New York Journal of Mathematics

New York J. Math. **21** (2015) 783–800.

Induced representations arising from a character with finite orbit in a semidirect product

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Abstract. Making use of a unified approach to certain classes of induced representations, we establish here a number of detailed spectral theoretic decomposition results. They apply to specific problems from noncommutative harmonic analysis, ergodic theory, and dynamical systems. Our analysis is in the setting of semidirect products, discrete subgroups, and solenoids. Our applications include analysis and ergodic theory of Bratteli diagrams and their compact duals; of wavelet sets, and wavelet representations.

Contents

1. Introduction

The purpose of the present paper is to demonstrate how a certain induction (from representation theory) may be applied to a number of problems in dynamics, in spectral theory and harmonic analysis; yielding answers in the form of explicit invariants, and equivalences. In more detail, we show that a certain family of induced representations forms a unifying framework. The setting is unitary representations, infinite-dimensional semidirect products and crossed products. While the details in our paper involve a variant of

Received April 19, 2015.

2010 *Mathematics Subject Classification.* Primary: 47L60, 46N30, 46N50, 42C15, 65R10, 05C50, 05C75, 31C20; Secondary: 46N20, 22E70, 31A15, 58J65, 81S25.

Key words and phrases. Hilbert space, unitary representation, induced representations, Frobenius reciprocity, semidirect products, imprimitivity, Pontryagin duality, direct integral decompositions, wavelet-sets, noncommutative harmonic analysis, Halmos–Rohlin, C^*-algebra, solenoid.

ISSN 1076-9803/2015

Mackey's theory, our approach is extended, and it is constructive and algorithmic. Our starting point is a family of imprimitivity systems. By this we mean a pair: a unitary representation, and a positive operator-valued mapping, subject to a covariance formula for the two (called imprimitivity). One of its uses is a characterization of those unitary representations of some locally compact group G "which arise" as an induction from a subgroup; i.e., as induced from of a representation of a specific subgroup of G. (Here, the notation "which arise" means "up to unitary equivalence.")

Below we give a summary of notation and terminologies, and we introduce a family of induced representations, see, e.g., [Jor88, Mac88, Ørs79].

The framework below is general; the context of locally compact (non-abelian) groups. But we shall state the preliminary results in the context of unimodular groups, although it is easy to modify the formulas and the results given to nonunimodular groups. One only needs to incorporate suitable factors on the respective modular functions, that of the ambient group and that of the subgroup. Readers are referred to [Ørs79] for additional details on this point. Another reason for our somewhat restricted setting is that our main applications below will be to the case of induction from suitable abelian subgroups. General terminology: In the context of locally compact abelian groups, say B, we shall refer to Pontryagin duality (see, e.g., [Rud62]); and so in particular, when the abelian group B is given, by "the dual" we mean the group of all continuous characters on B, i.e., the one-dimensional unitary representations of B. We shall further make use of Pontryagin's theorem to the effect that the double-dual of B is naturally isomorphic to B itself.

Our results are motivated in part by a number of noncommutative harmonic analysis issues involved in the analysis of wavelet representations, and wavelet sets; see especially [LiPT01]; and also [LaSST06, CM11, CM13, CMO14, MO14].

Definition 1.1. Let G be a locally compact group, $B \subset G$ a closed subgroup; both assumed unimodular. Given $V \in \mathrm{Rep}\,(B, \mathscr{H}_V)$, a representation of B in the Hilbert space $\mathscr{H}_V$, let

$$U = \mathrm{Ind}_B^G\,(V) \tag{1.1}$$

be the *induced representation.* U acts on the Hilbert space $\mathscr{H}_U$ as follows:

$\mathscr{H}_U$ consists of all measurable functions $f : G \to \mathscr{H}_V$ s.t.

$$f\,(bg) = V_b f\,(g)\,,\ \forall b \in B,\ \forall g \in G, \tag{1.2}$$

$$\|f\|^2_{\mathscr{H}_U} = \int_{B\backslash G} \|f\,(g)\|^2\, d\mu_{B\backslash G}\,(g) < \infty, \tag{1.3}$$

w.r.t. the measure on the homogeneous space $B\backslash G$, i.e., the Hilbert space carrying the induced representation. Set

$$(U_g f)\,(x) := f\,(xg)\,, \quad x, g \in G. \tag{1.4}$$

Remark 1.2. If the respective groups B and G are nonunimodular, we select respective right-invariant Haar measures db, dg; and corresponding modular functions $\triangle_B$ and $\triangle_G$. In this case, the modification to the above is that Equation (1.2) will instead be

$$f(bg) = \left(\frac{\triangle_B(b)}{\triangle_G(b)}\right)^{1/2} V_b f(g)\,;\; b \in B,\; g \in G, \tag{1.5}$$

thus a modification in the definition of $\mathscr{H}(\mathrm{Ind}_B^G(V))$.

We recall the following theorems of Mackey [Mac88, Ørs79].

Theorem 1.3 (Mackey). *$U = \mathrm{Ind}_B^G(V)$ is a unitary representation.*

Theorem 1.4 (Imprimitivity Theorem). *A representation $U \in \mathrm{Rep}(G, \mathscr{H})$ is induced iff $\exists$ a positive operator-valued mapping*

$$\pi : C_c(B\backslash G) \longrightarrow \mathscr{B}(\mathscr{H}) \text{ s.t.} \tag{1.6}$$

$$U_g \pi(\varphi) U_g^* = \pi(R_g \varphi) \tag{1.7}$$

$\forall g \in G$, $\forall \varphi \in C_c(B\backslash G)$; where π is nondegenerate, and R_g in (1.7) denotes the right regular action.

Proof. See [Ørs79, Jor88]. □

Theorem 1.5. *Given $U \in \mathrm{Rep}(G, \mathscr{H})$, the following are equivalent:*

1. *$\exists \pi$ s.t. (1.7) holds.*
2. *$\exists V \in \mathrm{Rep}(B, \mathscr{H}_V)$ s.t. $U \cong \mathrm{Ind}_B^G(V)$.*
3. *$L_G(U, \pi) \cong L_B(V)$.*
4. *If $V_i \in \mathrm{Rep}(B, \mathscr{H}_{V_i})$, $i = 1, 2$ and $U_i = \mathrm{Ind}_B^G(V_i)$, $i = 1, 2$, then*
$$L_G((U_1, \pi_1); (U_2, \pi_2)) \cong L_B(V_1, V_2);$$
i.e., all intertwining operators $V_1 \to V_2$ lift to the pair (U_i, π_i), $i = 1, 2$.

