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The Bergman space as a Banach algebra

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Abstract. In this paper we use the Duhamel product to provide a Banach algebra structure to each of a scale of Bergman spaces over the unit disk, and then carry out many interesting consequences. In particular we characterize cyclic vectors of the Volterra integration operator, and determine its extended eigenvalues and corresponding extended eigenoperators. We also identify its commutants and point out some intertwining relations between the Volterra integration operator and composition operators.

Contents

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let $dA(z)$ be the normalized Lebesgue measure on $\mathbb{D}$. The Lebesgue space of p-summable complex-valued functions is denoted by $L^p(\mathbb{D}, dA)$. The Bergman space $L^p_a(\mathbb{D}) = L^p_a$ is the Banach subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions with norm given by:

$$||f||_p = \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}},$$

while $\mathcal{H}^\infty(\mathbb{D})$ denotes the Banach algebra, (with respect to the pointwise product), of bounded analytic functions with supremum norm.

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The Duhamel product of two analytic functions in $\mathbb{D}$ is given by [W74]:

$$(f \circledast g)(z) := \frac{d}{dz} \int_0^z f(z-t)g(t)dt.$$

This contour integral from 0 to z is path independent as f and g are analytic in the simply connected domain $\mathbb{D}$, so we have chosen the line segment $[0, z]$ as a contour of integration. Wigley [W75, W74] elaborated at length on this product and used it to provide an algebra structure to the Frechet space $\mathcal{H}(\mathbb{D})$ of all holomorphic functions, as well as to the Hardy spaces $\mathcal{H}^p(\mathbb{D}), p \geq 1$, and he described their maximal ideal spaces. Merryfield and Watson [MW91] extended the matter to the context of vector-valued Hardy spaces of the polydisk. In the last decade, the Duhamel product has been extensively explored on various spaces, including $L^2[0,1]$, $\mathcal{C}^\infty[0,1]$, $W_p^{(n)}[0,1]$ and $\mathcal{C}^{(n)}(\mathbb{D})$, by M.T. Karaev and his collaborators; and many applications of it have been well investigated, see for instance [G09, K05, K06, K95, KS05, KT04] and the references therein.

Within these applications, the Volterra integration operator has played a central role; it is well-adapted to the Duhamel convolution calculus. On the one hand, the Volterra integration operator helps establish a key invertibility criterion in the convolution algebra with the Duhamel considered here, and the other hand, this Banach algebra structure, in turn, provides information about cyclic vectors of the Volterra integration operator, its extended eigenvalues and corresponding extended eigenvectors in the sense of Biswas et al. [BLP02]; and, hopefully, about invariant subspaces [K95]. Note that invariant subspaces of the Volterra operator on $\mathcal{H}^p(\mathbb{D})$ spaces have been investigated by Aleman and Korenblum [AK08] using different methods.

Certainly the Bergman space is one of the most important spaces of holomorphic functions, and therefore endowing it with a Banach algebra structure should be of great importance for operator theory on this space, and for function theory as well; this task motivates our present work. Accordingly, using the Duhamel product, we provide a Banach algebra structure to the Bergman spaces $L_a^p(\mathbb{D})$, for $p > 1$, that we named *Bergman–Duhamel algebra*. We show, in particular, that it is a commutative unital $*$-algebra, but not a C*-algebra. We determine its maximal ideal space, and establish an invertibility criterion in this algebra. Many facts about certain classes of operators, such as the Volterra integration operator, can be obtained in a uniform fashion in the framework of this Duhamel product. For instance, cyclic vectors, (and strong cyclic vectors), of that operator are shown to be those Bergman functions $f \in L_a^p$ with $f(0) \neq 0$. Moreover, the set $\mathbb{C} \setminus \{0\}$ is shown to be precisely the set of its extended eigenvalues; and a combination of Duhamel convolution integral operators and composition operators constitute the corresponding extended eigenoperators. Note that this fact has its roots in the work of Biswas et al. [BLP02]. Moreover, the set of all Duhamel integral operators is shown to constitute the commutants of the

Volterra operator. As a byproduct, we discuss some intertwining relations between the Volterra integration operator and composition operators, which makes contact with work of Tong and Zhou [TZ13].

Throughout the paper, C denotes a positive constant that may vary from one line to another.

