

Homotopy theory of labelled symmetric precubical sets

Philippe Gaucher

ABSTRACT. This paper is the third paper of a series devoted to higher-dimensional transition systems. The preceding paper proved the existence of a left determined model structure on the category of cubical transition systems. In this sequel, it is proved that there exists a model category of labelled symmetric precubical sets which is Quillen equivalent to the Bousfield localization of this left determined model category by the cubification functor. The realization functor from labelled symmetric precubical sets to cubical transition systems which was introduced in the first paper of this series is used to establish this Quillen equivalence. However, it is not a left Quillen functor. It is only a left adjoint. It is proved that the two model categories are related to each other by a zig-zag of Quillen equivalences of length two. The middle model category is still the model category of cubical transition systems, but with an additional family of generating cofibrations. The weak equivalences are closely related to bisimulation. Similar results are obtained by restricting the constructions to the labelled symmetric precubical sets satisfying the HDA paradigm.

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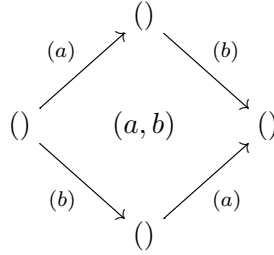
1. Introduction

1.1. Presentation of two combinatorial approaches of concurrency.

This paper is the third paper of a series devoted to higher-dimensional transition systems (HDTS). The first appearance of this notion dates back to [CS96] in which the concurrent execution of n action is modelled by a multiset of actions. The first paper of this series [Gau10b] proved that this approach is actually the same as the geometric approach of concurrency dating back to [Dij68] [Pra91] [Gun94]. It is really not possible to give an exhaustive list of references for this subject because this field of research is growing very fast but at least two surveys are available [Gou03] [VG06] presenting various topological and combinatorial models. In this theory, the concurrent execution of n actions is modelled by a labelled n -cube. Each coordinate represents the state of one of the n processes running concurrently, from 0 (not started) to 1 (complete). Figure 1 represents the concurrent execution of two actions a and b .

The formalism of labelled symmetric precubical sets is another example of combinatorial object encoding these ideas. The idea of modelling labelling *cubical sets* by working in a comma category is probably introduced in [Gou02]. Labelled *precubical* sets, meaning without the standard degeneracy maps coming from algebraic topology, and without symmetry operators, are actually sufficient to model the geometry of all process algebras for any synchronization algebra by [Gau08, Section 4]. Let us emphasize this fact. Not only are the standard degeneracy maps coming from algebraic topology useless for modelling the space of paths of a process algebra, but also new degeneracy maps, the *transverse degeneracy maps* of [Gau10a] seem to be required to better understand the semantics of process algebras. These non-standard degeneracy maps will not be used in this paper however. Every process algebra can then be viewed as a labelled *symmetric* precubical set (Definition 5.5) just by considering the *free labelled symmetric precubical set* generated by the associated labelled precubical set [Gau10a]. Thanks to the symmetry operators, the parallel composition of two processes P and Q is isomorphic to the parallel composition of two processes Q and P . Such an isomorphism just does not exist in general in the category of labelled precubical sets, except in degenerate situations like $P = Q$ of course.

A semantics of process algebras in terms of HDTS is given in [Gau10b]. Unlike labelled symmetric precubical sets, HDTS do not necessarily have face operators: they are not part of the definition indeed (see Definition 3.2). An immediate consequence is that the colimit of the cubes contained in a given HDTS (called its cubification, see Definition 8.8) does not necessarily give back the HDTS. A nonempty HDTS may even have an empty cubification. Even the cubical transition systems which are, by definition, equal to the union of their subcubes are not necessarily equal to their cubification (see below in this introduction). There is another striking difference between HDTS and labelled symmetric precubical sets: all Cattani–Sassone


 FIGURE 1. Concurrent execution of two actions a and b .

higher-dimensional transition systems satisfy the so-called HDA paradigm (see Section 9 of this paper, and [Gau10b, Section 7]). This implies that the formalization of the parallel composition, for any synchronization algebra, of two processes is much simpler with HDTS than with precubical sets, symmetric or not. Indeed, there is no need in the setting of HDTS to introduce tricky combinatorial constructions like the *directed coskeleton construction* of [Gau08],¹ or the *transverse degeneracy maps* of [Gau10a]. We just have to list all higher-dimensional transitions of a parallel composition by reading the definition from a computer science book and to put them in the set of transitions of the HDTS.

1.2. The salient mathematical facts of the preceding papers of this series. The first paper [Gau10b] is devoted to introducing a more convenient formalism to deal with HDTS. More precisely, the category of weak HDTS is introduced (Definition 3.2). It enjoys a lot of very nice properties: topological,² locally finitely presentable. The category of Cattani–Sassone higher-dimensional transition systems is interpreted as a full reflective subcategory of the category of weak HDTS [Gau10b, Corollary 5.7]. And it is proved in [Gau10b, Theorem 11.6] that the categorical localization of the category of Cattani–Sassone higher-dimensional transition systems by the cubification functor is equivalent to a full reflective subcategory of that of labelled symmetric precubical sets. The main tool is a realization functor from labelled symmetric precubical sets to HDTS, whose construction is presented in an improved form in Section 7 of this paper. Symmetry operators are required for this result since the group of automorphisms of the labelled n -cube in the category of Cattani–Sassone HDTS is the n -th symmetry group, not the singleton as in the category of labelled precubical sets of [Gau08]. In other terms, the category of HDTS has built-in symmetry operators which, of course, come from the action of the n -th symmetric

¹Let us just recall here that the choice of “directed” in “directed coskeleton construction” was a very bad idea.

²A topological category is a category equipped with a forgetful topological functor towards a power of the category of sets.

group on the set of n -dimensional transitions. The realization functor from labelled symmetric precubical sets to HDTS will be reused in this paper to get Quillen equivalences.

The second paper of the series [Gau11] is devoted to the study of the homotopy theory of HDTS. A left determined model structure is built on the topological and finitely presentable category of weak HDTS, and then restricted to the full subcategory of cubical transition systems [Gau11, Corollary 6.8], i.e., the weak HDTS which are equal to the union of their subcubes (Definition 3.8). This full coreflective subcategory of that of weak HDTS contains all examples coming from computer science even if the topological structure of the larger category of weak HDTS keeps playing an important role in the development of this theory. The class of weak equivalences of this left determined model structure is completely characterized. It appears that it is really too small to be interesting. It also turns out that all weak equivalences are bisimulations and it is tempting to Bousfield localize by all bisimulations. By [Gau11, Theorem 9.5], such a Bousfield localization exists but its study is out of reach at present. An intermediate Bousfield localization, by the cubification functor again, is proved to exist as well [Gau11, Section 8].

One word must be said about the notion of cubical transition system. Not all HDTS are equal to the colimit of their subcubes. For example the boundary $\partial C_2[x_1, x_2]$ of the full 2-cube $C_2[x_1, x_2]$, which is obtained by removing all its 2-transitions, that is to say $((0, 0), x_1, x_2, (1, 1))$ and $((0, 0), x_2, x_1, (1, 1))$.³ Indeed the HDTS $\partial C_2[x_1, x_2]$ has only two actions x_1 and x_2 , whereas its cubification has four distinct actions x_1, x'_1, x_2, x'_2 with the labelling map $\mu(x_k) = \mu(x'_k) = x_k$ for $k = 1, 2$. So $\partial C_2[x_1, x_2]$ is not isomorphic to its cubification, and therefore it cannot be a colimit of cubes. But it is cubical anyway. This is because of this subtle point that we are forced to use the category of cubical transition systems. It is not known whether a model category structure like the one of [Gau11] exists on the full subcategory of weak HDTS of colimits of cubes. The main problem consists of finding another set of generating cofibrations (instead of the set \mathcal{I} defined in Notation 4.5) without using the boundary of the labelled 2-cubes.

1.3. Presentation of this paper. This third paper of the series goes back to the link between labelled symmetric precubical sets and cubical transition systems. One of the main results of this paper is that a model category structure is constructed on the category of labelled symmetric precubical sets (Theorem 6.8) thanks to Marc Olschok's Ph.D. [Ols09b] [Ols09a]. And it is proved in Theorem 8.9 that there exists a Bousfield localization of the latter which is Quillen equivalent to the model category structure of cubical

³The n -cube $C_n[x_1, \dots, x_n]$ has actually and by definition n distinct actions (x_i, i) for $i = 1, \dots, n$, but it is assumed here that $x_1 \neq x_2$, so there is no need to overload the notations by writing $(x_1, 1), (x_2, 2)$; the n -cube viewed as a HDTS has exactly $n!$ n -dimensional transitions.

transition systems introduced in [Gau11] (not the left determined one, but its Bousfield localization by the cubification functor, which is studied in [Gau11, Section 8]). It is remarkable that like for the categorical equivalence of [Gau10b], the cubification functor is used once again, this time to obtain a Quillen equivalence. Theorem 9.6 is a similar theorem after restriction to the labelled symmetric precubical sets satisfying the HDA paradigm. Unfortunately, almost nothing is known about these Bousfield localizations.

Surprisingly, the realization functor from labelled symmetric precubical sets to cubical transition systems is not a left Quillen functor. It is only a left adjoint (Proposition 7.5). An intermediate model category must be used in the proofs to get the Quillen equivalences (Theorem 7.10). The cause of this problem is the family of cofibrations consisting of the inclusions $\partial\Box_S[x, y] \subset \Box_S[x, y]$ of the boundary of a labelled 2-cube to the full labelled 2-cube (Proposition 7.5) for x and y running over the set of labels Σ . The image by the realization functor is not a cofibration of HDTS. Indeed, the image of $\partial\Box_S[x, y]$ is precisely the cubification of the boundary of the 2-cube $\partial C_2[x, y]$ because every labelled symmetric precubical set is equal to the colimit of its cubes and because the realization functor is colimit-preserving. It has four actions (see above!) x, x', y, y' with the labelling map $\mu(x) = \mu(x') = x$ and $\mu(y) = \mu(y') = y$ whereas the realization of $\Box_S[x, y]$ is the 2-cube $C_2[x, y]$ which has two actions x and y : therefore the map from the realization of $\partial\Box_S[x, y]$ to the one of $\Box_S[x, y]$ cannot be one-to-one on actions, so it cannot be a cofibration of HDTS by definition. The intermediate model category is precisely obtained by adding this family of maps (so the realization of the inclusions $\partial\Box_S[x, y] \subset \Box_S[x, y]$) to the set of generating cofibrations of the model category of cubical transition systems! In other terms, we force the realization functor from labelled symmetric precubical sets to cubical transition systems to become a left Quillen functor. And the second surprise is that that just works fine.

Again the same family of inclusions $\partial\Box_S[x, y] \subset \Box_S[x, y]$ for x and y running over the set of labels Σ prevents the interval object of labelled symmetric precubical sets from being very good. It is only good (Proposition 6.6 and the remark after the proof). The realization as HDTS of the same family of cofibrations also prevents the interval object of cubical transition systems from being very good as well with respect to the augmented set of generating cofibrations (Theorem 7.10). As a consequence, the Olschok construction cannot tell us anything about the left determinedness of the model category of labelled symmetric precubical sets and of the augmented model category of cubical transition systems.