(Here, "$\cong$" denotes unitary equivalence.) Specifically,

$$\begin{aligned} L_G((U_1, \pi_1), (U_2, \pi_2)) &= \Big\{ W : \mathscr{H}_{U_1} \to \mathscr{H}_{U_2} \;\Big|\; W U_1(g) = U_2(g) W,\ \forall g \in G, \text{ and} \\ &\qquad U \pi_1(\varphi) = \pi_2(\varphi) W \Big\} \end{aligned} \tag{1.8}$$

and

$$L_B(V_1, V_2) = \left\{ w : \mathscr{H}_{V_1} \to \mathscr{H}_{V_2} \;\middle|\; w V_1(b) = V_2(b) w,\ \forall b \in B \right\}. \tag{1.9}$$

Proof. See [Ørs79, Jor88, Mac88]. □

We also recall the following result:

Lemma 1.6. *Let B be a lattice in $\mathbb{R}^d$, let α be an action of $\mathbb{Z}$ on B by automorphisms, and let $(K = \widehat{B}, \widehat{\alpha})$ represent the dual action of $\mathbb{Z}$ on the compact abelian group K. Fix a $d \times d$ matrix A preserving the lattice B with spec $(A) \subset \{\lambda : |\lambda| > 1\}$.*

If $\alpha = \alpha_A \in \mathrm{Aut}(B)$, then $\widehat{\alpha} \in \mathrm{Aut}(K)$ is ergodic, i.e., if ν is the normalized Haar measure on K and if $E \subset K$ is measurable s.t. $\widehat{\alpha}E = E$, then $\nu(E)(1 - \nu(E)) = 0$.

Proof. By Halmos–Rohlin's theorem (see [BreJ91]), we must show that if $b \in B$ and $A^n b = b$ for some $n \in \mathbb{N}$, then $b = 0$. This then follows from the assumption on the spectrum of A. □

2. The representation $\mathrm{Ind}_B^G(\chi)$

We now specify our notation and from this point on will restrict our study to the following setting:

- (i) B: a discrete abelian group (written additively);
- (ii) $\alpha \in \mathrm{Aut}(B)$;
- (iii) $G_\alpha := B \rtimes_\alpha \mathbb{Z}$;
- (iv) $K := \widehat{B}$, the compact dual group;
- (v) $\widehat{\alpha} \in \mathrm{Aut}(\widehat{B})$, dual action;
- (vi) $L_\alpha := K \rtimes_{\widehat{\alpha}} \mathbb{Z}$, the C^*-algebra crossed product [BreJ91, p.299]; also written as $C^*(K) \rtimes_{\widehat{\alpha}} \mathbb{Z}$.

More generally, let K be a compact Hausdorff space, and $\beta : K \to K$ a homeomorphism; then we study the C^*-crossed product $C^*(K) \rtimes_\beta \mathbb{Z}$ (see [BreJ91]).

We define the *induced representation*

$$U^\chi := \mathrm{Ind}_B^G(\chi) \tag{2.1}$$

where $\chi \in K$, i.e., a character on B, and

$$G := B \rtimes_\alpha \mathbb{Z} \tag{2.2}$$

as a semi-direct product. Note that U^χ in (2.1) is induced from a one-dimensional representation.

Below, $\{\delta_k\}_{k\in\mathbb{Z}}$ denotes the canonical basis in $l^2(\mathbb{Z})$, i.e.,

$$\delta_k = (\ldots, 0, 0, 1, 0, 0, \ldots)$$

with "1" at the k^{th} place.

Lemma 2.1. *The representation $\mathrm{Ind}_B^G(\chi)$ is unitarily equivalent to*

$$U^\chi : G \to \mathscr{B}\left(l^2(\mathbb{Z})\right), \text{ where} \tag{2.3}$$

$$U^\chi_{(j,b)}\delta_k = \chi(\alpha_{k-j}(b))\,\delta_{k-j}, \quad (j, b) \in G. \tag{2.4}$$

Proof. First note that both groups B and $G = B \rtimes_\alpha \mathbb{Z}$ are discrete and unimodular, and the respective Haar measures are the counting measure. Since $B \backslash G \simeq \mathbb{Z}$, so $\mathscr{H}^\chi = L^2 (B \backslash G) \simeq l^2 (\mathbb{Z})$. Recall the multiplication rule in G:

$$(k, c)(j, b) = (k + j, \alpha_k (b) + c) \tag{2.5}$$

where $j, k \in \mathbb{Z}$, and $b, c \in B$.

The representation $\mathrm{Ind}_B^G (\chi)$ acts on functions $f : G \to \mathbb{C}$ s.t.

$$f ((j, b)) = \chi (b) f (j, 0) \tag{2.6}$$

and

$$\|f\|^2_{\mathscr{H}^\chi} = \sum_{j \in \mathbb{Z}} |f (j, 0)|^2 . \tag{2.7}$$

Moreover, the mapping

$$l^2 \ni \xi \longmapsto W\xi \in \mathscr{H}^\lambda \tag{2.8}$$

given by

$$(W\xi)(j, b) = \chi (b) \xi_j \tag{2.9}$$

is a unitary intertwining operator, i.e.,

$$\mathrm{Ind}_B^G (\chi) W\xi = W U^\chi \xi \tag{2.10}$$

where

$$\left(U^\chi_{(j,b)} \xi\right)_k = \chi (\alpha_k (b)) \xi_{k+j} \tag{2.11}$$

$\forall j, k \in \mathbb{Z}$, $\forall b \in B$, $\forall \xi \in l^2 (\mathbb{Z})$. Note that (2.11) is equivalent to (2.4).

To verify (2.10), we have

$$\begin{aligned} \left(\mathrm{Ind}_B^G (\chi)_{(j,b)} W\xi\right)(k, c) &= W\xi ((k, c)(j, b)) \\ &\underset{(2.5)}{=} W\xi (k + j, \alpha_k (b) + c) \\ &\underset{(2.6)}{=} \chi (\alpha_k (b) + c) W\xi (k + j, 0) \\ &\underset{(2.9)}{=} \chi (\alpha_k (b) + c) \xi_{k+j} \end{aligned}$$

and

$$\begin{aligned} \left(W U^\chi_{(j,b)} \xi\right)(k, c) &\underset{(2.9)}{=} \chi (c) \left(U^\chi_{(j,b)} \xi\right)_k \\ &\underset{(2.11)}{=} \chi (c) \chi (\alpha_k (b)) \xi_{k+j} \\ &= \chi (\alpha_k (b) + c) \xi_{k+j}. \end{aligned}$$

In the last step we use that $\chi (\alpha_k (b) + c) = \chi (\alpha_k (b)) \chi (c)$, which is just the representation property of $\chi : B \to \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$,

$$\chi (b_1 + b_2) = \chi (b_1) \chi (b_2), \quad \forall b_1, b_2 \in B. \tag{2.12}$$

□

Remark 2.2. In summary, the representation $\mathrm{Ind}_B^G(\chi)$ has three equivalent forms:

(1) On $l^2(\mathbb{Z})$,

$$\left(\mathrm{Ind}_B^G(\chi)\,\xi\right)_k = \chi(\alpha_k(b))\,\xi_{k+j}, \quad (\xi_k) \in l^2(\mathbb{Z}). \tag{2.13}$$

(2) In the ONB $\{\delta_k \mid k \in \mathbb{Z}\}$,

$$\mathrm{Ind}_B^G(\chi)_{(j,k)}\,\delta_k = \chi(\alpha_{k-j}(b))\,\delta_{k-j}, \quad k \in \mathbb{Z}. \tag{2.14}$$

(3) On $\mathscr{H}^\chi$, consisting of functions $f : G \to \mathbb{C}$ s.t.

$$f(j,b) = \chi(b)\,f(j,0), \quad \|f\|^2_{\mathscr{H}^\chi} = \sum_{j\in\mathbb{Z}} |f(j,0)|^2 < \infty$$

where

$$\begin{aligned}\left(\mathrm{Ind}_B^G(\chi)_{(j,b)}\,f\right)(k,c) &= f((k,c)(j,b)) \\ &= f((k+j,\alpha_k(b)+c)) \\ &= \chi(\alpha_k(b)+c)\,f(k+j,0),\end{aligned}$$

with $(j,b),(k,c) \in G = B \rtimes_\alpha \mathbb{Z}$, i.e., $j,k \in \mathbb{Z}$, $b,c \in B$.