2. The Bergman–Duhamel algebra

For any $f \in L^p_a$, we have (see [Z07] p. 74):

$$|f^{(n)}(z)| \leq \frac{||f||_p}{(1-|z|^2)^{\frac{2}{p}}}, \quad \forall z \in \mathbb{D}. \tag{2.1}$$

In particular, on compacta we have (see [Z07] p. 99):

$$|f(z)| \leq C_p ||f||_p \quad \text{and} \quad |f'(z)| \leq C_p ||f||_p, \quad \forall z \in K \Subset \mathbb{D}. \tag{2.2}$$

Elementary calculation shows that the above Duhamel product can also be given by:

$$\begin{aligned}(f \circledast g)(z) &:= \frac{d}{dz}\int_0^z f(z-t)g(t)dt \\ &= \int_0^z f'(z-t)g(t)dt + f(0)g(z) \\ &= \int_0^z f(z-t)g'(t)dt + f(z)g(0).\end{aligned} \tag{2.3}$$

Moreover, it is obviously a commutative product.

If the integral line segment $[0, z]$ is halved, integration by parts leads to

$$\begin{aligned}(f \circledast g)(z) = &\int_0^{\frac{z}{2}} \left(f(z-t)g'(t) + g(z-t)f'(t)\right) dt \\ &+ f(z)g(0) + g(z)f(0) - f\left(\frac{z}{2}\right) g\left(\frac{z}{2}\right).\end{aligned} \tag{2.4}$$

Taking moduli in the latter and using Estimates (2.2), we get

$$\begin{aligned}|(f \circledast g)(z)| \leq C &\left\{ ||g||_p \int_0^{\frac{z}{2}} |f(z-t)||dt| + ||f||_p \int_0^{\frac{z}{2}} |g(z-t)||dt| \right\} \\ &+ C\{||g||_p|f(z)| + ||f||_p|g(z)| + ||f||_p||g||_p\}.\end{aligned} \tag{2.5}$$

Next, using polar coordinates, we majorize the two integrals in the R.H.S., (i.e., right hand side), of the latter by the Bergman norm:

Lemma 2.1. *For any Bergman function $f \in L^p_a$, we have*

$$\int_{\mathbb{D}} \left| \int_0^{\frac{z}{2}} |f(z-t)||dt| \right|^p dA(z) \leq C||f||_p^p, \quad 1 < p < \infty, \quad z \in \mathbb{D}. \tag{2.6}$$

Proof. In the integrand of the inside integral of (2.6), put $z = re^{i\theta},\ 0 \leq r < 1,\ t = \rho e^{i\theta},\ 0 \leq \rho \leq \frac{r}{2}$. Then, we obtain

$$\int_0^{\frac{z}{2}} |f(z-t)||dt| = \int_{\frac{r}{2}}^{r} \left| f\left(\rho e^{i\theta}\right)\right| d\rho.$$

Using Estimate (2.1), we obtain

$$\left| f\left(\rho e^{i\theta}\right)\right| \leq \frac{||f||_p}{(1-\rho^2)^{\frac{2}{p}}},$$

whence

$$\int_0^{\frac{z}{2}} |f(z-t)||dt| \leq ||f||_p \int_{\frac{r}{2}}^{r} \frac{d\rho}{(1-\rho)^{\frac{2}{p}}}.$$

Evaluating the definite integral of the R.H.S. of the latter, we obtain

$$\int_0^{\frac{z}{2}} |f(z-t)||dt| \leq \begin{cases} ||f||_p \left(\frac{(1-\frac{r}{2})^{1-\frac{2}{p}} - (1-r)^{1-\frac{2}{p}}}{1-\frac{2}{p}} \right), & p \neq 2 \\ ||f||_2 \left(\ln|1-\frac{r}{2}| - \ln|1-r| \right), & p = 2. \end{cases}$$

Integrating the p^{th} power of both sides of the latter over $\mathbb{D}$ and using Minkowski inequality for the R.H.S. we see that all integrals converge for $p \neq 1$, namely $\int_0^1 (1-r)^{-2+p} r dr < \infty$ for $p \neq 1$ and $\int_0^1 \ln^2|1-r| r dr < \infty$. The remaining terms behave well, and the θ-integrals do not effect the matter. So we get the desired estimate. □

Remark 2.2. One can use Marryfield & Watson's approach [MW91], which hinges on nontangential maximal functions. More precisely, for $z^* \in \partial\mathbb{D}$, denote by $\Gamma(z^*)$ a nontangential approach region with vertex at z^*, and consider the nontangential maximal function

$$\mathbf{N}f(z^*) = \sup_{z \in \Gamma(z^*)} |f(z)|.$$

When $z = re^{i\theta}$, $t = \rho e^{i\theta}$, $0 \leq \rho \leq \frac{r}{2}$, then $z - t \in \Gamma\left(e^{i\theta}\right)$; whence $|f(z-t)| \leq \mathbf{N}f\left(e^{i\theta}\right)$, and thus

$$\int_{\mathbb{D}} \left|\mathbf{N}f\left(e^{i\theta}\right)\right|^p dA \leq C||f||_p.$$

The point here is to observe that $\mathbf{N}f$ is only of weak 1-1 type on L^1_a, and this agrees with our results for $p = 1$.