This new model category structure on labelled symmetric precubical sets is very different from the ones coming from algebraic topology. Indeed, the class of cofibrations is strictly larger than the class of monomorphisms. Like the model category of flows [Gau03], it contains the generating cofibration $R : \{0, 1\} \rightarrow \{0\}$. This makes it impossible to use tools like Cisinski's

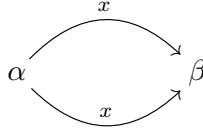


FIGURE 2. $\text{Cyl}(\square_S[x])$: the cylinder of $\square_S[x]$ is homotopy equivalent to the 1-cube $\square_S[x]$.

homotopy theory of toposes [Cis02] or Hirschhorn’s theory of Bousfield localization [Hir03]. The main technical tool of this paper is Marc Olschok’s Ph.D. thesis [Ols09b] [Ols09a] instead. Moreover, not only is the 1-cube not weakly equivalent to a point; it is in fact weakly equivalent to two copies of itself where the two initial (final resp.) states are identified as in Figure 2. This new model category is adapted, like the ones constructed on cubical transition systems in [Gau11], to the study of bisimulation [WN95] [JNW96]. In the case of Figure 2, the labelled symmetric precubical set has the same behavior as the 1-cube $\square_S[x]$ labelled by x . Indeed, the unique map from $\text{Cyl}(\square_S[x])$ to $C_1[x]$ is a bisimulation.

Outline of the paper. Section 2 is a reminder about the Olschok construction of combinatorial model categories, at least the first part of his Ph.D. devoted to the generalization of Cisinski’s work to the setting of locally presentable categories. Only what is used in this paper is recalled. So the statement of Theorem 2.6 is certainly less general than what is written in [Ols09a] and [Ols09b]. Section 3 is a reminder about weak HDTs and cubical transition systems. Several important basic examples of such objects are given. Section 4 recalls the homotopy theory of cubical transition systems. The exposition is improved, so it is more than just a reminder. In particular, an explicit set of generating cofibrations is given. Section 5 recalls the definition of labelled symmetric precubical set. Once again, several important basic examples are given. Section 6 constructs the new model category structure on labelled symmetric precubical sets (Theorem 6.8). Roughly speaking, we *really* just have to mimic the construction of the model category structure on cubical transition systems. Section 7 recalls the construction of the realization functor from labelled symmetric precubical sets to cubical transition systems. The exposition is much better than in [Gau10b] where it is introduced, so it is also more than just a reminder. It is also proved in the same section that the realization functor is not a left Quillen functor, and it is explained how to overcome this problem by adding one family of generating cofibrations to the category of cubical transition systems. And Section 8 proves one of the main result of this paper: there exists a model category of labelled symmetric precubical sets which is Quillen equivalent to the homotopy theory of cubical transition systems (Theorem 8.9). The

last section restricts the homotopy constructions to the full reflective subcategory of labelled symmetric precubical sets satisfying the HDA paradigm (Theorem 9.5) and proves a similar result (Theorem 9.6).

Preliminaries. The necessary bibliographical references and reminders are given throughout the text. The category of sets is denoted by **Set**. All categories are locally small. The set of maps in a category \mathcal{K} from X to Y is denoted by $\mathcal{K}(X, Y)$. The locally small category whose objects are the maps of \mathcal{K} and whose morphisms are the commutative squares is denoted by $\text{Mor}(\mathcal{K})$. The initial (final resp.) object, if it exists, is always denoted by \emptyset (**1**). The identity of an object X is denoted by Id_X . A subcategory is always *isomorphism-closed*. We refer to [AR94] for locally presentable categories, to [Ros09] for combinatorial model categories, and to [AHS06] for topological categories (i.e., categories equipped with a topological functor towards a power of the category of sets). We refer to [Hov99] and to [Hir03] for model categories. For general facts about weak factorization systems, see also [KR05]. We recommend the reading of Marc Olschok’s Ph.D. [Ols09b]. The first part, published in [Ols09a], is used in this paper.

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2. The Olschok construction of model categories

We want to review the Olschok construction of combinatorial model categories [Ols09a], as it is already used in [Gau11], and as it is used in this paper, i.e., by starting from a *good interval object*, i.e., a good cylinder functor $\text{Cyl}(-)$ of the form $\text{Cyl}(-) = V \times -$ where V is a bipointed object of the ambient category.

Let f and g be two maps of a locally presentable category \mathcal{K} . Write $f \square g$ when f satisfies the *left lifting property* with respect to g (or equivalently g satisfies the *right lifting property* with respect to f). Let us introduce the notations $\mathbf{inj}_{\mathcal{K}}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f \square g\}$, $\mathbf{proj}_{\mathcal{K}}(\mathcal{C}) = \{f \in \mathcal{K}, \forall g \in \mathcal{C}, f \square g\}$ and $\mathbf{cof}_{\mathcal{K}}(\mathcal{C}) = \mathbf{proj}_{\mathcal{K}}(\mathbf{inj}_{\mathcal{K}}(\mathcal{C}))$ where \mathcal{C} is a class of maps of a locally presentable category \mathcal{K} . The class of morphisms of \mathcal{K} that are transfinite compositions of pushouts of elements of \mathcal{C} is denoted by $\mathbf{cell}_{\mathcal{K}}(\mathcal{C})$.

2.1. Notation. For every map $f : X \rightarrow Y$ and every natural transformation $\alpha : F \rightarrow F'$ between two endofunctors of \mathcal{K} , the map $f \star \alpha$ is the canonical map

$$f \star \alpha : FY \sqcup_{FX} F'X \longrightarrow F'Y$$

induced by the commutative diagram of solid arrows

$$\begin{array}{ccc}
 FX & \xrightarrow{\alpha_X} & F'X \\
 Ff \downarrow & & \downarrow F'f \\
 FY & \xrightarrow{\alpha_Y} & F'Y
 \end{array}$$

and the universal property of the pushout. For a set of morphisms \mathcal{A} , let $\mathcal{A} \star \alpha = \{f \star \alpha, f \in \mathcal{A}\}$.

2.2. Definition. Let I be a set of maps of a locally presentable category \mathcal{K} . A good cylinder with respect to I is a functor $\text{Cyl} : \mathcal{K} \rightarrow \mathcal{K}$ together with two natural transformations $\gamma^k : \text{Id} \Rightarrow \text{Cyl}$ for $k = 0, 1$ and a natural transformation $\sigma : \text{Cyl} \Rightarrow \text{Id}$ such that the codiagonal $\text{Id} \sqcup \text{Id} \Rightarrow \text{Id}$ factors as a composite

$$\text{Id} \sqcup \text{Id} \xrightarrow{\gamma^0 \sqcup \gamma^1} \text{Cyl} \xrightarrow{\sigma} \text{Id}$$

and such that the left-hand natural transformation $\gamma^0 \sqcup \gamma^1$ induces for all $X \in \mathcal{K}$ a map

$$X \sqcup X \xrightarrow{\gamma_X^0 \sqcup \gamma_X^1} \text{Cyl}(X) \in \mathbf{cof}_{\mathcal{K}}(I).$$

When moreover the right-hand map σ_X belongs to $\mathbf{inj}_{\mathcal{K}}(I)$ for all $X \in \mathcal{K}$, the functor Cyl is called a very good cylinder.

2.3. Definition. Let I be a set of maps of a locally presentable category \mathcal{K} . A good cylinder $\text{Cyl} : \mathcal{K} \rightarrow \mathcal{K}$ with respect to I is cartesian if it is exponential and if there are the inclusions $\mathbf{cof}_{\mathcal{K}}(I) \star \gamma \subset \mathbf{cof}_{\mathcal{K}}(I)$ and $\mathbf{cof}_{\mathcal{K}}(I) \star \gamma^k \subset \mathbf{cof}_{\mathcal{K}}(I)$ for $k = 0, 1$ where $\gamma = \gamma^0 \sqcup \gamma^1$.

In this paper, all cylinders will be of the form the binary product by a bipointed object $\gamma^0, \gamma^1 : \mathbf{1} \rightrightarrows V$ called the *interval object*. The natural transformations $\gamma^k : \text{Id} \Rightarrow \text{Cyl}$ are equal to the natural transformations $(\mathbf{1} \times -) \Rightarrow (V \times -)$ induced by the two maps $\gamma^k : \mathbf{1} \rightarrow V$ for $k = 0, 1$. The natural transformation $\sigma : \text{Cyl} \Rightarrow \text{Id}$ is equal to the natural transformation $(V \times -) \Rightarrow (\mathbf{1} \times -)$ induced by the map $\sigma : V \rightarrow \mathbf{1}$ which is the unique map from the interval object to the terminal object. An interval object will be good (very good, cartesian resp.) if and only if the corresponding cylinder functor is good (very good, cartesian resp.).

2.4. Notation. Let I and S be two sets of maps of a locally presentable category \mathcal{K} . Let V be a good interval object with respect to I . Denote the sets of maps $\Lambda_{\mathcal{K}}^n(V, S, I)$ recursively by $\Lambda_{\mathcal{K}}^0(V, S, I) = S \cup (I \star \gamma^0) \cup (I \star \gamma^1)$

and $\Lambda_{\mathcal{K}}^{n+1}(V, S, I) = \Lambda_{\mathcal{K}}^n(V, S, I) \star \gamma$ for $n \geq 0$. Then let

$$\Lambda_{\mathcal{K}}(V, S, I) = \bigcup_{n \geq 0} \Lambda_{\mathcal{K}}^n(V, S, I).$$

Let \simeq be the homotopy relation associated with the cylinder $V \times -$, i.e., for all maps $f, g : X \rightarrow Y$, $f \simeq g$ is equivalent to the existence of a *homotopy* $H : V \times X \rightarrow Y$ with $H \circ \gamma^0 = f$ and $H \circ \gamma^1 = g$.

2.5. Notation. We denote by $\mathcal{W}_{\mathcal{K}}(V, S, I)$ the class of maps $f : X \rightarrow Y$ of \mathcal{K} such that for every $\Lambda_{\mathcal{K}}(V, S, I)$ -injective object T , the induced set map

$$\mathcal{K}(Y, T)/\simeq \xrightarrow{\cong} \mathcal{K}(X, T)/\simeq$$

is a bijection.

