New York Journal of Mathematics

New York J. Math. **21** (2015) 57–72.

On the topology of some Sasaki–Einstein manifolds

Charles P. Boyer and Christina W. Tønnesen-Friedman

Abstract. This is a sequel to our forthcoming paper (J. Geom. Anal., 2015) in which we concentrate on developing some of the topological properties of Sasaki–Einstein manifolds. In particular, we explicitly compute the cohomology rings for several cases not treated in that paper, and give formulæ for homotopy equivalence as well as homeomorphism equivalence in one particular 7-dimensional case.

Contents

1. Introduction

Recently the authors have been able to obtain many new results on extremal Sasakian geometry [BTF13c, BTF13a, BTF14a, BTF15] by giving a geometric construction that combines the 'join construction' of [BG00, BGO07] with the 'admissible construction of Hamiltonian 2-forms' for extremal Kähler metrics described in [ACG06, ACGTF04, ACGTF08b] and [ACGTF08a]. The current paper is a result of re-arranging the two previous ArXiv papers [BTF13b, BTF14b]. The basic analysis of both the constant scalar curvature and Sasaki–Einstein cases were combined in [BTF15] which also contains the foundational topological description. The current paper

Received June 17, 2014.

2010 *Mathematics Subject Classification.* Primary: 53D42; Secondary: 53C25.

Key words and phrases. Sasaki–Einstein metrics, join construction, Hamiltonian 2-form.

Both authors were partially supported by grants from the Simons Foundation, CPB by (#245002) and CWT-F by (#208799).

ISSN 1076-9803/2015

contains further results on the topology of the Sasaki–Einstein manifolds most of which appeared in [BTF13b], but were left out of [BTF15].

The main result concerning Sasaki–Einstein manifolds in [BTF15] is:

Theorem 1.1. *Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-join with a regular Sasaki manifold M which is an S^1-bundle over a compact positive Kähler–Einstein manifold N with a primitive Kähler class $[\omega_N] \in H^2(N,\mathbb{Z})$. Assume that the relatively prime positive integers (l_1,l_2) are the relative Fano indices given explicitly by*

$$l_1(\mathbf{w}) = \frac{\mathcal{I}_N}{\gcd(w_1+w_2,\mathcal{I}_N)}, \qquad l_2(\mathbf{w}) = \frac{w_1+w_2}{\gcd(w_1+w_2,\mathcal{I}_N)},$$

where $\mathcal{I}_N$ denotes the Fano index of N. Then for each vector $\mathbf{w} = (w_1,w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional $\mathbf{w}$-Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding Sasakian structure $\mathcal{S} = (\xi_{\mathbf{v}},\eta_{\mathbf{v}},\Phi,g)$ is Sasaki–Einstein (SE).

The procedure involved taking a join of a regular Sasaki–Einstein manifold M with the weighted 3-sphere $S^3_{\mathbf{w}}$, that is, S^3 with its standard contact structure, but with a weighted contact 1-form whose Reeb vector field generates rotations with generally different weights w_1, w_2 for the two complex coordinates z_1, z_2 of $S^3 \subset \mathbb{C}^2$. We call this the $S^3_{\mathbf{w}}$-join. By the $\mathbf{w}$-Sasaki cone we mean the two dimensional subcone of Sasaki cone induced by the Sasaki cone of $S^3_{\mathbf{w}}$. It is denoted by $\mathfrak{t}^+_{\mathbf{w}}$ and can be identified with the open first quadrant in $\mathbb{R}^2$.

Most of the SE structures in Theorem 1.1 are irregular. Such structures have irreducible transverse holonomy [HS12], implying there can be no generalization of the join procedure to the irregular case. We must deform within the Sasaki cone to obtain them. Furthermore, it follows from [RT11, CoS12] that constant scalar curvature Sasaki metrics (hence, SE) imply a certain K-semistability.

The SE metrics obtained from Theorem 1.1 were obtained earlier by physicists [GHP03, GaMSW04b, GaMSW04a, CvLPP05, MS05] working on the AdS/CFT correspondence. Their method, particularly that of [GaMSW04a], is very closely related to the Hamiltonian 2-form approach of [ACG06] (cf. Section 4.3 of [Spa11]). In fact Thorem 1.1 indicates that the physicist's results fit naturally into our geometric construction. Furthermore, we showed in [BTF15] that our geometric approach leads naturally to an algorithm for computing the cohomology ring of the $2n+3$-manifolds. In the present paper we explicitly compute the cohomology ring of all such examples of SE manifolds in dimension 7 showing that there are a countably infinite number of distinct homotopy types of such manifolds. The case that M is a standard odd dimensional sphere was computed in [BTF15], so here we give the cohomology rings of the joins when M is a circle bundle over one of the remaining del Pezzo surfaces. Explicitly, for $N = \mathbb{CP}^1 \times \mathbb{CP}^1$ we have:

Theorem 1.2. *For each relatively prime pair (w_1, w_2) of positive integers there exist Sasaki–Einstein metrics on the 7-manifolds $M^7_{l_1,l_2,\mathbf{w}}$ with cohomology ring*

$$\mathbb{Z}[x,y,u,z]/(x^2, l_2(\mathbf{w})xy, w_1w_2l_1(\mathbf{w})^2y^2, z^2, u^2, zu, zx, ux, uy)$$

with $(l_1,l_2) = (2,|\mathbf{w}|)$ if $\mathbf{w}$ is odd, or $(l_1,l_2) = (1, \frac{|\mathbf{w}|}{2})$ if $\mathbf{w}$ is even, where x, y are 2-classes, and z, u are 5-classes.

It is well-known that when N is the blow-up of $\mathbb{CP}^2$ at k generic points, namely $N = \mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ there is a Kähler–Einstein metric precisely for $k = 3,\dots,8$. Then our results give:

Theorem 1.3. *For each relatively prime pair (w_1, w_2) of positive integers there exist Sasaki–Einstein metrics on the 7-manifolds $M^7_{k,\mathbf{w}}$ with cohomology ring*

$$H^q(M^7_{k,\mathbf{w}},\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } q = 0,7, \\ \mathbb{Z}^{k+1} & \text{if } q = 2,5, \\ \mathbb{Z}^k_{w_1+w_2} \times \mathbb{Z}_{w_1w_2} & \text{if } q = 4, \\ 0 & \text{if otherwise,} \end{cases}$$

with the ring relations determined by

$$\alpha_i \cup \alpha_j = 0, \quad w_1w_2s^2 = 0, \quad (w_1+w_2)\alpha_i \cup s = 0,$$

where α_i, s are the $k+1$ two-classes with $i = 1,\dots,k$ where $k = 3,\dots,8$. Furthermore, when $4 \le k \le 8$ the local moduli space of Sasaki–Einstein metrics has real dimension $4(k-4)$.

