

# The motivic Adams vanishing line of slope $\frac{1}{2}$

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ABSTRACT. We establish a motivic version of Adams’ vanishing line of slope  $\frac{1}{2}$  in the cohomology of the motivic Steenrod algebra over  $\mathbb{C}$ .

## CONTENTS

1. Introduction	533
1.1. Organization	534
2. Preliminaries	535
2.1. Notation	535
2.2. Grading conventions	535
3. Margolis homology	536
4. The motivic Milnor basis	537
5. The module $\tilde{A}$ .	538
6. The cofiber of $\eta$	539
7. Multiplication by $h_1$ for $A(0)$ -free $A$ -modules	541
References	544

## 1. Introduction

One of the most powerful tools for computing stable homotopy groups of spheres is the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \hat{\pi}_{t-s},$$

where  $A$  denotes the Steenrod algebra of stable mod 2 cohomology operations and  $\hat{\pi}_*$  are the 2-completed stable homotopy groups. Adams established that these Ext groups vanish above a certain line of slope  $\frac{1}{2}$ , with the exception of the elements  $h_0^k$  in the 0-stem [A2].

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In the motivic context over  $\mathbb{C}$ , inspection of an Adams chart [I2] immediately shows that the analogous Ext groups do not vanish above the same line of slope  $\frac{1}{2}$ . (However, the motivic Ext groups do vanish above a line of slope 1, analogously to an earlier classical vanishing result of Adams [A1].) Further inspection shows that in a large range, all elements above the Adams line of slope  $\frac{1}{2}$  are  $h_1$ -local, in the sense that they are  $h_1$ -divisible and support infinitely many multiplications by  $h_1$ . This suggests the following theorem, whose proof is the goal of this article.

**Theorem 1.1.** *Let  $s > 0$ , and let  $A$  be the motivic mod 2 Steenrod algebra over  $\mathbb{C}$ . The map*

$$h_1 : \text{Ext}_A^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \rightarrow \text{Ext}_A^{s+1,f+1,w+1}(\mathbb{M}_2, \mathbb{M}_2)$$

*is an isomorphism if  $f \geq \frac{1}{2}s + 2$ , and it is a surjection if  $f \geq \frac{1}{2}s + \frac{1}{2}$ .*

In the grading  $(s, f, w)$  from Theorem 1.1,  $s$  is the stem and  $f$  is the Adams filtration so that  $(s, f)$  represent the traditional coordinates in an Adams chart. Meanwhile,  $w$  is the motivic weight, which is not relevant to the statement of the theorem.

**Proof.** In Section 7, we will show that

$$\text{Ext}_A^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \cong \text{Ext}_A^{s-1,f-1,w-1}(N, \mathbb{M}_2),$$

where  $N$  is a certain  $A$ -module that is free as a left  $A(0)$ -module. We will prove in Proposition 7.2 that multiplication by  $h_1$  has the desired property for such  $A$ -modules.  $\square$

Inspection shows that Theorem 1.1 is optimal in the following sense. Multiplication by  $h_1$  is not an isomorphism along the line  $f = \frac{1}{2}s + \frac{3}{2}$ , and it is not a surjection along the line  $f = \frac{1}{2}s$ .

This article is only concerned with algebraic calculations of Ext groups and does not discuss Adams spectral sequences for which these Ext groups are inputs. However, our interest in  $h_1$  multiplications is motivated by recent work [AM2, GI] on the  $\eta$ -local motivic sphere, where  $\eta$  is the first Hopf map that is detected by  $h_1$ .

Another possible approach to Theorem 1.1 uses the methods established by Andrews and Miller [AM2] (see especially [AM2, Lemma 5.2.1]). However, one must be careful about the slightly strange behavior of motivic Margolis homology (see Proposition 3.2). For the techniques of [AM2] to work motivically, one needs to verify that certain quotients of the motivic Steenrod algebra are  $\mathbb{M}_2$ -free. This is more or less the same as the homological results of Sections 3 and 4.

**1.1. Organization.** We review some background in Section 2. In Sections 3 through 6, we assemble various facts about motivic homological algebra. Finally, we give the main technical result in Proposition 7.2, from which Theorem 1.1 follows immediately.