We are now ready to recall the Olschok construction for this particular setting:

2.6. Theorem (Olschok). *Let \mathcal{K} be a locally presentable category. Let I be a set of maps of \mathcal{K} . Let $S \subset \mathbf{cof}_{\mathcal{K}}(I)$ be an arbitrary set of maps of \mathcal{K} . Let V be a good cartesian interval object with respect to I . Suppose also that for any object X of \mathcal{K} , the canonical map $\emptyset \rightarrow X$ belongs to $\mathbf{cof}_{\mathcal{K}}(I)$. Then there exists a unique combinatorial model category structure with class of cofibrations $\mathbf{cof}_{\mathcal{K}}(I)$ such that the fibrant objects are the $\Lambda_{\mathcal{K}}(V, S, I)$ -injective objects. The class of weak equivalences is $\mathcal{W}_{\mathcal{K}}(V, S, I)$. All objects are cofibrant.*

Proof. Since all objects are cofibrant, the class of weak equivalences is necessarily $\mathcal{W}_{\mathcal{K}}(V, S, I)$ by [Hir03, Theorem 7.8.6]. Hence the uniqueness. The existence is a consequence of [Ols09a, Theorem 3.16]. \square

2.7. Notation. For $S = \emptyset$, the model category is just denoted by \mathcal{K} .

If the interval is very good in Theorem 2.6, then $\mathcal{W}_{\mathcal{K}}(V, S, I)$ is the localizer generated by S (with respect to the class of cofibrations $\mathbf{cof}_{\mathcal{K}}(I)$) by [Ols09a, Theorem 4.5] and \mathcal{K} is then left determined in the sense of [RT03]. And the model category we obtain for $S \neq \emptyset$ is the Bousfield localization $\underline{\mathbf{L}}_S(\mathcal{K})$ of the left determined one by the set of maps S . If the interval is only good, then the Olschok construction can only tell us that the model category we obtained is the Bousfield localization $\underline{\mathbf{L}}_{\Lambda_{\mathcal{K}}(V, S, I)}(\mathcal{K})$ with respect to $\Lambda_{\mathcal{K}}(V, S, I)$ because, by [Ols09a, Lemma 4.4], the class of maps $\mathcal{W}_{\mathcal{K}}(V, S, I)$ is the localizer generated by $\Lambda_{\mathcal{K}}(V, S, I)$.

3. Cubical transition systems

3.1. Notation. A nonempty set of labels Σ is fixed.

3.2. Definition. A weak higher-dimensional transition system, or weak HDTS, consists of a triple

$$\left(S, \mu : L \rightarrow \Sigma, T = \bigcup_{n \geq 1} T_n \right)$$

where S is a set of states, where L is a set of actions, where $\mu : L \rightarrow \Sigma$ is a set map called the labelling map, and finally where $T_n \subset S \times L^n \times S$ for $n \geq 1$ is a set of n -transitions or n -dimensional transitions such that one has:

- (Multiset axiom) For every permutation σ of $\{1, \dots, n\}$ with $n \geq 2$, if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$ is a transition as well.
- (Composition axiom) For every $(n+2)$ -tuple $(\alpha, u_1, \dots, u_n, \beta)$ with $n \geq 3$, for every $p, q \geq 1$ with $p+q < n$, if the five tuples

$$(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \nu_1), (\nu_1, u_{p+1}, \dots, u_n, \beta), \\ (\alpha, u_1, \dots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \dots, u_n, \beta)$$

are transitions, then the $(q+2)$ -tuple $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$ is a transition as well.

A map of weak higher-dimensional transition systems

$$f : (S, \mu : L \rightarrow \Sigma, (T_n)_{n \geq 1}) \rightarrow (S', \mu' : L' \rightarrow \Sigma, (T'_n)_{n \geq 1})$$

consists of a set map $f_0 : S \rightarrow S'$, a commutative square

$$\begin{array}{ccc} L & \xrightarrow{\mu} & \Sigma \\ \tilde{f} \downarrow & & \parallel \\ L' & \xrightarrow{\mu'} & \Sigma \end{array}$$

such that if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then

$$(f_0(\alpha), \tilde{f}(u_1), \dots, \tilde{f}(u_n), f_0(\beta))$$

is a transition. The corresponding category is denoted by **WHDTS**. The n -transition $(\alpha, u_1, \dots, u_n, \beta)$ is also called a transition from α to β .

3.3. Notation. The labelling map from the set of actions to the set of labels will be very often denoted by μ .

The category **WHDTS** is locally finitely presentable by [Gau10b, Theorem 3.4]. The functor

$$\omega : \mathbf{WHDTS} \longrightarrow \mathbf{Set}^{\{s\} \cup \Sigma}$$

taking the weak higher-dimensional transition system $(S, \mu : L \rightarrow \Sigma, (T_n)_{n \geq 1})$ to the $(\{s\} \cup \Sigma)$ -tuple of sets $(S, (\mu^{-1}(x))_{x \in \Sigma}) \in \mathbf{Set}^{\{s\} \cup \Sigma}$ is topological by [Gau10b, Theorem 3.4] too.

There is a slight change in the terminology with respect to the one of [Gau10b] and [Gau11]. The *Coherence axiom* is called now the *Composition axiom* because this axiom really looks like a 5-ary composition even if it is not known what conclusion should be drawn from such an observation.

3.4. Notation. For $n \geq 1$, let $0_n = (0, \dots, 0)$ (n -times) and $1_n = (1, \dots, 1)$ (n -times). By convention, let $0_0 = 1_0 = ()$.

We give now some important examples of weak HDTS. In each of the following examples, the Multiset axiom and the Composition axiom are satisfied for trivial reasons.

- (1) Let $x_1, \dots, x_n \in \Sigma$, $n \geq 0$. The *pure n -transition* $C_n[x_1, \dots, x_n]^{\text{ext}}$ is the weak HDTS with the set of states $\{0_n, 1_n\}$, with the set of actions $\{(x_1, 1), \dots, (x_n, n)\}$ and with the transitions all $(n+2)$ -tuples

$$(0_n, (x_{\sigma(1)}, \sigma(1)), \dots, (x_{\sigma(n)}, \sigma(n)), 1_n)$$

for σ running over the set of permutations of the set $\{1, \dots, n\}$.

- (2) Every set X may be identified with the weak HDTS having the set of states X , with no actions and no transitions.
- (3) For every $x \in \Sigma$, let us denote by \underline{x} the weak HDTS with no states, one action x , and no transitions. Warning: the weak HDTS $\{x\}$ contains one state x and no actions whereas the weak HDTS \underline{x} contains no states and one action x .

The following example plays a special role in the theory:

3.5. Notation. For every $x \in \Sigma$, let us denote by $\uparrow x \uparrow$ the weak HDTS with four states $\{1, 2, 3, 4\}$, one action x and two transitions $(1, x, 2)$ and $(3, x, 4)$.

Another important example is the one of the n -cube which is recalled now.

3.6. Proposition ([Gau10b, Proposition 5.2]). *Let $x_1, \dots, x_n \in \Sigma$, $n \geq 0$. Let $T_d \subset \{0, 1\}^n \times \{(x_1, 1), \dots, (x_n, n)\}^d \times \{0, 1\}^n$ (with $d \geq 1$) be the subset of $(d+2)$ -tuples*

$$((\epsilon_1, \dots, \epsilon_n), (x_{i_1}, i_1), \dots, (x_{i_d}, i_d), (\epsilon'_1, \dots, \epsilon'_n))$$

such that:

- $i_m = i_n$ implies $m = n$, i.e., there are no repetitions in the list

$$(x_{i_1}, i_1), \dots, (x_{i_d}, i_d).$$

- For all i , $\epsilon_i \leq \epsilon'_i$.
- $\epsilon_i \neq \epsilon'_i$ if and only if $i \in \{i_1, \dots, i_d\}$.

Let $\mu : \{(x_1, 1), \dots, (x_n, n)\} \rightarrow \Sigma$ be the set map defined by $\mu(x_i, i) = x_i$. Then

$$C_n[x_1, \dots, x_n] = (\{0, 1\}^n, \mu : \{(x_1, 1), \dots, (x_n, n)\} \rightarrow \Sigma, (T_d)_{d \geq 1})$$

is a well-defined weak HDTS called the n -cube.

For $n = 0$, $C_0[]$, also denoted by C_0 , is nothing else but the weak HDTS $(\{\emptyset\}, \mu : \emptyset \rightarrow \Sigma, \emptyset)$. For every $x \in \Sigma$, one has $C_1[x] = C_1[x]^{\text{ext}}$.

3.7. Definition. Let $n \geq 1$ and $x_1, \dots, x_n \in \Sigma$. Let $\partial C_n[x_1, \dots, x_n]$ be the weak HDTS defined by removing from its set of transitions all n -transitions. It is called the boundary of $C_n[x_1, \dots, x_n]$.

We restricted our attention in [Gau11] to the so-called cubical transition systems, i.e., the weak HDTS which are equal to the union of their subcubes. These weak HDTS include all useful examples.

3.8. Definition. Let X be a weak HDTS. A cube of X is a map

$$C_n[x_1, \dots, x_n] \longrightarrow X.$$

A subcube of X is the image of a cube of X . A weak HDTS is a cubical transition system if it is equal to the union of its subcubes. The full subcategory of cubical transition systems is denoted by **CTS**.

Note that the weak HDTS $\partial C_2[x_1, x_2]$ is not a colimit of cubes but is cubical (see [Gau11, Corollary 3.12] and the discussion after it): it is obtained by identifying states in the cubical transition system $\uparrow x_1 \uparrow \sqcup \uparrow x_2 \uparrow$. This is the reason why we do not work in [Gau11] with the subcategory of *colimits* of cubes.

CTS is a small-injectivity class, and a full *coreflective* locally finitely presentable subcategory of **WHDTS** by [Gau11, Corollary 3.15]. More precisely, a weak HDTS is cubical if and only if it is injective with respect to the maps $\underline{x} \subset C_1[x]$ for all $x \in \Sigma$ and to the maps

$$C_n[x_1, \dots, x_n]^{\text{ext}} \subset C_n[x_1, \dots, x_n]$$

for all $n \geq 0$ and $x_1, \dots, x_n \in \Sigma$ by [Gau11, Theorem 3.6].

3.9. Definition. Let X be a weak HDTS. An action u of X is used if there exists a 1-transition (α, u, β) . All actions are used if X is injective with respect to the maps $\underline{x} \subset C_1[x]$ for all $x \in \Sigma$.

3.10. Definition. A weak HDTS X satisfies the Intermediate state axiom if for every $n \geq 2$, every p with $1 \leq p < n$ and every transition $(\alpha, u_1, \dots, u_n, \beta)$ of X , there exists a (not necessarily unique) state ν such that both $(\alpha, u_1, \dots, u_p, \nu)$ and $(\nu, u_{p+1}, \dots, u_n, \beta)$ are transitions.

By [Gau11, Proposition 6.6], a weak HDTS satisfies the Intermediate state axiom if and only if it is injective with respect to the maps

$$C_n[x_1, \dots, x_n]^{\text{ext}} \subset C_n[x_1, \dots, x_n]$$

for all $n \geq 0$ and $x_1, \dots, x_n \in \Sigma$. So a weak HDTS is cubical if and only if all actions are used and it satisfies the Intermediate state axiom.

4. The homotopy theory of cubical transition systems

Let us recall now the homotopy theory of **CTS**. This third paper about higher-dimensional transition systems contains some improvements in the exposition of this theory. In particular, a set of generating cofibrations can now be exhibited (in [Gau11], the existence of a set of generating cofibrations is proved using transfinite techniques).

4.1. Definition. A cofibration of cubical transition systems is a map of weak HDTS inducing an injection between the set of actions.

To make the reading of this paper easier, let us introduce a new notation (which will be used later in Proposition 7.8).

4.2. Notation. Let $\mathbb{S} : \mathbf{Set} \downarrow \Sigma \rightarrow \mathbf{CTS}$ be the functor given on objects as follows: if $\mu : L \rightarrow \Sigma$ is a set map then $\mathbb{S}(\mu)$ is the weak HDTS with set of states $\{0\}$, with labelling map μ , and with set of transitions

$$\{0\} \times \bigcup_{n \geq 1} L^n \times \{0\}.$$

Note here that $\mathbb{S}(\mu)$ is a cubical transition system because all actions are used and the Intermediate state axiom is satisfied.