Of particular interest is the join $M^{2r+3}_{\mathbf{w}} = S^{2r+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$ of the standard odd dimensional sphere with the weighted $S^3_{\mathbf{w}}$ where

$$(l_1,l_2) = \left(\frac{r+1}{\gcd(w_1+w_2, r+1)}, \frac{w_1+w_2}{\gcd(w_1+w_2, r+1)}\right). \tag{1}$$

By Theorem 4.5 of [BTF15] its cohomology ring is

$$\mathbb{Z}[x,y]/(w_1w_2l_1(\mathbf{w})^2x^2, x^{r+1}, x^2y, y^2) \tag{2}$$

where x, y are classes of degree 2 and $2r+1$, respectively. Let k be the length of the prime decomposition of w_1w_2. Then for arbitrary r we show that there are 2^{k-1} Sasaki–Einstein manifolds of the form $M^{2r+3}_{\mathbf{w}}$ with cohomology ring given by Equation (2). For the manifolds $M^7_{\mathbf{w}}$ of dimension 7 ($r = 2$) much more is known about the topology. These are special cases of what are called generalized Witten spaces in [Esc05]. In particular, the homotopy type was given in [Kru97], while the homeomorphism and diffeomorphism type was given in [Esc05]. For our subclass admitting Sasaki–Einstein metrics we give necessary and sufficient conditions on $\mathbf{w}$ for homotopy equivalence when the order of H^4 is odd in Proposition 3.7 below. Thus, we answer in the affirmative the existence of Einstein metrics on certain generalized Witten manifolds.

Acknowledgements. The authors would like to thank David Calderbank for discussions and Matthias Kreck for providing us with a proof that the first Pontrjagin class is a homeomorphism invariant.

2. The w-Sasaki cone when $c_1(\mathcal{D}) = 0$

In this section we describe some of the properties of the $\mathbf{w}$-Sasaki cone when $c_1(\mathcal{D}) = 0$, ending with some examples. Since we require that N be a positive Kähler–Einstein manifold, we have $c_1(N) = \mathcal{I}_N[\omega_N]$ where $\mathcal{I}_N$ is the Fano index. Recall from [BTF15] that the cohomological Einstein condition $c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}) = 0$ implies:

Lemma 2.1. *Necessary conditions for the Sasaki manifold $M_{l_1,l_2,\mathbf{w}}$ to admit a Sasaki–Einstein metric is that $\mathcal{I}_N > 0$, and that*

$$l_2 = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|, \mathcal{I}_N)}, \qquad l_1 = \frac{\mathcal{I}_N}{\gcd(|\mathbf{w}|, \mathcal{I}_N)}.$$

The integers l_1, l_2 in Lemma 2.1 were called *relative Fano indices* in [BG00]. For the remainder of the paper we assume that these integers take the values given by Lemma 2.1 unless explicitly stated otherwise. Note that the Fano index $\mathcal{I}_N$ of a Fano manifold of complex dimension r is bounded by $r+1$, thus, l_1 is also bounded by $r+1$. Moreover, $\mathcal{I}_N = r+1$ if and only if the universal cover of the regular Sasaki manifold M is the standard sphere S^{2r+1}.

We shall make use of the following easily verified proposition for low values of $\mathcal{I}_N$.

Proposition 2.2. *Let M be a regular Sasaki–Einstein manifold and consider the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$. Then:*

(1) *If $\mathcal{I}_N = 1$ then $(l_1, l_2) = (1, |\mathbf{w}|)$.*

(2) *If $\mathcal{I}_N = 2$, then* $(l_1, l_2) = \begin{cases} (2, |\mathbf{w}|) & \text{if } |\mathbf{w}| \text{ is odd,} \\ (1, \frac{|\mathbf{w}|}{2}) & \text{if } |\mathbf{w}| \text{ is even.} \end{cases}$

(3) *If $\mathcal{I}_N = 3$, then* $(l_1, l_2) = \begin{cases} (3, |\mathbf{w}|) & \text{if } 3 \text{ does not divide } |\mathbf{w}|, \\ (1, \frac{|\mathbf{w}|}{3}) & \text{if } 3 \text{ divides } |\mathbf{w}|. \end{cases}$

A natural question that arises is whether the $\mathbf{w}$-cone contains a regular Reeb vector field.

Proposition 2.3. *Assume $\mathbf{w} \neq (1,1)$ and let $K = \gcd(\mathcal{I}_N, |\mathbf{w}|)$. Then there are exactly $K-1$ different $\mathbf{w}$-Sasaki cones that have a regular Reeb vector field. These are given by*

$$\mathbf{w} = \Big(\frac{K+n}{\gcd(K+n, K-n)}, \frac{K-n}{\gcd(K+n, K-n)}\Big), \tag{3}$$

where $1 \leq n < K$.

Proof. By Proposition 3.4 of [BTF15] a $\mathbf{w}$-Sasaki cone contains a regular Reeb vector field if and only if there is $n \in \mathbb{Z}^+$ such that

$$w_1 - w_2 = n\frac{w_1 + w_2}{\gcd(\mathfrak{I}_N, w_1 + w_2)}.$$

Clearly, for a solution we must have $n < \gcd(\mathfrak{I}_N, w_1 + w_2)$. Then we have a solution if and only if

$$(K - n)w_1 = (K + n)w_2$$

for all $1 \leq n < K$. Since $w_1 > w_2$ and they are relatively prime we have the unique solution Equation (3) for each integer $1 \leq n < K$. □

We have an immediate corollary to Proposition 2.3:

Corollary 2.4. *If $\mathfrak{I}_N = 1$ there are no regular Reeb vector fields in any $\mathbf{w}$-Sasaki cone with $\mathbf{w} \neq (1, 1)$.*

Example 2.5. Let us determine the $\mathbf{w}$-joins with regular Reeb vector field for $\mathfrak{I}_N = 2, 3$. For example, if $\mathfrak{I}_N = 2$ for a solution to Equation (3) we must have $K = 2$ which gives $n = 1$ and $\mathbf{w} = (3, 1)$. This has as a consequence Corollary 2.7 below. Similarly if $\mathfrak{I}_N = 3$ we must have $K = 3$, which gives two solutions $\mathbf{w} = (2, 1)$ and $\mathbf{w} = (5, 1)$.

Example 2.6. Let p and q be relatively prime positive integers satisfying $p > 1$ and $1 \leq q < p$. Recall that the contact structures $Y^{p,q}$ on $S^2 \times S^3$ were discovered in [GaMSW04b], where it is shown that there is a unique Sasaki–Einstein metric in the Sasaki cone of each such $Y^{p,q}$. These SE metrics are most often irregular. From the viewpoint of the present work, $Y^{p,q}$ is a join $M_{l_1,l_2,\mathbf{w}} = M^3 \star_{l_1,l_2} S^3_{\mathbf{w}}$ where $N = S^2$ with its standard (Fubini–Study) Kähler structure. Hence, $\mathfrak{I}_N = 2$. This example has been treated in more detail elsewhere [BP14, B11, BTF15] so we shall be very brief here[1]. As in [BTF15] we have using Lemma 2.1

$$\mathbf{w} = \frac{(p + q, p - q)}{\gcd(p + q, p - q)}, \qquad l_1 = \gcd(p + q, p - q), \qquad l_2 = p. \tag{4}$$

It follows from Proposition 2.2 that there are two cases depending on whether $|\mathbf{w}|$ is odd or even. In the former case $p = |\mathbf{w}|$, and in the latter $p = \frac{|\mathbf{w}|}{2}$. From Example 2.5 we have:

Corollary 2.7. *For $Y^{p,q}$ the $\mathbf{w}$-Sasaki cone has a regular Reeb vector field if and only if $p = 2, q = 1$ or equivalently $\mathbf{w} = (3, 1)$.*

We remark that the quotient of $Y^{2,1}$ by the regular Reeb vector field is $\mathbb{CP}^2$ blown-up at a point; whereas, we have arrived at it from the $\mathbf{w}$-Sasaki cone of an S^1 orbibundle over $S^2 \times \mathbb{CP}^1[3, 1]$.

