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Weak-L^∞ inequalities for BMO functions

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Abstract. Let I be an interval contained in $\mathbb{R}$ and let $\varphi : I \to \mathbb{R}$ be a given function. The paper contains the proof of the sharp estimate

$$\left|\left|\varphi - \frac{1}{|I|}\int_I \varphi\right|\right|_{W(I)} \leq 2||\varphi||_{BMO(I)},$$

where $W(I)$ is the weak-L^∞ space introduced by Bennett, DeVore and Sharpley. The proof exploits Bellman function method: the above inequality is deduced from the existence of a special function possessing certain majorization and concavity properties.

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1. Introduction

A locally integrable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to belong to BMO, the space of functions of bounded mean oscillation, if

$$\sup_Q \big\langle |\varphi - \langle\varphi\rangle_Q| \big\rangle_Q < \infty. \tag{1.1}$$

Here the supremum is taken over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes and

$$\langle\varphi\rangle_Q = \frac{1}{|Q|}\int_Q \varphi(x)\mathrm{d}x$$

stands for the average of φ over Q. The space BMO is equipped with a quasinorm, given by the left-hand side of (1.1), and denoted by $||\cdot||_{BMO^1}$.

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One can consider a slightly less restrictive setting in which only the cubes Q within a given Q^0 are considered; then the corresponding class of functions is denoted by $BMO(Q^0)$.

The space BMO, introduced by John and Nirenberg in [7], plays a prominent role in analysis and probability and turns up in numerous contexts in various analytic branches of mathematics (properties of Hardy spaces; boundedness of singular integral operators; interpolation theory; etc.). It is well-known that the functions of bounded mean oscillation enjoy strong integrability properties; this was actually observed by John and Nirenberg in their pioneering paper [7]. In particular, one can show that for any $0 < p < \infty$, the p-oscillation

$$||\varphi||_{BMO^p} := \sup_Q \big\langle |\varphi - \langle\varphi\rangle_Q|^p \big\rangle_Q^{1/p}$$

is finite for any $\varphi \in BMO$. It is not difficult to see that for $p \geq 1$, $||\cdot||_{BMO^p}$ forms an equivalent seminorm on $BMO(\mathbb{R}^n)$ (with the equivalence constants depending only on p). In the sequel, we will work with $||\cdot||_{BMO^2}$ and denote it simply by $||\cdot||_{BMO}$. One of the reasons for this choice is the identity

$$||\varphi||_{BMO^2} = \sup_Q \big\{\langle\varphi^2\rangle_Q - \langle\varphi\rangle_Q^2\big\}^{1/2}, \tag{1.2}$$

which enables a very careful and efficient control of the seminorm; see below.

From now on, we will restrict our considerations to dimension one. Then the cubes are simply intervals, and we will switch the notation from Q to I to stress that we consider the case $n = 1$. Our primary goal is to study some sharp estimates for the BMO class. In the recent years, there has been considerable interest in obtaining inequalities of this type. Probably the first result in this direction was that of Slavin [15] and Slavin and Vasyunin [16], which introduced the efficient setup for the study of various results of this type, and identified the optimal constants in the so-called integral form of John–Nirenberg inequality. More precisely, it was shown there that if $\varphi : I \to \mathbb{R}$ satisfies $||\varphi||_{BMO(I)} < 1$, then

$$\langle e^{\varphi}\rangle_I \leq \frac{\exp(-||\varphi||_{BMO(I)})}{1 - ||\varphi||_{BMO(I)}}\, e^{\langle\varphi\rangle_I}.$$

This result is sharp: for each $\varepsilon < 1$ there is a function φ which satisfies $||\varphi||_{BMO(I)} = \varepsilon$ and $\langle e^{\varphi}\rangle_I = e^{-\varepsilon} e^{\langle\varphi\rangle_I}/(1-\varepsilon)$. As a by-product, this proves that there is no exponential estimate of the above type when $||\varphi||_{BMO(I)} \geq 1$.

The following sharp version of the related classical weak form of John–Nirenberg inequality was obtained by Vasyunin [19] and Vasyunin and Volberg [21]: if $\varepsilon := ||\varphi||_{BMO(I)} < \infty$, then

$$\frac{1}{|I|}\big|\{s \in I : |\varphi(s) - \langle\varphi\rangle_I| \geq \lambda\}\big| \leq \begin{cases} 1 & \text{if } 0 \leq \lambda \leq \varepsilon, \\ \varepsilon^2/\lambda^2 & \text{if } \varepsilon \leq \lambda \leq 2\varepsilon, \\ e^{2-\lambda/\varepsilon}/4 & \text{if } \lambda \geq 2\varepsilon, \end{cases}$$

and for each value of ε and λ, equality can be attained. This easily yields the following weak type bounds, by optimizing over λ:

$$||\varphi - \langle\varphi\rangle_I||_{L^{p,\infty}(I)} \leq C_p||\varphi||_{BMO(I)}, \tag{1.3}$$

where

$$C_p = \begin{cases} 1 & \text{if } 0 < p < 2, \\ \dfrac{pe^{2/p-1}}{2^{2/p}} & \text{if } p \geq 2, \end{cases} \tag{1.4}$$

and

$$||\varphi||_{L^{p,\infty}(I)} = \sup_{\lambda>0} \lambda \left[\frac{1}{|I|}\big|\{s \in I : |\varphi(s)| \geq \lambda\}\big|\right]^{1/p}$$

is the usual weak p-th quasinorm. See also Korenovskii [8], Slavin and Vasyunin [17], Slavin and Volberg [18] and Osękowski [14] for related sharp estimates for BMO functions. We would also like to mention here the recent work of Ivanishvili et. al. [6], which is devoted to the unified treatment of the above problems. More precisely, it introduces the machinery which can be applied to prove a general estimate in the BMO setting (under some regularity conditions on the boundary value function). Consult also [5] for the short communication on the subject.