4.3. Definition. Let us call $V := \mathbb{S}(\Sigma \times \{0, 1\} \rightarrow \Sigma)$ the interval object of **CTS** where $\Sigma \times \{0, 1\} \rightarrow \Sigma$ is the projection map.

Note that $\mathbb{S}(\text{Id}_\Sigma)$ is the terminal object $\mathbf{1}$ of **CTS**. For $k \in \{0, 1\}$, denote by $\gamma^k : \mathbf{1} \rightarrow V$ the map of cubical transition systems induced by the composite set map $\Sigma \cong \Sigma \times \{k\} \subset \Sigma \times \{0, 1\}$. And denote by $\sigma : V \rightarrow \mathbf{1}$ the canonical map, also induced by the projection map $\Sigma \times \{0, 1\} \rightarrow \Sigma$. Let $\gamma = \gamma^0 \sqcup \gamma^1 : \mathbf{1} \sqcup \mathbf{1} \rightarrow V$.

The interval V is exponential by [Gau11, Proposition 5.8]. It is very good by [Gau11, Proposition 5.7] and cartesian by [Gau11, Proposition 5.10]. We are going to use the following fact which is already implicitly present in [Gau10b] and [Gau11].

4.4. Proposition. *Let $f : A \rightarrow B$ be a map of weak HDTS which is bijective on states and actions. Then it is one-to-one on transitions.*

Proof. Let $(\alpha, u_1, \dots, u_m, \beta)$ and $(\alpha', u'_1, \dots, u'_{m'}, \beta')$ be two transitions of A such that

$$(f_0(\alpha), \tilde{f}(u_1), \dots, \tilde{f}(u_m), f_0(\beta)) = (f_0(\alpha'), \tilde{f}(u'_1), \dots, \tilde{f}(u'_{m'}), f_0(\beta')).$$

Then $m = m'$, $f_0(\alpha) = f_0(\alpha')$, $\tilde{f}(u_i) = \tilde{f}(u'_i)$ for $1 \leq i \leq n$ and $f_0(\beta) = f_0(\beta')$. So by hypothesis, $\alpha = \alpha'$, $\beta = \beta'$ and $u_i = u'_i$ for $1 \leq i \leq n$. Hence

$$(\alpha, u_1, \dots, u_m, \beta) = (\alpha', u'_1, \dots, u'_{m'}, \beta'). \quad \square$$

$$\left\{ \begin{array}{c} C_1[x] \sqcup C_1[x] \\ \xrightarrow{x_1} \\ \xrightarrow{x_2} \end{array} \right\} \xrightarrow{p_x} \left\{ \begin{array}{c} \varinjlim(C_1[x] \xleftarrow{x} \rightarrow C_1[x]) \\ \xrightarrow{x} \\ \xrightarrow{x} \end{array} \right\}$$

FIGURE 3. Monomorphism and epimorphism in **CTS** with $\mu(x_1) = \mu(x_2) = x$

4.5. Notation. Let \mathcal{I} be the set of maps of cubical transition systems given by⁴

$$\begin{aligned} \mathcal{I} = & \{C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\}\} \\ & \cup \{\partial C_n[x_1, \dots, x_n] \rightarrow C_n[x_1, \dots, x_n] \mid n \geq 1 \text{ and } x_1, \dots, x_n \in \Sigma\} \\ & \cup \{C_1[x] \rightarrow \uparrow x \uparrow \mid x \in \Sigma\}. \end{aligned}$$

Let us recall again that the cubical HDTS $\uparrow x \uparrow$ is not a colimit of cubes by [Gau11, After Definition 3.13 and before Remark 3.14]. The colimit of all cubes (left-hand cubical transition system of Figure 3) of $\uparrow x \uparrow$ (right-hand cubical transition system of Figure 3) is equal to the coproduct of two copies of $C_1[x]$ and it has two actions x_1 and x_2 with $\mu(x_1) = \mu(x_2) = x$ whereas $\uparrow x \uparrow$ has only one action x . Figure 3 is an example of a map of **CTS** which is not an isomorphism, but which is a mono and an epi. Hence the additional family of cofibrations $C_1[x] \rightarrow \uparrow x \uparrow$ for x running over Σ cannot be deduced from the other generating cofibrations.

4.6. Theorem. *The class of cofibrations of cubical transition systems is equal to*

$$\mathbf{cell}_{\mathbf{CTS}}(\mathcal{I}) = \mathbf{cof}_{\mathbf{CTS}}(\mathcal{I}).$$

Proof. Let $f : A \rightarrow B$ be a cofibration of cubical transition systems, i.e., a map of cubical transition systems which is one-to-one on actions. Let us factor f as a composite $A \rightarrow Z \rightarrow B$ where the left-hand map belongs to $\mathbf{cell}_{\mathbf{CTS}}(\{C, R\})$ and where the right-hand map belongs to $\mathbf{inj}_{\mathbf{CTS}}(\{C, R\})$. Then the sets of states of Z and B coincide, therefore we can suppose without lack of generality that f induces a bijection between the sets of states. For every action u of B which does not belong to A , the map $\underline{u} \rightarrow B$ factors (not in a unique way) as a composite $\underline{\mu(u)} \rightarrow C_1[\underline{\mu(u)}] \rightarrow B$ because B is cubical. The map $f : A \rightarrow B$ is bijective on states, therefore the composite $C_0 \sqcup C_0 \cong \partial C_1[\underline{\mu(u)}] \subset C_1[\underline{\mu(u)}] \rightarrow B$ factors as a composite $C_0 \sqcup C_0 \rightarrow A \rightarrow B$. Then for every action u of B not in A , there exists a commutative

⁴ The notations $C : \emptyset \rightarrow \{0\}$ and $R : \{0, 1\} \rightarrow \{0\}$ are already used in [Gau03] and in [Gau09] for the same generating cofibrations (in different categories of course). We will stick to this notation here, and for the model category of labelled symmetric precubical sets as well.

square

$$\begin{array}{ccc}
 C_0 \sqcup C_0 \cong \partial C_1[\mu(u)] & \xrightarrow{g_u} & A \\
 \downarrow f_u & & \downarrow \\
 C_1[\mu(u)] & \longrightarrow & B.
 \end{array}$$

Then consider the pushout diagram

$$\begin{array}{ccc}
 \bigsqcup_u \partial C_1[\mu(u)] & \xrightarrow{g_u} & A \\
 \downarrow f_u & & \downarrow \\
 \bigsqcup_u C_1[\mu(u)] & \longrightarrow & Z.
 \end{array}$$

The canonical map $Z \rightarrow B$ induced by the pushout is bijective both on states and on actions, and by Proposition 4.4, injective on transitions. Let us now factor the map $Z \rightarrow B$ as a composite $Z \rightarrow D \rightarrow B$ where the left-hand map belongs to

$$\begin{aligned}
 \mathbf{cell}_{\mathbf{CTS}}(\{ & \partial C_n[x_1, \dots, x_n] \rightarrow C_n[x_1, \dots, x_n] \mid n \geq 2 \text{ and } x_1, \dots, x_n \in \Sigma \} \\
 & \cup \{ C_0 \sqcup C_0 \sqcup C_1[x] \rightarrow \uparrow x \uparrow \mid x \in \Sigma \})
 \end{aligned}$$

and the right-hand map belongs to

$$\begin{aligned}
 \mathbf{inj}_{\mathbf{CTS}}(\{ & \partial C_n[x_1, \dots, x_n] \rightarrow C_n[x_1, \dots, x_n] \mid n \geq 2 \text{ and } x_1, \dots, x_n \in \Sigma \} \\
 & \cup \{ C_0 \sqcup C_0 \sqcup C_1[x] \rightarrow \uparrow x \uparrow \mid x \in \Sigma \}).
 \end{aligned}$$

where the map $C_0 \sqcup C_0 \sqcup C_1[x] \rightarrow \uparrow x \uparrow$ is defined so that it is bijective on states. It is important to notice that the maps $\partial C_n[x_1, \dots, x_n] \rightarrow C_n[x_1, \dots, x_n]$ for every $n \geq 2$ and every $x_1, \dots, x_n \in \Sigma$ and the maps $C_0 \sqcup C_0 \sqcup C_1[x] \rightarrow \uparrow x \uparrow$ for every $x \in \Sigma$ are bijective on states and actions. Therefore the two maps of cubical transition systems $Z \rightarrow D$ and $D \rightarrow B$ are bijective on states and actions, and by Proposition 4.4 injective on transitions.

Let (α, u, β) be a 1-transition of B . Then u is an action of B and therefore of D as well and α and β are two states of D . Since D is cubical, there exists a 1-transition (α', u, β') of D giving rise to a map $C_1[\mu(u)] \rightarrow D$. Then

consider the commutative diagram

$$\begin{array}{ccc}
 \{\alpha\} \sqcup \{\beta\} \sqcup C_1[\mu(u)] & \xrightarrow{\quad} & D \\
 \downarrow & \nearrow k & \downarrow \\
 \uparrow \mu(u) \uparrow & \xrightarrow{\quad} & B.
 \end{array}$$

The existence of the lift k implies that the transition (α, u, β) belongs to D , hence the map $D \rightarrow B$ is onto on 1-transitions. Let us prove by induction on $n \geq 1$ that the map $D \rightarrow B$ is onto on p -transitions for $p \leq n$.

Let $(\alpha, u_1, \dots, u_{n+1}, \beta)$ be a $(n+1)$ -transition of B , giving rise to a map

$$C_{n+1}^{\text{ext}}[\mu(u_1), \dots, \mu(u_{n+1})] \rightarrow B,$$

which factors as a composite

$$C_{n+1}^{\text{ext}}[\mu(u_1), \dots, \mu(u_{n+1})] \rightarrow C_{n+1}[\mu(u_1), \dots, \mu(u_{n+1})] \rightarrow B$$

because B is cubical. The composite

$$\partial C_{n+1}[\mu(u_1), \dots, \mu(u_{n+1})] \subset C_{n+1}[\mu(u_1), \dots, \mu(u_{n+1})] \rightarrow B$$

factors uniquely as a composite $\partial C_{n+1}[\mu(u_1), \dots, \mu(u_{n+1})] \rightarrow D \rightarrow B$ by the induction hypothesis. We obtain a commutative diagram of cubical transition systems

$$\begin{array}{ccc}
 \partial C_{n+1}[\mu(u_1), \dots, \mu(u_{n+1})] & \xrightarrow{\quad} & D \\
 \downarrow & \nearrow k & \downarrow \\
 C_{n+1}[\mu(u_1), \dots, \mu(u_{n+1})] & \xrightarrow{\quad} & B.
 \end{array}$$

The existence of the lift k implies that the transition $(\alpha, u_1, \dots, u_{n+1}, \beta)$ belongs to D , hence the map $D \rightarrow B$ is onto on $(n+1)$ -transitions. So we obtain $D \cong B$.