Now, integrating the p^{th}-power of (2.5), using Minkowski's inequality as well as (2.6), we arrive at

$$||f \circledast g||_p \leq C_p ||f||_p ||g||_p. \tag{2.7}$$

This can be rephrased as follows, where we have incorporated the extreme case $p = \infty$, due to Wigley [W75], which requires slight different techniques.

Theorem 2.3. *The Bergman space $L^p_a(\mathbb{D})$, $1 < p \leq \infty$, is a unital, (the unit here is the constant function $\mathbb{1}$, defined by $\mathbb{1}(z) = 1, \forall z \in \mathbb{D}$), commutative Banach algebra with respect to the Duhamel convolution product $\circledast$, which will be called the Bergman–Duhamel algebra.*

Actually, it turns out that $(L^p_a(\mathbb{D}), \circledast)$ is a commutative unital $*$-algebra:

Proposition 2.4. *The map $f \longrightarrow f^*$, where $f^*(z) = \overline{f(\overline{z})}$ is an isometric involution on $(L^p_a(\mathbb{D}), \circledast)$.*

Proof. First, observe that $f \in L^p_a$ if and only if $f^* \in L^p_a$ with $||f||_p = ||f^*||_p$. Moreover, all requirements of involution are obviously satisfied but the product one. For, we have

$$\begin{aligned}(f \circledast g)^*(z) &= \overline{\int_0^{\overline{z}} f'(\overline{z} - t)g(t)dt + f(0)g(\overline{z})} \\ &= \int_0^{w} \overline{f'(w-t)}\;\overline{g(t)}d\overline{t} + \overline{f(0)}\;\overline{g(\overline{z})} \\ &= \int_0^{z} f^{*\prime}(z-s)g^*(s)ds + f^*(0)g^*(z) \\ &= (f^* \circledast g^*)(z) = (g^* \circledast f^*)(z).\end{aligned}$$

□

Remark 2.5. However $(L^p_a(\mathbb{D}), \circledast)$, $1 < p \leq \infty$ is not a C*-algebra. For, just observe that the C*-algebra identity is not satisfied for the function $f(z) = z$. Indeed, for $1 < p < \infty$, we have

$$||f \circledast f^*||_p = \frac{1}{2(p+1)^{\frac{1}{p}}} \neq \frac{2^{\frac{2}{p}}}{(p+2)^{\frac{2}{p}}} = ||f||_p^2.$$

Similarly, for $p = \infty$, we have

$$||f \circledast f^*||_\infty = \operatorname*{ess\,sup}_{\mathbb{D}} \left|\frac{z^2}{2}\right| \neq \left(\operatorname*{ess\,sup}_{\mathbb{D}} |z|\right)^2 = ||f||_\infty^2.$$

Next, we establish a pioneering $\circledast$-invertibility criterion in the Bergman–Duhamel algebra. The proof involves the classical Volterra integration operator, namely

$$(Vg)(z) = \int_0^z g(t)dt,$$

which is known to be compact on $L^p_a(\mathbb{D})$, see [AS97]. Another easy but useful property of it is the following intimate relationship with convolutions:

$$(V^n g)(z) = \int_0^z \frac{(z-t)^{n-1}}{(n-1)!}\, g(t)dt = \left(\frac{w^{n-1}}{(n-1)!} * g\right)(z), \tag{2.8}$$

where the symbol $*$ stands for the usual "Mikusinski" convolution given by

$$(f * g)(z) := \int_0^z f(z-t)g(t)dt.$$

Proposition 2.6. *Let $1 < p \leq \infty$. Then, $f \in L^p_a(\mathbb{D})$ is $\circledast$-invertible if and only if $f(0) \neq 0$.*

Proof. The extreme case $p = \infty$ is included in Wigley's theorem [W75]. So, we prove only the case $1 < p < \infty$.

Sufficiency: If $f \in L^p_a(\mathbb{D})$ is invertible, there is a $g \in L^p_a(\mathbb{D})$, with

$$(f \circledast g)(z) = 1.$$

Thus, $(f \circledast g)(0) = f(0)g(0) = 1$; whence $f(0) \neq 0$.