Except for Korenovskii's result, all the estimates formulated above were established with the use of a powerful technique, the so-called Bellman function method. This approach, roughly speaking, translates the problem of proving a given estimate for BMO class into that of constructing a certain special function, which possesses appropriate majorization and concavity. The method has its origins in certain extremal problems in the dynamic programming. As observed by Burkholder [3], [4] in the eighties, this type of approach can be modified appropriately to work in a martingale context: Burkholder applied it successfully to provide a sharp L^p estimate for martingale transforms. In the nineties, in the sequence of works [10], [11] and [12], Nazarov, Treil and Volberg noticed that the technique can be used to study a wide range of problems arising in harmonic analysis, and formulated the general, modern framework of the method. Since then, the approach has been efficiently applied in numerous papers, both in harmonic analysis and probability. We refer the reader to the works [9], [13], [20], the papers mentioned above and references therein.

We turn our attention to the main results of this paper. Our main objective is to provide the extension of (1.3) to the case $p = \infty$. To achieve this, we need an appropriate definition of weak L^∞ spaces. For this, we need some more notation. For a given measurable function $\varphi : I \to \mathbb{R}$, we define φ^*, the decreasing rearrangement of φ, by the formula

$$\varphi^*(t) = \inf\{\lambda \geq 0 : |\{x \in I : |\varphi(x)| > \lambda\}| \leq t\}.$$

Then $\varphi^{**} : (0, |I|] \to [0, \infty)$, the maximal function of φ^*, is given by

$$\varphi^{**}(t) = \frac{1}{t} \int_0^t \varphi^*(s) \mathrm{d}s, \qquad t \in (0, |I|].$$

It is not difficult to check that φ^{**} can be alternatively defined by

$$\varphi^{**}(t) = \frac{1}{t} \sup \left\{ \int_E |\varphi(x)| \mathrm{d}x \, : \, E \subset I, \, |E| = t \right\}.$$

We are ready to introduce the weak-L^∞ space. Following Bennett, DeVore and Sharpley [1], we let

$$||\varphi||_{W(I)} = \sup_{t \in (0, |I|]} \big(\varphi^{**}(t) - \varphi^*(t) \big)$$

and define $W(I) = \{\varphi : ||\varphi||_{W(I)} < \infty\}$. Let us describe the motivation behind the definition of this class. Note that for each $1 \leq p < \infty$, the usual weak space $L^{p,\infty}$ properly contains L^p, but for $p = \infty$, the two spaces coincide. Thus, there is no Marcinkiewicz interpolation theorem between L^1 and L^∞ for operators which are unbounded on L^∞. The space W was invented to fill this gap. It contains L^∞, can be understood as an appropriate limit of $L^{p,\infty}$ as $p \to \infty$, and possesses appropriate interpolation property: if an operator T is bounded from L^1 to $L^{1,\infty}$ and from L^∞ to W, then it can be extended to a bounded operator on all L^p spaces, $1 < p < \infty$. For further evidence that the space W can be understood as a weak-L^∞, we refer the reader to the paper [1] and the monograph [2] by Bennett and Sharpley.

Our main result can be stated as follows.

Theorem 1.1. *For any $\varphi \in BMO(I)$ we have the estimate*

$$||\varphi - \langle \varphi \rangle_I||_{W(I)} \leq 2||\varphi||_{BMO} \tag{1.5}$$

and the constant 2 is the best possible.

Our proof rests on the Bellman function method. We would like to point out here that the desired estimate does not fall into the scope of the (general) bounds covered by [5] and [6], since the corresponding boundary value function is not sufficiently regular.

We have organized this paper as follows. The next section is devoted to the proof of (1.5). In Section 3, we will exhibit an example which shows that equality can hold in (1.5); thus the constant 2 appearing in this estimate cannot be replaced by a smaller number. In the final part of the paper we describe some informal steps which have led us to the appropriate Bellman function.

2. A locally concave function and the proof of (1.5)

The proof of (1.5) depends heavily on the following intermediate result.

Theorem 2.1. *Suppose that $\lambda \geq 0$ is a fixed parameter. Then for any function $\varphi : I \to \mathbb{R}$ satisfying $\langle \varphi \rangle_I = 0$ and $||\varphi||_{BMO} \leq 1$ we have*

$$\int_I (|\varphi(s)| - \lambda - 2)\chi_{(\lambda,\infty)}(|\varphi(s)|)\,ds \leq 0. \tag{2.1}$$

Remark 2.2. The above inequality is sharp, in the sense that the constant 2 cannot be replaced by any smaller number. Otherwise, as we will see below, the improvement of the constant 2 in (1.5) would be possible; but this is not true, as we will show later.