## 2. Preliminaries

**2.1. Notation.** We continue with notation from [I1] as follows:

- (1)  $\mathbb{M}_2$  is the motivic cohomology of  $\mathbb{C}$  with  $\mathbb{F}_2$  coefficients.
- (2)  $A$  is the mod 2 motivic Steenrod algebra over  $\mathbb{C}$  generated by elements  $Sq^1$  of bidegree  $(1, 0)$  and  $Sq^{2^n}$  of bidegree  $(2^n, 2^{n-1})$  for  $n \geq 1$ .
- (3)  $A(0)$  is the (exterior)  $\mathbb{M}_2$ -subalgebra of  $A$  generated by  $Sq^1$ .
- (4)  $A(1)$  is the  $\mathbb{M}_2$ -subalgebra of  $A$  generated by  $Sq^1$  and  $Sq^2$ .
- (5)  $\text{Ext}_A$  is the trigraded ring  $\text{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$ .

The following two theorems of Voevodsky are the starting points of our calculations.

**Theorem 2.1** ([V1]).  $\mathbb{M}_2$  is the bigraded ring  $\mathbb{F}_2[\tau]$ , where  $\tau$  has bidegree  $(0, 1)$ .

It is often more convenient to work with the dual  $A_{*,*} = \text{Hom}_{\mathbb{M}_2}(A, \mathbb{M}_2)$ , which was described by Voevodsky. (See also [B] for a clean description.)

**Theorem 2.2** ([V3], [V2, Theorem 12.6]). The dual motivic Steenrod algebra  $A_{*,*}$  is generated as an  $\mathbb{M}_2$ -algebra by  $\xi_i \in A_{2(2^i-1), 2^i-1}$  and  $\tau_i \in A_{2^{i+1}-1, 2^i-1}$  subject to the relations

$$\tau_i^2 = \tau \xi_{i+1}.$$

The coproduct is given on the generators by the following formulas, in which  $\xi_0 = 1$ .

$$\begin{aligned} \Delta(\tau_k) &= \tau_k \otimes 1 + \sum_i \xi_{k-i}^{2^i} \otimes \tau_i, \\ \Delta(\xi_k) &= \sum_i \xi_{k-i}^{2^i} \otimes \xi_i. \end{aligned}$$

**Remark 2.3.** The quotient  $A_{*,*}/\tau = A_{*,*} \otimes_{\mathbb{M}_2} \mathbb{F}_2$  is analogous to the odd-primary classical dual Steenrod algebra, in the sense that there is an infinite family of exterior generators  $\tau_i$  and an infinite family of polynomial generators  $\xi_i$ . On the other hand, the localization  $A_{*,*}[\tau^{-1}]$  is analogous to the mod 2 classical dual Steenrod algebra, which has only polynomial generators  $\tau_i$ .

**2.2. Grading conventions.** We follow [I1] in grading  $\text{Ext}_A$  according to  $(s, f, w)$ , where:

- (1)  $f$  is the Adams filtration, i.e., the homological degree.
- (2)  $s + f$  is the internal degree, i.e., corresponds to the first coordinate in the bidegrees of  $A$ .
- (3)  $s$  is the stem, i.e., the internal degree minus the Adams filtration.
- (4)  $w$  is the weight.

### 3. Margolis homology

Recall that  $Sq^1 Sq^1 = 0$ , so that  $Sq^1$  acts as a differential on any  $A$ -module. We write  $H^{**}(M; Sq^1)$  for the resulting *Margolis homology* groups [AM1]. We say that an  $A$ -module is *bounded below* if  $M^{p,q} = 0$  for sufficiently small  $p$ . A bounded below  $A$ -module is *of finite type* if each  $M^{p,*}$  is a finitely generated  $\mathbb{M}_2$ -module.

**Remark 3.1.** We will need the following fact about finitely generated  $\mathbb{M}_2$ -modules. Such modules are of the form

$$\bigoplus_i \Sigma^{p_i, q_i} \mathbb{M}_2 \oplus \bigoplus_j \Sigma^{p_j, q_j} (\mathbb{M}_2 / \tau^{r_j}).$$

This is a graded version of the classification of finitely generated modules over a principal ideal domain, since every homogeneous ideal of  $\mathbb{M}_2$  is generated by an element of the form  $\tau^{r_i}$ . Consequently, a finitely generated  $\mathbb{M}_2$ -module is free if and only if it has no  $\tau$  torsion.

