterated cables. \square

4. Cabling knots with constant $a(n)$

In Section 3 we studied the behavior of $d_+[J_K(n)]$ under knot cabling for knots of period at most two. In this section we study knots with period greater than two under the additional hypothesis that, for $n \gg 0$ we have $a(n) = a$, where a is a constant. In this case, by abusing the terminology, we will say that $a(n)$ is constant. The main result in this section is the following.

Theorem 4.1. *Let K be a knot such that for $n \gg 0$ we have*

$$d_+[J_K(n)] = an^2 + b(n)n + d(n)$$

where a is a constant, $b(n)$ and $d(n)$ are periodic functions with $b(n) \leq 0$. Let

$$\begin{aligned} M_1 &= \max\{|b(i) - b(j)| : i \equiv j \pmod{2}\}, \\ M_2 &= \max\{2b(i) + |b(i) - b(j)| + |d(i) - d(j)| : i \equiv j \pmod{2}\}. \end{aligned}$$

Suppose $p - (4a - M_1)q < 0$ or $p - (4a + M_1)q > \max\{0, M_2\}$. Then for $n \gg 0$ we have

$$d_+[J_{K_{p,q}}(n)] = An^2 + B(n)n + D(n)$$

where A is a constant, and $B(n), D(n)$ are periodic functions with $B(n) \leq 0$. Moreover, if $js_K \subset bs_K$ then $js_{K_{p,q}} \subset bs_{K_{p,q}}$.

As a corollary of Theorem 4.1 and Theorem 2.2 we obtain the following result which implies, in particular, that for knots with constant $a(n)$ the Slope Conjecture is closed under cabling for infinitely many pairs (p, q) .

Corollary 4.2. *Let K be a knot such that for $n \gg 0$ we have*

$$d_+[J_K(n)] = an^2 + b(n)n + d(n)$$

where a is a constant, $b(n)$ and $d(n)$ are periodic functions with $b(n) \leq 0$. Let M_1, M_2 be as in the statement of Theorem 4.1. If K satisfies the Slope Conjecture, then $K_{p,q}$ satisfies the conjecture provided that $p - (4a - M_1)q < 0$ or $p - (4a + M_1)q > \max\{0, M_2\}$.

Example 4.3. As an illustration we apply Theorem 4.1 and Corollary 4.2 to the knots $8_{20}, 9_{43}, 9_{44}$. By [10, Section 4], these are the only knots, with at most nine crossings, that have Jones period larger than 2. Indeed, the period of these knots is 3. Another application of Theorem 4.1 will be illustrated in Example 6.3.

For $K = 8_{20}$ we have

$$d_+[J_K(n)] = \begin{cases} 2n^2/3 - n/2 - 1/6 & \text{if } n \not\equiv 0 \pmod{3} \\ 2n^2/3 - 5n/6 - 1/2 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

Hence $K_{p,q}$ satisfies the Slope Conjecture if $p - \frac{7}{3}q < 0$ or $p - 3q > 0$.

For $K = 9_{43}$ we have

$$d_+[J_K(n)] = \begin{cases} 8n^2/3 - n/2 - 13/6 & \text{if } n \not\equiv 0 \pmod{3} \\ 8n^2/3 - 5n/6 - 7/2 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

Hence $K_{p,q}$ satisfies the Slope Conjecture if $p - \frac{31}{3}q < 0$ or $p - 11q > \frac{2}{3}$.

For $K = 9_{44}$ we have

$$d_+[J_K(n)] = \begin{cases} 7n^2/6 - n - 1/6 & \text{if } n \not\equiv 0 \pmod{3} \\ 7n^2/6 - 4n/3 - 1/2 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

Hence $K_{p,q}$ satisfies the Slope Conjecture if $p - \frac{13}{3}q < 0$ or $p - 5q > 0$.

Theorem 4.1 follows from Theorem 1.2 and the following proposition.

Proposition 4.4. *Let K be a knot such that for $n \gg 0$ we have*

$$d_+[J_K(n)] = an^2 + b(n)n + d(n)$$

where a is a constant, $b(n)$ and $d(n)$ are periodic functions with $b(n) \leq 0$. Let

$$M_1 = \max\{|b(i) - b(j)| : i \equiv j \pmod{2}\},$$

$$M_2 = \max\{2b(i) + |b(i) - b(j)| + |d(i) - d(j)| : i \equiv j \pmod{2}\}.$$

Suppose $p - (4a - M_1)q < 0$ or $p - (4a + M_1)q > \max\{0, M_2\}$. Then for $n \gg 0$ we have

$$d_+[J_{K_{p,q}}(n)] = An^2 + B(n)n + D(n)$$

where A is a constant with $A \in \{q^2a, pq/4\}$, and $B(n), D(n)$ are periodic functions with $B(n) \leq 0$.

Proof. Fix $n \gg 0$. Recall the cabling formula (3.2) of the colored Jones polynomial

$$J_{K_{p,q}}(n) = v^{pq(n^2-1)/4} \sum_{k \in \mathcal{S}_n} v^{-pk(qk+1)} J_K(2qk + 1).$$

In the above formula, there is a sum. As in the proof of Proposition 3.2 we will show that, under the assumption of the proposition, there is a unique term of the sum whose highest degree is strictly greater than those of the other terms. This implies that the highest degree of the sum is exactly equal to the highest degree of that unique term.

For $k \in \mathcal{S}_n$ let

$$f(k) := d_+[v^{-pk(qk+1)} J_K(2qk + 1)] = -pk(qk + 1) + d_+[J_K(|2qk + 1|)].$$

The goal is to show that $f(k)$ attains its maximum on \mathcal{S}_n at a unique k .