To study this estimate, we will need some auxiliary objects. Suppose that $\lambda > 0$ is a fixed parameter and consider the parabolic strip

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq x^2 + 1\}.$$

Let us split Ω into the union of the following three sets (see Figure 1 below):

$$\begin{aligned} D_1 &= \big\{(x, y) \in \Omega : |x| \leq \lambda + 1,\ y \geq 2(\lambda + 1)|x| - \lambda^2 - 2\lambda\big\}, \\ D_2 &= \big\{(x, y) \in \Omega : y < 2(\lambda + 1)|x| - \lambda^2 - 2\lambda\big\}, \\ D_3 &= \big\{(x, y) \in \Omega : |x| > \lambda + 1,\ y \geq 2(\lambda + 1)|x| - \lambda^2 - 2\lambda\big\}. \end{aligned}$$

Next, consider the function $B_\lambda : \Omega \to [0, \infty)$ given by

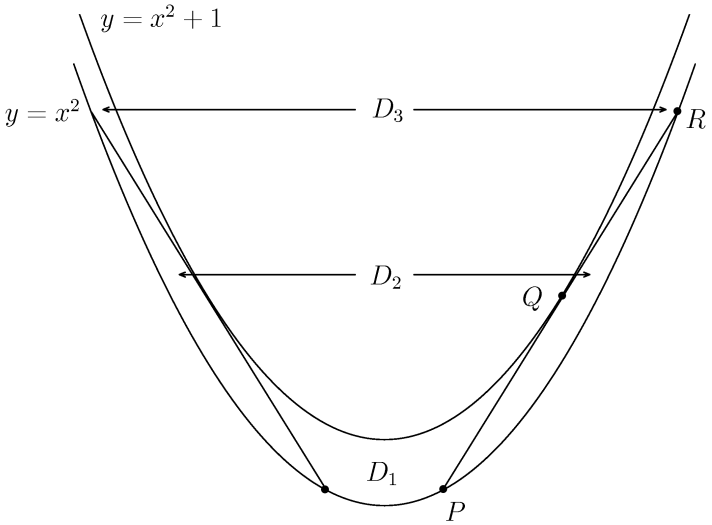

FIGURE 1. The regions $D_1 - D_3$. The points P, Q, R have coordinates (λ, λ^2), $(\lambda + 1, (\lambda + 1)^2 + 1)$ and $(\lambda + 2, (\lambda + 2)^2)$, respectively.

$$B_\lambda(x,y)=\begin{cases}0 & \text{on } D_1,\\ |x|-\lambda-\dfrac{2(|x|-\lambda)^2}{y-2\lambda|x|+\lambda^2} & \text{on } D_2,\\ |x|-\lambda-2 & \\ \quad+\big(1-\sqrt{x^2+1-y}\big)\exp\big[-x+\sqrt{x^2+1-y}+\lambda+1\big] & \text{on } D_3.\end{cases}$$

One easily checks that B_λ is continuous on $\Omega\setminus\{(\pm\lambda,\lambda^2)\}$ and upper semicontinuous on Ω. The key property of B_λ is studied in a separate lemma below.

Lemma 2.3. *The function B_λ is locally concave, i.e., it is concave along any line segment contained in Ω.*

Proof. Let us first verify the local concavity in the interior of each set D_i. For D_1 there is nothing to prove, so we may assume that $i\in\{2,3\}$. By the symmetry condition $B_\lambda(x,y)=B_\lambda(-x,y)$, we may restrict ourselves to the sets $D_i^+=D_i\cap\{(x,y):x\geq 0\}$. To show the concavity of B_λ in the interior of D_2^+, it suffices to prove that the Hessian matrix of B_λ is nonpositive-definite. To accomplish this, observe first that for each $(x,y)\in D_2^+$, there is a line segment passing through (x,y) along which B_λ is linear. Indeed, we have

$$B_\lambda\big(x+h(x-\lambda),y+h(y-\lambda^2)\big)=\left[x-\lambda-\frac{2(x-\lambda)^2}{y-2\lambda x+\lambda^2}\right](1+h)$$

for all h sufficiently close to 0. This implies that the Hessian has determinant zero; so, to obtain the concavity in the interior of D_2^+, it is enough to check that $\frac{\partial^2 B_\lambda}{\partial y^2}(x,y)\leq 0$ on this set. But this is evident: we have

$$-\frac{4(|x|-\lambda)^2}{(y-2\lambda|x|+\lambda^2)^3}\leq 0.$$

Next, let us verify the concavity on D_3^+. As previously, we take a look at the Hessian matrix. Again, note that for each (x,y) lying in the interior of D_3^+, the function B_λ is linear along some line segment containing (x,y). To be more precise, we easily check that