Necessity: If $f(0) \neq 0$, then put $F(z) = f(z) - f(0)$, and consider the operator

$$\mathfrak{D}_F : \ L^p_a(\mathbb{D}) \to L^p_a(\mathbb{D})$$

defined by

$$\mathfrak{D}_F g = (F \circledast g)(z) = \int_0^z F'(z-t)g(t)dt = \int_0^z f'(z-t)g(t)dt, \quad g \in L^p_a(\mathbb{D}).$$

We claim that such $\mathfrak{D}_F$ is compact. For, consider the partial Taylor series:

$$F_N = \sum_{n=0}^{N} \widehat{F}(n)z^n = \sum_{n=1}^{N} \widehat{f}(n)z^n.$$

Then, we have

$$\begin{aligned}\mathfrak{D}_{F_N} g(z) &= \int_0^z F_N'(z-t)g(t)dt = \int_0^z F_N'(t)g(z-t)dt \\ &= \sum_{n=1}^{N} n!\widehat{f}(n) \int_0^z \frac{(z-t)^{n-1}}{(n-1)!} \ g(t)dt.\end{aligned}$$

Therefore, by (2.8), we get

$$\mathfrak{D}_{F_N} = \sum_{n=1}^{N} n!\widehat{f}(n)V^n.$$

Hence $\mathfrak{D}_{F_N}$ is compact on $L^p_a(\mathbb{D})$, $\forall N > 0, \ 1 < p < \infty$.

Now, by (2.7), we have

$$||\mathfrak{D}_F - \mathfrak{D}_{F_N}|| = ||\mathfrak{D}_{F-F_N}|| \leq C||F - F_N||_p.$$

Passing to the limit in the latter as $N \to \infty$ and invoking Corollary 4 of [Z91], we see that $\mathfrak{D}_F$ is compact.

Next, consider $\mathfrak{D}_f$, (with symbol f), and assume that $g \in \ker \mathfrak{D}_F$, i.e.,

$$\mathfrak{D}_f g(z) = \int_0^z f'(z-t)g(t)dt + f(0)g(z) = 0, \quad \forall z \in \mathbb{D}.$$

Evaluation at zero gives $g(0) = 0$. Similarly, we could get

$$0 = \frac{d}{dz}(\mathfrak{D}_f g)(z) = \int_0^z f''(z-t)g(t)dt + f'(0)g(z) + f(0)g'(z), \quad \forall z \in \mathbb{D};$$

and evaluation at zero gives $g'(0) = 0$. Repeating this process for iterative derivatives, we obtain $g^{(n)}(0) = 0,\ n \geq 1$, whence we see that $g \equiv 0$. Hence, we infer that $\ker \mathfrak{D}_f = \{0\}$.

Further, observe that we can write

$$\mathfrak{D}_f = \mathfrak{D}_{f(0)+F} = f(0)I + \mathfrak{D}_F.$$

Now, $\mathfrak{D}_F$ is compact, $\mathfrak{D}_f$ is injective and $f(0) \neq 0$. The Fredholm alternative implies that $\mathfrak{D}_f$ is invertible in $L^p_a(\mathbb{D})$. □

Remark 2.7. The latter proof gives rise to an interesting convolution integral operator, namely the following Duhamel convolution operator with analytic symbol φ:

$$\mathfrak{D}_\varphi f(z) := (\varphi \circledast f)(z) = \int_0^z \varphi'(z-t)f(t)dt + \varphi(0)f(z),\ \ f \in L^p_a(\mathbb{D}).$$

At least for analogy sake, (with integration, multiplication and composition operators), this operator should be interesting in its own right. Investigating various properties of it might be worth considering; see for instance Theorem 2.9 below.

The previous invertibility criterion has many interesting consequences:

Corollary 2.8. *The structure space of the algebra* $(L^p_a(\mathbb{D}),\ \circledast)$, $1 < p \leq \infty$, *consists of one character, namely* $\mu_0 :\ f \in L^p_a(\mathbb{D}) \to \mu_0(f) = f(0)$. *Thus the Gelfand transform is trivial.*

Proof. It is clear from the definition of $\circledast$ that the set of functions vanishing at zero form an ideal, (in fact, these are the only noninvertible elements). Since a proper ideal cannot contain an invertible element, we infer that this algebra admits only that maximal ideal, which should be the kernel of the evaluation at zero functional. Thus, the maximal ideal space $\mathcal{M}\left((L^p_a(\mathbb{D}),\ \circledast)\right)$ is a singleton, whose unique element is μ_0. □

We are able to establish a certain Young type property for the $\circledast$-product:

Theorem 2.9. *Let* $f \in L^q_a(\mathbb{D})$. *Then, we have:*

(i) *If* $1 < q < 2$, *then* $\mathfrak{D}_f \in \mathscr{B}(L^p_a)$ *for any* p *satisfying* $1 < p < \frac{q}{2-q}$.
(ii) *If* $q \geq 2$, *then* $\mathfrak{D}_f \in \mathscr{B}(L^p_a)$ *for all* $p > 1$.