The map $C_0 \sqcup C_0 \sqcup C_1[x] \rightarrow \uparrow x \uparrow$ is the composite

$$C_0 \sqcup C_0 \sqcup C_1[x] \rightarrow C_0 \sqcup C_0 \sqcup \uparrow x \uparrow \rightarrow \uparrow x \uparrow$$

where the left-hand map is a pushout of the generating cofibration

$$C_1[x] \rightarrow \uparrow x \uparrow$$

and where the right-hand map is a pushout of the generating cofibration $R : \{0, 1\} \rightarrow \{0\}$ twice. So $\mathbf{cell}_{\mathbf{CTS}}(\mathcal{I})$ is the class of cofibrations. Therefore $\mathbf{cell}_{\mathbf{CTS}}(\mathcal{I})$ is closed under retract and $\mathbf{cell}_{\mathbf{CTS}}(\mathcal{I}) = \mathbf{cof}_{\mathbf{CTS}}(\mathcal{I})$. \square

4.7. Corollary. *Let S be an arbitrary set of maps in \mathbf{CTS} . There exists a unique combinatorial model category structure on \mathbf{CTS} such that \mathcal{I} is the set of generating cofibrations and such that the fibrant objects are the $\Lambda_{\mathbf{CTS}}(V, S^{\text{cof}}, \mathcal{I})$ -injective objects where S^{cof} is a set of cofibrant replacements of the maps of S . All objects are cofibrant. The class of weak equivalences is the localizer generated by S .*

Proof. This corollary is a consequence of Theorem 4.6 and Theorem 2.6. \square

5. Labelled symmetric precubical sets

Let $[n] = \{0, 1\}^n$ for $n \geq 0$. The unique member of the singleton set $[0]$ is denoted by $()$. The set $[n]$ is equipped with the partial ordering $\{0 < 1\}^n$. Let $\delta_i^\alpha : [n-1] \rightarrow [n]$ be the set map defined for $1 \leq i \leq n$ and $\alpha \in \{0, 1\}$ by $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$. These maps are called the *face maps*. Let $\sigma_i : [n] \rightarrow [n]$ be the set map defined for $1 \leq i \leq n-1$ and $n \geq 2$ by $\sigma_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \epsilon_i, \epsilon_{i+2}, \dots, \epsilon_n)$. These maps are called the *symmetry maps*. The subcategory of \mathbf{Set} generated by the composites of face maps and symmetry maps is denoted by \square_S . A presentation by generators and relations of \square_S is given in [GM03, Section 6]: they consist of the usual cocubical relations, together with the Moore relations for symmetry operators and an additional family of relations relating the face operators and the symmetry operators. It will not be used in this paper.

5.1. Definition ([GM03]). A symmetric precubical set is a presheaf over \square_S . The corresponding category is denoted by $\square_S^{\text{op}}\mathbf{Set}$. If K is a symmetric precubical set, then let $K_n := K([n])$ and for every set map $f : [m] \rightarrow [n]$ of \square_S , denote by $f^* : K_n \rightarrow K_m$ the corresponding set map.

Let $\square_S[n] := \square_S(-, [n])$. It is called the *n -dimensional (symmetric) cube*. By the Yoneda lemma, one has the natural bijection of sets

$$\square_S^{\text{op}}\mathbf{Set}(\square_S[n], K) \cong K_n$$

for every precubical set K . The *boundary* of $\square_S[n]$ is the symmetric precubical set denoted by $\partial\square_S[n]$ defined by removing the interior of

$$\square_S[n] : (\partial\square_S[n])_k := (\square_S[n])_k$$

for $k < n$ and $(\partial\square_S[n])_k = \emptyset$ for $k \geq n$. In particular, one has $\partial\square_S[0] = \emptyset$. An *n -dimensional symmetric precubical set* K is a symmetric precubical set such that $K_p = \emptyset$ for $p > n$ and $K_n \neq \emptyset$. If K is a symmetric precubical set, then $K_{\leq n}$ is the symmetric precubical set given by $(K_{\leq n})_p = K_p$ for $p \leq n$ and $(K_{\leq n})_p = \emptyset$ for $p > n$.

5.2. Notation. Let $f : K \rightarrow L$ be a morphism of symmetric precubical sets. Let $n \geq 0$. The set map from K_n to L_n induced by f will be sometimes denoted by f_n .

5.3. Notation. Let $\partial_i^\alpha = (\delta_i^\alpha)^*$. And let $s_i = (\sigma_i)^*$.

The precubical nerve of any topological space can be endowed with such a structure: the s_i maps are given by permuting the coordinates: see [GM03] again.

5.4. Proposition ([Gau10a, Proposition A.4]). *The following data define a symmetric precubical set denoted by $!^S\Sigma$:*

- $(!^S\Sigma)_0 = \{()\}$ (the empty word).
- For $n \geq 1$, $(!^S\Sigma)_n = \Sigma^n$.
- $\partial_i^0(a_1, \dots, a_n) = \partial_i^1(a_1, \dots, a_n) = (a_1, \dots, \widehat{a}_i, \dots, a_n)$ where the notation \widehat{a}_i means that a_i is removed.
- $s_i(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n)$ for $1 \leq i \leq n$.

The map $!^S : \mathbf{Set} \rightarrow \square_S^{\text{op}} \mathbf{Set}$ yields a well-defined functor from the category of sets to the category of symmetric precubical sets.

5.5. Definition. A labelled symmetric precubical set (over Σ) is an object of the comma category $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S\Sigma$. That is, an object is a map of symmetric precubical sets $\ell : K \rightarrow !^S\Sigma$ and a morphism is a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\quad} & L \\ & \searrow & \swarrow \\ & & !^S\Sigma. \end{array}$$

The map ℓ is called the labelling map. The symmetric precubical set K is sometimes called the underlying symmetric precubical set of the labelled symmetric precubical set. A labelled symmetric precubical set $K \rightarrow !^S\Sigma$ will be denoted by $(K//\Sigma)$. And the set of n -cubes K_n will be also denoted by $(K//\Sigma)_n$.

The link between labelled symmetric precubical sets and process algebra is detailed in [Gau08] and in the appendix of [Gau10a].

5.6. Notation. Let $n \geq 1$. Let a_1, \dots, a_n be labels of Σ . Let us denote by $\square_S[a_1, \dots, a_n] : \square_S[n] \rightarrow !^S\Sigma$ the labelled symmetric precubical set corresponding by the Yoneda lemma to the n -cube (a_1, \dots, a_n) . And let us denote by $\partial \square_S[a_1, \dots, a_n] : \partial \square_S[n] \rightarrow !^S\Sigma$ the labelled symmetric precubical set defined as the composite

$$\partial \square_S[a_1, \dots, a_n] : \partial \square_S[n] \subset \square_S[n] \xrightarrow{\square_S[a_1, \dots, a_n]} !^S\Sigma.$$

Every set can be identified with a sum of 0-cubes $C_0[]$ (also denoted by C_0).

Since colimits are calculated objectwise for presheaves, the n -cubes are finitely accessible. Since the set of cubes is a dense (and hence strong) generator, the category of labelled symmetric precubical sets is locally finitely presentable by [AR94, Theorem 1.20 and Proposition 1.57]. When the set of labels Σ is the singleton $\{\tau\}$, the category $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S\{\tau\}$ is isomorphic to the category of (unlabelled) symmetric precubical sets because $!^S\{\tau\}$ is the terminal symmetric precubical set.

6. The homotopy theory of labelled symmetric precubical sets

This section is devoted to the construction of a model category structure on the category $\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma$ of labelled symmetric precubical sets. Note that if Σ is a singleton, then the category is isomorphic to the category of unlabelled symmetric precubical sets, and what follows applies as well.

6.1. Definition. The interval object of $\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma$ is the labelled symmetric precubical set $(!^S(\Sigma \times \{0, 1\})//\Sigma)$ induced by the projection map $\Sigma \times \{0, 1\} \rightarrow \Sigma$. Let

$$\text{Cyl}(K//\Sigma) = (!^S(\Sigma \times \{0, 1\})//\Sigma) \times (K//\Sigma).$$

Note that $\text{Id}_{!^S\Sigma} : (!^S\Sigma//\Sigma)$ is the terminal object $\mathbf{1}$ of $\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma$. For $k \in \{0, 1\}$, denote by $\gamma^k : (!^S\Sigma//\Sigma) \rightarrow (!^S(\Sigma \times \{0, 1\})//\Sigma)$ the map of labelled symmetric precubical sets induced by the composite set map $\Sigma \cong \Sigma \times \{k\} \subset \Sigma \times \{0, 1\}$. And denote by $\sigma : (!^S(\Sigma \times \{0, 1\})//\Sigma) \rightarrow (!^S\Sigma//\Sigma)$ the canonical map, also induced by the projection map $\Sigma \times \{0, 1\} \rightarrow \Sigma$. Let $\gamma = \gamma^0 \sqcup \gamma^1 : (!^S\Sigma//\Sigma) \sqcup (!^S\Sigma//\Sigma) \rightarrow (!^S(\Sigma \times \{0, 1\})//\Sigma)$.

If $(K//\Sigma)$ and $(L//\Sigma)$ are two labelled symmetric precubical sets, then their binary product in $\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma$ is the labelled symmetric precubical set $(K \times_{!^S\Sigma} L//\Sigma)$.

6.2. Proposition. *The interval object $(!^S(\Sigma \times \{0, 1\})//\Sigma)$ is exponential.*

Proof. Let $(K//\Sigma)$ be a labelled symmetric precubical set. Then

$$(\text{Cyl}(K//\Sigma))_n = K_n \times_{\Sigma^n} (\Sigma \times \{0, 1\})^n \cong K_n \times \{0, 1\}^n,$$

i.e., $(!^S(\Sigma \times \{0, 1\})//\Sigma) \times (K//\Sigma) \cong ((K_* \times \{0, 1\}^*)//\Sigma)$ with an obvious definition of the face and symmetry maps. So the associated cylinder functor $(!^S(\Sigma \times \{0, 1\})//\Sigma) \times -$ is colimit-preserving because the category of sets is cartesian-closed and because colimits are calculated objectwise in the category $\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma$. Since $\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma$ is well-copowered by [AR94, Theorem 1.58], the cylinder is a left adjoint by the dual of the Special Adjoint Functor Theorem [ML98]. Hence the interval object is exponential. \square

6.3. Definition. A map of labelled symmetric precubical sets

$$f : (K//\Sigma) \longrightarrow (L//\Sigma)$$

is a cofibration if for every $n \geq 1$, the set map $K_n \longrightarrow L_n$ is one-to-one.