$$\begin{aligned}&B_\lambda\big(x+h,y+2\big(x-\sqrt{x^2+1-y}\big)h\big)\\ &=x+h-\lambda-2+\big(1-\sqrt{x^2+1-y}-h\big)\exp\big[-x+\sqrt{x^2+1-y}+\lambda+1\big],\end{aligned}$$

provided h is sufficiently close to 0. Thus, the Hessian has determinant zero and it suffices to show that $\frac{\partial^2 B_\lambda}{\partial y^2}(x,y)\leq 0$ in the interior of D_3^+. A little calculation shows that this partial derivative equals

$$-\frac{1}{2}(x^2+1-y)^{-1/2}\exp\big(-x+\sqrt{x^2+1-y}+\lambda+1\big),$$

which is nonpositive. This yields the local concavity of B_λ in the interiors of D_1, D_2 and D_3. To get this property in the interior of Ω, we need to check what happens at the common boundaries of the sets D_i. Again, we

may restrict our analysis to the subdomain $\Omega^+ = \Omega \cap \{(x,y) : x > 0\}$. Let us look at the boundary $\partial D_1^+ \cap \partial D_2^+$. If $(x,y) \in D_1^+$, then $B_\lambda(x,y) = 0$; on the other hand, if (x,y) lies in the interior of D_2^+, then

$$\frac{\partial B_\lambda(x,y)}{\partial y} = \frac{2(x-\lambda)^2}{(y-2\lambda x+\lambda^2)^2} > 0$$

and hence, in particular, $B_\lambda \leq 0$ on D_2^+. Thus the local concavity of B_λ propagates to the whole $D_1 \cup D_2$. Finally, note that the partial derivatives of B_λ match at the common boundary of D_2^+ and D_3^+ (i.e., B_λ is of class C^1 in the interior of $D_2^+ \cup D_3^+$).

It remains to show the local concavity on the whole Ω (i.e., extend the concavity to the boundary of Ω), and to accomplish this, we will show that B_λ is continuous along line segments contained in Ω. This is simple: first, note that B_λ is continuous on $\Omega \setminus \{(-\lambda, \lambda^2)\}$. Furthermore, if we take any line segment $J \subset \Omega$, with one of its endpoints equal to (λ, λ^2), then

$$\lim_{X \to (\lambda,\lambda^2),\, X \in J} B_\lambda(X) = B_\lambda(\lambda, \lambda^2).$$

A similar statement is valid for the point $(-\lambda, \lambda^2)$. This completes the proof. □

In what follows, we will require the following auxiliary statement, which can be found in [16] (it appears as Lemma 4c there).

Lemma 2.4. *Fix $\varepsilon < 1$. Then for every interval I and every $\varphi : I \to \mathbb{R}$ with $||\varphi||_{BMO(I)} \leq \varepsilon$, there exists a splitting $I = I_- \cup I_+$ such that the whole straight-line segment with the endpoints $(\langle\varphi\rangle_{I_\pm}, \langle\varphi^2\rangle_{I_\pm})$ is contained within Ω. Moreover, the splitting parameter $\alpha = |I_+|/|I|$ can be chosen uniformly (with respect to φ and I) separated from 0 and 1.*

Proof of (2.1)**.** We may assume that $\lambda > 0$, by a straightforward limiting argument. The reasoning splits naturally into three parts.

Step 1. Some auxiliary objects. Pick an arbitrary $(x,y) \in \Omega$ and let $\varphi : I \to \mathbb{R}$ be an arbitrary function as in the statement. Next, let $\varepsilon \in (0,1)$ be a fixed parameter and put $\tilde{\varphi} = \varepsilon\varphi$; then, clearly, $||\tilde{\varphi}||_{BMO(I)} \leq \varepsilon$. We will require the following family $\{\mathcal{I}^n\}_{n\geq 0}$ of partitions of I, constructed by the inductive use of Lemma 2.4. We start with $\mathcal{I}^0 = \{I\}$; then, given $\mathcal{I}^n = \{I^{n,1}, I^{n,2}, \ldots, I^{n,2^n}\}$, we split each $I^{n,k}$ according to Lemma 2.4, applied to the function $\tilde{\varphi}$, and define

$$\mathcal{I}^{n+1} = \big\{I_-^{n,1}, I_+^{n,1}, I_-^{n,2}, I_+^{n,2}, \ldots, I_-^{n,2^n}, I_+^{n,2^n}\big\}.$$

Since the splitting parameter is uniformly separated from 0 and 1, the diameter of the partitions converges to 0: $\sup_{1\leq k\leq 2^n} |I^{n,k}| \to 0$ as $n \to \infty$. The next step is to define functional sequences $(\varphi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ by the formulas

$$\varphi_n(x) = \langle\tilde{\varphi}\rangle_{I^n(x)} \quad \text{and} \quad \psi_n(x) = \langle\tilde{\varphi}^2\rangle_{I^n(x)}.$$

Here $I^n(x) \in \mathcal{I}^n$ denotes an interval containing x; if there are two such intervals, we pick the one which has x as its right endpoint. A crucial observation is that for each n the pair (φ_n, ψ_n) takes values in Ω. Indeed, for any $J \in \mathcal{I}^n$ we have

$$0 \le \langle \tilde{\varphi}^2 \rangle_J - \langle \tilde{\varphi} \rangle_J^2 \le 1,$$

where the left estimate follows from Schwarz inequality, and the right is due to (1.2) and the assumption $||\tilde{\varphi}||_{BMO(I)} = \varepsilon ||\varphi||_{BMO(I)} \le 1$.