Proof. Let $f \in L^q_a(\mathbb{D}), q > 1$, and $g \in L^p_a, p > 1$. Since (2.1) and (2.2) are valid for any p with $1 \leq p \leq \infty$, repeating the calculation steps preceding Lemma 2.1, we arrive at a twin estimate of (2.5):

$$|(f \circledast g)(z)| \leq C\left\{||g||_p \int_0^{\frac{z}{2}} |f(z-t)||dt| + ||f||_q \int_0^{\frac{z}{2}} |g(z-t)||dt|\right\}$$
$$+ C\{||g||_p|f(z)| + ||f||_q|g(z)| + ||f||_q||g||_p\}.$$

Now the proof of Lemma 2.1 yields

$$\int_0^{\frac{z}{2}} |f(z-t)||dt| \leq \begin{cases} ||f||_q \left(\frac{(1-\frac{r}{2})^{1-\frac{2}{q}} - (1-r)^{1-\frac{2}{q}}}{1-\frac{2}{q}} \right), & q \neq 2 \\ ||f||_2 \left(\ln|1-\frac{r}{2}| - \ln|1-r| \right), & q = 2. \end{cases}$$

Integrating the p^{th} power of both sides of the latter over $\mathbb{D}$ and using Minkowski's inequality for the R.H.S. we get the desired estimate involving improper integrals of the form $\int_0^1 (1-r)^{p-\frac{2p}{q}} r dr$ and $\int_0^1 \ln^p |1-r| r dr$, which converge for the claimed ranges of p and q. □

Remark 2.10. Our techniques exclude the case $p = 1$. It is however interesting to see whether the $\circledast$-product on the Bergman space satisfies the following Young property: if $f \in L^1_a(\mathbb{D})$ and $g \in L^p_a(\mathbb{D}),\ 1 < p \leq \infty$, then $f \circledast g \in L^p_a(\mathbb{D})$.

3. Applications to the Volterra integration operator

One more consequence is the characterization of cyclic vectors of the Volterra operator we have already met. Before stating it, let us recall that if $\mathbb{X}$ is a seprable Banach space and $A \in \mathscr{B}(\mathbb{X})$, then $x \in \mathbb{X}$ is said to be cyclic for A if

$$\text{Span}\{A^n x : n \geq 0\} = \mathbb{X},$$

and x is said to be strong cyclic for A if

$$\{A\}' x = \{Tx : T \in \mathscr{B}(\mathbb{X}),\ TA = AT\} = \mathbb{X}.$$

Theorem 3.1. *For $1 < p < \infty$, one has* $Cycl(V) = \{f \in L^p_a(\mathbb{D}) : f(0) \neq 0\}$.

Proof. By (2.8) we know that $V^n f = \mathfrak{D}_f \left(\frac{z^n}{n!}\right)$, $n \geq 0$. Hence, we see that

$$\begin{aligned} \text{Span}\{V^n f : n \geq 0\} &= \text{Span}\left\{\mathfrak{D}_f \left(\frac{z^n}{n!}\right) : n \geq 0\right\} \\ &= \overline{\mathfrak{D}_f \ \text{Span}\left\{\left(\frac{z^n}{n!}\right) : n \geq 0\right\}} = \overline{\mathfrak{D}_f \ L^p_a(\mathbb{D})}. \end{aligned}$$

If $f(0) \neq 0$, then $\mathfrak{D}_f$ is seen above to be invertible. In particular it is with dense range; whence $f \in Cycl(V)$. Conversely, if $f \in Cycl(V)$, then $f(0) = 0$ cannot be. □

The following result says that the set of strong cyclic vectors of the Volterra operator V coincides with the set of its cyclic vectors.