6.4. Proposition. *The class of cofibrations is generated by the set*

$$\begin{aligned} \mathcal{I} = \{ & \partial \square_S[a_1, \dots, a_n] \subset \square_S[a_1, \dots, a_n] \mid n \geq 1 \text{ and } a_1, \dots, a_n \in \Sigma \} \\ & \cup \{ C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\} \}, \end{aligned}$$

i.e., the class of cofibrations is exactly $\mathbf{cof}_{\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma}(\mathcal{I})$. Moreover, one has

$$\mathbf{cell}_{\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma}(\mathcal{I}) = \mathbf{cof}_{\square_S^{\text{op}}\mathbf{Set}\downarrow!^S\Sigma}(\mathcal{I}).$$

Proof. This kind of proof is standard. Let $f : (K//\Sigma) \rightarrow (L//\Sigma)$ be a cofibration of labelled symmetric precubical sets. Let

$$\mathcal{I}_0 = \{C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\}\},$$

and for $n \geq 1$, let

$$\mathcal{I}_n = \{\partial \square_S[a_1, \dots, a_n] \subset \square_S[a_1, \dots, a_n] \mid a_1, \dots, a_n \in \Sigma\}.$$

Let $f = f_0$. Factor f_0 as a composite $(K//\Sigma) \rightarrow (K^1//\Sigma) \xrightarrow{f^1} (L//\Sigma)$ where the left-hand map belongs to $\mathbf{cell}_{\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma}(\mathcal{I}_0)$ and where the right-hand map belongs to $\mathbf{inj}_{\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma}(\mathcal{I}_0)$. Then f^1 is bijective on 0-cubes and by hypothesis is one-to-one on n -cubes with $n \geq 1$. Let us suppose $f^n : (K^n//\Sigma) \rightarrow (L//\Sigma)$ constructed for $n \geq 1$ and let us suppose that it is bijective on k -cubes for $k < n$ and one-to-one on k -cubes for $k \geq n$. Consider the pushout diagram of labelled symmetric precubical sets

$$\begin{array}{ccc} \bigsqcup \partial \square_S[a_1, \dots, a_n] & \longrightarrow & (K^n//\Sigma) \\ \downarrow & & \downarrow \\ \bigsqcup \square_S[a_1, \dots, a_n] & \longrightarrow & (K^{n+1}//\Sigma) \end{array}$$

where the sum is over all commutative squares of the form

$$\begin{array}{ccc} \partial \square_S[a_1, \dots, a_n] & \longrightarrow & (K^n//\Sigma) \\ \downarrow & & \downarrow \\ \square_S[a_1, \dots, a_n] & \longrightarrow & (L//\Sigma). \end{array}$$

Then the map $f^{n+1} : (K^{n+1}//\Sigma) \rightarrow (L//\Sigma)$ is bijective on k -cubes for $k < n + 1$ and one-to-one on k -cubes for $k \geq n + 1$. Hence $f = \varinjlim f_n$. \square

6.5. Remark. The sets of generating cofibrations of $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$ and \mathbf{CTS} are both denoted by \mathcal{I} . The context will always enable the reader to avoid any confusion.

6.6. Proposition. *The codiagonal $(!^S \Sigma // \Sigma) \sqcup (!^S \Sigma // \Sigma) \rightarrow (!^S \Sigma // \Sigma)$ factors as a composite*

$$(!^S \Sigma // \Sigma) \sqcup (!^S \Sigma // \Sigma) \rightarrow (!^S(\Sigma \times \{0, 1\}) // \Sigma) \rightarrow (!^S \Sigma // \Sigma)$$

such that the left-hand map induces a cofibration

$$(K//\Sigma) \sqcup (K//\Sigma) \rightarrow \text{Cyl}(K//\Sigma)$$

for any labelled symmetric precubical set $(K//\Sigma)$. In other terms, the interval object $(!^S(\Sigma \times \{0, 1\}) // \Sigma)$ is good.

Proof. The left-hand map is induced by the two inclusions

$$\Sigma \cong \Sigma \times \{\epsilon\} \subset \Sigma \times \{0, 1\}$$

with $\epsilon = 0, 1$. The right-hand map is induced by the projection

$$\Sigma \times \{0, 1\} \longrightarrow \Sigma.$$

For $n \geq 1$, the left-hand map induces on the sets of n -cubes the one-to-one set map $(\Sigma \times \{0\})^n \sqcup (\Sigma \times \{1\})^n \subset (\Sigma \times \{0, 1\})^n$. So for any labelled symmetric precubical set $(K//\Sigma)$, and any $n \geq 1$, the map

$$(K//\Sigma) \sqcup (K//\Sigma) \rightarrow \text{Cyl}(K//\Sigma)$$

induces on the sets of n -cubes the one-to-one set map

$$(K_n \times \{0\}^n) \sqcup (K_n \times \{1\}^n) \subset (K_n \times \{0, 1\})^n.$$

Note that the set map $(!^S \Sigma \sqcup !^S \Sigma)_0 \longrightarrow (!^S(\Sigma \times \{0, 1\}))_0$ is not one-to-one because it is isomorphic to the set map $R : \{0, 1\} \rightarrow \{0\}$. \square

The interval object $(!^S(\Sigma \times \{0, 1\})//\Sigma)$ is not very good. It is easy to prove that the right-hand map satisfies the right lifting property with respect to all generating cofibrations except the cofibrations $\partial \square_S[x, y] \rightarrow \square_S[x, y]$ for $x, y \in \Sigma$. Indeed, in the commutative square of solid arrows of labelled symmetric precubical sets

$$\begin{array}{ccc} \partial \square_S[x, y] & \xrightarrow{g} & !^S(\Sigma \times \{0, 1\}) \\ \downarrow \subset & \nearrow k & \downarrow \\ \square_S[x, y] & \xrightarrow{\quad} & !^S \Sigma \end{array}$$

the lift k exists if and only if two opposite faces of $\partial \square_S[x, y]$ are labelled by g in $!^S(\Sigma \times \{0, 1\})$ by the same element of $\Sigma \times \{0, 1\}$.

6.7. Proposition. *For every cofibration $f : (K//\Sigma) \rightarrow (L//\Sigma)$ of labelled symmetric precubical sets, the maps $f \star \gamma$ and $f \star \gamma^\epsilon$ for $\epsilon = 0, 1$ are cofibrations as well. In other terms, the interval object $(!^S(\Sigma \times \{0, 1\})//\Sigma)$ is cartesian.*

Proof. The map

$$f \star \gamma : ((L//\Sigma) \sqcup (L//\Sigma)) \sqcup_{(K//\Sigma) \sqcup (K//\Sigma)} \text{Cyl}(K//\Sigma) \rightarrow \text{Cyl}(L//\Sigma)$$

is a cofibration because for $n \geq 1$, the set map

$$(f \star \gamma)_n : (L_n \sqcup L_n) \sqcup_{K_n \sqcup K_n} (K_n \times \{0, 1\}^n) \rightarrow L_n \times \{0, 1\}^n$$

is one-to-one. Indeed, it consists of the inclusions

$$K_n \subset K_n \times \{0\}^n \subset L_n \times \{0\}^n \subset L_n \times \{0, 1\}^n$$

and $K_n \subset K_n \times \{1\}^n \subset L_n \times \{1\}^n \subset L_n \times \{0, 1\}^n$. The map

$$f \star \gamma^\epsilon : (L//\Sigma) \sqcup_{(K//\Sigma)} \text{Cyl}(K//\Sigma) \rightarrow \text{Cyl}(L//\Sigma)$$

with $\epsilon \in \{0, 1\}$ is a cofibration because for $n \geq 1$, the map (with K_n embedded in $K_n \times \{\epsilon\}^n$ and L_n embedded in $L_n \times \{\epsilon\}^n$)

$$(f \star \gamma^\epsilon)_n : L_n \sqcup_{K_n} (K_n \times \{0, 1\}^n) \longrightarrow L_n \times \{0, 1\}^n$$

is one-to-one. Indeed, it consists of the inclusions

$$K_n \subset K_n \times \{\epsilon\}^n \subset L_n \times \{\epsilon\}^n \subset L_n \times \{0, 1\}^n. \quad \square$$

Hence the theorem:

6.8. Theorem. *There exists a unique combinatorial model category structure on $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$ such that the class of cofibrations is generated by \mathcal{I} and such that the fibrant objects are the $\Lambda_{\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma}(!^S(\Sigma \times \{0, 1\})//\Sigma), \emptyset, \mathcal{I}$ -injective objects. All objects are cofibrant.*

7. Realizing labelled precubical sets as cubical transition systems

We want now to recall the construction of the realization functor from labelled symmetric precubical sets to cubical transition systems, as expounded in Section 8 and Section 9 of [Gau10b]. In the same way as for the exposition of the homotopy theory of cubical transition systems, this third paper of the series contains an improvement. We also explain in this section how to make this functor a Quillen functor by adding one generating cofibration to the model category of cubical transition systems.

7.1. Notation. $\text{CUBE}(\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma)$ is the full subcategory of that of labelled symmetric precubical sets containing the labelled cubes $\square_S[a_1, \dots, a_n]$ with $n \geq 0$ and $a_1, \dots, a_n \in \Sigma$.

7.2. Notation. $\text{CUBE}(\mathbf{WHDTs})$ is the full subcategory of \mathbf{WHDTs} containing the labelled cubes $C_n[a_1, \dots, a_n]$ with $n \geq 0$ and with $a_1, \dots, a_n \in \Sigma$.

The following theorem is new and is an improvement of [Gau10b, Theorem 8.5].

7.3. Theorem. *There exists one and only one functor*

$$\mathbb{T} : \text{CUBE}(\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma) \longrightarrow \text{CUBE}(\mathbf{WHDTs})$$

such that $\mathbb{T}(\square_S[a_1, \dots, a_n]) := C_n[a_1, \dots, a_n]$ for all $a_1, \dots, a_n \in \Sigma$, $n \geq 0$, and such that for any map $f : \square_S[a_1, \dots, a_m] \rightarrow \square_S[b_1, \dots, b_n]$ of labelled symmetric precubical sets, the map $\{0, 1\}^m \rightarrow \{0, 1\}^n$ between the sets of states induced by $\mathbb{T}(f)$ is the map induced by f between the sets of 0-cubes. Moreover this functor yields an isomorphism of categories

$$\text{CUBE}(\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma) \cong \text{CUBE}(\mathbf{WHDTs}).$$

Sketch of proof. Let $m, n \geq 0$ and $a_1, \dots, a_m, b_1, \dots, b_n \in \Sigma$. A map of labelled symmetric precubical sets $f : \square_S[a_1, \dots, a_m] \rightarrow \square_S[b_1, \dots, b_n]$ gives rise to a set map $f_0 : \{0, 1\}^m \rightarrow \{0, 1\}^n$ from the set of states of $C_m[a_1, \dots, a_m]$ to the set of states of $C_n[b_1, \dots, b_n]$ which belongs to $\square_S([m], [n]) = \square_S^{\text{op}} \mathbf{Set}(\square_S[m], \square_S[n])$. By [Gau10b, Lemma 8.1], there exists a unique set map $\widehat{f} : \{1, \dots, n\} \rightarrow \{1, \dots, m\} \cup \{-\infty, +\infty\}$ such that $f_0(\epsilon_1, \dots, \epsilon_m) = (\epsilon_{\widehat{f}(1)}, \dots, \epsilon_{\widehat{f}(n)})$ for every $(\epsilon_1, \dots, \epsilon_m) \in [m]$ with the conventions $\epsilon_{-\infty} = 0$ and $\epsilon_{+\infty} = 1$. Moreover, the restriction

$$\overline{f} : \widehat{f}^{-1}(\{1, \dots, m\}) \rightarrow \{1, \dots, m\}$$

is a bijection. Since $f : \square_S[a_1, \dots, a_m] \rightarrow \square_S[b_1, \dots, b_n]$ is compatible with the labelling, one necessarily has $a_i = b_{\overline{f}^{-1}(i)}$ for every $i \in \{1, \dots, m\}$. One deduces a set map $\widetilde{f} : \{(a_1, 1), \dots, (a_m, m)\} \rightarrow \{(b_1, 1), \dots, (b_n, n)\}$ from the set of actions of $C_m[a_1, \dots, a_m]$ to the set of actions of $C_n[b_1, \dots, b_n]$ by setting $\widetilde{f}(a_i, i) = (b_{\overline{f}^{-1}(i)}, \overline{f}^{-1}(i)) = (a_i, \overline{f}^{-1}(i))$. By [Gau10b, Lemma 8.2], if $f : \square_S[a_1, \dots, a_m] \rightarrow \square_S[b_1, \dots, b_n]$ and $g : \square_S[b_1, \dots, b_n] \rightarrow \square_S[c_1, \dots, c_p]$ are two maps of labelled symmetric precubical sets, then one has $\widehat{g \circ f} = \widehat{f} \circ \widehat{g}$. And by [Gau10b, Lemma 8.3], the two set maps f_0 and \widetilde{f} above defined by starting from a map of labelled symmetric precubical sets

$$f : \square_S[a_1, \dots, a_m] \rightarrow \square_S[b_1, \dots, b_n]$$

yield a map of weak higher-dimensional transition systems

$$\mathbb{T}(f) : C_m[a_1, \dots, a_m] \rightarrow C_n[b_1, \dots, b_n].$$

Hence the proof is complete with [Gau10b, Proposition 8.4] and [Gau10b, Theorem 8.5]. \square

7.4. Theorem ([Gau10b, Theorem 9.2]). *There exists a unique colimit-preserving functor*

$$\mathbb{T} : \square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma \rightarrow \mathbf{WHDTs}$$

extending the functor \mathbb{T} previously constructed on the full subcategory of labelled cubes. Moreover, this functor is a left adjoint.