Since $d_+[J_K(n)]$ is a quadratic quasi-polynomial, $f(k)$ is a piece-wise quadratic polynomial. The above goal will be achieved in 2 steps. In the first step we show that $f(k)$ attains its maximum on each piece at a unique k . Then in the second step we show that the maximums of $f(k)$ on all the pieces are distinct.

Step 1. Let π be the period of $d_+[J_K(n)]$. For $\varepsilon \in \{\pm 1\}$ and $0 \leq i < \pi$, let $h_i^\varepsilon(x)$ be the quadratic real polynomial defined by

$$h_i^\varepsilon(x) := (-pq + 4q^2a)x^2 + (-p + 4qa + 2qb(i)\varepsilon)x + a + b(i)\varepsilon + d(i).$$

For each $k \in \mathcal{S}_n$, we have $f(k) = h_{i_k}^{\varepsilon_k}(k)$ for a unique pair (ε_k, i_k) . Let

$$I_n := \{(\varepsilon_k, i_k) \mid k \in \mathcal{S}_n\}.$$

Then $f(k)$ is a piece-wise quadratic polynomial of exactly $|I_n|$ pieces, each of which is associated with a unique pair (ε, i) in I_n .

For each $(\varepsilon, i) \in I_n$, let

$$\mathcal{S}_{n,\varepsilon,i} := \{k \in \mathcal{S}_n \mid (\varepsilon_k, i_k) = (\varepsilon, i)\}$$

which is the set of all k on the piece associated with (ε, i) .

The quadratic polynomial $h_i^\varepsilon(x)$ is concave up if $p - 4qa < 0$, and concave down if $p - 4qa > 0$. Hence, for $n \gg 0$, $h_i^\varepsilon(k)$ is maximized on the set $\mathcal{S}_{n,\varepsilon,i}$ at a unique $k = k_{n,\varepsilon,i}$, where

$$k_{n,\varepsilon,i} := \begin{cases} \max \mathcal{S}_{n,\varepsilon,i} & \text{if } (p - 4qa)\varepsilon < 0, \\ \min \mathcal{S}_{n,\varepsilon,i} & \text{if } (p - 4qa)\varepsilon > 0. \end{cases}$$

Note that, as in the proof of Proposition 3.2, we use the assumption that $b(i) \leq 0$ when $(p - 4qa)\varepsilon > 0$. Moreover we have

$$\begin{cases} |k_{n,\varepsilon,i}| \rightarrow \infty \text{ as } n \rightarrow \infty, & \text{if } p - 4qa < 0 \\ |k_{n,\varepsilon,i}| \leq \pi, & \text{if } p - 4qa > 0. \end{cases}$$

Step 2. Let

$$\text{Max}_n := \max\{f(k) \mid k \in \mathcal{S}_n\}.$$

From Step 1 we have $\text{Max}_n = \max\{h_i^\varepsilon(k_{n,\varepsilon,i}) \mid (\varepsilon, i) \in I_n\}$. We claim that

$$h_{i_1}^{\varepsilon_1}(k_{n,\varepsilon_1,i_1}) \neq h_{i_2}^{\varepsilon_2}(k_{n,\varepsilon_2,i_2})$$

for $(\varepsilon_1, i_1) \neq (\varepsilon_2, i_2)$.

Indeed, let $k_1 := k_{n,\varepsilon_1,i_1}$ and $k_2 := k_{n,\varepsilon_2,i_2}$. Note that $k_1 \neq k_2$. Moreover, k_1 and k_2 are both in \mathbb{Z} or $\frac{1}{2} + \mathbb{Z}$. As a result we have $k_1 \pm k_2 \in \mathbb{Z}$, and $i_1 - i_2 \equiv 2q(k_1 - k_2) \equiv 0 \pmod{2}$. Let

$$\sigma := h_{i_1}^{\varepsilon_1}(k_1) - h_{i_2}^{\varepsilon_2}(k_2).$$

Without loss of generality, we can assume that $|k_1| \geq |k_2|$. Then we write $\sigma = \sigma' + d(i_1) - d(i_2)$ where

$$\sigma' := \begin{cases} (k_1 - k_2) \left((-p + 4qa)(q(k_1 + k_2) + 1) + 2qb(i_1)\varepsilon_1 \right) \\ \quad + (b(i_1) - b(i_2))\varepsilon_1(2qk_2 + 1) & \text{if } \varepsilon_1 = \varepsilon_2, \\ \left((-p + 4qa)(k_1 - k_2) + 2b(i_1)\varepsilon_1 \right) (q(k_1 + k_2) + 1) \\ \quad - (b(i_1) - b(i_2))\varepsilon_1(2qk_2 + 1) & \text{if } \varepsilon_1 \neq \varepsilon_2. \end{cases}$$

We consider the following 2 cases.

Case 1. Suppose that $p - (4a - M_1)q < 0$. In particular, we have $p - 4qa < 0$. There are 2 subcases.

Subcase 1.1. We have $\varepsilon_1 = \varepsilon_2$. Since k_1 and k_2 have the same sign, we have

$$|q(k_1 + k_2) + 1| - |2qk_2 + 1| = 2q(|k_1| - |k_2|) \geq 0.$$

Hence

$$\begin{aligned} |\sigma'| &\geq |(-p + 4qa)(q(k_1 + k_2) + 1) + 2qb(i_1)\varepsilon_1| \\ &\quad - |(b(i_1) - b(i_2))(q(k_1 + k_2) + 1)| \\ &\geq (-p + 4qa - |b(i_1) - b(i_2)|)|q(k_1 + k_2) + 1| + 2qb(i_1). \end{aligned}$$

Since $|q(k_1 + k_2) + 1| \rightarrow \infty$ as $n \rightarrow \infty$, and

$$-p + 4qa - |b(i_1) - b(i_2)| \geq -p + 4qa - M_1 > 0$$

we get $|\sigma'| \rightarrow \infty$ as $n \rightarrow \infty$.