Theorem 3.2. *A Bergman function $g \in L^p_a(\mathbb{D})$ is a strong cyclic vector for the Volterra operator V if and only if $g(0) \neq 0$.*

Proof. Suppose that g is a strong cyclic vector for V, then by Proposition 2.6 and Corollary 3.1 there is some $f \in L^p_a(\mathbb{D})$ such that $\mathfrak{D}_f g = \mathbb{1}$. This means that $f \circledast g = g \circledast f = \mathbb{1}$; whence g is invertible. By Proposition 2.6, $g(0) \neq 0$.

Conversely, if $g(0) \neq 0$, then by Proposition 2.6 again, there is some $h \in L^p_a(\mathbb{D})$ such that $h \circledast g = \mathbb{1}$. Inserting an arbitrary $f \in L^p_a(\mathbb{D})$ to the latter, we get

$$(f \circledast g) \circledast h = f \circledast \mathbb{1} = f, \ \forall f \in L^p_a(\mathbb{D}).$$

This implies that $f \in L^p_a(\mathbb{D})$ can be written as $f = \mathfrak{D}_{f\circledast h} g$, while $\mathfrak{D}_{f\circledast h} \in \{V\}'$ by Corollary 3.1. Thus $\{V\}' g = L^p_a(\mathbb{D})$. $\square$

The next assertion characterizes extended eigenvectors (in the sense of Biswas–Lambert–Petrovic [BLP02]) of the Volterra integration operator V on the Bergman space L^p_a. For such purpose, we make appeal to composition operators $\mathfrak{C}_\varphi$, (with disk self-map symbols φ), on the Bergman space L^p_a. The composition operator with symbol $\varphi_\lambda(z) = \lambda z$ is denoted $\mathfrak{C}_\lambda$.

Note that the spectrum of the above quasinilpotent Volterra operator V is $\sigma(V) = \{0\}$, but this unique spectral value $\lambda = 0$ is not an eigenvalue. It turns out that it is not an extended eigenvalue either, since its kernel is trivial. We will show that the set of extended eigenvalues of V on L^2_a is precisely $\mathbb{C} \setminus \{0\}$.

Theorem 3.3. *Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ and let $A \in \mathscr{B}(L^p_a(\mathbb{D}))$, and recall the above notation $\mathbb{1}$ for the function with constant value 1. Then, we have:*

(i) *If $|\lambda| \leq 1$, then: $AV = \lambda V A$ if and only if $A = \mathfrak{D}_{A\mathbb{1}} \mathfrak{C}_\lambda$.*
(ii) *If $|\lambda| > 1$, then: $AV = \lambda V A$ if and only if $A \mathfrak{C}_{\frac{1}{\lambda}} = \mathfrak{D}_{A\mathbb{1}}$.*

Proof. Let us prove (i). Starting from the operator equation $AV = \lambda V A$ and using induction, we obtain $AV^n = \lambda^n V^n A$. In particular, we have $AV^n \mathbb{1} = \lambda^n V^n A\mathbb{1}$, which by (2.8) yields

$$A\frac{z^n}{n!} = \lambda^n \frac{z^{n-1}}{(n-1)!} * A\mathbb{1}.$$

Hence, we get

$$\mathbb{1} * A\frac{z^n}{n!} = \frac{(\lambda z)^n}{n!} * A\mathbb{1}.$$

So, for any polynomial P, we get

$$\mathbb{1} * AP(z) = P(\lambda z) * A\mathbb{1}.$$

Owing to density of polynomials in the Bergman space L^p_a, we obtain

$$\mathbb{1} * Af(z) = f(\lambda z) * A\mathbb{1}, \ \forall f \in L^p_a.$$

Hence, we infer that

$$VAf(z) = \int_0^z A\mathbb{1}(z-t)\mathfrak{C}_\lambda f(t)dt, \ \forall f \in L^p_a. \tag{3.1}$$

Differentiating Equation (3.1), we obtain

$$Af(z) = \frac{d}{dz}\int_0^z A\mathbb{1}(z-t)f(\lambda t)dt = \mathfrak{D}_{A\mathbb{1}}\mathfrak{C}_\lambda f(z);$$

whence $A = \mathfrak{D}_{A\mathbb{1}}\mathfrak{C}_\lambda$.

Conversely, suppose that $A = \mathfrak{D}_{A\mathbb{1}}\mathfrak{C}_\lambda$. Thus, we see that

$$\begin{aligned} AVf &= \mathfrak{D}_{A\mathbb{1}}\mathfrak{C}_\lambda Vf = A\mathbb{1} \circledast (Vf)(\lambda z) \\ &= \lambda z \circledast A\mathbb{1} \circledast (Vf)(\lambda z) = \lambda\left(z \circledast \mathfrak{D}_{A\mathbb{1}}\mathfrak{C}_\lambda f(z)\right) = \lambda VAf(z). \end{aligned}$$

Thus, we infer that $AV = \lambda VA$.