One has $\mathbb{T}(\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma) \subset \mathbf{CTS}$ since every colimit of cubes is cubical by [Gau11, Theorem 3.11] (see also [Gau10b, Proposition 9.8]). But we have the surprising negative result:

7.5. Proposition. *The restriction $\mathbb{T} : \square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma \rightarrow \mathbf{CTS}$ is not a left Quillen functor.*

Proof. Consider the cofibration $\partial \square_S[x, y] \subset \square_S[x, y]$ where x and y are two elements of Σ . The map of cubical transition systems

$$\mathbb{T}(\partial \square_S[x, y] \subset \square_S[x, y])$$

is not a cofibration of cubical transition systems because the set of actions of $\mathbb{T}(\partial \square_S[x, y])$ is the set with four elements $\{x_1, x_2, y_1, y_2\}$ with $\mu(x_1) =$

$\mu(x_2) = x$ and $\mu(y_1) = \mu(y_2) = y$ and the set of actions of $\mathbb{T}(\square_S[x, y])$ is the set with two elements $\{x, y\}$. \square

To overcome the problem, we are going to add the maps

$$\mathbb{T}(\partial \square_S[x, y] \subset \square_S[x, y])$$

with x, y running over Σ to the set of generating cofibrations of **CTS**. Surprisingly, the Olschok construction can be used again with the same interval object.

7.6. Notation. Let $\mathcal{I}^+ = \mathcal{I} \cup \{\mathbb{T}(\partial \square_S[x, y] \subset \square_S[x, y]) \mid x, y \in \Sigma\}$.

7.7. Proposition. *For any labelled symmetric precubical set $(K//\Sigma)$, one has*

$$\mathbb{T}(!^S(\Sigma \times \{0, 1\})//\Sigma \times (K//\Sigma)) = V \times \mathbb{T}((K//\Sigma)).$$

Proof. One can suppose without loss of generality that $K = \square_S[x_1, \dots, x_n]$ for $n \geq 0$ and $x_1, \dots, x_n \in \Sigma$ because all involved functors are colimit-preserving. For $n = 0$, the two members of the equality are isomorphic to the 1-state cubical transition system $\{()\}$. Now suppose that $n \geq 1$. The sets of states of the two members of the equality are equal to $\{0, 1\}^n$. Since the two weak HDTS are cubical, all actions are used. So it suffices to check that the two members of the equality have the same set of transitions by using the fact that a cubical transition system is the union of its subcubes by Definition 3.8.

By the Yoneda lemma, the p -cubes of $(K//\Sigma)$ for $p \geq 1$ are in bijection with the maps of labelled symmetric precubical sets

$$\square_S[x_{\phi(1)}, \dots, x_{\phi(p)}] \rightarrow (K//\Sigma)$$

where $\phi : \{1, \dots, p\} \rightarrow \{1, \dots, n\}$ is a one-to-one map. The image of the p -transition

$$(0_p, (x_{\phi(1)}, \phi(1)), \dots, (x_{\phi(p)}, \phi(p)), 1_p)$$

of $\mathbb{T}(\square_S[x_{\phi(1)}, \dots, x_{\phi(p)}]) = C_p[x_{\phi(1)}, \dots, x_{\phi(p)}]$ is a p -transition of the form⁵

$$((\alpha_1, \dots, \alpha_n), (x_{\phi(1)}, \phi(1)), \dots, (x_{\phi(p)}, \phi(p)), (\beta_1, \dots, \beta_n))$$

where $\alpha_i \leq \beta_i$ for all $i \in \{1, \dots, n\}$ with equality if and only if i does not belong to the image of ϕ .

Similarly, using the calculation in the proof of Proposition 6.2, the p -cubes of $(!^S(\Sigma \times \{0, 1\})//\Sigma \times (K//\Sigma))$ for $p \geq 1$ are in bijection with the maps of labelled symmetric precubical sets

$$\square_S[(x_{\phi(1)}, \epsilon_1), \dots, (x_{\phi(p)}, \epsilon_p)] \rightarrow (!^S(\Sigma \times \{0, 1\})//\Sigma \times (K//\Sigma))$$

where $\phi : \{1, \dots, p\} \rightarrow \{1, \dots, n\}$ is a one-to-one map and where $\epsilon_1, \dots, \epsilon_p \in \{0, 1\}$. The image of the p -transition

$$(0_p, (x_{\phi(1)}, \phi(1), \epsilon_1), \dots, (x_{\phi(p)}, \phi(p), \epsilon_p), 1_p)$$

⁵Remember that a p -cube $C_p[u_1, \dots, u_p]$ has the set of actions $\{(u_1, 1), \dots, (u_p, p)\}$.

of $\mathbb{T}(\square_S[(x_{\phi(1)}, \epsilon_1), \dots, (x_{\phi(p)}, \epsilon_p)]) = C_p[(x_{\phi(1)}, \epsilon_1), \dots, (x_{\phi(p)}, \epsilon_p)]$ is a p -transition of the form

$$((\alpha_1, \dots, \alpha_n), (x_{\phi(1)}, \phi(1), \epsilon_1), \dots, (x_{\phi(p)}, \phi(p), \epsilon_p), (\beta_1, \dots, \beta_n))$$

where $\alpha_i \leq \beta_i$ for all $i \in \{1, \dots, n\}$ with equality if and only if i does not belong to the image of ϕ . Since $(!^S(\Sigma \times \{0, 1\})//\Sigma) \times (K//\Sigma)$ is cubical, it is equal to the union of its subcubes by Definition 3.8. So all transitions of $(!^S(\Sigma \times \{0, 1\})//\Sigma) \times (K//\Sigma)$ are of the form above.

For similar reasons, the set of transitions of the cubical transition system

$$\mathbb{T}((K//\Sigma))$$

consists of the tuples of the form

$$((\alpha_1, \dots, \alpha_n), (x_{\phi(1)}, \phi(1)), \dots, (x_{\phi(p)}, \phi(p)), (\beta_1, \dots, \beta_n))$$

with $\phi : \{1, \dots, p\} \rightarrow \{1, \dots, n\}$ one-to-one and where $\alpha_i \leq \beta_i$ for all $i \in \{1, \dots, n\}$ with equality if and only if i does not belong to the image of ϕ . Hence the set of transitions of $V \times \mathbb{T}((K//\Sigma))$ is equal to the one of $\mathbb{T}(!^S(\Sigma \times \{0, 1\})//\Sigma) \times (K//\Sigma)$ by [Gau11, Proposition 5.5] (see also the description of the cylinder in the proof of [Gau11, Proposition 5.8]) and the proof is complete. \square

7.8. Proposition. *The functor $!^S : \mathbf{Set} \rightarrow \square_S^{\text{op}} \mathbf{Set}$ of Proposition 5.4 induces a well-defined functor from $\mathbf{Set} \downarrow \Sigma$ to $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$. And one has the equality of functors*

$$\mathbb{T} \circ !^S = \mathbb{S} : \mathbf{Set} \downarrow \Sigma \rightarrow \mathbf{CTS}.$$

Proof. Let $\mu : L \rightarrow \Sigma$ be a set map. The two cubical transition systems $(\mathbb{T} \circ !^S)(L \rightarrow \Sigma)$ and $\mathbb{S}(L \rightarrow \Sigma)$ have, for any set map $L \rightarrow \Sigma$, the same set of states $\{0\}$ and the same set of actions L . So, by definition of a cubical transition system, the set of transitions of $(\mathbb{T} \circ !^S)(L \rightarrow \Sigma)$ is a subset of $\{0\} \times \bigcup_{n \geq 1} L^n \times \{0\}$. The latter set turns out to be the set of transitions of $!^S(L \rightarrow \Sigma)$ by definition of $!^S$. Therefore the identity maps of $\{0\}$ and of L induce a well-defined map of cubical transition systems

$$(\mathbb{T} \circ !^S)(L \rightarrow \Sigma) \rightarrow \mathbb{S}(L \rightarrow \Sigma)$$

which is bijective on states and actions, and one-to-one on transitions by Proposition 4.4. To complete the proof, note that any tuple $(0, u_1, \dots, u_n, 0)$ is a transition of the image by \mathbb{T} of the n -cube (u_1, \dots, u_n) of $!^S(L \rightarrow \Sigma)$. Therefore the map $(\mathbb{T} \circ !^S)(L \rightarrow \Sigma) \rightarrow \mathbb{S}(L \rightarrow \Sigma)$ is surjective on transitions. \square

7.9. Proposition. *One has $\mathbb{T}(!^S(\Sigma \times \{0, 1\})//\Sigma) = V$. Moreover, for $k = 0, 1$, the image by \mathbb{T} of the map of labelled symmetric precubical sets $\gamma^k : \mathbf{1} \rightarrow (!^S(\Sigma \times \{0, 1\})//\Sigma)$ induced by $\Sigma \cong \Sigma \times \{k\} \subset \Sigma \times \{0, 1\}$ is the map of cubical transition systems $\mathbf{1} \rightarrow V$ induced by $\Sigma \cong \Sigma \times \{k\} \subset \Sigma \times \{0, 1\}$.*

Proof. The equality $\mathbb{T}(!^S(\Sigma \times \{0, 1\})//\Sigma) = V$ comes from Proposition 7.8 applied to the projection map $\Sigma \times \{0, 1\} \rightarrow \Sigma$. The last part about γ^k is a corollary of Proposition 7.8 applied to the set map

$$\Sigma \cong \Sigma \times \{k\} \subset \Sigma \times \{0, 1\}. \quad \square$$

We can now introduce the “augmented” model category structure of cubical transition systems.