3. Irreducibility

Let $G = B \rtimes_\alpha \mathbb{Z}$, and $\mathrm{Ind}_B^G(\chi)$ be the induced representation. Our main results are summarized as follows:

(1) If $\chi \in K = \widehat{B}$ has infinite order, then $\mathrm{Ind}_B^G(\chi) \in \mathrm{Rep}_{\mathrm{irr}}\left(G, l^2(\mathbb{Z})\right)$, i.e., $\mathrm{Ind}_B^G(\chi)$ is irreducible. We have

$$\mathrm{Ind}_B^G(\chi)_{(j,b)} = D_\chi(b)\,T_j \tag{3.1}$$

where $D_\chi(b)$ is diagonal, and $T_j : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, $(T_j\xi)_k = \xi_{k+j}$, for all $\xi \in l^2(\mathbb{Z})$.

(2) Suppose χ has finite order p, i.e., $\exists p$ s.t. $\widehat{\alpha}^p\chi = \chi$, but $\widehat{\alpha}^k\chi \neq \chi$, $\forall 1 \leq k < p$. Set $U_p^\chi = \mathrm{Ind}_B^G(\chi)$, then

$$U_p^\chi(j,b) = D_\chi(b)\,P^j \tag{3.2}$$

where $T_j\delta_k = \delta_{k-j}$ and $P =$ the $p\times p$ permutation matrix. See (3.9).

Details below.

Lemma 3.1. *Set* $(T_j\xi)_k = \xi_{k+j}$, *for* $k,j \in \mathbb{Z}$, $\xi \in l^2$; *equivalently,* $T_j\delta_k = \delta_{k-1}$. *Then*

(1) *The following identity holds:*

$$T_nU^\chi = U^{\widehat{\alpha}^n\chi}T_n, \quad \forall n \in \mathbb{Z}, \forall \chi \in K. \tag{3.3}$$

(2) *For* $\omega = (\omega_i) \in l^\infty(\mathbb{Z})$, *set*

$$(\pi(\omega)\xi)_k = \omega_k \xi_k, \quad \xi \in l^2(\mathbb{Z});\ or$$
$$\pi(\omega)\delta_k = \omega_k \delta_k. \tag{3.4}$$

Then

$$U^\chi_{(j,b)} \pi(\omega) \left(U^\chi_{(j,b)}\right)^* = \pi(T_j \omega) \tag{3.5}$$

$\forall (j,b) \in G$, $\forall \chi \in K$.

(3) *Restriction to the representation:*

$$U^\chi \Big|_B = \sum_{j \in \mathbb{Z}}^{\oplus} \left(\widehat{\alpha}^j \chi\right). \tag{3.6}$$

Recall that $G := B \rtimes_\alpha \mathbb{Z}$, *and so the subgroup* B *corresponds to* $j = 0$ *in* $\{(j,b) \mid j \in \mathbb{Z}, b \in B\}$. *Note that* $\widehat{\alpha}^j \chi$ *is a one-dimensional representation of* B.

Proof. One checks that

$$\left(T_n U^\chi_{(j,b)} \xi\right)_k = \left(U^\chi_{(j,b)} \xi\right)_{k+n} = \chi(\alpha_{k+n}(b)) \xi_{k+n+j}, \text{ and}$$

$$\left(U^{\widehat{\alpha}^n \chi}_{(j,b)} T_n \xi\right)_k = (\widehat{\alpha}^n \chi)(\alpha_k(b)) (T_n \xi)_{k+j} = \chi(\alpha_{k+n}(b)) \xi_{k+n+j}.$$

The other assertions are immediate. ☐

Corollary 3.2. *If* $\widehat{\alpha}^p \chi = \chi$, *then* T_p *commutes with* U^χ. *In this case,*

$$U^\chi_{(j,b)} \delta_k = \widehat{\alpha}^{k-j}(\chi)(b) \delta_{k-j} \tag{3.7}$$

induces an action on $l^2(\mathbb{Z}/p\mathbb{Z})$. *If* m *is fixed, the same representation is repeated, where the same action occurs in the sub-band*

$$l^2\left(\{\delta_{j+mp} \mid 0 \le j < p\}\right). \tag{3.8}$$

Proof. Immediate from Lemma 3.1. ☐

The p-dimensional representation in each band $\{\delta_{j+mp} \mid 0 \le j < p\}$ may be given in matrix form:

Lemma 3.3. *Let* $\chi \in K$, *and assume the orbit of* χ *under the action of* $\widehat{\alpha}$ *has finite order* p, *i.e.,* $\widehat{\alpha}^p \chi = \chi$ *and* $\widehat{\alpha}^k \chi \ne \chi$, $1 \le k < p$. *For* $b \in B$, *set*

$$D_\chi(b) = \begin{pmatrix} \chi(b) & & & \\ & \chi(\alpha(b)) & & \Large 0 \\ \Large 0 & & \ddots & \\ & & & \chi\left(\alpha^{p-1}(b)\right) \end{pmatrix} = \operatorname{diag}\left(\chi\left(\alpha^i(b)\right)\right)_0^{p-1},$$

a $p \times p$ *diagonal matrix. If* $\chi = (\chi_1, \dots, \chi_n)$, *set* $D_\chi(b) = \operatorname{diag}(\chi_i(b))_{n \times n}$.

For the permutation matrix P, we shall use its usual $p\times p$ matrix representation

$$P=\begin{pmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & 0 & \dots & \vdots\\ \vdots & \vdots & \vdots & \dots & \vdots\\ \vdots & \vdots & \vdots & & 0\\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}_{p\times p}. \tag{3.9}$$

We have that:

(1) *The following identity holds:*

$$PD_{\chi}(b)=D_{\chi}(\alpha(b))\,P. \tag{3.10}$$

(2) *For all $(j,b)\in G=B\rtimes_{\alpha}\mathbb{Z}$, let*

$$U_p^{(\chi)}(j,b):=D_{\chi}(b)\,P^{j}. \tag{3.11}$$

Then $U_p^{\chi}\in\mathrm{Rep}(G,\mathbb{C}^p)$, i.e.,

$$U_p^{\chi}(j,b)\,U_p^{\chi}(j',b')=U_p^{\chi}\left(j+j',\alpha_j(b')+b\right)$$

for all $(j,b),(j',b')$ in G.