Let us prove (ii). If $|\lambda| > 1$, then the equation $Av = \lambda VA$ implies that $V^{n+1}A = V\left(\frac{1}{\lambda^n}AV^n\right)$. Applying the latter to the function $\mathbb{1}$ and making use of Formula (2.8), we see that $\dfrac{z^n}{n!} \circledast \mathbb{1} = \mathbb{1} \circledast A\left(\dfrac{\left(\frac{z}{\lambda}\right)^n}{n!}\right)$. Thus the above density argument yields $\mathbb{1} \circledast Af\left(\dfrac{z}{\lambda}\right) = f(z) \circledast A\mathbb{1}$; which can be rewritten as $VA\mathfrak{C}_{\frac{1}{\lambda}}f = A\mathbb{1} \circledast f$. Hence, by differentiation, we get $A\mathfrak{C}_{\frac{1}{\lambda}} = \mathfrak{D}_{A\mathbb{1}}$.

Conversely, if $A\mathfrak{C}_{\frac{1}{\lambda}} = \mathfrak{D}_{A\mathbb{1}}$, then

$$\lambda VAz = \lambda VA\mathfrak{C}_{\frac{1}{\lambda}}\mathfrak{C}_\lambda z = \lambda V\mathfrak{D}_{A\mathbb{1}}\mathfrak{C}_\lambda z.$$

So for polynomials, we get

$$\begin{aligned} \lambda VAP(z) &= \lambda V\mathfrak{D}_{A\mathbb{1}}P(\lambda z) = \lambda\mathfrak{D}_{A\mathbb{1}}VP(\lambda z) \\ &= \lambda A\mathfrak{C}_{\frac{1}{\lambda}}VP(\lambda z) = A\mathfrak{C}_{\frac{1}{\lambda}}(\lambda z \circledast P(\lambda z)) = A(z \circledast P) = AVP(z). \end{aligned}$$

By density, we get $\lambda VA = AV$. □

This yields in particular a characterization of the commutant of the Volterra operator on the Bergman space, which says in particular that the operators $\mathfrak{D}_f$, $f \in L_a^p$, are in fact nontrivial ($\neq I$) eigenoperators corresponding to the "trivial" extended eigenvalue $\lambda = 1$ of V.

Corollary 3.4. *The commutants of the Volterra integration operator on the Bergman space $L_a^p, 1 < p \leq \infty$ is characterized by: $\{V\}' = \{\mathfrak{D}_f,\ f \in L_a^p\}$.*

The following result says that a nonzero λ, satisfying $|\lambda| < 1$, can be an extended eigenvalue of the Volterra operator with a composition operator as its extended eigenoperator only rarely; this agrees "almost" with the more general conclusion preceding Proposition 3.5 in [TZ13].

Corollary 3.5. *Let φ be a disk self map and suppose that $|\lambda| \leq 1,\ \lambda \neq 0$. Then, the composition operator and the Volterra integration operator satisfy the following intertwining relation $\mathfrak{C}_\varphi V = \lambda V\mathfrak{C}_\varphi$ if and only if $\varphi(z) = \lambda z$.*

Proof. If $\varphi(z) = \lambda z$, then clearly

$$(\mathfrak{C}_\varphi Vf)(z) = \int_0^{\lambda z} f(t)dt = (\lambda V\mathfrak{C}_\varphi f)(z), \forall f \in L_a^p.$$

Conversely, if $\mathfrak{C}_\varphi V = \lambda V \mathfrak{C}_\varphi$, then by the first item of Theorem 3.3 we have $\mathfrak{C}_\varphi = \mathfrak{D}_{\mathfrak{C}_\varphi \mathbb{1}} \mathfrak{C}_\lambda = \mathfrak{D}_{\mathbb{1}} \mathfrak{C}_\lambda$. Thus, we get

$$\begin{aligned}(\mathfrak{C}_\varphi f)(z) = (\mathfrak{D}_{\mathbb{1}} \mathfrak{C}_\lambda f)(z) &= \frac{d}{dz}\int_0^z \mathbb{1}(z-t)(\mathfrak{C}_\lambda f)(t)dt \\ &= \frac{d}{dz}\int_0^z f(\lambda t)dt = f(\lambda z) = (\mathfrak{C}_\lambda f)(z),\ \forall f \in L_a^p.\end{aligned}$$

Thus, we infer that $\mathfrak{C}_\varphi = \mathfrak{C}_\lambda$. □

References

[AK08] Aleman, Alexandru; Korenblum, Boris. Volterra invariant subspaces of $\mathcal{H}^p$. *Bull. Sci. Math.* **132** (2008), no. 6, 510–528. MR2445578 (2009h:47014), Zbl 1168.46011, doi: 10.1016/j.bulsci.2007.08.001.