**Proposition 3.2.** *Let  $M$  be a bounded below  $A$ -module of finite type. Then  $M$  is free as an  $A(0)$ -module if and only if  $M$  is free as an  $\mathbb{M}_2$ -module and  $H^{*,*}(M; Sq^1) = 0$ .*

**Proof.** The forward implication is clear. Thus suppose that  $M$  is free over  $\mathbb{M}_2$  and  $H^{*,*}(M; Sq^1) = 0$ . Suppose that  $M$  is concentrated in degrees  $(p, q)$  with  $p \geq n_0$ . Let  $x$  be a nonzero element of  $M^{n_0,*}$  of smallest weight, and let  $y = Sq^1(x)$  in  $M^{n_0+1,*}$ . The map  $Sq^1 : M^{n_0,*} \rightarrow M^{n_0+1,*}$  is injective because  $H^{*,*}(M; Sq^1) = 0$ , so  $y$  is non-zero.

Let  $N$  be the  $\mathbb{M}_2$ -submodule of  $M$  generated by  $x$  and  $y$ , and let  $P$  be the quotient  $M/N$ . Then  $N$  is a free  $A(0)$ -module generated by  $x$ .

We will next argue that  $P$  is  $\mathbb{M}_2$ -free; by Remark 3.1, this is the same as showing that  $P$  has no  $\tau$  torsion. Equivalently, we will show that if  $\tau z$  belongs to  $N$ , then so does  $z$ . The main point is that  $y$  is not divisible by  $\tau$ . Suppose for sake of contradiction that  $\tau z = y$ . Then  $\tau Sq^1(z) = Sq^1(y) = 0$ . Since  $M$  is  $\mathbb{M}_2$ -free, Remark 3.1 implies that  $Sq^1(z) = 0$ . On the other hand,  $z$  cannot be in the image of  $Sq^1$  because its weight is less than the weight of  $x$ . This contradicts the assumption that  $H^{*,*}(M; Sq^1)$  is zero.

This establishes that  $M$  is isomorphic to  $N \oplus P$  as an  $A(0)$ -module. The long exact sequence associated to the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

shows that  $H^{*,*}(P; Sq^1) = 0$ .

Having split  $M$  as  $N \oplus P$ , we may now apply the same argument to  $P$  to split off another free  $A(0)$ -module. The finite type assumption on  $M$  guarantees that this process eventually splits all of  $M$  into free  $A(0)$ -modules. □

**Corollary 3.3.** *Suppose that*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence of bounded below  $A$ -modules of finite type which are free over  $\mathbb{M}_2$ . If any two modules in this sequence are free over  $A(0)$ , then so is the third.

**Proof.** Two of the modules have no Margolis homology by Proposition 3.2. The long exact sequence in Margolis homology shows that the third also has no Margolis homology. Then Proposition 3.2 again implies that the third module is free over  $A(0)$ .  $\square$

#### 4. The motivic Milnor basis

Let  $E = (\epsilon_0, \epsilon_1, \epsilon_2, \dots)$  be a sequence of ones and zeros, almost all zero, and let  $R = (r_1, r_2, \dots)$  be a sequence of nonnegative integers, almost all zero. Then, according to Theorem 2.2, the elements

$$\tau(E)\xi(R) := \prod_{i \geq 0} \tau_i^{\epsilon_i} \prod_{j \geq 1} \xi_j^{r_j}$$

give an  $\mathbb{M}_2$ -basis for  $A_{*,*}$ . We follow [DuI] in writing  $P^{(\epsilon_0+2r_1, \epsilon_1+2r_2, \dots)}$  for the corresponding elements of the dual  $\mathbb{M}_2$ -basis for  $A$ . These elements form the *motivic Milnor basis* for  $A$ . The resulting  $\mathbb{M}_2$ -basis for  $A(1)$  consists of the elements  $P^{(s_1, s_2, 0, 0, \dots)}$  such that  $0 \leq s_1 \leq 3$  and  $0 \leq s_2 \leq 1$ .

Let  $S = (s_1, s_2, \dots)$ . The *excess* of the Milnor basis element  $P^S$  is defined to be  $e(P^S) = \sum_i s_i$ . We extend this to arbitrary elements of  $A$  by taking the excess of an element  $\theta$  to be the maximal excess of any Milnor basis element appearing in an expression for  $\theta$ .

For two sequences  $R = (r_1, r_2, \dots)$  and  $S = (s_1, s_2, \dots)$ , we write  $R + S$  for the termwise sum  $(r_1 + s_1, r_2 + s_2, \dots)$ .