Proof. It suffices to verify (3.10), and the rest of the lemma is straightforward. Set $\chi_k:=\chi\left(\alpha^k(b)\right)$, $0\le k<p$, and

$$D_{\chi}(b)=\mathrm{diag}\left(\chi_0,\chi_1,\dots,\chi_{p-1}\right). \tag{3.12}$$

The assertion in (3.10) follows from a direct calculation. We illustrate this with $p=3$: see Example 3.4 below. $\square$

Example 3.4. For $p=3$, (3.10) reads:

$$PD_{\chi}(b)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}\chi_0&0&0\\0&\chi_1&0\\0&0&\chi_2\end{pmatrix}=\begin{pmatrix}0&\chi_1&0\\0&0&\chi_2\\\chi_0&0&0\end{pmatrix}.$$

$$D_{\chi}(\alpha(b))\,P=\begin{pmatrix}\chi_1&0&0\\0&\chi_2&0\\0&0&\chi_0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}=\begin{pmatrix}0&\chi_1&0\\0&0&\chi_2\\\chi_0&0&0\end{pmatrix}.$$

Lemma 3.5. *Let $\chi\in K$, assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p. Let U_p^{χ} be the corresponding representation. Then $U_p^{\chi}\in\mathrm{Rep}_{\mathrm{irr}}\left(G,l^2(\mathbb{Z}/p\mathbb{Z})\right)$, i.e., U_p^{χ} is irreducible.*

Proof. Recall $U_p^{\chi}(j,b)=D_{\chi}(b)\,P^{j}$, $\forall\,(j,b)\in G$; see (3.11).

Let $A : l^2(\mathbb{Z}_p) \to l^2(\mathbb{Z}_p)$ be in the commutant of U_p^χ, so A commutes with P. It follows that A is a Toeplitz matrix

$$A = \begin{pmatrix} A_0 & A_2 & \ddots & \ddots & A_{p-1} \\ A_{-1} & A_0 & A_2 & \ddots & \ddots \\ \ddots & A_{-1} & A_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ A_{-p+1} & \ddots & \ddots & \ddots & A_0 \end{pmatrix} \tag{3.13}$$

relative to the ONB $\{\delta_0, \delta_1, \ldots, \delta_{p-1}\}$, where $A_{i,j} := \langle \delta_i, A\delta_j \rangle_{l^2}$. Note that

$$\begin{aligned} A_{i,j} &= \langle \delta_i, A\delta_j \rangle = \langle \delta_i, AP\delta_{j+1} \rangle = \langle \delta_i, PA\delta_{j+1} \rangle \\ &= \langle P^*\delta_i, A\delta_{j+1} \rangle = \langle \delta_{i+1}, A\delta_{j+1} \rangle = A_{i+1,j+1} \end{aligned}$$

where $i+1$, $j+1$ are the additions in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, i.e., $+ \mod p$.

If A also commutes with D, then

$$A_k \left(\chi(b) - \chi\left(\alpha^k(b) \right) \right) = 0 \tag{3.14}$$

for all $k = 1, 2, \ldots, p-1$, and all $b \in B$. But since $\chi, \widehat{\alpha}\chi, \ldots, \widehat{\alpha}^{p-1}\chi$ are distinct, $A_k = 0$, $\forall k \neq 0$, and so $A = A_0 I_{p\times p}$, $A_0 \in \mathbb{C}$. □

Proposition 3.6. *The orbit of $\chi \in K$ under the action of $\widehat{\alpha}$ has finite order p if and only if*

$$\mathrm{Ind}_B^G(\chi) = \sum^{\oplus} (p\text{-}\textit{dimensional irreducibles}).$$

Proof. This follows from an application of the general theory; more specifically from an application of Mackey's imprimitivity theorem, in the form given to it in [Ørs79]; see also Theorem 1.4 above. □

Lemma 3.7. $\mathrm{Ind}_B^G(\chi)$ *is irreducible iff the orbit of χ under the action of $\widehat{\alpha}$ has finite order p, i.e., iff the set $\left\{ \widehat{\alpha}^k\chi \mid k \in \mathbb{Z} \right\}$ consists of distinct points.*

Proof. We showed that if χ has finite order p, then the translation T_p operator commutes with $U^\chi := \mathrm{Ind}_B^G(\chi)$, where $T_p\delta_k = \delta_{k-p}$, $k \in \mathbb{Z}$. Hence assume χ has infinite order.

Consider the cases $U^\chi(j, b)$ of

(1) $b = 0$. $U^\chi(j, 0) = T_j$, $j \in \mathbb{Z}$;
(2) $j = 0$. $U^\chi(0, b) = D_\chi(b)$, $\delta_k \longmapsto \chi(\alpha_k(b))\,\delta_k$, where $\chi(\alpha_k(b)) = \left(\widehat{\alpha}^k\chi\right)(b)$, and $D_\chi = \left(\widehat{\alpha}^k\chi\right) =$ diagonal matrix.

Thus if $A \in \mathscr{B}\left(l^2(\mathbb{Z})\right)$ is in the commutant of U^χ, then

$$(A\xi)_k = \sum_{j\in\mathbb{Z}} \eta_{k-j}\xi_j, \quad \eta \in l^\infty.$$

But

$$(\chi(\alpha_s(b)) - \chi(\alpha_k(b)))\,\eta_{k-s} = 0 \tag{3.15}$$

$\forall b$, $\forall k, s$, so $\eta_t = 0$ if $t \in \mathbb{Z}\setminus\{0\}$ and $A = \eta_0 I$.

Note from (3.15) that if $k - s \neq 0$ then $\exists b$ s.t. $\chi(\alpha_s(b)) - \chi(\alpha_k(b)) \neq 0$, since $\widehat{\alpha}^s\chi \neq \widehat{\alpha}^k\chi$. (Compare with (3.14) in Lemma 3.5.) □

Theorem 3.8. *Let* $\mathrm{Ind}_B^G(\chi)$ *be the induced representation in* (2.1)–(2.2).

1. $\mathrm{Ind}_B^G(\chi)$ *is irreducible iff* χ *has no finite periods.*
2. *Suppose the orbit of* χ *under the action of* $\widehat{\alpha}$ *has finite order* p, *i.e.,* $\widehat{\alpha}^p\chi = \chi$, *and* $\widehat{\alpha}^k\chi \neq \chi$, $1 \leq k < p$. *Then the commutant is as follows:*

$$M_p := \left\{\mathrm{Ind}_B^G(\chi)\right\}' \cong \left\{f(z^p) \;\middle|\; f \in L^\infty(\mathbb{T})\right\} \tag{3.16}$$

where $\cong$ *in* (3.16) *denotes unitary equivalence.*

Theorem 3.9. *Let* $G = B \rtimes_\alpha \mathbb{Z}$, $K = \widehat{B}$, $\widehat{\alpha} \in \mathrm{Aut}(K)$. *Assume the orbit of* χ *under the action of* $\widehat{\alpha}$ *has finite order* p, *i.e.,* $\widehat{\alpha}^p\chi = \chi$, $\widehat{\alpha}^k\chi \neq \chi$, *for all* $1 \leq k < p$. *Then the representation* $\mathrm{Ind}_B^G(\chi)$ *has abelian commutant*

$$M_p = \left\{\mathrm{Ind}_B^G(\chi)\right\}'$$

and M_p *does not contain minimal projections.*

Proof. Follows from the fact that M_p is L^∞ (Lebesgue) as a von Neumann algebra, and this implies the conclusion.