Step 2. Bellman induction. Now we will show that for any $n \ge 0$ and any $1 \le k \le 2^n$ we have

$$\int_{I^{n,k}} B_\lambda(\varphi_n(s), \psi_n(s)) \mathrm{d}s \ge \int_{I^{n,k}} B_\lambda(\varphi_{n+1}(s), \psi_{n+1}(s)) \mathrm{d}s. \tag{2.2}$$

To do this, observe that φ_n, ψ_n are constant on $I^{n,k}$, while φ_{n+1}, ψ_{n+1} are constant on the intervals $I_\pm^{n,k}$ into which $I^{n,k}$ splits. Hence, if we divide both sides by $|I^{n,k}|$, we see that the above bound reads

$$\begin{aligned} B_\lambda\big(\langle \tilde{\varphi} \rangle_{I^{n,k}}, \langle \tilde{\varphi}^2 \rangle_{I^{n,k}}\big) \ge{}& \frac{|I_-^{n,k}|}{|I^{n,k}|} B_\lambda \left(\langle \tilde{\varphi} \rangle_{I_-^{n,k}}, \langle \tilde{\varphi}^2 \rangle|_{I_-^{n,k}} \right) \\ &+ \frac{|I_+^{n,k}|}{|I^{n,k}|} B_\lambda \left(\langle \tilde{\varphi} \rangle_{I_+^{n,k}}, \langle \tilde{\varphi}^2 \rangle_{I_+^{n,k}} \right). \end{aligned}$$

This is a consequence of the local concavity of B_λ and the fact that the whole line segment with endpoints $\big(\langle \tilde{\varphi} \rangle_{I_\pm^{n,k}}, \langle \tilde{\varphi}^2 \rangle_{I_\pm^{n,k}}\big)$ is contained in Ω (which is guaranteed by Lemma 2.4). Summing (2.2) over all $k = 1, 2, \ldots, 2^n$, we obtain

$$\int_I B_\lambda(\varphi_n(s), \psi_n(s)) \mathrm{d}s \ge \int_I B_\lambda(\varphi_{n+1}(s), \psi_{n+1}(s)) \mathrm{d}s$$

and hence, by induction,

$$\int_I B_\lambda(\varphi_0(s), \psi_0(s)) \mathrm{d}s \ge \int_I B_\lambda(\varphi_n(s), \psi_n(s)) \mathrm{d}s$$

for any $n = 0, 1, 2, \ldots$. Observe that

$$\int_I B_\lambda(\varphi_0(s), \psi_0(s)) \mathrm{d}s = |I| \cdot B_\lambda(\langle \tilde{\varphi} \rangle_I, \langle \tilde{\varphi}^2 \rangle_I) = |I| \cdot B_\lambda(0, \langle \tilde{\varphi}^2 \rangle_I) = 0$$

and therefore the previous estimate implies

$$\int_I B_\lambda(\varphi_n(s), \psi_n(s)) \mathrm{d}s \le 0. \tag{2.3}$$

Step 3. A limiting argument. To deal with the left-hand side of (2.3), let n go to infinity. As we have already mentioned above, the diameter of the partition $\mathcal{I}^n$ tends to 0. Consequently, by Lebesgue's differentiation theorem, we have $\varphi_n \to \tilde{\varphi}$ and $\psi_n \to \tilde{\varphi}^2$ almost everywhere on I. Unfortunately, this does not say anything about the limit behavior of $B_\lambda(\varphi_n, \psi_n)$, since the

function B_λ is not continuous on the whole Ω. To overcome this difficulty, note that B_λ majorizes the lower semi-continuous function

$$\tilde{B}_\lambda(x,y)=\begin{cases}B(x,y) & \text{if } (x,y)\neq(\pm\lambda,\lambda^2),\\ -2 & \text{if } (x,y)=(\pm\lambda,\lambda^2),\end{cases}$$

which, in turn, is bounded from below by -2. Consequently, by Fatou's lemma applied to $\tilde{B}_\lambda$, we get

$$\begin{aligned}\liminf_{n\to\infty}\int_I B_\lambda(\varphi_n(s),\psi_n(s))\mathrm{d}s &\geq \liminf_{n\to\infty}\int_I \tilde{B}_\lambda(\varphi_n(s),\psi_n(s))\mathrm{d}s\\ &\geq \int_I \tilde{B}_\lambda(\tilde{\varphi}(s),\tilde{\varphi}^2(s))\mathrm{d}s\\ &= \int_I (|\varepsilon\varphi(s)|-\lambda-2)\chi_{[\lambda,\infty)}(|\varepsilon\varphi(s)|)\mathrm{d}s.\end{aligned}$$

Hence, by (2.3), we have proved that

$$\int_I (|\varepsilon\varphi(s)|-\lambda-2)\chi_{[\lambda,\infty)}(|\varepsilon\varphi(s)|)\mathrm{d}s\leq 0.$$

It remains to let $\varepsilon\to 0$ and apply Fatou's lemma again to get the desired assertion. □

We turn our attention to the inequality of Theorem 1.1.