**Lemma 4.1.** *Let  $R = (r_1, r_2, \dots)$  and  $S = (s_1, s_2, \dots)$ . Then*

$$P^R \cdot P^S = \prod_i \binom{r_i}{s_i} \cdot P^{R+S} + \text{terms of lower excess.}$$

**Proof.** This follows from the description of the dual motivic Steenrod algebra, similarly to the classical case [M].  $\square$

We will use the Milnor basis to establish the following fact about the right action of  $A(1)$  on  $A$ .

**Proposition 4.2.** *The motivic Steenrod algebra  $A$  is free as a right  $A(1)$ -module.*

**Proof.** Define a filtration on  $A$  by  $F_s(A) = \{\theta \in A \mid e(\theta) \leq s\}$ . Lemma 4.1 implies that the associated graded object

$$\text{gr}_* A = \bigoplus_s F_s(A)/F_{s-1}(A)$$

inherits the structure of an  $\mathbb{M}_2$ -algebra. Let  $M$  be the  $\mathbb{M}_2$ -submodule of  $A$  generated by Milnor basis elements of the form  $P^{(4s_1, 2s_2, s_3, s_4, \dots)}$ . Consider the composition

$$M \otimes_{\mathbb{M}_2} A(1) \longrightarrow A \longrightarrow \text{gr}_* A,$$

in which the first map is induced by the multiplication of  $A$  and the second map is an isomorphism of  $\mathbb{M}_2$ -modules.

Lemma 4.1 implies that the composition is an isomorphism of  $\mathbb{M}_2$ -modules. Therefore, the first map is also an isomorphism. This shows that  $A$  is free as a right  $A(1)$ -module with basis consisting of elements of the form  $P^{(4s_1, 2s_2, s_3, s_4, \dots)}$ .  $\square$

**Remark 4.3.** The reader may wonder why Proposition 4.2 is not an immediate consequence of results in [MM]. The problem is that it is not obvious that the projection  $A \rightarrow A//A(1)$  is split as an  $\mathbb{M}_2$ -map. The proof of Proposition 4.2 is essentially the same as showing that the projection is in fact split.

**Example 4.4.** Let  $B$  be the Hopf subalgebra of  $A$  that is generated over  $\mathbb{M}_2$  by  $\tau\text{Sq}^1$ . The projection  $A \rightarrow A//B$  is not split as an  $\mathbb{M}_2$ -map, since  $\text{Sq}^1$  projects to an element that is  $\tau$  torsion. Moreover,  $A$  is not free as a right  $B$ -module.

### 5. The module $\tilde{A}$ .

**Definition 5.1.** Let  $\tilde{A}(1)$  be the left  $A(1)$ -module on two generators  $a$  and  $b$  of degrees  $(0, 0)$  and  $(2, 0)$  respectively, subject to the relations

$$\text{Sq}^2 a = \tau b, \quad \text{Sq}^1 \text{Sq}^2 \text{Sq}^1 a = \text{Sq}^2 b.$$

The relation  $\text{Sq}^2 a = \tau b$  implies that  $\text{Sq}^2 \text{Sq}^2 a = \tau \text{Sq}^2 b$ , so

$$\tau \text{Sq}^1 \text{Sq}^2 \text{Sq}^1 a = \tau \text{Sq}^2 b.$$

However, the first relation does not imply that  $\text{Sq}^1 \text{Sq}^2 \text{Sq}^1 a = \text{Sq}^2 b$ . This explains why we need a second relation in the definition of  $\tilde{A}(1)$ .

Figure 1 represents  $\tilde{A}(1)$ , according to the following key:

- (1) Each circle represents a copy of  $\mathbb{M}_2$ .
- (2) Each straight line represents multiplication by  $\text{Sq}^1$ .
- (3) Each curved line represents multiplication by  $\text{Sq}^2$ .
- (4) Each dashed line indicates that the squaring operation hits  $\tau$  times an  $\mathbb{M}_2$ -generator, but not the generator itself.

Analogous to the Milnor basis for  $A(1)$ , we have a basis for  $\tilde{A}(1)$  consisting of the elements  $\tilde{P}^{(s_1, s_2)}$  such that  $0 \leq s_1 \leq 3$  and  $0 \leq s_2 \leq 1$ . The elements  $\tilde{P}^2, \tilde{P}^3, \tilde{P}^{2,1}$ , and  $\tilde{P}^{3,1}$  have weight one less than the corresponding Milnor basis elements for  $A(1)$ . We define the excess in  $\tilde{A}(1)$  using this basis.