Details: Recall that $l^2(\mathbb{Z}) \simeq L^2(\mathbb{T})$, where

$$l^2 \ni \xi \longmapsto f_\xi(z) = \sum_{j\in\mathbb{Z}} \xi_j z^j \in L^2(\mathbb{T}) \tag{3.17}$$

$$T\xi \longmapsto z^{-j} f_\xi(z), \quad z \in \mathbb{T}. \tag{3.18}$$

Note that $\mathrm{Ind}_B^G(\chi)$ may be realized on $l^2(\mathbb{Z})$ or equivalently on $L^2(\mathbb{T})$ via (3.17), where

$$\|f_\xi\|_{L^2}^2 = \int_{\mathbb{T}} |f_\xi|^2 = \sum_{k\in\mathbb{Z}l} |\xi_k|^2 .$$

On $l^2(\mathbb{Z})$, we have

$$\mathrm{Ind}_B^G(\chi)_{(j,b)} = D_\chi(b)\, T_j;$$

See (3.1), and Lemma 3.3.

And on $L^2(\mathbb{T})$, we have

$$\mathrm{Ind}_B^G(\chi)_{(j,b)} = \widehat{D}_\chi(b)\,\widehat{T}_j, \text{ where}$$

$\widehat{D}_\chi(b)$ denotes rotation on $\mathbb{T}$, extended to the solenoid; and $\widehat{T}_j$ = multiplication by z^{-j} acting on $L^2(\mathbb{T})$.

By (3.16), M_p is abelian and has no minimal projections. Note projections in L^∞ are given by P_E = multiplication by $\chi_E(z^p)$, where E is measurable in $\mathbb{T}$. □

4. Super-representations

By “super-representation” we will refer here to a realization of noncommutative relations “inside” certain unitary representations of suitable groups acting in enlargement Hilbert spaces; this is in the sense of “dilation” theory [FK02], but now in a wider context than is traditional. Our present use of “super-representations” is closer to that of [BiDP05, DJ08, DJP09, DLS11].

Definition 4.1. Let G be a locally compact group, B a given subgroup of G, and let $\mathscr{H}_0$ be a Hilbert space. Let $U_0 : G \to \mathscr{B}(\mathscr{H}_0)$ be a positive definite operator-valued mapping, i.e., for all finite systems $\{c_j\}_{j=1}^n \subset \mathbb{C}$, $\{g_j\}_{j=1}^n \subset G$, we have

$$\sum_j \sum_k \overline{c}_j c_k U_0(g_j^{-1} g_k) \geq 0$$

in the usual ordering of Hermitian operators.

If there is a Hilbert space $\mathscr{H}$, an isometry $V : \mathscr{H}_0 \to \mathscr{H}$, and a unitary representation $U : G \to$ (unitary operators on $\mathscr{H}$) such that

(1) U is induced from a unitary representation of B, and
(2) $U_0(g) = V^* U(g) V$, $g \in G$,

then we say that U is a *super-representation.*

Let B be discrete, abelian as before, $K = \widehat{B}$, $\alpha \in \mathrm{Aut}(B)$, $\widehat{\alpha} \in \mathrm{Aut}(K)$, and $G = B \rtimes_\alpha \mathbb{Z}$. Set:

- $\mathrm{Rep}(G)$: unitary representations of G;
- $\mathrm{Rep}_{\mathrm{irr}}(G)$: irreducible representations in $\mathrm{Rep}(G)$.

For $\chi \in K$, let $O(\chi)$ be the orbit of χ, i.e.,

$$O(\chi) = \left\{ \widehat{\alpha}^j(\chi) \mid j \in \mathbb{Z} \right\}. \tag{4.1}$$

Given $U \in \mathrm{Rep}(G)$, let $\mathrm{Class}(U)$ = the equivalence class of all unitary representations equivalent to U; i.e.,

$$\begin{aligned} \mathrm{Class}(U) &= \left\{ V \in \mathrm{Rep}(G) \;\middle|\; V \simeq U \right\} \\ &= \left\{ V \in \mathrm{Rep}(G) \mid \exists W, \text{ unitary s.t. } WV_g = U_g W,\ g \in G \right\}. \end{aligned} \tag{4.2}$$

For $U_1, U_2 \in \mathrm{Rep}(G)$, set

$$L\big(U^{(1)}, U^{(2)}\big) = \Big\{ W : \mathscr{H}(U_1) \to \mathscr{H}(U_2) \mid W \text{ bounded s.t. } WU_g^{(1)} = U_g^{(2)} W,\ g \in G \Big\}. \tag{4.3}$$

Theorem 4.2. *The mapping* $\{$*set of all orbits* $O(\chi)\} \longrightarrow \mathrm{Class}(\mathrm{Rep}(G))$

$$K \ni \chi \longmapsto U^\chi := \mathrm{Ind}_B^G(\chi) \in \mathrm{Rep}(G)$$

passes to

$$O(\chi) \longmapsto \mathrm{Class}(U^\chi). \tag{4.4}$$

Proof. (Sketch) By Lemma 3.1 Equation (3.3), if $\chi \in K$, $j \in \mathbb{Z}$, then $T_j U^{\chi} T_j^* = U^{\widehat{\alpha}^j \chi}$, and so Class$(U^{\chi})$ depends only on $O(\chi)$ and not on the chosen point in $O(\chi)$.

Using (3.4)–(3.6), we can show that U^{χ} is irreducible, but if $O(\chi)$ is finite then we must pass to the quotient $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ and realize U^{χ} on $l^2(\mathbb{Z}_p)$, as a finite-dimensional representation. □

Theorem 4.3. *There is a natural isomorphism:*

$$\begin{aligned} L_G(U^{\chi_1}, U^{\chi_2}) &\cong L_B\left(\chi_2, U^{\chi_1}\big|_B\right) \\ &= \sum_{j\in\mathbb{Z}} L_B\left(\chi_2 \mid \widehat{\alpha}^j(\chi_1)\right) = \#\left\{j \mid \chi_2 = \widehat{\alpha}^j(\chi_1)\right\}. \end{aligned}$$

Proof. Follows from Frobenius reciprocity. □

Another application of Frobenius reciprocity:

Theorem 4.4. *Let $\chi \in K$, and assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p, i.e., $\widehat{\alpha}^p\chi = \chi$, and $\widehat{\alpha}^k\chi \neq \chi$ if $1 \leq k < p$. Let $U_p^{\chi} \in \mathrm{Rep}\left(G, l^2(\mathbb{Z}_p)\right)$. (Recall that U_p^{χ} is irreducible, $\dim U_p^{\chi} = p$.) Then*

$$L_G\left(U_p^{\chi}, \mathrm{Ind}_B^G(\chi)\right) \simeq L_B\left(\chi, U_p^{\chi}\big|_B\right)$$

and

$$\dim L_B\left(\chi, U_p^{\chi}\big|_B\right) = 1. \tag{4.5}$$

Proof. Since $U_p^{(\chi)}(j,b) := D_{\chi}(b)P^j$ (see (3.11)), where $P =$ the permutation matrix on $\mathbb{Z}_p$, we get

$$U_p^{\chi}\big|_B = \sum_{0\leq k<p}^{\oplus} \widehat{\alpha}^k\chi;$$

see (3.6). But notice that all the p characters $\chi, \widehat{\alpha}(\chi), \ldots, \widehat{\alpha}^{p-1}(\chi)$ are distinct, so (4.5) holds since $L_B\left(\chi, \widehat{\alpha}^k(\chi)\right) = 0$ if $k \neq 0 \mod p$. □