Subcase 1.2. We have $\varepsilon_1 \neq \varepsilon_2$. Since k_1 and k_2 have opposite signs, we have

$$(q|k_1 - k_2| + 1) - |2qk_2 + 1| \geq 2q(|k_1| - |k_2|) \geq 0.$$

Hence

$$\begin{aligned} |\sigma'| &\geq |(-p + 4qa)(k_1 - k_2) + 2b(i_1)\varepsilon_1| - |b(i_1) - b(i_2)|(q|k_1 - k_2| + 1) \\ &\geq (-p + 4qa - q|b(i_1) - b(i_2)|)|k_1 - k_2| - |b(i_1) - b(i_2)| + 2b(i_1). \end{aligned}$$

Since $|k_1 - k_2| \rightarrow \infty$ as $n \rightarrow \infty$, and

$$-p + 4qa - q|b(i_1) - b(i_2)| \geq -p + 4qa - qM_1 > 0,$$

we get $|\sigma'| \rightarrow \infty$ as $n \rightarrow \infty$.

Case 2. Suppose that $p - (4a + M_1)q > \max\{0, M_2\}$. There are 2 subcases.

Subcase 2.1. We have $\varepsilon_1 = \varepsilon_2$. Note that $q(k_1 + k_2) + 1$ and ε_1 have the same sign. Moreover, both $-p + 4qa$ and $2qb(i_1)$ are nonpositive. As in Subcase 1.1 we have

$$\begin{aligned} |\sigma'| &\geq |(-p + 4qa)(q(k_1 + k_2) + 1) + 2qb(i_1)\varepsilon_1| \\ &\quad - |(b(i_1) - b(i_2))(q(k_1 + k_2) + 1)| \\ &= (p - 4qa - |b(i_1) - b(i_2)|) |q(k_1 + k_2) + 1| - 2qb(i_1). \end{aligned}$$

Since $p - 4qa - |b(i_1) - b(i_2)| \geq p - 4qa - M_1 > \max\{0, M_2\}$, we get

$$|\sigma'| > M_2 - 2qb(i_1) \geq |d(i_1) - d(i_2)|.$$

Subcase 2.2. We have $\varepsilon_1 \neq \varepsilon_2$. Note that $k_1 - k_2$ and ε_1 have the same sign. Moreover, both $-p + 4qa$ and $2qb(i_1)$ are nonpositive. As in Subcase 1.2 we have

$$\begin{aligned} |\sigma'| &\geq |(-p + 4qa)(k_1 - k_2) + 2b(i_1)\varepsilon_1| - |b(i_1) - b(i_2)| (q|k_1 - k_2| + 1) \\ &= (p - 4qa - q|b(i_1) - b(i_2)|) |k_1 - k_2| - |b(i_1) - b(i_2)| - 2b(i_1). \end{aligned}$$

Since $p - 4qa - q|b(i_1) - b(i_2)| \geq p - 4qa - qM_1 > \max\{0, M_2\}$, we get

$$|\sigma'| > M_2 - |b(i_1) - b(i_2)| - 2b(i_1) \geq |d(i_1) - d(i_2)|.$$

In all cases, for $n \gg 0$ we have $|\sigma'| > |d(i_1) - d(i_2)|$. Hence

$$\sigma = \sigma' + d(i_1) - d(i_2) \neq 0.$$

We have proved that $f(k)$ attains its maximum on \mathcal{S}_n at a unique k . More precisely, there exists a unique $(\varepsilon_n, i_n) \in I_n$ such that $h_{i_n}^\varepsilon(k_{n, \varepsilon_n, i_n}) = \text{Max}_n$.

Equation (3.2) then implies that

$$d_+[J_{K_{p,q}}(n)] = pq(n^2 - 1)/4 + h_{i_n}^{\varepsilon_n}(k_{n, \varepsilon_n, i_n}).$$

If $p - 4qa < 0$ then $k_{n, \varepsilon_n, i_n} = \varepsilon_n(n/2 + s_n)$, where s_n is a periodic sequence and $s_n \leq -1/2$. We have

$$\begin{aligned} d_+[J_{K_{p,q}}(n)] &= q^2an^2 + ((-p + 4qa)(qs_n + \varepsilon_n/2) + qb(i_n))n - pq/4 \\ &\quad + (-p + 4qa)s_n(qs_n + \varepsilon_n) + 2qb(i_n)s_n + a + b(i_n)\varepsilon_n + d(i_n). \end{aligned}$$

In this case we have

$$B(n) = (-p + 4qa)(qs_n + \varepsilon_n/2) + qb(i_n) < 0,$$

since $qs_n + \varepsilon_n/2 \leq -q/2 + 1/2 < 0$ and $b(i_n) \leq 0$.

If $p - 4qa > 0$ then $k_{n, \varepsilon_n, i_n} = s_n$, where s_n is a periodic function. We have

$$\begin{aligned} d_+[J_{K_{p,q}}(n)] &= pq(n^2 - 1)/4 + (-p + 4qa)s_n(qs_n + 1) + 2qb(i_n)\varepsilon_n s_n \\ &\quad + a + b(i_n)\varepsilon_n + d(i_n). \end{aligned}$$

In this case we have $B(n) = 0$.