Proof of (1.5)**.** With no loss of generality, we may assume that $\langle\varphi\rangle_I=0$, replacing φ by $\varphi-\langle\varphi\rangle_I$, if necessary. Furthermore, by homogeneity of (1.5), we may assume that $||\varphi||_{BMO(I)}\leq 1$. Pick $t\in(0,|I|]$ and recall the alternative definition of φ^{**}:

$$\varphi^{**}(t)=\sup\left\{\frac{1}{|E|}\int_E|\varphi(s)|\mathrm{d}s\,:\,E\subset I,\,|E|=t\right\}.$$

This identity yields

$$\varphi^{**}(t)-\varphi^*(t)=\sup\left\{\frac{1}{|E|}\int_E\big(|\varphi(s)|-\varphi^*(t)\big)\mathrm{d}s\,:\,E\subset I,\,|E|=t\right\}.$$

By the very definition of φ^*, we have $|\{s\,:\,|\varphi(s)|>\varphi^*(t)\}|\leq t$. Consequently, the above formula implies

$$\varphi^{**}(t)-\varphi^*(t)\leq\frac{1}{|\{s:|\varphi(s)|>\varphi^*(t)\}|}\int_I\big(|\varphi(s)|-\varphi^*(t)\big)_+\mathrm{d}s\leq 2,$$

where the latter bound follows from (2.1), with $\lambda=\varphi^*(t)$. Since the number $t\in(0,|I|]$ was arbitrary, the proof is complete. □

3. Sharpness

Now we will prove that equality in (1.5) can be attained. Consider the following example: let $\varphi : [0,1] \to \mathbb{R}$ by given by

$$\varphi(s) = -2\chi_{[0,1/8]}(s) + 2\chi_{[7/8,1]}(s).$$

Clearly, we have $\langle\varphi\rangle_{[0,1]} = 0$. Furthermore, we easily check that $\varphi^*(s) = 2\chi_{[0,1/4]}(s)$ and

$$\varphi^{**}(t) = \frac{1}{t}\int_0^t \varphi^*(s)\mathrm{d}s = \begin{cases} 2 & \text{if } t \leq 1/4, \\ (2t)^{-1} & \text{if } t > 1/4. \end{cases}$$

Consequently, we see that $||\varphi||_{W([0,1])} = \sup_{t\in(0,1]}(\varphi^{**}(t) - \varphi^*(t)) = 2$. Next, we will show that $||\varphi||_{BMO([0,1])} \leq 1$; this will yield the claim. To this end, we need to verify that for all a, $b \in [0,1]$ with $a < b$, we have

$$\Delta_{[a,b]} := \langle\varphi^2\rangle_{[a,b]} - \langle\varphi\rangle^2_{[a,b]} \leq 1. \tag{3.1}$$

Set $J = [a,b]$ and put $\alpha_1 = |J \cap [0,1/8]|/|J|$, $\alpha_2 = |J \cap (1/8,7/8)|/|J|$ and $\alpha_3 = |J \cap [7/8,1]|/|J|$. Then $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and

$$\Delta_J = 4(\alpha_1 + \alpha_3 - (\alpha_3 - \alpha_1))^2.$$

If one of α_1 and α_3 vanishes - say, $\alpha_3 = 0$ - then $\Delta_J = 4\alpha_1(1-\alpha_1) \leq 1$. If $\alpha_1 \neq 0$ and $\alpha_3 \neq 0$, then $\alpha_2 \geq 3/4$ and so $\alpha_1 + \alpha_3 \leq 1/4$. Consequently, $\Delta_J \leq 4(\alpha_1 + \alpha_3) \leq 1$. This establishes (3.1) and hence completes the proof of Theorem 1.1.

4. On the size of the weak-type constant and the search of appropriate Bellman function

We conclude the paper by giving some reasoning which has led us to the discovery of the best constant 2 and the function B_λ. We would like to stress that the arguments will be informal and should be rather treated as an intuitive search for these objects. Actually, as we will see, we will *guess* the formula for Bellman function basing on several auxiliary assumptions.

So, suppose that we want to show the validity of (1.5) with some constant c. A reasoning similar to that used in Section 2 shows that it is enough to establish the bound

$$\int_I (|\varphi(s)| - \lambda - c)\chi_{(\lambda,\infty)}(|\varphi(s)|)\mathrm{d}s \leq 0 \tag{4.1}$$

for all $\lambda \geq 0$ and all $\varphi : I \to \mathbb{R}$ satisfying $\langle\varphi\rangle_I = 0$ and $||\varphi||_{BMO} \leq 1$. As we have seen above, the key to the study of this estimate is a locally concave function $\mathbb{B}_\lambda : \Omega \to \mathbb{R}$, which satisfies $\mathbb{B}_\lambda(x,x^2) = (|x| - \lambda - c)\chi_{(\lambda,\infty)}(|x|)$ for all $x \in \mathbb{R}$ and $\mathbb{B}_\lambda(0,y) \leq 0$ for all $y \in [0,1]$. A beautiful feature of the Bellman function approach is that this implication can be reversed: the