[1]Unfortunately, the conventions are slightly different. In [BP14, B11] the convention $w_1 \leq w_2$ is used; whereas, here and in [BTF15] the opposite convention, $w_1 \geq w_2$, is used.

3. The topology of the Sasaki–Einstein manifolds

We briefly recall the method used in [BTF15] to prove Theorem 1.1. The idea is that if we know the differentials in the spectral sequence of the fibration

$$M \longrightarrow N \longrightarrow \mathsf{B}S^1, \tag{5}$$

we can use the commutative diagram of fibrations

$$\begin{array}{ccccc} M \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\ \downarrow{=} & & \downarrow & & \downarrow{\psi} \\ M \times S^3_{\mathbf{w}} & \longrightarrow & N \times \mathsf{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1 \times \mathsf{B}S^1 \end{array} \tag{6}$$

to compute the cohomology ring of the join $M_{l_1,l_2,\mathbf{w}}$. Here $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space [Hae84] of an orbifold if G is an orbifold.

3.1. Examples in general dimension. In this section we mainly give partial topological results for some examples of general dimension.

3.1.1. M is a standard sphere. The topology of the join when M is a regular Sasakian sphere S^{2r+1} was worked out in [BTF15] and further studied in [BTF14c]. We shall treat the 7-dimensional case in more detail in Section 3.2.1 below; however, before doing so we give the following result for $M^{2r+3}_{\mathbf{w}} = S^{2r+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$ with $(l_1(\mathbf{w}), l_2(\mathbf{w}))$ satisfying Equations (1).

Lemma 3.1. *If $H^4(M^{2r+3}_{\mathbf{w}}, \mathbb{Z}) = H^4(M^{2r+3}_{\mathbf{w}'}, \mathbb{Z})$, then $w'_1 w'_2 = w_1 w_2$ and $l_1(\mathbf{w}') = l_1(\mathbf{w})$.*

Proof. The equality of the 4th cohomology groups together with the definition of l_1 imply

$$w'_1 w'_2 l_1 \gcd(|\mathbf{w}|, r+1)^2 = w_1 w_2 \gcd(|\mathbf{w}'|, r+1)^2.$$

Set $g_{\mathbf{w}} = \gcd(|\mathbf{w}|, r+1)$ and $g_{\mathbf{w}'} = \gcd(|\mathbf{w}'|, r+1)$. Assume $g_{\mathbf{w}'} > 1$. Since $\gcd(w'_1, w'_2) = 1$, $g_{\mathbf{w}'}$ does not divide $w'_1 w'_2$. Thus, $g^2_{\mathbf{w}'}$ divides $g^2_{\mathbf{w}}$. Interchanging the roles of $\mathbf{w}'$ and $\mathbf{w}$ gives $g_{\mathbf{w}'} = g_{\mathbf{w}}$ which implies $l_1(\mathbf{w}') = l_1(\mathbf{w})$, and hence, the lemma in the case that $g_{\mathbf{w}'} > 1$. Now assume $g_{\mathbf{w}'} = 1$. Then we have $w_1 w_2 = w'_1 w'_2 g^2_{\mathbf{w}}$ which implies that $g_{\mathbf{w}}$ divides $w_1 w_2$. But then since w_1, w_2 are relatively prime, we must have $g_{\mathbf{w}} = 1$. □

Let us set $W = w_1 w_2$, and write the prime decomposition of

$$W = w_1 w_2 = p_1^{a_1} \dots p_k^{a_k}$$

Let P_k be the number of partitions of W into the product $w_1 w_2$ of unordered relatively prime integers, including the pair $(w_1 w_2, 1)$. Then a counting argument gives $P_k = 2^{k-1}$. Once counted we then order the pair $w_1 > w_2$ as before. Let $\mathcal{P}_W$ denote the set of $(2r+3)$-manifolds $M^{2r+3}_{\mathbf{w}}$ with isomorphic

cohomology rings. Then Lemma 3.1 implies that the cardenality of $\mathcal{P}_W$ is $P_k = 2^{k-1}$. This proves:

Proposition 3.2. *Let k denote the length of the prime decomposition of w_1w_2, then there are 2^{k-1} simply connected Sasaki–Einstein manifolds*

$$M^{2r+3}_{\mathbf{w}} = S^{2r+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$$

of dimension $2r+3$ with isomorphic cohomology rings such that H^4 has order $w_1w_2l_1(\mathbf{w})^2$.

3.1.2. M is a rational homology sphere. If we replace the standard odd dimensional sphere by a rational homology sphere V^{2r+1} with a regular Sasakian structure the computations in [BTF15] immediately give:

Proposition 3.3. *The rational cohomology ring of the $S^3_{\mathbf{w}}$-join*

$$V^{2r+1} \star_{l_1,l_2,\mathbf{w}} S^3_{\mathbf{w}}$$

of a rational homology sphere V^{2r+1} is

$$\mathbb{Q}[x,y]/(x^2,y^2)$$

where x, y are classes of degree 2 and $2r+1$, respectively. Here the l_1, l_2 are any positive integers satisfying $\gcd(l_2, w_1w_2l_1) = 1$.

Examples of rational homology spheres with regular Sasaki–Einstein metrics are given in [BGN02]. They are the Sasakian homogeneous Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ of 2-frames in $\mathbb{R}^{2n+1}$ and the 3-Sasakian homogeneous 11-manifold[2] $G_2/Sp(1)_+$. Since we want the join to have a Sasaki–Einstein metric somewhere it the Sasaki cone, we require that the pair (l_1, l_2) to be the relative Fano indices of Lemma 2.1.

Example 3.4. The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ of dimension $4n-1$. It is a circle bundle over the odd complex quadric $Q_{2n-1}(\mathbb{C})$. Its Fano index $\mathcal{I}$ is $2n-1$. So the relative Fano indices are

$$l_1(\mathbf{w}) = \frac{2n-1}{\gcd(|\mathbf{w}|, 2n-1)}, \qquad l_2(\mathbf{w}) = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|, 2n-1)}. \tag{7}$$

Moreover, the cohomology of $V_2(\mathbb{R}^{2n+1})$ is

$$H^p(V_2(\mathbb{R}^{2n+1}), \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } p = 0, 4n-1, \\ \mathbb{Z}_2 & \text{if } p = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

From the long exact homotopy sequence and the commutative diagram one easily obtains the partial results for the join $M_{l_1,l_2,\mathbf{w}}(V) = V_2(\mathbb{R}^{2n+1}) \star_{l_1,l_2} S^3_{\mathbf{w}}$ when $n > 2$, namely $M_{l_1,l_2,\mathbf{w}}(V)$ is simply connected,

$$H^3(M_{l_1,l_2,\mathbf{w}}(V), \mathbb{Z}) = H^5(M_{l_1,l_2,\mathbf{w}}(V), \mathbb{Z}) = 0,$$

[2]The reason for the subscript + on $Sp(1)$ is that there are two nonconjugate $Sp(1)$ subgroups in the exceptional Lie group G_2 which we denote by the subscripts $\pm$. The quotient by $Sp(1)_-$ is equivalent to $V_2(\mathbb{R}^7)$.

and

$$\pi_1(M_{l_1,l_2,\mathbf{w}}(V)) = H^2(M_{l_1,l_2,\mathbf{w}}(V),\mathbb{Z}) \approx \mathbb{Z}, \qquad H^4(M_{l_1,l_2,\mathbf{w}}(V),\mathbb{Z}) \approx \mathbb{Z}_{w_1w_2l_1^2}. \tag{8}$$

Since the Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ are S^1-bundles over a complex quadric, they are special cases of 3.1.3 below.