This completes the proof of Proposition 4.4. □

5. Conjectures

Recall that for every nontrivial knot $K \subset S^3$, there is an integer $N_K > 0$ and periodic functions $a_K(n), b_K(n), c_K(n)$ such that

$$d_+[J_K(n)] = a_K(n)n^2 + b_K(n)n + c_K(n)$$

for $n \geq N_K$. In Propositions 3.2 and 4.4 we made the assumption that $b(n) \leq 0$. Then we concluded that, under the appropriate hypotheses, this property is preserved under cabling. As we will discuss below, the property that $b(n) \leq 0$ is known to hold for all nontrivial knots, of any period, for which $b_K(n)$ has been calculated.

We propose the following conjecture.

Conjecture 5.1. *For every nontrivial knot $K \subset S^3$, we have*

$$b_K(n) \leq 0.$$

Moreover, if $b_K(n) = 0$ then K is a composite knot or a cable knot or a torus knot.

Note that $b_U(n) = 1/2$ for the trivial knot U .

Remark 5.2. It is known that a knot K is composite or cable or a torus knot if and only if its complement M_K contains embedded essential annuli [17, Lemma V.1.3.4]. Thus the last part of Conjecture 5.1 can alternatively be stated as follows: If $b_K(n) = 0$, then M_K contains an embedded essential annulus.

By Theorem 3.9, for B -adequate knots and their iterated cables we have $b_K(n) \leq 0$. Moreover, if K is a B -adequate knot and

$$2b = v_B - c = 0,$$

then by Lemma 3.8 K is a torus knot. Thus Conjecture 5.1 holds for B -adequate knots and their cables. Notice that, as shown in the proof of Theorem 3.9, the case $b_K(n) = 0$ occurs quite often for cables of B -adequate knots. Conjecture 5.1 holds for the knots of Table 1. In the next section we will check that Conjecture 5.1 holds true for 2-fusion knots. Thus the conjecture holds for all the classes of knots for which $d_+[J_K(n)]$ and $d_-[J_K(n)]$ have been calculated to date.

We now turn our attention to the Strong Slope Conjecture (Conjecture 1.6) stated in the Introduction. By Theorem 3.9, Conjecture 1.6 is true for iterated cables of B -adequate knots (and in particular iterated torus knots). Furthermore, the arguments in the proofs of Corollary 2.8 and Theorem 3.9 generalize easily to show that, under the hypothesis of Proposition 3.2 or Proposition 4.4, the Strong Slope Conjecture is closed under knot cabling. For instance we have the following:

Corollary 5.3. *Let K be a knot such that for $n \gg 0$ we have $d_+[J_K(n)]$ is a quadratic quasi-polynomial of period ≤ 2 . Suppose p/q is not a Jones slope of K . Then if K satisfies Conjecture 1.6 so does $K_{p/q}$.*

K	js_K	jx_K	$\chi(S)$	$ \partial S $	js_K^*	$jx_K^*(n)$	$\chi(S^*)$	$ \partial S^* $
8 ₁₉	{12}	{0}	0	2	{0}	{5}	-5	1
8 ₂₀	{8/3}	{-1, -5/3}	-3	1	{-10}	{4}	-4	1
8 ₂₁	{1}	{-2}	-4	2	{-12}	{3}	-3	1
9 ₄₂	{6}	{-1}	-2	2	{-8}	{5}	-5	1
9 ₄₃	{32/3}	{-1, -5/3}	-3	1	{-4}	{5}	-5	1
9 ₄₄	{14/3}	{-2, -8/3}	-6	1	{-10}	{4}	-4	1
9 ₄₅	{1}	{-2}	-4	2	{-14}	{4}	-4	1
9 ₄₆	{2}	{-1}	-2	2	{-12}	{5}	-5	1
9 ₄₈	{11}	{-3}	-6	2	{-4}	{3}	-3	1

TABLE 2. Nonalternating Montesinos knots up to nine crossings.

Remark 5.4. Similar ones to Conjectures 5.1 and 1.6 can also be formulated for the lowest degree of the colored Jones polynomial by noting that $J_K(n, v) = J_{K^*}(n, v^{-1})$, where K^* is the mirror image of K . To illustrate this point we discuss the example of the knot 9₄₉. This knot has genus two and is not B -adequate since the leading coefficient of its colored Jones polynomial is 2 [10] and not ± 1 . By Table 1, $js_K^* = \{0\}$ and $2b_K^*(n) = 3$. A genus two Seifert surface S , has boundary slope $0 = a_K^*(n)$ and $\chi(S) = -3 = -2b_K^*(n)$. Thus Conjectures 5.1 and 1.6 hold for the mirror image 9₄₉^{*}. We also mention that the same is true for 9₄₉ since it is known to be A -adequate.

Next we discuss more families of knots, not covered by Theorem 3.9, for which the above conjectures are true.

5.1. Nonalternating Montesinos knots up to nine crossings. Table 2 summarizes the relevant information about these knots. The Jones slopes and the sets of cluster points $\{2b_K(n)\}'$, $\{2b_K^*(n)\}'$ were obtained from Garoufalidis' paper [10]. The corresponding boundary slopes together with the values $\chi(S)$ and $|\partial S|$ were obtained using Dunfield's program for calculating boundary slopes of Montesinos knots [2]. In all cases, Conjectures 5.1 and 1.6 are easily verified for the knots and their mirror images. Note that, for example, for 9₄₄ we have $a/b = 14/3 \in js_K$, $\{2b_K(n)\}' = \{-2, -8/3\}$, and $\frac{\chi(S)}{|\partial S|b} = \frac{-6}{3} = -2$ as predicted by Conjecture 1.6. However, this assertion alone doesn't guarantee that $b_K(n) \leq 0$. Thus, in general, Conjecture 1.6 does not imply Conjecture 5.1.