We have also proved the following:

Theorem 4.5. *Let $\chi \in K$, and assume the orbit of χ under the action of $\widehat{\alpha}$ has finite order p. Assume G is compact. Let $U_p^{\chi} \in \mathrm{Rep}_{\mathrm{irr}}\left(G, l^2(\mathbb{Z}_p)\right)$. Then U_p^{χ} is contained in $\mathrm{Ind}_B^G(\chi)$ precisely once, i.e.,*

$$\dim L_G\left(U_p^{\chi}, \mathrm{Ind}_B^G(\chi)\right) = 1.$$

But this form of Frobenius reciprocity only holds for certain groups G, e.g., when G is compact. Now for $G = B \rtimes_{\alpha} \mathbb{Z}$, the formal Frobenius reciprocity breaks down, and in fact:

Theorem 4.6. $L_G\left(U_p^{\chi}, \mathrm{Ind}_B^G(\chi)\right) = 0$ *if $\chi \in K$ is an element of finite order p.*

Proof. We sketch the details for $p = 3$ to simplify notation. For $p = 3$,

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D_\chi(b) = \begin{pmatrix} \chi(b) & 0 & 0 \\ 0 & \chi(\alpha(b)) & 0 \\ 0 & 0 & \chi(\alpha_2(b)) \end{pmatrix}$$

and

$$U_p^\chi(j,b) = D_\chi(b) P^j \tag{4.6}$$

while

$$\left(\mathrm{Ind}_B^G(\chi)_{(j,b)}\,\xi\right)_k = \chi(\alpha_k(b))\,\xi_{k+j}$$

$\forall (j,b) \in G$, $\forall k \in \mathbb{Z}$, $\forall \xi \in l^2(\mathbb{Z}_p)$.

Let $W \in L_G\left(U_p^\chi, \mathrm{Ind}_B^G(\chi)\right)$, and u_0, u_1, u_2 be the canonical basis in $\mathscr{H}(U_p^\chi) = \mathbb{C}^3$, where

$$U_p^\chi(j,b)\,u_k = \chi(\alpha_{k+2j}(b))\,u_{k+2j} \mod 3$$

$\forall (j,b) \in G$, $k \in \{0,1,2\} \simeq \mathbb{Z}/3\mathbb{Z}$.

Set $Wu_k = \xi^{(k)} = (\xi_s^{(k)})_{s\in\mathbb{Z}} \in l^2(\mathbb{Z})$, where $\|\xi^{(k)}\|^2 = \sum_{s\in\mathbb{Z}} |\xi_s^{(k)}|^2 < \infty$.

It follows that

$$WU_p^\chi(j,b)\,u_k = \mathrm{Ind}_B^G(\chi)_{(j,b)}\,Wu_k$$

$\forall (j,b) \in G$, $k \in \{0,1,2\}$. Thus

$$\chi(\alpha_{k+2j}(b))\,\xi_s^{(k+2j)_3} = \chi(\alpha_s(b))\,\xi_{s+j}^{(k)}, \quad \forall s,j \in \mathbb{Z}.$$

Now set $j = 3t \in 3\mathbb{Z}$, and we get

$$\chi(\alpha_k(b))\,\xi_s^{(k)} = \chi(\alpha_s(b))\,\xi_{s+3t}^{(k)}$$

and

$$\left|\xi_s^{(k)}\right| = \left|\xi_{s+3t}^{(k)}\right|, \quad \forall s,t \in \mathbb{Z}. \tag{4.7}$$

Since $\xi^{(k)} \in l^2(\mathbb{Z})$, $\lim_{t\to\infty} \xi_{s+3t}^{(k)} = 0$. We conclude from (4.7) that $\xi^{(k)} = 0$ in $l^2(\mathbb{Z})$. □

Remark 4.7. The decomposition of $\mathrm{Ind}_B^G(\chi) : G \longrightarrow B\left(l^2(\mathbb{Z})\right)$ is still a bit mysterious. Recall this representation commutes with $T_3 : (\xi_k) \longmapsto (\xi_{k+3})$; or equivalently via $f(z) \longmapsto z^3 f(z)$. So it is not irreducible.

The reason for $L_G\left(U_p^\chi, \mathrm{Ind}_B^G(\chi)\right) = 0$, e.g., in the case of $p = 3$, is really that there is no isometric version of the 3×3 permutation matrix P in T, where $T\delta_k = \delta_{k-1}$.

If $W : \mathbb{C}^3 \longrightarrow l^2(\mathbb{Z})$ is bounded, $WP = TW$, then applying the polar decomposition

$$W = (W^*W)^{1/2}\,V$$

with $V : \mathbb{C}^3 \to l^2$ isometric, and $VP = TV$, or

$$P = V^*TV, \quad P^j = V^*T_jV.$$

So T_j has the form

$$T_j = \begin{pmatrix} P^j & * \\ 0 & * \end{pmatrix}.$$

Pick $u \in \mathbb{C}^3$; then $\langle u, T_j u\rangle = \left\langle u, P^j u\right\rangle$. However, $\langle u, T_j u\rangle \to 0$ by Riemann–Lebesgue; while $\left\langle u, P^j u\right\rangle \nrightarrow 0$ since $P^3 = I$.

Definition 4.8. A group L acts on a set S if there is a mapping

$$L \times S \longrightarrow S, \quad (\lambda, s) \longmapsto \lambda\left[s\right], \quad \lambda\lambda'[s] = \lambda[\lambda'[s]]$$

and λ^{-1} is the inverse of $S \ni s \longmapsto \lambda\left[s\right] \in S$, a bijection. Often S will have the structure of a topological space or will be equipped with a σ-algebra of measurable sets.

Remark 4.9. While we have stressed discrete decompositions of induced representations, the traditional literature has stressed direct integral decompositions, see, e.g., [Mac88]. A more recent use of continuous parameters in decompositions is a construction by J. Packer et al. [LiPT01] where "wavelet sets" arise as sets of support for direct integral measures. In more detail, starting with a wavelet representation of a certain discrete wavelet group (an induced representation of a semidirect product), the authors in [LiPT01] establish a direct integral where the resulting measure is a subset of R called "wavelet set." These wavelet sets had been studied earlier, but independently of representation theory. In the dyadic case, a wavelet set is a subset of $\mathbb{R}$ which tiles $\mathbb{R}$ itself by a combination of $\mathbb{Z}$-translations, and scaling by powers of 2. In the Packer et al. case, $\mathbb{R} \hookrightarrow K$, and

$$E \text{ wavelet set} \Longleftrightarrow \int_E^{\oplus} \mathrm{Ind}_B^G\left(\chi_t\right) dt \text{ is the wavelet representation in } L^2\left(\mathbb{R}\right).$$

Theorem 4.10. *The group* $L := K \rtimes_{\widehat{\alpha}} \mathbb{Z}$ *acts on the set* $\mathrm{Rep}\left(G\right)$ *by the following assignment:*

Arrange it so that all the representations $U^{\chi} := \mathrm{Ind}_B^G\left(\chi\right)$, $\chi \in K$, *acts on the same* l^2*-space. Then* $\mathrm{Rep}\left(G\right) \sim \mathrm{Rep}\left(G, l^2\right)$.