Example 3.5. The homogeneous 3-Sasakian 11-manifold $G_2/Sp(1)_+$. It is a rational homology sphere with a $\mathbb{Z}_3$ in cohomological degrees 4 and 8. By Proposition 2.3 in [BG00] the Fano index $\mathfrak{I}$ associated to $G_2/Sp(1)_+$ is 3. Then by Proposition 2.2 we have Sasaki–Einstein metrics on the simply connected 13-manifolds, $G_2/Sp(1)_+ \star_{3,|\mathbf{w}|} S^3_{\mathbf{w}}$ if 3 does not divide $|\mathbf{w}|$, and $G_2/Sp(1)_+ \star_{1,\frac{|\mathbf{w}|}{3}} S^3_{\mathbf{w}}$ if 3 divides $|\mathbf{w}|$. These 13-manifolds are simply connected with $\pi_2 = \mathbb{Z}$ and torsion in H^4.

3.1.3. *M* is the link of a Fermat hypersurface. The projective Fermat hypersurface $F_{d,n+1}$ of degree d in $\mathbb{CP}^{n+1}$ is described in homogeneous coordinates by the equation

$$z_0^d + z_1^d + \cdots + z_{n+1}^d = 0. \tag{9}$$

It is Fano when $d \leq n+1$ with index $\mathfrak{I}_{F_{d,n+1}} = n+2-d$ when $d \leq n+1$. Moreover, they have a Kähler–Einstein metric when $\frac{n+1}{2} \leq d \leq n+1$. So for this range of d the Sasakian circle bundle $\mathcal{S}_{d,n+1}$ over $F_{d,n+1}$ has a Sasaki–Einstein metric [BG00]. Note that $F_{2,2n}$ is the complex quadric $Q_{2n-1} \subset \mathbb{CP}^{2n}$ and $\mathcal{S}_{2,2n} = V_2(\mathbb{R}^{2n+1})$ described in Example 3.4 and they are endowed with KE and SE metrics, respectively although d is outside of the range given above. The integral cohomology ring of $F_{d,n+1}$ is well understood [KW80]. It is torsion free with $H_*(F_{d,n+1},\mathbb{Z}) = H_*(\mathbb{P}^n,\mathbb{Z})$ except in the middle dimension n where the nth cohomology group of $F_{d,n+1}$ is $\mathbb{Z}^{b_n}$ when n is odd, and $\mathbb{Z}^{b_n+1}$ when n is even, where

$$b_n = (-1)^n\Big(1 + \frac{(1-d)^{n+2}-1}{d}\Big).$$

Then if $n > 4$ we see as in Example 3.4 that the join $M_{l_1,l_2,\mathbf{w}} = \mathcal{S}_{d,n+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$ is simply connected satisfying the conditions of Equation (8). In order that the join has a Sasaki–Einstein metric in its $\mathbf{w}$-Sasaki cone, we must choose the relative Fano indices to be

$$l_1(\mathbf{w}) = \frac{n+2-d}{\gcd(|\mathbf{w}|, n+2-d)}, \qquad l_2(\mathbf{w}) = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|, n+2-d)} \tag{10}$$

with $\frac{n+1}{2} \leq d \leq n+1$ or $d = 2$. For $2 < d < \frac{n+1}{2}$ it is unknown whether there is such an SE metric.

3.2. Examples in dimension 7. We focus attention to dimension seven in which case N is a del Pezzo Surface, namely $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$, and $\mathbb{CP}^2$ blown-up at k generic points with $1 \leq k \leq 8$. Then the $S^3_{\mathbf{w}}$-join a Sasakian circle bundle over N will be a Sasaki 7-manifold.

3.2.1. $M = S^5, N = \mathbb{CP}^2$. For $\mathbb{CP}^2$ with its standard Fubini–Study Kählerian structure, we have $\mathcal{I}_N = 3$. From Example 2.5 we see that we have a regular Reeb vector field in the $\mathbf{w}$-Sasaki cone in precisely two cases, either $\mathbf{w} = (2,1)$, or $\mathbf{w} = (5,1)$. In the first case the relative Fano indices are $(l_1, l_2) = (1,1)$ while in the second case they are $(l_1, l_2) = (1,2)$. In the former case our 7-manifold $M^7_{(2,1)} = S^5 \star_{1,1} S^3_{(2,1)}$ is an S^3-bundle over $\mathbb{CP}^2$; whereas, in the latter case the 7-manifold $M^7_{(5,1)} = S^5 \star_{1,2} S^3_{(5,1)}$ is an $L(2;5,1)$ bundle over $\mathbb{CP}^2$. Moreover, it follows from standard lens space theory that $L(2;5,1)$ is diffeomorphic to the real projective space $\mathbb{RP}^3$. For general $\mathbf{w}$ we have two cases by Proposition 2.2, $H^4(M^7_{\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}_{w_1 w_2}$ if 3 divides $|\mathbf{w}|$ and $H^4(M^7_{\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}_{9w_1 w_2}$ if 3 does not divide $|\mathbf{w}|$. In both cases the cohomology ring is given by

$$\mathbb{Z}[x,y]/(w_1 w_2 l_1^2 x^2, x^3, x^2 y, y^2)$$

where x, y are classes of degree 2 and 5, respectively. Notice that since 3 must divide $w_1 + w_2$ in the first case and $w_1 w_2$ are relatively prime, the cohomology rings are never isomorphic for the two different cases.

Remark 3.6. Let us make a brief remark about the homogeneous case $\mathbf{w} = (1,1)$ with symmetry group $SU(3) \times SU(2) \times U(1)$. There is a unique solution with a Sasaki–Einstein metric as shown in [BG00]. However, dropping both the Einstein and Sasakian conditions, Kreck and Stolz [KreS88] gave a diffeomorphism and homeomorphism classification. Furthermore, using the results of [WZ90], they show that in certain cases each of the 28 diffeomorphism types admits an Einstein metric. If we drop the Einstein condition and allow contact bundles with nontrivial c_1 we can apply the classification results of [KreS88] to the Sasakian case. This will be studied elsewhere.

For dimension 7 we see from Proposition 2.2 that if 3 divides $w_1 + w_2$ then the order $|H^4|$ is W. However, if 3 does not divide $w_1 + w_2$ then the order of $|H^4|$ is $9W$. So by Lemma 3.1 $\mathcal{P}_W$ splits into two cases, $\mathcal{P}^0_W$ if $W+1$ is divisible by 3, and $\mathcal{P}^1_W$ if $W+1$ is not divisible by 3. Of course, in either case the cardenality of $\mathcal{P}_W$ is 2^{k-1} where k is the number of prime powers in the prime decomposition of W.