Corollary 5.5. *Suppose that $K \in \{8_{19}, 8_{21}, 9_{42}, 9_{45}, 9_{46}, 9_{48}, 9_{49}\}$ and let K' be an iterated cable of K . Then, Conjectures 5.1 and 1.6 hold true for K' .*

Proof. We first note that $d_+[J_K(n)] = an^2 + bn + d(n)$ is a quadratic quasi-polynomial of period ≤ 2 , with $4a \in \mathbb{Z}$, $2b \in \mathbb{Z}$ and $b \leq 0$. Suppose $K_{p,q}$ is a

cable of K . Theorem 3.4 and the proof of Proposition 3.2 then imply that

$$d_+[J_{K_{p,q}}(n)] = An^2 + Bn + D(n)$$

is a quadratic quasi-polynomial of period ≤ 2 , with $4A \in \mathbb{Z}$, $2B \in \mathbb{Z}$ and $B \leq 0$. Moreover, the proof of Proposition 3.2 shows that one of the following is true:

- (1) We have $4A = pq$ and $B = 0$.
- (2) We have $4A = 4q^2a$ and $2B = 2|q|b + (1 - |q|)|4aq - p|$.

In case (1), a surface S with boundary slope pq is the cabling annulus; thus $\chi(S) = 0 = 2B$ and Conjectures 5.1 and 1.6 are satisfied. In case (2), let S be a surface that satisfies Conjecture 1.6 for K . We view $M_{K_{p,q}}$ as the union of M_K and a cable space $C_{p,q}$. An essential surface S' realizing the boundary slope $4A = 4q^2a$ for $K_{p,q}$ is obtained by Theorem 2.2. This surface is constructed as in the proof of Lemma 2.6. By Corollary 2.8 we see that $|\partial S'| = |\partial S|$, $|S' \cap T_+| = |q| |\partial S|$ and

$$\chi(S') = \chi(M_K \cap S') + \chi(C_{p,q} \cap S') = |q|\chi(S) + |\partial S| |4aq - p|(1 - |q|).$$

By assumption, $\chi(S) = |\partial S|(2b)$. Combining the last two equations with the formula in (2) above, we have

$$\chi(S') = |\partial S| (2|q|b + (1 - |q|)|4aq - p|) = |\partial S'| (2B),$$

which shows that S' satisfies Conjecture 1.6.

Applying the above argument repeatedly we obtain the result for iterated cables. □

5.2. A family of pretzel knots. Let p be an odd integer and consider the pretzel knot $K_p = (-2, 3, p)$. It is known that K_p is A -adequate if $p > 0$, and B -adequate if $p < 0$. Moreover, K_p is a torus knot if $p \in \{1, 3, 5\}$.

Suppose now that $p \geq 5$. Then K_p is A -adequate and by above discussion Conjecture 1.6 holds for the mirror image K_p^* .

By [10, Section 4.7] and Example 6.3 below we have

$$4a_{K_p}(n) = 2(p^2 - p - 5)/(p - 3) \quad \text{and} \quad 2b_{K_p}(n) = -(p - 5)/(p - 3).$$

Since $4a_{K_p}(n)$ is not an integer, in fact it is easily checked that

$$\gcd(2(p^2 - p - 5), p - 3) = 1,$$

K_p is not B -adequate. Thus we can't apply Theorem 3.9. Note moreover that $\gcd(p - 5, p - 3) = 2$. By [1, Theorem 3.3], K_p has an essential surface with boundary slope $2(p^2 - p - 5)/(p - 3)$, with two boundary components, and Euler characteristic $-(p - 5)$, which is equal to $(p - 3)(2b_{K_p}(n))$. Note that for $p = 5$ we get $b_{K_p}(n) = 0$. The knot $K = (2, -3, 5)$ is known to be a torus knot, which is in agreement with Conjecture 5.1.

Suppose now that $p \leq -1$. Then K is B -adequate. By [10, Section 4.7] and Example 6.3 we have $4a_{K_p}^*(n) = 2(p + 1)^2/p$ and

$$2b_{K_p}^*(n) = \begin{cases} 1 & \text{if } n \not\equiv 0 \pmod{p} \\ 1 - 2/p & \text{if } n \equiv 0 \pmod{p}. \end{cases}$$

Again since $4a_{K_p}^*(n)$ is not an integer, K_p is not A -adequate and Theorem 3.9 doesn't apply to K_p^* . According to [1, Theorem 3.3], K_p has an essential surface with boundary slope $2(p + 1)^2/p$ and Euler characteristic p , which is equal to $p(2b_{K_p}^*(n))$ when $n \not\equiv 0 \pmod{p}$. Thus we have:

Corollary 5.6. *For an odd integer p , the pretzel knots $K_p = (-2, 3, p)$ and K_p^* satisfy Conjectures 5.1 and 1.6.*

6. Two-fusion knots

The family of 2-fusion knots is a two-parameter family of closed 3-braids denoted by

$$\{K(m_1, m_2) \mid m_1, m_2 \in \mathbb{Z}\}.$$

For the precise definition and description see [11, 3]. The purpose of this section is to prove the following.

Theorem 6.1. *Conjecture 5.1 holds for 2-fusion knots.*

Note that $K(m_1, m_2)$ is a torus knot if $m_2 \in \{-1, 0\}$. In fact,

$$K(m_1, 0) = T(2, 2m_1 + 1) \quad \text{and} \quad K(m_1, -1) = T(2, 2m_1 - 3).$$

It is known that $K(m_1, m_2)$ is hyperbolic if $m_1 \notin \{0, 1\}$, $m_2 \notin \{-1, 0\}$ and $(m_1, m_2) \neq (-1, 1)$. See [11]. Note that $K(-1, 1) = T(2, 5)$.