validity of (4.1) implies the existence of a function $\mathbb{B}_\lambda$ which enjoys the above properties. For instance, one can take

$$\mathbb{B}_\lambda(x,y)=\sup\left\{\int_I(|\varphi(s)|-\lambda-c)\chi_{(\lambda,\infty)}(|\varphi(s)|)\mathrm{d}s\right\}, \tag{4.2}$$

where the supremum is taken over all functions φ on I satisfying $\langle\varphi\rangle_I = x$, $\langle\varphi^2\rangle_I = y$ and $||\varphi||_{BMO}\leq 1$. See e.g. [6], [16] or [17] for a detailed explanation of this phenomenon. In particular, the formula (4.2) shows that we may search for $\mathbb{B}_\lambda$ in the class of functions satisfying the symmetry condition

$$\mathbb{B}_\lambda(x,y)=\mathbb{B}_\lambda(-x,y),\qquad (x,y)\in\Omega. \tag{4.3}$$

From now on, we assume that this property holds.

So, we have translated the problems of finding the best c and showing (4.1) into the new setting: for which c is there a family $(\mathbb{B}_\lambda)_{\lambda\geq 0}$ of functions satisfying the above conditions? To shed some light at this question, let us fix c, $\lambda>0$ and try to construct an appropriate $\mathbb{B}_\lambda$. Let $P=(\lambda,\lambda^2)$, $P'=(-\lambda,\lambda^2)$, $O=(0,1)$ and suppose that A consists of all points from Ω which lie below the lines OP and OP'. The function $\mathbb{B}_\lambda$ vanishes at the set $\{(x,x^2):|x|\leq\lambda\}$ and is nonpositive at the vertical segment $\{0\}\times[0,1]$. By (4.3) and the local concavity of $\mathbb{B}_\lambda$, we see that this function must be nonpositive on A. Next, take points $P_1=(x,x^2)$, $P_2\in OP$ lying close to P (with $x<\lambda$), and draw the line passing through P_1, P_2; this line intersects the lower boundary of Ω at P_1 and some other point, say, P_3. From the above discussion, we know that $\mathbb{B}_\lambda(P_1)=0$, $\mathbb{B}_\lambda(P_2)\leq 0$; hence, by the local concavity of $\mathbb{B}_\lambda$, we see that $\mathbb{B}_\lambda(P_3)\leq 0$. However, if we let P_1, $P_2\to P$, then $P_3\to(\lambda+2,(\lambda+2)^2)$ and therefore,

$$2-c=\mathbb{B}_\lambda(\lambda+2,(\lambda+2)^2)=\lim_{P_1,P_2\to P}\mathbb{B}_\lambda(P_3)\leq 0.$$

This shows that $c\geq 2$. We *assume* that $c=2$ and take a look at the line segment with endpoints P and $R=(\lambda+2,(\lambda+2)^2)$. The function $\mathbb{B}_\lambda$ vanishes at both endpoints; if it took a positive value at some point from the segment, then for any S lying in the interior of PR we would have $\mathbb{B}_\lambda(S)>0$. This would give a contradiction, by taking S sufficiently close to P and exploiting the concavity of $\mathbb{B}_\lambda$ along the segment joining S with the point P' (which has coordinates $(-\lambda,\lambda^2)$). Indeed, $\mathbb{B}_\lambda$ would be nonnegative at the endpoints of the segment, and, on the other hand, for any $X\in P'S\cap A$ we have $\mathbb{B}_\lambda(X)\leq 0$, as shown above.

Hence, we have proved that $\mathbb{B}_\lambda$ vanishes along the segment PR. Similar arguments to those used above show that this enforces $\mathbb{B}_\lambda$ to vanish on the whole region D_1 (defined in Section 2; see Figure 1). To find the formula for $\mathbb{B}_\lambda$ on the sets D_2 and D_3, we will use the following fact which is true for any Bellman function in the BMO setting. Namely, each of the sets D_2, D_3 can be foliated, i.e., there exists a family of pairwise disjoint line segments whose union is $D_2\cup D_3$, such that $\mathbb{B}_\lambda$ is linear along each segment

(see [6], [16], [17] for details). In what follows, we will *guess* the appropriate foliation, basing on foliations presented in the aforementioned papers. As we will see, this will almost immediately lead us to the desired function $\mathbb{B}_\lambda$. By symmetry, we may restrict our analysis to D_2^+ and D_3^+ (where, as in Section 2, $A^+ = A \cap \{(x, y) : x \geq 0\}$).