Proposition 3.7. *Suppose the order of H^4 is odd. The elements $M^7_{\mathbf{w}}$ and $M^7_{\mathbf{w}'}$ in $\mathcal{P}^0_W$ are homotopy inequivalent if and only if either*

$$\left(\frac{w_1' + w_2'}{3}\right)^3 \equiv \pm \left(\frac{w_1 + w_2}{3}\right)^3 \mod \mathbb{Z}_W.$$

The elements $M^7_{\mathbf{w}}$ and $M^7_{\mathbf{w}'}$ in $\mathcal{P}^1_W$ are homotopy inequivalent if and only if

$$(w_1' + w_2')^3 \equiv \pm(w_1 + w_2)^3 \mod \mathbb{Z}_{9W}.$$

Proof. For $r = 2$ consider the E_6 differential $d_6(\beta) = l_2(\mathbf{w})^3 s^3$ in the spectral sequence of Theorem 4.5 of [BTF15]. Since l_2 is relatively prime to $l_1(\mathbf{w})^2 w_1 w_2$, this takes values in the multiplicative group $\mathbb{Z}^*_{l_1^2 W}$ of units in $\mathbb{Z}_{l_1^2 W}$. Taking into account the choice of generators, it takes its values in $\mathbb{Z}^*_{l_1^2 W}/\{\pm 1\}$. According to Theorem 5.1 of [Kru97] $M^7_{\mathbf{w}}, M^7_{\mathbf{w}'} \in \mathcal{P}_W$ are homotopy equivalent if and only if $l_2(\mathbf{w}')^3 = l_2(\mathbf{w})^3$ in $\mathbb{Z}^*_{l_1^2 W}/\{\pm 1\}$. Of course, this means that $l_2(\mathbf{w}')^3 = \pm l_2(\mathbf{w})^3$ in $\mathbb{Z}^*_{l_1^2 W}$. Note that the the other two conditions of Theorem 5.1 of [Kru97] are automatically satisfied in our case. □

Using a Maple program we have checked some examples for homotopy equivalence which appears to be quite sparse. So far we haven't found any examples of a homotopy equivalence. However, we have not done a systematic computer search which we leave for future work.

Example 3.8. Our first example is an infinite sequence of pairs with the same cohomology ring. Set $W = 3p$ with p an odd prime not equal to 3, which gives $P_k = 2$. Then for each odd prime $p \neq 3$ there are two manifolds in $\mathcal{P}^1_W$, namely $M^7_{(3p,1)}$ and $M^7_{(p,3)}$. The order of H^4 is $27p$. We check the conditions of Proposition 3.7. We find

$$(3p+1)^3 \equiv 9p + 1 \mod 27p, \qquad (p+3)^3 \equiv p^3 + 9p^2 + 27 \mod 27p.$$

First we look for integer solutions of $p^3 + 9p^2 - 9p + 26 \equiv 0 \mod 27p$. By the rational root test the solutions could only be $p = 2, 13, 26$ none of which are solutions. Next we check the second condition of Proposition 3.7, namely, $p^3 + 9p^2 + 9p + 28 \equiv 0 \mod 27p$. Again by the rational root test we find the only possibilities are $p = 2, 7, 14, 28$, from which we see that there are no solutions. Thus, we see that $M^7_{(3p,1)}$ and $M^7_{(p,3)}$ are not homotopy equivalent for any odd $p \neq 3$.

By the same arguments one can also show that the infinite sequence of pairs of the form $M^7_{(9p,1)}$ and $M^7_{(p,9)}$, with p an odd prime relatively prime to 3, are never homotopy equivalent.

Remark 3.9. In Example 3.8 we do not need to have p a prime, but we do need it to be relatively prime to 3. In this more general case, there will be more elements in $\mathcal{P}^1_W$. For example, if $p = 55$ we have $P_k = 4$ and the pair $M^7_{(165,1)}, M^7_{(55,3)}$ has the same cohomology ring as $M^7_{(33,5)}$ and $M^7_{(15,11)}$. However, they are not homotopy equivalent to either member of the pair nor to each other.

Example 3.10. A somewhat more involved example is obtained by setting $W = 5 \cdot 7 \cdot 11 \cdot 17$. Here $P_k = 8$, so this gives eight 7-manifolds in $\mathcal{P}^0_W$,

namely,

$$M^7_{(6545,1)}, \quad M^7_{(1309,5)}, \quad M^7_{(935,7)}, \quad M^7_{(595,11)},$$
$$M^7_{(385,17)}, \quad M^7_{(187,35)}, \quad M^7_{(119,55)}, \quad M^7_{(85,77)}.$$

One can check that these do not satisfy the conditions for homotopy equivalence of Proposition 3.7. So they are all homotopy inequivalent.

It is easy to get a necessary condition for homeomorphism.

Proposition 3.11. *Suppose $w_1'w_2' = w_1w_2$ is odd and that $M^7_{\mathbf{w}}$ and $M^7_{\mathbf{w}'}$ are homeomorphic. Then in addition to the conditions of Proposition* 3.7*, we must have*

$$2(w_1' + w_2')^2 \equiv 2(w_1 + w_2)^2 \mod 3w_1w_2.$$

Proof. This is because the first Pontrjagin class p_1 is actually a homeomorphism invariant[3]. From Kruggel [Kru97] we see that if 3 does not divide $|\mathbf{w}|$

$$p_1(M^7_{\mathbf{w}}) \equiv 3|\mathbf{w}|^2 - 9w_1^2 - 9w_2^2 \equiv -6|\mathbf{w}|^2 \mod 9w_1w_2, \tag{11}$$

which implies the result in this case. If 3 divides $|\mathbf{w}|$ we have

$$p_1(M^7_{\mathbf{w}}) \equiv -6\left(\frac{|\mathbf{w}|}{3}\right)^2 \mod w_1w_2 \tag{12}$$

and this implies the same result. □

Note that Equations (11) and (12) both imply the third condition of Theorem 5.1 in [Kru97] holds in our case. To determine a full homeomorphism and diffeomorphism classification requires the Kreck–Stolz invariants [KreS88] $s_1, s_2, s_3 \in \mathbb{Q}/\mathbb{Z}$. These can be determined as functions of $\mathbf{w}$ in our case by using the formulae in [Esc05, Kru05]; however, they are quite complicated and the classification requires computer programing which we leave for future work.

It is interesting to compare the Sasaki–Einstein 7-manifolds described in this section with the 3-Sasakian 7-manifolds studied in [BGM94, BG99] for their cohomology rings have the same form. Seven dimensional manifolds whose cohomology rings are of this type were called 7-manifolds of type r in [Kru97] where r is the order of H^4. First recall that the 3-Sasakian 7-manifolds in [BGM94] are given by a triple of pairwise relatively prime positive integers (p_1, p_2, p_3) and H^4 is isomorphic to $\mathbb{Z}_{\sigma_2(\mathbf{p})}$ where

$$\sigma_2(\mathbf{p}) = p_1p_2 + p_1p_3 + p_2p_3$$

is the second elementary symmetric function of $\mathbf{p} = (p_1, p_2, p_3)$. It follows that σ_2 is odd. The following theorem is implicit in [Kru97], but we give its simple proof here for completeness.