From now on we consider $m_2 \notin \{-1, 0\}$ only. For $n \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{Z}$ such that $0 \leq k_1 \leq n$ and $|n - 2k_1| \leq n + 2k_2 \leq n + 2k_1$, let

$$\begin{aligned} Q(n, k_1, k_2) = & \frac{k_1}{2} - \frac{3k_1^2}{2} - 3k_1k_2 - k_2^2 - k_1m_1 - k_1^2m_1 - k_2m_2 \\ & - k_2^2m_2 - 6k_1n - 3k_2n + 2m_1n + 4m_2n \\ & - k_2m_2n - 2n^2 + m_1n^2 + 2m_2n^2 \\ & + \frac{1}{2} \left((1 + 8k_1 + 4k_2 + 8n) \min\{l_1, l_2, l_3\} - 3 \min\{l_1, l_2, l_3\}^2 \right) \end{aligned}$$

where

$$l_1 = 2k_1 + n, \quad l_2 = 2k_1 + k_2 + n, \quad l_3 = k_2 + 2n.$$

6.1. The highest degree. The quantity $Q(n, k_1, k_2)$ is closely related to the degree $\delta_K(n)$. According to [11] for the 2-fusion knot $K = K(m_1, m_2)$, with $m_2 \notin \{-1, 0\}$, we have the following possibilities:

Case A. Suppose that $m_1, m_2 \geq 1$. Then

$$\delta_K(n) = Q(n, k_1, -k_1),$$

where

$$c_1 = \frac{1 - m_1 + m_2 + m_2 n}{2(-1 + m_1 + m_2)},$$

and k_1 is the integers closest to c_1 satisfying $k_1 \leq n/2$.

Case B. Suppose that $m_1 \leq 0, m_2 \geq 1$. There are 2 subcases.

Subcase B.1. We have $(1 + m_1 + m_2 \leq 0)$ or

$$(1 + m_1 + m_2 > 0 \quad \text{and} \quad 1 + 2m_1 + m_2 < 0).$$

Then

$$\delta_K(n) = Q(n, n, 0).$$

Subcase B.2. We have $1 + m_1 + m_2 > 0$ and $1 + 2m_1 + m_2 \geq 0$. Then

$$\delta_K(n) = Q(n, k_1, k_1 - n),$$

where

$$c_2 = \frac{1 - m_1 - m_2 + (1 + m_2)n}{2(1 + m_1 + m_2)}.$$

and k_1 is one of the integers closest to c_2 .

Case C. Suppose that $m_2 \leq -2$. There are 2 subcases.

Subcase C.1. We have $m_1 \leq -3m_2/2$. Then

$$\delta_K(n) = Q(n, n, n).$$

Subcase C.2. We have $m_1 > -3m_2/2$. Let

$$c_3 = \frac{-3/2 + m_1 + m_2 + (1 + m_2)n}{1 - 2m_1 - 2m_2}$$

and let k_1 be one of the integers closest to c_3 . Then

$$\delta_K(n) = \begin{cases} Q(n, k_1, k_1) & \text{if } c_3 \notin \frac{1}{2} + \mathbb{Z}, \\ Q(n, k_1, k_1) - (c_3 + 1/2) & \text{if } c_3 \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

6.2. Calculating the linear term. In this subsection we will prove the following.

Theorem 6.2. *For the 2-fusion knot $K = K(m_1, m_2)$, with $m_2 \notin \{-1, 0\}$, we have*

$$b_K(n) = \begin{cases} \frac{m_2(1-m_1)}{2(-1+m_1+m_2)} & \text{if } (m_1, m_2) \in I_1 \cup I_2, \\ 1 + m_1 & \text{if } (m_1, m_2) \in I_3, \\ \frac{m_1(m_2-1)}{2(1+m_1+m_2)} & \text{if } (m_1, m_2) \in I_4, \\ 5/2 + m_1 + 3m_2 & \text{if } (m_1, m_2) \in I_5, \\ \frac{(-5+2m_1)(1+m_2)}{2(-1+2m_1+2m_2)} & \text{if } (m_1, m_2) \in I_6 \text{ and } \frac{-1+(1+m_2)(n-1)}{-1+2m_1+2m_2} \notin \mathbb{Z}, \\ \frac{(-3+2m_1)(1+m_2)}{2(-1+2m_1+2m_2)} & \text{if } (m_1, m_2) \in I_6 \text{ and } \frac{-1+(1+m_2)(n-1)}{-1+2m_1+2m_2} \in \mathbb{Z}. \end{cases}$$

In particular we have $b_K(n) \leq 0$. Moreover $b_K(n) = 0$ if and only if $m_1 \in \{0, 1\}$ and $m_2 \geq 1$, or $(m_1, m_2) = (-1, 1)$.

Proof. As in the previous subsection, there are 3 cases.

Case A. $m_1, m_2 \geq 1$. Recall that

$$c_1 = \frac{1 - m_1 + m_2 + m_2n}{2(-1 + m_1 + m_2)}$$

and k_1 is one of the integers closest to c_1 satisfying $k_1 \leq \frac{n}{2}$. We have

$$\begin{aligned} \delta_K(n) = Q(n, k_1, -k_1) &= (1 - m_1 - m_2)k_1^2 + (1 - m_1 + m_2 + m_2n)k_1 \\ &\quad + 2m_1n + 4m_2n + m_1n^2 + 2m_2n^2 + \frac{n}{2} + \frac{n^2}{2}. \end{aligned}$$

Write $k_1 = c_1 + r_n$ where r_n is a periodic sequence with

$$\begin{cases} |r_n| \leq 1/2 & \text{if } m_1 \geq 2, \\ r_n \in \{-1/2, -1\} & \text{if } m_1 = 1. \end{cases}$$