[AS97] Aleman, Alexandru; Siskakis, Aristomenis G. Integration operators on Bergman spaces. *Indiana Univ. Math. J.* **46** (1997), no. 2, 337–356. MR1481594 (99b:47039), Zbl 0951.47039.

[BLP02] Biswas, Animikh; Lambert, Alan; Petrovic, Srdjan. Extended eigenvalues and the Volterra operator. *Glasgow Math. J.* **44** (2002), no. 3, 521–534. MR1956558 (2004c:47039), Zbl 1037.47013, doi: 10.1017/S001708950203015X.

[G09] Gürdal, M. Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra. *Expo. Math.* **27** (2009), no. 2, 153–160. MR2518622 (2010d:47041), Zbl 1172.47002, doi: 10.1016/j.exmath.2008.10.006.

[K95] Karaev, M.T. Usage of the Duhamel product in description of invariant subspaces. *Proc. Inst. Math., Mech. Academy. Sci. Azerb. (Trudy IMM Azerb. Akad. Nauk.)*, **3 (11)**, (1995), 137–146.

[K05] Karaev, M.T. On some applications of the ordinary and extended Duhamel products. *Sibirsk. Mat. Zh.* **46** (2005), no. 3, 553–566. Translation in *Siberian Math. J.* **46** (2005), no. 3, 431–442. MR2164560 (2006f:30062), Zbl 1224.46100, doi: 10.1007/s11202-005-0046-6.

[K06] Karaev, M.T. On extended eigenvalues and extended eigenvectors of some operator classes. *Proc. Amer. Math. Soc.* **134** (2006), no. 8, 2382–2392. MR2213712 (2007f:47007), Zbl 1165.47302, doi: 10.1090/S0002-9939-06-08258-X.

[KS05] Karaev, M.T.; Saltan, S. A Banach algebra structure for the Wiener algebra $\mathscr{W}(\mathbb{D})$ on the disc. *Complex Var. Theory Appl.* **50** (2005), no. 4, 299–305. MR2126322 (2005k:47067), Zbl 1082.47028, doi: 10.1080/02781070500032911.

[KT04] Karaev, M.T.; Tuna, H. Description of maximal ideal space of some Banach algebra with multiplication as Duhamel product. *Complex Var. Theory Appl.* **49** (2004), no. 6, 449–457. MR2073174 (2005d:46109), Zbl 1069.47034.

[MW91] Merryfield, Kent G.; Watson, Saleem. A local algebra structure for $\mathcal{H}^p$ of the polydisc. *Colloquium Math.* **62** (1991), no. 1, 73–79. MR1114621 (92k:46084), Zbl 0766.46040.

[TZ13] Tong, Ce-Zhong; Zhou, Ze-Hua. Intertwining relations for Volterra operators on the Bergman space. *Illinois J. Math.* **57** (2013), no. 1, 195–211. MR3224567, Zbl 06322989.

[W74] Wigley, Neil M. The Duhamel product of analytic functions. *Duke Math. J.* **41** (1974), 211–217. MR0335830 (49 #608), Zbl 0283.30036, doi: 10.1215/S0012-7094-74-04123-4.

[W75] Wigley, Neil M. A Banach algebra structure for $\mathcal{H}^p$. *Canad. Math. Bull.* **18** (1975), no. 4, 597–603. MR0397413 (53 #1272), Zbl 0324.46051, doi: 10.4153/CMB-1975-106-4.

[Z91] Zhu, Kehe. Duality of Bloch spaces and norm convergence of Taylor series. *Michigan Math. J.* **38** (1991), no. 1, 89–101. MR1091512 (92h:30004), Zbl 0728.30026, doi: 10.1307/mmj/1029004264.

[Z07] Zhu, Kehe. Operator theory in function spaces. Second edition. Mathematical Surveys and Monographs, 138. *American Mathematical Society, Providence R.I.*, 2007. xvi+348 pp. ISBN: 978-0-8218-3965-2. MR2311536 (2008i:47064), Zbl 1123.47001, doi: 10.1090/surv/138.

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