[3]This appears to be a folklore result with no proof anywhere in the literature. It is stated without proof on page 2828 of [Kru97] and on page 31 of [KreL05]. We thank Matthias Kreck for providing us with a proof that p_1 is a homeomorphism invariant.

Theorem 3.12. *The 7-manifolds $M^7_{\mathbf{p}}$ and $M^7_{\mathbf{w}}$ are not homotopy equivalent for any admissible* $\mathbf{p}$ *or* $\mathbf{w}$.

Proof. These manifolds are distinguished by π_4. Our manifolds $M^7_{\mathbf{w}}$ are quotients of $S^5 \times S^3$ by a free S^1-action, whereas, the manifolds $M^7_{\mathbf{p}}$ of [BGM94] are free S^1 quotients of $SU(3)$. So from their long exact homotopy sequences we have $\pi_i(M^7_{\mathbf{w}}) \approx \pi_i(S^5 \times S^3)$ and $\pi_i(M^7_{\mathbf{p}}) \approx \pi_i(SU(3))$ for all $i > 2$. But it is known that $\pi_4(SU(3)) \approx 0$ whereas, $\pi_4(S^5 \times S^3) \approx \mathbb{Z}_2$. □

3.2.2. $M = S^2 \times S^3, N = \mathbb{CP}^1 \times \mathbb{CP}^1$. Note that this is Example 3.1.3 with $n = d = 2$. We have $\mathfrak{I}_N = 2$, so there are two cases: $|\mathbf{w}|$ is odd impying $l_2 = |\mathbf{w}|$ and $l_1 = 2$; and $|\mathbf{w}|$ is even with $l_2 = \frac{|\mathbf{w}|}{2}$ and $l_1 = 1$. In both cases the smoothness condition $\gcd(l_2, l_1 w_i) = 1$ is satisfied. The E_2 term of the Leray–Serre spectral sequence of the top fibration of diagram (6) is

$$E_2^{p,q} = H^p(\mathsf{B}S^1, H^q(S^2 \times S^3 \times S^3_{\mathbf{w}}, \mathbb{Z})) \approx \mathbb{Z}[s] \otimes \mathbb{Z}[\alpha]/(\alpha^2) \otimes \Lambda[\beta, \gamma],$$

which by the Leray–Serre Theorem converges to $H^{p+q}(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z})$. Here α is a 2-class and β, γ are 3-classes. From the bottom fibration in Diagram (6) we have $d_2(\beta) = \alpha \otimes s_1$ and $d_4(\gamma) = w_1 w_2 s_2^2$. From the commutativity of diagram (6) we have $d_2(\beta) = l_2 s$ and $d_4(\gamma_{\mathbf{w}}) = w_1 w_2 l_1^2 s^2$ which gives $E_4^{4,0} \approx \mathbb{Z}_{w_1 w_2 l_1^2}$, $E_4^{0,3} \approx \mathbb{Z}$, $E_4^{2,2} \approx \mathbb{Z}_{l_2}$, and $E_\infty^{0,3} = 0$. Then using Poincaré duality and universal coefficients we obtain:

Proposition 3.13. *In this case $M^7_{l_1,l_2,\mathbf{w}}$ with either $(l_1, l_2) = (2, |\mathbf{w}|)$ or $(1, \frac{|\mathbf{w}|}{2})$ has the cohomology ring given by*

$$H^*(M^7_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}[x, y, u, z]/(x^2, l_2 xy, w_1 w_2 l_1^2 y^2, z^2, u^2, zu, zx, ux, uy)$$

where x, y are 2-classes, and z, u are 5-classes.

There is only one case with a regular Reeb vector field, and that is $\mathbf{w} = (3, 1)$ in which case the relative Fano indices are $(1, 2)$. Then the 7-manifold is $(S^2 \times S^3) \star_{1,2} S^3_{(3,1)}$ can be realized as an $L(2; 3, 1) \approx \mathbb{RP}^3$ lens space bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Proposition 3.13 and Theorem 1.1 prove Theorem 1.2.

3.2.3. $M = k(S^2 \times S^3), N = \mathbb{CP}^2$ blown-up at k generic points with $k = 1, \ldots, 8$. Equivalently we write $N = N_k = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$. All the Kähler structures have an extremal representative, but for $k = 1, 2$ they are not CSC. However, for $k = 3, \ldots, 8$ they are CSC, and hence, Kähler–Einstein. Notice that when $4 \leq k \leq 8$ the complex automorphism group has dimension 0, so the $\mathbf{w}$-Sasaki cone is the entire Sasaki cone. Moreover, if $5 \leq k \leq 8$ the local moduli space has positive dimension, and we can choose any of the complex structures. By a theorem of Kobayashi and Ochiai [KO73] we have $\mathfrak{I}_{N_k} = 1$ for all $k = 1, \ldots, 8$. So $l_1 = 1, l_2 = |\mathbf{w}|$, and by Corollary 2.4 there are no regular Reeb vector fields in the $\mathbf{w}$-Sasaki cone with $\mathbf{w} \neq (1, 1)$. In particular, if $4 \leq k \leq 8$, there are no regular Reeb vector fields in the Sasaki cone. Generally, these are $L(|\mathbf{w}|; w_1, w_2)$ lens

space bundles over N_k. Of course, the case $\mathbf{w} = (1,1)$ is just an S^1-bundle over $N_k \times \mathbb{CP}^1$ with the product complex structure which is automatically regular. These were studied in [BG00]. Let $\mathcal{S}_k$ denote the total space of the principal S^1-bundle over N_k corresponding to the anticanonical line bundle K^{-1} on N_k. By a well-known result of Smale $\mathcal{S}_k$ is diffeomorphic to the k-fold connected sum $k(S^2 \times S^3)$. We consider the join $\mathcal{S}_k \star_{1,|\mathbf{w}|} S^3_{\mathbf{w}}$. The case $\mathbf{w} = (1,1)$ was studied in [BG00] where it is shown to have a Sasaki–Einstein metric when $3 \leq k \leq 8$. Moreover, in this case we have determined the integral cohomology ring (see Theorem 5.4 of [BG00]). Here we generalize this result.

Proposition 3.14. *The integral cohomology ring of the 7-manifolds* $M^7_{k,\mathbf{w}} = \mathcal{S}_k \star_{1,|\mathbf{w}|} S^3_{\mathbf{w}}$ *is given by*

$$H^q(M^7_{k,\mathbf{w}}, \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } q = 0, 7, \\ \mathbb{Z}^{k+1} & \text{if } q = 2, 5, \\ \mathbb{Z}^k_{|\mathbf{w}|} \times \mathbb{Z}_{w_1 w_2} & \text{if } q = 4, \\ 0 & \text{otherwise}, \end{cases}$$

with the ring relations determined by $\alpha_i \cup \alpha_j = 0, w_1 w_2 s^2 = 0, |\mathbf{w}|\alpha_i \cup s = 0$, *where* α_i, s *are the* $k+1$ *two-classes with* $i = 1, \ldots, k$.