We have

$$\delta_K(n) = Q(n, c_1 + r_n, -c_1 - r_n) = Q(n, c_1, -c_1) + (1 - m_1 - m_2)r_n^2$$

and

$$\begin{aligned} Q(n, c_1, -c_1) &= 2m_1n + 4m_2n + m_1n^2 + 2m_2n^2 + \frac{n}{2} + \frac{n^2}{2} \\ &\quad - \frac{(1 - m_1 + m_2 + m_2n)^2}{4(1 - m_1 - m_2)} \\ &= \left(m_1 + 2m_2 + \frac{1}{2} + \frac{m_2^2}{4(-1 + m_1 + m_2)} \right) n^2 \\ &\quad + \left(2m_1 + 4m_2 + \frac{1}{2} + \frac{m_2(1 - m_1 + m_2)}{2(-1 + m_1 + m_2)} \right) n \\ &\quad - \frac{(1 - m_1 + m_2)^2}{4(1 - m_1 - m_2)}. \end{aligned}$$

Since $d_+[J_K(n)] = \delta_K(n-1) + (n-1)/2$ we obtain

$$\begin{aligned} d_+[J_K(n)] &= \left(m_1 + 2m_2 + \frac{1}{2} + \frac{m_2^2}{4(-1 + m_1 + m_2)} \right) n^2 \\ &\quad + \frac{m_2(1 - m_1)}{2(-1 + m_1 + m_2)} n \\ &\quad - \left(m_1 + 2m_2 + \frac{1}{2} - \frac{(1 - m_1)^2}{4(-1 + m_1 + m_2)} \right) \\ &\quad + (1 - m_1 - m_2)r_{n-1}^2. \end{aligned}$$

Case B. $m_1 \leq 0, m_2 \geq 1$. There are 2 subcases.

Subcase B.1. $(1 + m_1 + m_2 \leq 0)$ or $(1 + m_1 + m_2 > 0$ and $1 + 2m_1 + m_2 < 0)$. Then

$$\delta_K(n) = Q(n, n, 0) = \left(\frac{1}{2} + 2m_2 \right) n^2 + \left(\frac{3}{2} + m_1 + 4m_2 \right) n.$$

Hence

$$d_+[J_K(n)] = \left(\frac{1}{2} + 2m_2 \right) n^2 + (1 + m_1)n - \left(\frac{3}{2} + m_1 + 2m_2 \right).$$

Subcase B.2. $1 + m_1 + m_2 > 0$ and $1 + 2m_1 + m_2 \geq 0$. Recall that

$$c_2 = \frac{1 - m_1 - m_2 + (1 + m_2)n}{2(1 + m_1 + m_2)}$$

and k_1 is one of the integers closest to c_2 . We have

$$\begin{aligned} \delta_K(n) &= Q(n, k_1, k_1 - n) \\ &= (-1 - m_1 - m_2)k_1^2 + (1 - m_1 - m_2 + (1 + m_2)n)k_1 \\ &\quad + 2m_1n + 5m_2n + m_1n^2 + 2m_2n^2 + \frac{n}{2} + \frac{n^2}{2}. \end{aligned}$$

Write $k_1 = c_2 + r_n$ where r_n is a periodic sequence with $|r_n| \leq 1/2$. As in Case A we have

$$\delta_K(n) = Q(n, c_2, c_2 - n) + (-1 - m_1 - m_2)r_n^2$$

and

$$\begin{aligned} Q(n, c_2, c_2 - n) &= 2m_1n + 5m_2n + m_1n^2 + 2m_2n^2 + \frac{n}{2} + \frac{n^2}{2} \\ &\quad - \frac{(1 - m_1 - m_2 + (1 + m_2)n)^2}{4(-1 - m_1 - m_2)} \\ &= \left(\frac{3}{4} + \frac{3m_1}{4} + \frac{9m_2}{4} + \frac{m_1^2}{4(1 + m_1 + m_2)} \right) n^2 \\ &\quad + \left(1 + 2m_1 + \frac{9m_2}{2} - \frac{m_1}{1 + m_1 + m_2} \right) n \\ &\quad - \frac{(1 - m_1 - m_2)^2}{4(-1 - m_1 - m_2)}. \end{aligned}$$

Hence

$$\begin{aligned} d_+[J_K(n)] &= \left(\frac{3}{4} + \frac{3m_1}{4} + \frac{9m_2}{4} + \frac{m_1^2}{4(1 + m_1 + m_2)} \right) n^2 \\ &\quad + \frac{m_1(m_2 - 1)}{2(1 + m_1 + m_2)}n - \left(\frac{3}{4} + \frac{3m_1}{4} + \frac{9m_2}{4} - \frac{(m_2 - 1)^2}{4(1 + m_1 + m_2)} \right) \\ &\quad + (-1 - m_1 - m_2)r_n^2. \end{aligned}$$

Case C. $m_2 \leq -2$. There are 2 subcases.

Subcase C.1. $m_1 \leq -3m_2/2$. Then

$$\delta_K(n) = Q(n, n, n) = (2 + m_1 + 3m_2)n.$$

Hence

$$d_+[J_K(n)] = (5/2 + m_1 + 3m_2)(n - 1).$$

Subcase C.2. $m_1 > -3m_2/2$. Recall that

$$c_3 = \frac{-3/2 + m_1 + m_2 + (1 + m_2)n}{1 - 2m_1 - 2m_2}$$

and let k_1 be one of the integers closest to c_3 . We have

$$\delta_K(n) = \begin{cases} Q(n, k_1, k_1) & \text{if } c_3 \notin \frac{1}{2} + \mathbb{Z} \\ Q(n, k_1, k_1) - (c_3 + 1/2) & \text{if } c_3 \in \frac{1}{2} + \mathbb{Z} \end{cases}$$

and

$$\begin{aligned} Q(n, k_1, k_1) &= (1/2 - m_1 - m_2)k_1^2 - (-3/2 + m_1 + m_2 + (1 + m_2)n)k_1 \\ &\quad + 2m_1n + 4m_2n + m_1n^2 + 2m_2n^2 + \frac{n}{2} + \frac{n^2}{2}. \end{aligned}$$