5. Induction and Bratteli diagrams

A Bratteli diagram is a group G with vertices V, and edges

$$E \subset V \times V \setminus \left\{\text{diagonal}\right\}.$$

It is assumed that

$$V = \bigcup_{n=0}^{\infty} V_n,$$

as a disjoint union in such a way that the edges E in G can be arranged in a sequnce of lines $V_n \to V_{n+1}$, so no edge links pairs of vertices at the same level V_n. With a system of inductions and restrictions one then creates these diagrams.

Let G be created as follows: For a given V_n, let the vertices in V_n represent irreducible representations of some group G_n, and assume G_n is a subgroup in a bigger group G_{n+1}. For the vertices in V_{n+1}, i.e., in the next level in the Bratteli diagram G, we take the irreducible representations occurring in the decomposition of each of the restrictions:

$$\operatorname{Ind}_{G_n}^{G_{n+1}}(L)\Big|_{G_n},$$

so the decomposition of the restriction of the induced representation; see Figure 5.1.

Multiple lines in a Bratteli diagram count the occurrence of irreducible representations with multiplicity; these are called *multiplicity lines.*

Counting multiplicity lines at each level $V_n \to V_{n+1}$ we get a so called *incidence matrix.* For more details on the use of Bratteli diagrams in representation theory, see [BraJKR00, BraJKR01, BraJKR02, BraJO04, BeKwY14, BeKa14, BeJ15].

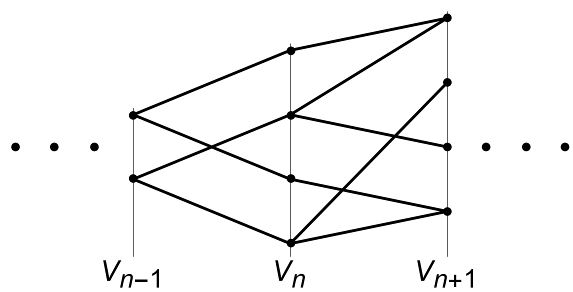

FIGURE 5.1. Part of a Bratteli diagram.

While there is a host of examples from harmonic analysis and dynamical systems; see the discussion above, we shall concentrate below on a certain family of examples when the discrete group B in the construction (from Sections 2 and 3) arises as a discrete Bratteli diagram, and, as a result, its Pontryagin dual compact group K is a solenoid; also often called a compact Bratteli diagram.

Example 5.2 below illustrates this in the special case of constant incidence diagrams, but much of this discussion applies more generally.

Example 5.1. Let $B = \mathbb{Z}\left[\frac{1}{2}\right]$, dyadic rationals in $\mathbb{R}$. Let $\alpha_2 =$ multiplication by 2, so that $\alpha_2 \in \operatorname{Aut}(B)$, and set $G_2 := B \rtimes_{\alpha_2} \mathbb{Z}$.

Recall the Baumslag–Solitar group G_2 with two generators $\{u, t\}$, satisfying $utu^{-1} = t^2$. With the correspondence,

$$u \longleftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \longleftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

α_2 acts by conjugation,

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

In particular, $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k = 1, 2, 3, \dots$.

We have

$$\mathrm{Ind}_{\mathbb{Z}\left[\frac{1}{2}\right]}^{G_2} (\chi) \begin{pmatrix} 2^j & b \\ 0 & 1 \end{pmatrix} = D_\chi (b)\, T^j.$$

Moreover, $\left\{2^j \,:\, j \in \mathbb{Z}\right\} \simeq \mathbb{Z}\left[\frac{1}{2}\right] \backslash G_2 \simeq \mathbb{Z}$. Set $(T\xi)_k := \xi_{k+1}$, $\xi \in l^2 (\mathbb{Z})$, and let

$$D_\chi (b) = \begin{pmatrix} \ddots & & & & \\ & \chi (b) & & & \text{\huge 0} \\ & & \chi (2b) & & \\ & & & \chi (3b) & \\ \text{\huge 0} & & & & \ddots \end{pmatrix}, \quad \forall b \in \mathbb{Z}\left[\tfrac{1}{2}\right].$$

Note that

$$T D_\chi (b) = D_\chi (\alpha_2 (b))\, T = D_{\widehat{\alpha}_2 \chi} (b)\, T, \quad \forall b \in \mathbb{Z}\left[\tfrac{1}{2}\right].$$

Example 5.2. Let A be a $d \times d$ matrix over $\mathbb{Z}$, $\det A \neq 0$; then

$$\mathbb{Z}^d \hookrightarrow A^{-1}\mathbb{Z}^d \hookrightarrow A^{-2}\mathbb{Z}^d \hookrightarrow \cdots$$

and set

$$B_A := \bigcup_{k \geq 0} A^{-k} \mathbb{Z}^d.$$

Let $\alpha_A =$ multiplication by A, so that $\alpha_A \in \mathrm{Aut}(B_A)$, and $G_A := B_A \rtimes_{\alpha_A} \mathbb{Z}$.

The corresponding standard wavelet representation U_A will now act on the Hilbert space $L^2\left(\mathbb{R}^d\right)$ with d-dimensional Lebesgue measure; and U_A is a unitary representation of the discrete matrix group $G_A = \{(j, \beta)\}_{j \in \mathbb{Z}, \beta \in B_A}$, specified by

$$(j, \beta)(k, \gamma) = \left(j + k, \beta + A^j \gamma\right), \text{ and} \tag{5.1}$$

defined for all $j, k \in \mathbb{Z}$, and $\beta, \gamma \in B_A$. Alternatively, the group from (5.1) may be viewed in matrix form as follows:

$$(j, \beta) \longrightarrow \begin{pmatrix} A^j & \beta \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad (k, \gamma) \longrightarrow \begin{pmatrix} A^k & \gamma \\ 0 & 1 \end{pmatrix}.$$

The wavelet representation U_A of G_A acting on $L^2\left(\mathbb{R}^d\right)$ is now

$$U_A (A)(x) := (\det A)^{-\frac{j}{2}} f\left(A^{-j}(x - \beta)\right), \tag{5.2}$$

defined for all $f \in L^2\left(\mathbb{R}^d\right)$, $\forall j \in \mathbb{Z}$, $\forall \beta \in B_A$, and $x \in \mathbb{R}^d$.