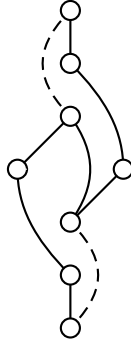


FIGURE 1.  $\tilde{A}(1)$ .

**Remark 5.2.** Just like in the classical case [DaM], the  $A(1)$ -modules  $A(1)$  and  $\tilde{A}(1)$  each extend to  $A$ -modules in four different ways, determined by the action of  $Sq^4$ . Adams spectral sequence computations verify that all eight of these  $A$ -modules arise as the cohomology of a 2-complete motivic spectrum. These constructions are the subject of work in progress on motivic  $v_1$ -self maps.

**Definition 5.3.** Let  $\tilde{A}$  be the left  $A$ -module  $A \otimes_{A(1)} \tilde{A}(1)$ .

**Remark 5.4.** Proposition 4.2 implies that  $\tilde{A}$  is the left  $A$ -module generated by two elements  $a$  and  $b$  of degrees  $(0, 0)$  and  $(2, 0)$  subject to the relations

$$Sq^2a = \tau b, \quad Sq^3Sq^1a = Sq^2b.$$

**Proposition 5.5.** *The  $A$ -module  $\tilde{A}$  is free as a left  $A(0)$ -module.*

**Proof.** We have a Milnor-style basis for  $\tilde{A}$  consisting of elements of the form  $P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1, s_2}$  such that  $0 \leq s_1 \leq 3$  and  $0 \leq s_2 \leq 1$ . We also have a filtration by excess.

Lemma 4.1 implies that

$$P^1 P^{(4r_1, 2r_2, r_3, \dots)} \equiv P^{(4r_1+1, 2r_2, r_3, \dots)} \equiv P^{(4r_1, 2r_2, r_3, \dots)} P^1$$

in  $A$ , modulo terms of lower excess. Also,  $P^1 \tilde{P}^{s_1, s_2} = \tilde{P}^{s_1+1, s_2}$  in  $\tilde{A}(1)$  if  $s_1$  is even. Therefore, if  $s_1$  is even, then

$$P^1 (P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1, s_2}) \equiv P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1+1, s_2}$$

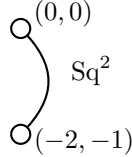
in  $\tilde{A}$ , modulo terms of lower excess. It follows that an  $A(0)$ -basis for  $\tilde{A}$  consists of elements of the form  $P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1, s_2}$  such that  $s_1$  is even.  $\square$

## 6. The cofiber of $\eta$

**Definition 6.1.** Let  $C(\eta)$  denote the cofiber of the first Hopf map

$$\eta : S^{1,1} \longrightarrow S^{0,0}.$$

We then write  $\mathbf{C}_\eta$  for the  $A$ -module  $\Sigma^{-2,-1}\mathbf{H}^{*,*}(C(\eta))$ . Thus  $\mathbf{C}_\eta$  has a bottom cell in bidegrees  $(-2, -1)$  and a top cell in  $(0, 0)$ , connected by a  $\text{Sq}^2$  (see figure).



The following result implies that the groups  $\text{Ext}^*(M, \mathbf{C}_\eta)$  may be computed using a resolution  $F^* \rightarrow M$  whose terms are of the form  $F^n \cong A^r \oplus \tilde{A}^s$ . Here  $M$  is an arbitrary  $A$ -module.

**Proposition 6.2.**  $\text{Ext}_A^{s,f}(\tilde{A}, \mathbf{C}_\eta) = 0$  for  $f > 0$ .

**Proof.** The kernel of the defining quotient  $A(1) \oplus \Sigma^{2,0}A(1) \rightarrow \tilde{A}(1)$  is isomorphic to  $\Sigma^{2,1}\tilde{A}(1)$ . It follows that we can define a periodic free  $A(1)$ -resolution

$$\tilde{A}(1) \leftarrow F_0 \leftarrow F_1 \leftarrow \dots,$$

where  $F_n \cong A(1)\{x_n, y_n\}$ , with  $|x_n| = (2n, n)$ ,  $|y_n| = (2n + 2, n)$ ,

$$d(x_n) = \text{Sq}^2(x_{n-1}) + \tau \cdot y_{n-1}, \quad \text{and} \quad d(y_n) = \text{Sq}^3\text{Sq}^1(x_{n-1}) + \text{Sq}^2(y_{n-1}).$$