Write $k_1 = c_3 + r_n$ where r_n is a periodic sequence with $|r_n| \leq 1/2$. As in Case A we have

$$Q(n, k_1, k_1) = Q(n, c_3, c_3) + (1/2 - m_1 - m_2)r_n^2$$

and

$$\begin{aligned}
 Q(n, c_3, c_3) &= 2m_1n + 4m_2n + m_1n^2 + 2m_2n^2 + \frac{n}{2} + \frac{n^2}{2} \\
 &\quad - \frac{(-3/2 + m_1 + m_2 + (1 + m_2)n)^2}{4(1/2 - m_1 - m_2)} \\
 &= \frac{(2m_1 + 3m_2)^2}{2(-1 + 2m_1 + 2m_2)}n^2 \\
 &\quad + \left(\frac{1}{2} + 2m_1 + \frac{9m_2}{2} + \frac{-3 + 2m_1}{2(-1 + 2m_1 + 2m_2)} \right)n \\
 &\quad - \frac{(-3/2 + m_1 + m_2)^2}{4(1/2 - m_1 - m_2)}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 d_+[J_K(n)] &= \frac{(2m_1 + 3m_2)^2}{2(-1 + 2m_1 + 2m_2)}n^2 \\
 &\quad + \begin{cases} \frac{(-5+2m_1)(1+m_2)}{2(-1+2m_1+2m_2)}n & \text{if } \frac{-1+(1+m_2)(n-1)}{-1+2m_1+2m_2} \notin \mathbb{Z} \\ \frac{(-3+2m_1)(1+m_2)}{2(-1+2m_1+2m_2)}n & \text{if } \frac{-1+(1+m_2)(n-1)}{-1+2m_1+2m_2} \in \mathbb{Z} \end{cases} \\
 &\quad - \left(\frac{1}{2} + m_1 + 2m_2 - \frac{(2m_1 - 5)^2}{8(2m_1 + 2m_2 - 1)} \right) \\
 &\quad + (1/2 - m_1 - m_2)r_{n-1}^2.
 \end{aligned}$$

This completes the proof of Theorem 6.2. □

6.3. Proof of Theorem 6.1. Theorem 6.2 implies that Conjecture 5.1 holds true for 2-fusion knots: The fact that $b_K(n) \leq 0$ is clear by the statement of Theorem 6.2. Moreover, $b_K(n) = 0$ if and only if $m_1 \in \{0, 1\}$ and $m_2 \geq 1$, or $(m_1, m_2) = (-1, 1)$. As noted in [11] we have

$$K(1, m_2) = K^*(0, -m_2 - 1).$$

On the other hand, by definition the knot $K(0, m_2)$ is a torus knot. Finally $K(-1, 1)$ is the torus knot $T(2, 5)$. Thus if $b_K(n) = 0$, and $K = K(m_1, m_2)$, then K is a torus knot. □

Example 6.3. Consider the 2-fusion knot $K(m, 1)$, also known as the $(-2, 3, 2m + 3)$ -pretzel knot. It is known that $K(m, 1)$ is B -adequate if $m \leq -2$ and is A -adequate if $m \geq -1$. Moreover $K(m, 1)$ is a torus knot if $|m| \leq 1$, and $K(-2, 1)$ is the twist knot 5_2 which is an adequate knot. Hence we consider the two cases $m \geq 2$ and $m \leq -3$ only.

Note that $K(m_1, m_2)$ is the mirror image of $K(1 - m_1, -1 - m_2)$. In particular, $K(m, 1)$ is the mirror image of $K(1 - m, -2)$.

Case 1. $m \geq 2$. From the proof of Theorem 6.2 we have

$$d_+[J_{K(m,1)}(n)] = \left(\frac{5}{2} + m + \frac{1}{4m}\right)n^2 + \left(\frac{1}{2m} - \frac{1}{2}\right)n - \left(3 + \frac{3m}{4} - \frac{1}{4m}\right) - mr_{n-1}^2$$

where r_n is a periodic sequence with $|r_n| \leq 1/2$.

Hence, by Theorem 4.1, the (p, q) -cable of $K(m, 1)$ satisfies Conjecture 5.1 and the Slope Conjecture if

$$p - \left(10 + 4m + \frac{1}{m}\right)q < 0 \quad \text{or} \quad p - \left(10 + 4m + \frac{1}{m}\right)q > \frac{m}{4} + \frac{1}{m} - 1.$$

Case 2. $m \leq -3$. From the proof of Theorem 6.2 we have

$$d_+[J_{K(1-m,-2)}(n)] = -\frac{2(m+2)^2}{2m+3}n^2 + b(n)n + (6m+17)/8 + (m+3/2)r_{n-1}^2$$

where r_n is a periodic sequence with $|r_n| \leq 1/2$, and

$$b(n) = \begin{cases} -\frac{1}{2} & \text{if } (2m+3) \nmid n, \\ -\frac{2m+1}{2(2m+3)} & \text{if } (2m+3) \mid n. \end{cases}$$

Hence, by Theorem 4.1, the (p, q) -cable of $K(1 - m, -2) = (K(m, 1))^*$ satisfies Conjecture 5.1 and the Slope Conjecture if

$$p + \left(10 + 4m + \frac{1}{2m+3}\right)q < 0$$

or

$$p + \left(10 + 4m + \frac{3}{2m+3}\right)q > -\left(\frac{m}{4} + \frac{1}{2m+3} + \frac{11}{8}\right).$$

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