Acknowledgements. The co-authors thank the following colleagues for helpful and enlightening discussions: Professors Daniel Alpay, Sergii Bezuglyi, Ilwoo Cho, Ka Sing Lau, Azita Mayeli, Paul Muhly, Myung-Sin Song, Wayne Polyzou, Keri Kornelson, and members in the Math Physics seminar at the University of Iowa.

The authors are very grateful to an anonymous referee for a list of corrections and constructive suggestions. The revised paper is now much better, both in substance, and in its presentation.

References

[BeJ15] Bezuglyi, Sergey; Jorgensen, Palle E.T. Monopoles, dipoles, and harmonic functions on Bratteli diagrams. Preprint, 2015. arXiv:1508.01253.

[BeKa14] Bezuglyi, S.; Karpel, O. Orbit equivalent substitution dynamical systems and complexity. *Proc. Amer. Math. Soc.* **142** (2014), no. 12, 4155–4169. MR3266986, Zbl 06371039.

[BeKwY14] Bezuglyi, Sergey; Kwiatkowski, Jan; Yassawi, Reem. Perfect orderings on finite rank Bratteli diagrams. *Canad. J. Math.* **66** (2014), no. 1, 57–101. MR3150704, Zbl 06265844.

[BiDP05] Bildea, Stefan; Dutkay, Dorin Ervin; Picioroaga, Gabriel. MRA super-wavelets. *New York J. Math.* **11** (2005), 1–19. MR2154344 (2006b:42048), Zbl 1079.42022.

[BraJKR00] Bratteli, Ola; Jørgensen, Palle E. T; Kim, Ki Hang; Roush, Fred. Non-stationarity of isomorphism between AF algebras defined by stationary Bratteli diagrams. *Ergodic Theory Dynam. Systems* **20** (2000), no. 6, 1639–1656. MR1804950 (2001k:46104), Zbl 0982.46042.

[BraJKR01] Bratteli, Ola; Jørgensen, Palle E. T; Kim, Ki Hang; Roush, Fred. Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups. *Ergodic Theory Dynam. Systems* **21** (2001), no. 6, 1625–1655. MR1869063 (2002h:46088), Zbl 1007.46046.

[BraJKR02] Bratteli, Ola; Jørgensen, Palle E. T; Kim, Ki Hang; Roush, Fred. Computation of isomorphism invariants for stationary dimension groups. *Ergodic Theory Dynam. Systems* **22** (2002), no. 1, 99–127. MR1889566 (2004c:37019), Zbl 1067.46060.

[BraJO04] Bratteli, Ola; Jørgensen, Palle E. T; Ostrovs′kyĭ, Vasyl′. Representation theory and numerical AF-invariants. The representations and centralizers of certain states on $\mathscr{O}_d$. *Mem. Amer. Math. Soc.* **168** (2004), no. 797, xviii+178. MR2030387 (2005i:46069), Zbl 1060.46048.

[BreJ91] Brenken, Berndt; Jorgensen, Palle E. T. A family of dilation crossed product algebras. *J. Operator Theory* **25** (1991), no. 2, 299–308. MR1203035 (94m:46103), Zbl 0789.46056.

[CM11] Currey, Bradley; Mayeli, Azita. Gabor fields and wavelet sets for the Heisenberg group. *Monatsh. Math.* **162** (2011), no. 2, 119–142. MR2769882 (2012d:42069), Zbl 1213.42106.

[CM13] Currey, Bradley; Mayeli, Azita. The orthonormal dilation property for abstract Parseval wavelet frames. Canad. Math. Bull. **56** (2013), no. 4, 729–736. MR3121682, Zbl 1287.43002.

[CMO14] Currey, Bradley; Mayeli, Azita; Oussa, Vignon. Characterization of shift-invariant spaces on a class of nilpotent Lie groups with applications. *J. Fourier Anal. Appl.* **20** (2014), no. 2, 384–400. MR3200927, Zbl 1308.43006.

[DJ08] Dutkay, Dorin Ervin; Jorgensen, Palle E. T. A duality approach to representations of Baumslag–Solitar groups. *Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey.* Contemp. Math., 449, 99–127. *Amer. Math. Soc., Providence, RI*, 2008. MR2391800 (2009j:22007) Zbl 1156.22006.

[DJP09] Dutkay, Dorin Ervin; Jorgensen, Palle E. T.; Picioroaga, Gabriel. Unitary representations of wavelet groups and encoding of iterated function systems in solenoids. *Ergodic Theory Dynam. Systems* **29** (2009), no. 6, 1815–1852. MR2563094 (2011e:22009), Zbl 1190.37007.

[DLS11] Dutkay, Dorin Ervin; Larson, David R.; Silvestrov, Sergei. Irreducible wavelet representations and ergodic automorphisms on solenoids. *Oper. Matrices* **5** (2011), no. 2, 201–219. MR2830593 (2012k:42074), Zbl 1226.42028.

[FK02] Frazho, Arthur E.; Kaashoek, Marinus A. A band method approach to a positive expansion problem in a unitary dilation setting. *Integral Equations Operator Theory* **42** (2002), no. 3, 311–371. MR1875185 (2003e:47035), Zbl 1037.47007.

[Jor88] Jorgensen, Palle E. T. Operators and representation theory. Canonical models for algebras of operators arising in quantum mechanics. North-Holland Mathematics Studies, 147. Notas de Matemática [Mathematical Notes], 120. *North-Holland Publishing Co., Amsterdam*, 1988. viii+337 pp. ISBN: 0-444-70321-7. MR919948 (89e:47001), Zbl 0678.46050.

[LaSST06] Larson, David; Schulz, Eckart; Speegle, Darrin; Taylor, Keith F. Explicit cross-sections of singly generated group actions. *Harmonic analysis and applications*, 209–230. Appl. Numer. Harmon. Anal. *Birkhäuser Boston, Boston, MA*, 2006. MR2249311 (2007e:42045), Zbl 1129.42437.

[LiPT01] Lim, Lek-Heng; Packer, Judith A.; Taylor, Keith F. A direct integral decomposition of the wavelet representation. *Proc. Amer. Math. Soc.* **129** (2001), no. 10, 3057–3067. MR1840112 (2002c:47146), Zbl 0976.65125.

[Mac88] Mackey, George W. Induced representations and the applications of harmonic analysis. *Harmonic analysis* (Luxembourg, 1987), 16–51. Lecture Notes in Math., 1359. *Springer, Berlin*, 1988. MR974302 (90c:22021), Zbl 0672.43001.

[MO14] Mayeli, Azita; Oussa, Vignon. Regular representations of time-frequency groups. *Math. Nachr.* **287** (2014), no. 11-12, 1320–1340. MR3247020, Zbl 1304.43002.

[Ørs79] Ørsted, Bent. Induced representations and a new proof of the imprimitivity theorem. *J. Funct. Anal.* **31** (1979), no. 3, 355–359. MR531137 (80d:22007), Zbl 0402.22004.

[Rud62] Rudin, Walter. Fourier analysis on groups. Interscience Tracts in Pure and Applied Mathematics, 12. *Interscience Publishers (a division of John Wiley and Sons), New York-London*, 1962. ix+285 pp. MR0152834 (27 #2808), Zbl 0107.09603.

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This paper is available via http://nyjm.albany.edu/j/2015/21-34.html.