Using this resolution, it is simple to verify that

$$\text{Ext}_{A(1)}^{s,f}(\tilde{A}(1), \mathbf{C}_\eta) = 0, \quad \text{for } f > 0.$$

Since  $A$  is free as a right  $A(1)$ -module by Proposition 4.2, we get that

$$\text{Ext}_A^{s,f}(\tilde{A}, \mathbf{C}_\eta) \cong \text{Ext}_{A(1)}^{s,f}(\tilde{A}(1), \mathbf{C}_\eta) = 0 \quad \text{for } f > 0. \quad \square$$

The next lemma translates properties of  $h_1$  multiplication into homological properties of the  $A$ -module  $\mathbf{C}_\eta$ .

**Lemma 6.3.** *Let  $n \geq 0$  and let  $M$  be an  $A$ -module. The following are equivalent:*

(1) *The map*

$$h_1 : \text{Ext}_A^{s,f,w}(M, \mathbb{M}_2) \rightarrow \text{Ext}_A^{s+1,f+1,w+1}(M, \mathbb{M}_2)$$

*is injective when  $s < 2f - 2$  and  $f \leq n$ , and it is surjective when  $s \leq 2f$  and  $f < n$ .*

(2)  $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta) = 0$  when  $s < 2f$  and  $f \leq n$ .

**Proof.** The  $A$ -module  $\mathbf{C}_\eta$  sits in a short exact sequence

$$0 \rightarrow \mathbb{M}_2 \rightarrow \mathbf{C}_\eta \rightarrow \Sigma^{-2,-1}\mathbb{M}_2 \rightarrow 0.$$



This gives a long exact sequence

$$\begin{aligned} \xrightarrow{h_1} \text{Ext}_A^{s+1, f-1, w}(M, \mathbb{M}_2) &\rightarrow \text{Ext}_A^{s+1, f-1, w}(M, \mathbf{C}_\eta) \\ &\rightarrow \text{Ext}_A^{s-1, f-1, w-1}(M, \mathbb{M}_2) \xrightarrow{h_1} \end{aligned}$$

in which the connecting homomorphism is multiplication by  $h_1$ . The result then follows easily.  $\square$

Our next goal is to carry out an explicit low-dimensional Ext computation. This result will be critical for a later argument. We will consider  $A(0)$  to be an  $A$ -module with the obvious action by  $A$ , i.e.,  $\text{Sq}^1$  acts by multiplication in the subalgebra  $A(0)$ , and  $\text{Sq}^n$  acts trivially for  $n \geq 2$ .

**Proposition 6.4.**  $\text{Ext}^{s, f, w}(A(0), \mathbf{C}_\eta) = 0$  when  $s < 2f$  and  $f \leq 4$ .

**Proof.** The short exact sequence

$$0 \rightarrow \Sigma^{1, 0}\mathbb{M}_2 \rightarrow A(0) \rightarrow \mathbb{M}_2 \rightarrow 0$$

gives a calculation of  $\text{Ext}(A(0), \mathbb{M}_2)$  in low degrees, starting from knowledge of  $\text{Ext}(\mathbb{M}_2, \mathbb{M}_2)$  in a similar range. The short exact sequence

$$0 \rightarrow \mathbb{M}_2 \rightarrow \mathbf{C}_\eta \rightarrow \Sigma^{-2, -1}\mathbb{M}_2 \rightarrow 0$$

then yields a calculation of  $\text{Ext}(A(0), \mathbf{C}_\eta)$ . The results of these calculations are shown in Figures 2 and 3.  $\square$

Figures 2 and 3 represent low-dimensional Ext calculations necessary for the proof of Proposition 6.4. Here is a key for reading the charts.

- (1) Black dots indicate copies of  $\mathbb{M}_2$ .
- (2) Vertical lines indicate multiplications by  $h_0$ .
- (3) Lines of slope 1 indicate multiplications by  $h_1$ .
- (4) Lines of slope 1/3 indicate multiplications by  $h_2$ .
- (5) Blue lines indicate that the multiplication hits  $\tau$  times a generator.
- (6) Red arrows indicate infinitely many copies of  $\mathbb{M}_2/\tau$  connected by  $h_1$  multiplications.