Proof. As before the E_2 term of the Leray–Serre spectral sequence of the top fibration of diagram (6) is

$$E_2^{p,q} = H^p(\mathsf{B}S^1, H^q(\mathcal{S}_k \times S^3_{\mathbf{w}}, \mathbb{Z})) \approx \mathbb{Z}[s] \otimes \prod_i \Lambda[\alpha_i, \beta_i, \gamma]/\mathfrak{I},$$

where $\alpha_i, \beta_j, \gamma$ have degrees $2, 3, 3$, respectively, and $\mathfrak{I}$ is the ideal generated by the relations $\alpha_i \cup \beta_i = \alpha_j \cup \beta_j, \alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0$ for all i, j, $\alpha_i \cup \beta_j = 0$ for $i \neq j$ and $\gamma^2 = 0$.

Consider the lower product fibration of diagram (6). As in the previous case the first nonvanishing differential of the second factor is d_4, and as in that case $d_4(\gamma) = w_1 w_2 s_2^2$. For the first factor we know from Smale's classification of simply connected spin 5-manifolds that $\mathcal{S}_k$ is diffeomorphic to the k-fold connected sum $k(S^2 \times S^3)$. Moreover, since $N = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, the first factor fibration is

$$k(S^2 \times S^3) \longrightarrow \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 \longrightarrow \mathsf{B}S^1.$$

Here the first nonvanishing differential is $d_2(\beta_i) = \alpha_i \otimes s$. Again from the commutativity of diagram (6) for the top fibration we have $d_2(\beta_i) = |\mathbf{w}|\alpha_i \otimes s$ at the E_2 level and $d_4(\gamma) = w_1 w_2 s^2$ at the E_4 level. One easily sees that the $k+1$ 2-classes $\alpha_i \in E_2^{2,0}$ and $s \in E_2^{0,2}$ live to E_∞ and there is no torsion in degree 2. Moreover, there is nothing in degree 1, and the 3-classes $\beta_i \in E_2^{3,0}$ and $\gamma \in E_4^{3,0}$ die, so there is nothing in degree 3. However, there is torsion in degree 4, namely $\mathbb{Z}^k_{|\mathbf{w}|} \times \mathbb{Z}_{w_1 w_2}$. The remainder follows from Poincaré duality and dimensional considerations. □

This generalizes Theorem 5.4 of [BG00] where the case $\mathbf{w} = (1,1)$ is treated and together with Theorem 1.1 proves Theorem 1.3.

Remark 3.15. Since $|\mathbf{w}|$ and w_1w_2 are relatively prime,

$$H^4(M^7_{k,\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}^{k-1}_{|\mathbf{w}|} \times \mathbb{Z}_{w_1w_2|\mathbf{w}|}.$$

We can ask the question: when can $M^7_{k,\mathbf{w}}$ and $M^7_{k',\mathbf{w}'}$ have isomorphic cohomology rings? It is interesting and not difficult to see that there is only one possibility, namely $M^7_{1,(3,2)}$ and $M^7_{1,(5,1)}$ in which case $H^4 \approx \mathbb{Z}_{30}$.

References

[ACG06] Apostolov, Vestislav; Calderbank, David M. J.; Gauduchon, Paul. Hamiltonian 2-forms in Kähler geometry. I. General theory. *J. Differential Geom.* **73** (2006), no. 3, 359–412. MR2228318 (2007b:53149), Zbl 1101.53041, arXiv:math/0202280.

[ACGTF04] Apostolov, Vestislav; Calderbank, David M. J.; Gauduchon, Paul; Tønnesen–Friedman, Christina W. Hamiltonian 2-forms in Kähler geometry. II. Global classification. *J. Differential Geom.* **68** (2004), no. 2, 277–345. MR2144249 (2006g:53054), Zbl 1079.32012, arXiv:math/0401320.

[ACGTF08a] Apostolov, Vestislav; Calderbank, David M. J.; Gauduchon, Paul; Tønnesen–Friedman, Christina W. Extremal Kähler metrics on ruled manifolds and stability. Géométrie différentielle, physique mathématique, mathématiques et société. II. *Astérisque* **322** (2008), 93–150. MR2521655 (2010h:32029), Zbl 1204.53028.

[ACGTF08b] Apostolov, Vestislav; Calderbank, David M. J.; Gauduchon, Paul; Tønnesen–Friedman, Christina W. Hamiltonian 2-forms in Kähler geometry. III. Extremal metrics and stability. *Invent. Math.* **173** (2008), no. 3, 547–601. MR2425136 (2009m:32043), Zbl 1145.53055, arXiv:math/0511118, doi: 10.1007/s00222-008-0126-x.

[B11] Boyer, Charles P. Completely integrable contact Hamiltonian systems and toric contact structures on $S^2 \times S^3$. *SIGMA Symmetry Integrability Geom. Methods Appl.* **7** (2011), Paper 058, 22 pp. MR2861218, Zbl 1242.53112, arXiv:1101.5587, doi: 10.3842/SIGMA.2011.058.

[BG99] Boyer, Charles; Galicki, Krzysztof. 3-Sasakian manifolds. *Surveys in differential geometry: essays on Einstein manifolds*, 123–184, Surv. Differ. Geom., VI. *Int. Press, Boston, MA*, 1999. MR2001m:53076 (2001m:53076), Zbl 1008.53047, arXiv:hep-th/9810250, doi: 10.3792/pjaa.69.335.

[BG00] Boyer, Charles P.; Galicki, Krzysztof. On Sasakian–Einstein geometry. *Internat. J. Math.* **11** (2000), no. 7, 873–909. MR2001k:53081 (2001k:53081), Zbl 1022.53038, arXiv:math/9811098, doi: 10.1142/S0129167X00000477.

[BGM94] Boyer, Charles P.; Galicki, Krzysztof; Mann, Benjamin M. The geometry and topology of 3-Sasakian manifolds. *J. Reine Angew. Math.* **455** (1994), 183–220. MR1293878 (96e:53057), Zbl 0889.53029, doi: 10.1515/crll.1994.455.183.

[BGN02] Boyer, Ch. P.; Galicki, K.; Nakamaye, M. Einstein metrics on rational homology 7-spheres. *Ann. Inst. Fourier (Grenoble)* **52** (2002), no. 5, 1569–1584. MR1935557 (2003j:53056), Zbl 1023.53029, arXiv:math/0108113, doi: 10.5802/aif.1925.

[BGO07] Boyer, Charles P.; Galicki, Krzysztof; Ornea, Liviu. Constructions in Sasakian geometry. *Math. Z.* **257** (2007), no. 4, 907–924. MR2342558

(2008m:53103), Zbl 1134.53021, arXiv:math/0602233, doi: 10.1007/s00209-007-0151-2.

[BP14] Boyer, Charles P.; Pati, Justin. On the equivalence problem for toric contact structures on S^3-bundles over S^2. *Pacific J. Math.* **267** (2014), no. 2, 277–324. MR3207586, Zbl 06323648, arXiv:1204.2209, doi: 10.2140/pjm.2014.267.277.

[BTF13a] Boyer, Charles P.; Tønnesen–Friedman, Christina W. Extremal Sasakian geometry on $T^2 \times S^3$ and related manifolds. *Compos. Math.* **149** (2013), no. 8, 1431–1456. MR3103072, Zbl 1288.53033, arXiv:1108.2005, doi: 10.1112/S0010437X13007094.