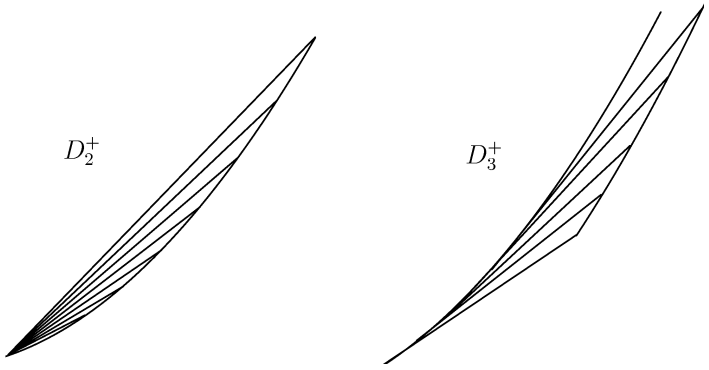

FIGURE 2. The foliations of D_2^+ and D_3^+

We start with D_2^+. Keeping the papers [6], [16] and [17] in mind, it seems plausible to conjecture that the appropriate split of this region is the family $(J_x)_{x\in(\lambda,\lambda+2)}$, where J_x is a line segment joining (λ, λ^2) and the point (x, x^2) (see Figure 2). This immediately leads us to the formula for $\mathbb{B}_\lambda$ on D_2^+. Indeed, given $(x, y) \in D_2^+$, we easily check that $(x, y) \in J_{(y-\lambda x)/(x-\lambda)}$, and by the linearity of $\mathbb{B}_\lambda$ along this segment (and the fact that we know the values of $\mathbb{B}_\lambda$ at its endpoints), we compute the value of $\mathbb{B}_\lambda$ at (x, y):

$$\mathbb{B}_\lambda(x, y) = x - \lambda - \frac{2(x-\lambda)^2}{y - 2\lambda x + \lambda^2}.$$

We turn our attention to the set D_3^+. As previously, a little thought and a careful examination of examples appearing in the literature suggest to consider the foliation $(K_x)_{x\in(\lambda+1,\infty)}$, where for any $x > \lambda + 1$, K_x is the line segment with endpoints $(x, x^2 + 1)$ and $(x + 1, (x + 1)^2)$, tangent to the upper boundary of Ω. See Figure 2. To compute the formula for $\mathbb{B}_\lambda$ on D_3^+, let us first take the point $(x, x^2 + 1)$ (where $x > \lambda + 1$), belonging to the upper boundary of D_3^+. By our choice of foliation, $\mathbb{B}_\lambda$ is linear along the line segment with endpoints $(x, x^2 + 1)$ and $(x + 1, (x + 1)^2)$. Let us lengthen this segment a little "to the left", i.e., consider the segment with endpoints $(x - \delta, x^2 + 1 - 2x\delta)$, $(x + 1, (x + 1)^2)$ for some small positive δ. Assuming that $\mathbb{B}_\lambda$ is regular (say, of class C^1), it follows that the difference

$$\mathbb{B}_\lambda(x, x^2 + 1) - \frac{1}{1+\delta} \cdot \mathbb{B}_\lambda(x - \delta, x^2 + 1 - 2x\delta) - \frac{\delta}{1+\delta}\mathbb{B}_\lambda(x + 1, (x + 1)^2)$$

is of order $o(\delta)$. Furthermore, by our choice of foliation, we have

$$\begin{aligned}&\mathbb{B}_\lambda(x-\delta, x^2+1-2x\delta)\\&= (1-\delta)\mathbb{B}_\lambda(x-2\delta,(x-2\delta)^2+1)+\delta\mathbb{B}_\lambda(x-2\delta+1,(x-2\delta+1)^2).\end{aligned}$$

However,

$$\mathbb{B}_\lambda(x+1,(x+1)^2)=x-\lambda-1,\quad \mathbb{B}_\lambda(x-2\delta+1,(x-2\delta+1)^2)=x-2\delta-\lambda-1,$$

so if we substitute $F(x)=\mathbb{B}_\lambda(x,x^2)$ and combine the above observations, we get

$$\frac{F(x)-F(x-2\delta)}{2\delta}=-\frac{F(x-2\delta)}{1+\delta}+\frac{x-\lambda-1}{1+\delta}+\frac{\delta}{1+\delta}+O(\delta).$$

So, F satisfies the differential equation $F'(x)=-F(x)+x-\lambda-1$ and hence $F(x)=x-\lambda-2+\kappa e^{-x}$ for some constant κ. Since $F(\lambda+1)=0$, as we have computed above, this implies $\kappa=e^{-\lambda-1}$ and hence

$$\mathbb{B}_\lambda(x,x^2+1)=x-\lambda-2+\exp(-x+\lambda+1).$$

To compute the formula on the whole D_3^+, pick a point (x,y) belonging to this set. We easily compute that (x,y) belongs to the segment $K_{x-\sqrt{x^2+1-y}}$ from our foliation. Since we know the values of $\mathbb{B}_\lambda$ at the endpoints of this segment, we easily compute that

$$\mathbb{B}_\lambda(x,y)=x-\lambda-2+\big(1-\sqrt{x^2+1-y}\big)\exp\big(-x+\sqrt{x^2+1-y}+\lambda+1\big).$$

Thus we have arrived at the function introduced in Section 2. We would like to stress that at this point of the analysis, the function $\mathbb{B}_\lambda$ is only *a candidate* for the Bellman function: its discovery was based on a series of conjectures. To complete the reasoning, one needs to verify rigorously that this function indeed enjoys all the required properties. This was carried out successfully in Section 2.

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