### 7. Multiplication by $h_1$ for $A(0)$ -free $A$ -modules

Recall that  $A//A(0)$  denotes the Hopf algebra quotient  $A \otimes_{A(0)} \mathbb{M}_2$ . There is a short exact sequence

$$0 \leftarrow \mathbb{M}_2 \xleftarrow{\varepsilon} A//A(0) \leftarrow I \leftarrow 0$$

of  $A$ -modules, where  $\varepsilon$  is the augmentation and  $I$  is the augmentation ideal. This short exact sequence gives rise to a long exact sequence

$$\text{Ext}_A^{s, f, w} \xrightarrow{\varepsilon^*} \text{Ext}_A^{s, f, w}(A//A(0), \mathbb{M}_2) \rightarrow \text{Ext}_A^{s, f, w}(I, \mathbb{M}_2) \xrightarrow{\partial} \text{Ext}_A^{s-1, f+1, w}$$

of modules over  $\text{Ext}_A$ . We have a change-of-rings isomorphism

$$\text{Ext}_A(A//A(0), \mathbb{M}_2) \cong \text{Ext}_{A(0)}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathbb{M}_2[h_0],$$

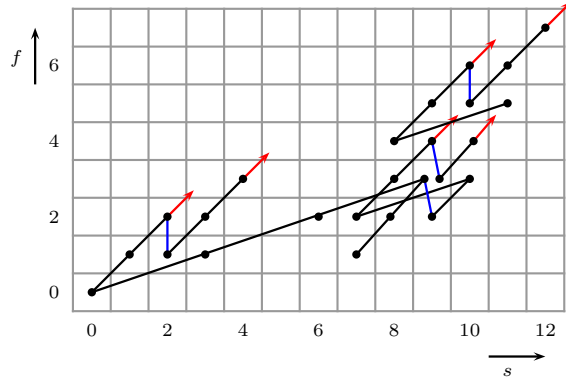


FIGURE 2.  $\text{Ext}_A(A(0), \mathbb{M}_2)$ .

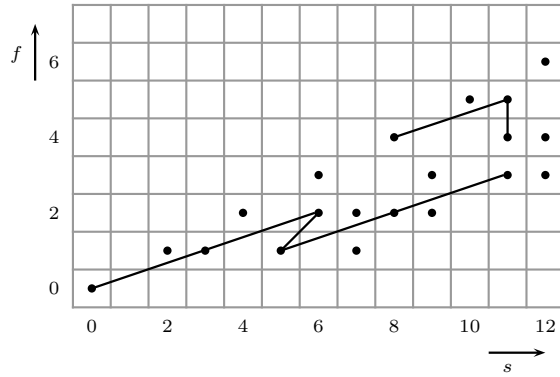


FIGURE 3.  $\text{Ext}_A(A(0), \mathbb{C}_\eta)$ .

so  $\varepsilon^*$  is an isomorphism when  $s = 0$  and  $\partial$  an isomorphism when  $s > 1$ .

The bottom class of  $I$  is  $\text{Sq}^2$ , which occurs in bidegree  $(2, 1)$ , so that we may write  $I \cong \Sigma^{2,1}N$  for a connective  $A$ -module  $N$ . It follows that for  $s > 0$ , we have a commutative square

$$\begin{array}{ccc}
 \text{Ext}_A^{s,f,w} & \xrightarrow{h_1} & \text{Ext}_A^{s+1,f+1,w+1} \\
 \cong \uparrow \partial & & \partial \uparrow \cong \\
 \text{Ext}_A^{s+1,f-1,w}(I, \mathbb{M}_2) & \xrightarrow{h_1} & \text{Ext}_A^{s+2,f,w+1}(I, \mathbb{M}_2) \\
 \cong \uparrow & & \uparrow \cong \\
 \text{Ext}_A^{s-1,f-1,w-1}(N, \mathbb{M}_2) & \xrightarrow{h_1} & \text{Ext}_A^{s,f,w}(N, \mathbb{M}_2)
 \end{array}$$

of  $\text{Ext}_A$ -module maps. Taking  $N = \Sigma^{-2,-1}I$  in the proof of Theorem 1.1, the above diagram and the following two results combine to yield the main theorem.

**Lemma 7.1.** *The  $A$ -module  $N$  is free as a left  $A(0)$ -module.*

**Proof.** This argument is identical to the classical case.

We use the admissible  $\mathbb{M}_2$ -basis for  $A$ , which consists of monomials of the form  $\text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$  such that  $r_i \geq 2r_{i+1}$ . Then  $A//A(0)$  has an  $\mathbb{M}_2$ -basis consisting of the admissible monomials  $\text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$  such that  $r_n > 1$ .