[BTF13b] Boyer, Charles P.; Tønnesen–Friedman, Christina W. The Sasaki join, Hamiltonian 2-forms, and Sasaki–Einstein metrics. Preprint, 2013. arXiv:1309.7067.

[BTF13c] Boyer, Charles P.; Tønnesen–Friedman, Christina W. Sasakian manifolds with perfect fundamental groups. *Afr. Diaspora J. Math.* **14** (2013), no. 2, 98–117. MR3093238, Zbl 1285.53034, arXiv:1210.2639.

[BTF14a] Boyer, Charles P.; Tønnesen–Friedman, Christina W. Extremal Sasakian geometry on S^3-bundles over Riemann surfaces. *Int. Math. Res. Not. IMRN* **2014**, no. 20, 5510–5562. MR3271180, Zbl 06369440, arXiv:1302.0776, doi: 10.1093/imrn/rnt139.

[BTF14b] Boyer, Charles P.; Tønnesen–Friedman, Christina W. The Sasaki join, Hamiltonian 2-forms, and constant scalar curvature. Preprint, 2014. arXiv:1402.2546,

[BTF14c] Boyer, Charles P.; Tønnesen–Friedman, Christina W. Simply connected manifolds with infinitely many toric contact structures and constant scalar curvature Sasaki metrics. Preprint, 2014. arXiv:1404.3999.

[BTF15] Boyer, Charles P.; Tønnesen–Friedman, Christina W. The Sasaki join, Hamiltonian 2-forms, and constant scalar curvature. J. Geom. Anal., to appear, 2015.

[CoS12] Collins, Tristan C.; Székelyhidi, Gábor. K-semistability for irregular Sasakian manifolds. Preprint, 2012. arXiv:1204.2230.

[CvLPP05] Cvetič, M.; Lü, H.; Page, Don N.; Pope, C. N. New Einstein–Sasaki spaces in five and higher dimensions. *Phys. Rev. Lett.* **95** (2005), no. 7, 071101, 4 pp. MR2167018, arXiv:hep-th/0504225, doi: 10.1103/PhysRevLett.95.071101.

[Esc05] Escher, Christine M. A diffeomorphism classification of generalized Witten manifolds. *Geom. Dedicata* **115** (2005), 79–120. MR2180043 (2006i:57058), Zbl 1080.55011, doi: 10.1007/s10711-005-3087-x.

[GaMSW04a] Gauntlett, Jerome P.; Martelli, Dario; Sparks, James; Waldram, Daniel. A new infinite class of Sasaki–Einstein manifolds. *Adv. Theor. Math. Phys.* **8** (2004), no. 6, 987–1000. MR2194373 (2007e:53049), Zbl 1095.53034, arXiv:hep-th/0403038, doi: 10.4310/ATMP.2004.v8.n6.a3.

[GaMSW04b] Gauntlett, Jerome P.; Martelli, Dario; Sparks, James; Waldram, Daniel. Sasaki–Einstein metrics on $S^2 \times S^3$. *Adv. Theor. Math. Phys.* **8** (2004), no. 4, 711–734. MR2141499 (2006m:53067), Zbl 1136.53317, arXiv:hep-th/0403002.

[GHP03] Gibbons, G. W.; Hartnoll, Sean A.; Pope, C. N. Bohm and Einstein–Sasaki metrics, black holes, and cosmological event horizons. *Phys. Rev. D* (3) **67** (2003), no. 8, 084024, 24 pp. MR1995313 (2004i:83070), arXiv:hep-th/0208031, doi: 10.1103/PhysRevD.67.084024.

[Hae84] Haefliger, André. Groupoïdes d'holonomie et classifiants. Transversal structure of foliations (Toulouse, 1982). *Astérisque* **116** (1984), 70–97. MR86c:57026a (86c:57026a), Zbl 0562.57012.

[HS12] He, Weiyong; Sun, Song. The generalized Frankel conjecture in Sasaki geometry. Preprint, 2012. arXiv:1209.4026, doi: 10.1093/imrn/rnt185.

[KO73] Kobayashi, Shoshichi; Ochiai, Takushiro. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.* **13** (1973), 31–47. MR0316745 (47 #5293), Zbl 0261.32013.

[KreL05] Kreck, Matthias; Lück, Wolfgang. The Novikov conjecture. Geometry and algebra, Oberwolfach Seminars, 33. *Birkhäuser Verlag, Basel*, 2005. xvi+267 pp. ISBN: 3-7643-7141-2. MR2117411 (2005i:19003), Zbl 1058.19001, doi: 10.1007/b137100.

[KreS88] Kreck, Matthias; Stolz, Stephan. A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$-symmetry. *Ann. of Math.* (2) **127** (1988), no. 2, 373–388. MR932303 (89c:57042), Zbl 0649.53029, doi: 10.2307/2007059.

[Kru97] Kruggel, Bernd. A homotopy classification of certain 7-manifolds. *Trans. Amer. Math. Soc.* **349** (1997), no. 7, 2827–2843. MR1422611 (97m:55012), Zbl 0880.57013, doi: 10.1090/S0002-9947-97-01962-4.

[Kru05] Kruggel, Bernd. Homeomorphism and diffeomorphism classification of Eschenburg spaces. *Q. J. Math.* **56** (2005), no. 4, 553–577. MR2182466 (2006h:53045), Zbl 1104.53028, doi: 10.1093/qmath/hah031.

[KW80] Kulkarni, Ravindra S.; Wood, John W. Topology of nonsingular complex hypersurfaces. *Adv. in Math.* **35** (1980), no. 3, 239–263. MR563926 (82j:57008), Zbl 0439.57007, doi: 10.1016/0001-8708(80)90051-1.

[MS05] Martelli, Dario; Sparks, James. Toric Sasaki–Einstein metrics on $S^2 \times S^3$. *Phys. Lett. B* **621** (2005), no. 1–2, 208–212. MR2152673 (2006c:53041), Zbl 1247.53055, arXiv:hep-th/0505027, doi: 10.1016/j.physletb.2005.06.059.

[RT11] Ross, Julius; Thomas, Richard. Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics. *J. Differential Geom.* **88** (2011), no. 1, 109–159. MR2819757, Zbl 1244.32013, arXiv:0907.5214.

[Spa11] Sparks, James. Sasaki–Einstein manifolds. *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, 265–324, Surv. Differ. Geom., 16, *Int. Press, Somerville, MA*, 2011. MR2893680 (2012k:53082), Zbl 1256.53037, arXiv:1004.2461.

[WZ90] Wang, McKenzie Y.; Ziller, Wolfgang. Einstein metrics on principal torus bundles. *J. Differential Geom.* **31** (1990), no. 1, 215–248. MR1030671 (91f:53041), Zbl 0691.53036, `http://projecteuclid.org/euclid.jdg/1214444095`.

(Charles P. Boyer) Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.
`cboyer@math.unm.edu`

(Christina W. Tønnesen-Friedman) Department of Mathematics, Union College, Schenectady, New York 12308, USA
`tonnesec@union.edu`

This paper is available via `http://nyjm.albany.edu/j/2015/21-2.html`.