When  $r_1$  is even,  $\text{Sq}^1 \cdot \text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$  equals  $\text{Sq}^{r_1+1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$ . Therefore, the augmentation ideal  $I$  of  $A//A(0)$  is a free left  $A(0)$ -module with basis consisting of admissible monomials of the form  $\text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$  such that  $r_1$  is even and  $r_n > 1$ .  $\square$

**Proposition 7.2.** *Let  $M$  be an  $A$ -module that is free as an  $A(0)$ -module and concentrated in nonnegative degrees. Then the map*

$$h_1 : \text{Ext}_A^{s,f,w}(M, \mathbb{M}_2) \rightarrow \text{Ext}_A^{s+1,f+1,w+1}(M, \mathbb{M}_2)$$

*is an isomorphism if  $s < 2f - 2$ , and it is a surjection if  $s \leq 2f$ .*

We will mimic the classical argument of Adams [A2], with some variations to account for motivic phenomena.

**Proof.** By Lemma 6.3, it suffices to show that  $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta)$  vanishes when  $s < 2f$ .

We begin by recalling from Proposition 6.4 that  $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta)$  vanishes when  $s < 2f$  and  $f \leq 4$ .

An arbitrary module  $M$  can be built up iteratively as an extension

$$0 \rightarrow \bigoplus \Sigma^{t_i, w_i} A(0) \rightarrow M \rightarrow M' \rightarrow 0$$

of  $A$ -modules, where  $\text{Ext}_A^{s,f,w}(M', \mathbf{C}_\eta)$  vanishes for  $s < 2f$  and  $f \leq 4$  by induction. The long exact sequence in  $\text{Ext}$  then shows that  $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta)$  also vanishes for  $s < 2f$  and  $f \leq 4$ .

We have now established the proposition for  $f \leq 4$ . The next step is to extend the result to larger values of  $f$ . As in the previous step, we start with the special case  $M = A(0)$ .

In order to compute  $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta)$ , we must construct a resolution for  $A(0)$ . Proposition 6.2 says that this resolution can be built from copies of  $A$  or from copies of  $\tilde{A}$ .

We construct a resolution

$$A(0) \leftarrow R_0 \leftarrow R_1 \leftarrow R_2 \leftarrow R_3 \leftarrow \cdots$$

of  $A(0)$  in the usual way by adding a copy of  $A$  to  $R_{n+1}$  for each indecomposable in the kernel  $K_n$  of the boundary map  $R_n \rightarrow R_{n-1}$ . However, when we find two indecomposable elements  $x$  and  $y$  of  $K_n$  such that  $\text{Sq}^2x = \tau y$ , we add one copy of  $\tilde{A}$  to  $R_{n+1}$  rather than two copies of  $A$ .

As a result of this process, one can verify that the kernel  $K_3$  of the boundary map  $R_3 \rightarrow R_2$  vanishes in degrees less than 12. So we can write  $K_3 = \Sigma^{12,0}D$ , where  $D$  is concentrated in non-negative degrees. Note that  $D$  is  $\mathbb{M}_2$ -free since it is a submodule of the  $\mathbb{M}_2$ -free module  $\Sigma^{-12,0}R_3$ .

Proposition 5.5 implies that each  $R_n$  is  $A(0)$ -free. Then Corollary 3.3 implies that  $D$  is  $A(0)$ -free as well.

We already know from an earlier step that  $\text{Ext}_A^{s,f,w}(D, \mathbf{C}_\eta)$  vanishes for  $s < 2f$  and  $f \leq 4$ . We have isomorphisms

$$\text{Ext}_A^{s,f,w}(D, \mathbf{C}_\eta) \cong \text{Ext}_A^{s+12,f,w}(K_3, \mathbf{C}_\eta) \cong \text{Ext}_A^{s+8,f+4,w}(A(0), \mathbf{C}_\eta)$$

for  $f > 0$ , so it follows that  $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta)$  vanishes for  $s < 2f$  and  $f \leq 8$ .

As before, we can then show that  $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta)$  vanishes for  $s < 2f$  and  $f \leq 8$  for an arbitrary module  $M$ . This establishes the proposition for  $f \leq 8$ .

We can repeat this process to establish the proposition for all  $f$ .  $\square$

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