7.10. Theorem. *There exists a unique combinatorial model category structure on \mathbf{CTS} such that the set of maps \mathcal{I}^+ is the set of generating cofibrations and such that the fibrant objects are the $\Lambda_{\mathbf{CTS}}(V, \emptyset, \mathcal{I}^+)$ -injective objects. All objects are cofibrant. This model category will be denoted by \mathbf{CTS}^+ .*

Proof. The interval object V is still good⁶ with respect to the maps of \mathcal{I}^+ . We already know that the interval object V is exponential. To prove that it is cartesian with respect to \mathcal{I}^+ , we just have to prove that for any $x, y \in \Sigma$, $\mathbb{T}(f_{x,y}) \star \gamma^k$ and $\mathbb{T}(f_{x,y}) \star \gamma$ belong to $\mathbf{cof}_{\mathbf{CTS}}(\mathcal{I}^+)$ for $k = 0, 1$ and where $f_{x,y} : \partial \square_S[x, y] \subset \square_S[x, y]$ is the inclusion. By Proposition 6.7, $f_{x,y} \star \gamma^k$ and $f_{x,y} \star \gamma$ are cofibrations of $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$. By Proposition 7.7 and Proposition 7.9, $\mathbb{T}(f_{x,y} \star \gamma^k) = \mathbb{T}(f_{x,y}) \star \gamma^k$ and $\mathbb{T}(f_{x,y} \star \gamma) = \mathbb{T}(f_{x,y}) \star \gamma$, hence the interval object V is cartesian with respect to \mathcal{I}^+ . All cubical transition systems are still cofibrant for this new class of cofibrations. Hence the result by Theorem 2.6. \square

7.11. Definition. A cubical transition system satisfies CSA1 (the First Cattani–Sassone axiom) if for every transition (α, u, β) and (α, u', β) such that the actions u and u' have the same label in Σ , one has $u = u'$.

The full subcategory of cubical transition systems satisfying CSA1 is reflective by [Gau10b, Corollary 5.7]. The reflection is denoted by

$$\mathbf{CSA}_1 : \mathbf{CTS} \rightarrow \mathbf{CTS}.$$

7.12. Proposition. *Let T be a cubical transition system satisfying CSA1. Let I and S be arbitrary sets of maps of \mathbf{CTS} . Then T is $\Lambda_{\mathbf{CTS}}(V, S, I)$ -injective if and only if it is S -orthogonal.*

⁶The interval object V is not very good with respect to the maps of \mathcal{I}^+ because there is no lift k in the following diagram with $x, y \in \Sigma$ if the set of actions of $\mathbb{T}(\partial \square_S[x, y])$ is taken to be $\{(x, 0), (x, 1), (y, 0), (y, 1)\}$.

$$\begin{array}{ccc} \mathbb{T}(\partial \square_S[x, y]) & \longrightarrow & \mathbb{T}(!^S(\Sigma \times \{0, 1\})//\Sigma) \\ \downarrow & \nearrow k & \\ \mathbb{T}(\square_S[x, y]) & & \end{array}$$

The lift k would exist if and only if two opposite faces of the empty square $\mathbb{T}(\partial \square_S[x, y])$ were labelled by the same action.

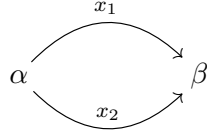


FIGURE 4. Cubical transition system $\text{Cyl}(C_1[x])$ homotopy equivalent to the 1-cube $C_1[x]$ with $\mu(x_1) = \mu(x_2) = x$.

Proof. In the case in which $I = \mathcal{I}$, the set of generating cofibrations for **CTS**, this is [Gau11, Proposition 7.7], and the same proof works for arbitrary sets. \square

7.13. Proposition. *Every cubical transition system satisfying CSA1 is fibrant both in **CTS** and in **CTS**⁺.*

Proof. The first part is [Gau11, Proposition 7.8]. Every cubical transition system satisfying CSA1 is \emptyset -orthogonal, so fibrant in **CTS**⁺ as well by Proposition 7.12 and Theorem 7.10. \square

7.14. Proposition. *For every cubical transition system X , the unit*

$$X \rightarrow \text{CSA}_1(X)$$

*is a weak equivalence of both **CTS** and **CTS**⁺.*

Proof. The argument of the proof of [Gau11, Theorem 7.10] must be used: the unit $X \rightarrow \text{CSA}_1(X)$ is a transfinite composition of pushouts of maps of the form $\sigma_{C_1[x]} : \text{Cyl}(C_1[x]) \rightarrow C_1[x]$ (the source is depicted in Figure 4) with $x \in \Sigma$. Consider such a pushout:

$$\begin{array}{ccc}
 \text{Cyl}(C_1[x]) & \xrightarrow{\phi} & X \\
 \sigma_{C_1[x]} \downarrow & & \downarrow f \\
 C_1[x] & \xrightarrow{\quad} & Y
 \end{array}$$

The map $\sigma_{C_1[x]}$ is *never* a cofibration of course. The point is that ϕ is either a cofibration of cubical transition systems or it takes the two actions of $\text{Cyl}(C_1[x])$ to the same action of X : in the first case, f is a weak equivalence in **CTS** and **CTS**⁺ because of the left properness; in the second case, f is just an isomorphism. \square

7.15. Theorem. *The two model categories **CTS** and **CTS**⁺ have the same class of weak equivalences: the class of maps of cubical transition systems f such that $\text{CSA}_1(f)$ is an isomorphism.*

Proof. Two maps $f, g : X \rightarrow Y$ with Y satisfying CSA1 are homotopic with respect to the cylinder $V \times -$ if and only if they are equal by [Gau11, Proposition 7.4]. Therefore two weakly equivalent cubical transition systems in \mathbf{CTS} or \mathbf{CTS}^+ satisfying CSA1 are isomorphic: this is [Gau11, Proposition 7.9]. The proof is complete with Proposition 7.14. \square

7.16. Proposition. *The functor $\mathbb{T} : \square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma \rightarrow \mathbf{CTS}$ preserves weak equivalences.*

Proof. Let $f : (K//\Sigma) \rightarrow (L//\Sigma)$ be a weak equivalence of labelled symmetric precubical sets. Let T be a fibrant object of the left determined model category structure of \mathbf{CTS} . We have to prove that the set map $\mathbf{CTS}(\mathbb{T}((L//\Sigma)), T) / \simeq \rightarrow \mathbf{CTS}(\mathbb{T}((K//\Sigma)), T) / \simeq$ induced by f is a bijection of sets where \simeq means the homotopy relation induced by the cylinder of \mathbf{CTS} . By adjunction, we have to check that the set map

$$(\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma)((L//\Sigma), \mathbb{R}(T)) / \simeq \longrightarrow (\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma)((K//\Sigma), \mathbb{R}(T)) / \simeq$$

induced by f is a bijection of sets where \mathbb{R} is the right adjoint to \mathbb{T} . Since f is a weak equivalence of labelled symmetric precubical sets, it suffices to check that $\mathbb{R}(T)$ is $\Lambda_{\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma}(!^S(\Sigma \times \{0, 1\})//\Sigma, \emptyset, \mathcal{I})$ -injective. By Proposition 7.7 and Proposition 7.9, this is equivalent to T being $\Lambda_{\mathbf{CTS}}(V, \emptyset, \mathcal{I})$ -injective, which holds because T is fibrant. \square

7.17. Corollary. *There exists a zig-zag of left Quillen functors*

$$\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma \xrightarrow{\mathbb{T}} \mathbf{CTS}^+ \xleftarrow{\text{Id}_{\mathbf{CTS}}} \mathbf{CTS}.$$

Moreover, the right-hand left Quillen functor induces a Quillen equivalence $\mathbf{CTS} \simeq \mathbf{CTS}^+$.

Proof. The functor $\mathbb{T} : \square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma \rightarrow \mathbf{CTS}^+$ is a left Quillen functor by Proposition 7.15 and Proposition 7.16 and by definition of the cofibrations of \mathbf{CTS}^+ . The functor $\text{CSA}_1 : \mathbf{CTS} \rightarrow \mathbf{CTS}^+$ is a cofibrant-fibrant replacement for the two model categories \mathbf{CTS} and \mathbf{CTS}^+ by Proposition 7.13 and by Proposition 7.14. They have the same class of weak equivalences so the left Quillen functor $\mathbf{CTS} \rightarrow \mathbf{CTS}^+$ induces an adjoint equivalence of categories between the homotopy categories. \square

8. Homotopical property of the realization functor

We are now ready to compare labelled symmetric precubical sets and cubical transition systems from a homotopy point of view.

8.1. Notation. Let $\mathcal{S} = \{p_x : C_1[x] \sqcup C_1[x] \rightarrow \uparrow x \uparrow \mid x \in \Sigma\}$. Let $\mathcal{S}^{\text{cof}} = \{p_x^{\text{cof}} \mid x \in \Sigma\}$ where $(-)^{\text{cof}}$ is a cofibrant replacement in \mathbf{CTS} .

Let X be a cubical transition system. Let us factor in \mathbf{CTS} the canonical map $X \rightarrow \mathbf{1}$ as a composite $X \rightarrow \underline{\mathbf{L}}_{\mathcal{S}}(X) \rightarrow \mathbf{1}$ where the left-hand map belongs to $\mathbf{cell}_{\mathbf{CTS}}(\mathcal{S})$ and the right-hand map belongs to $\mathbf{inj}_{\mathbf{CTS}}(\mathcal{S})$. The

functor $\underline{\mathbf{L}}_{\mathcal{S}} : \mathbf{CTS} \rightarrow \mathbf{CTS}$ is studied in [Gau11]. The next proposition explains the image of the functor $\underline{\mathbf{L}}_{\mathcal{S}}$ on objects.

8.2. Proposition. *For a cubical transition system*

$$X = \left(S, \mu : L \rightarrow \Sigma, T = \bigcup_{n \geq 1} T_n \right),$$

the following statements are equivalent:

- (1) *The labelling map μ is one-to-one.*
- (2) *X is \mathcal{S} -injective.*
- (3) *X is \mathcal{S} -orthogonal.*

If one of these statements is true, then X satisfies CSA1.

Proof. The equivalence (2) \Leftrightarrow (3) is due to the fact that all maps of \mathcal{S} are epimorphisms. We have to prove now that (1) \Leftrightarrow (2). Let us suppose (2). Let x_1 and x_2 be two actions of X with $\mu(x_1) = \mu(x_2) = x$. Since X is injective with respect to $\underline{x}_i \rightarrow C_1[x]$ for $i = 1, 2$, the two maps $\underline{x}_i \subset X$ factors as a composite $\underline{x}_i \rightarrow C_1[x] \rightarrow X$. Hence there is a map

$$C_1[x] \sqcup C_1[x] \rightarrow X$$

sending one action of the source to x_1 and the other one to x_2 . By hypothesis, X is p_x -injective. Therefore the latter map factors as a composite

$$C_1[x] \sqcup C_1[x] \rightarrow \uparrow x \uparrow \rightarrow X.$$

So $x_1 = x_2$, and μ is one-to-one. Conversely, suppose that μ is one-to-one. Let $C_1[x] \sqcup C_1[x] \rightarrow X$ be a map. Then the two actions of the source are taken to the same action of X because they have the same labelling. Therefore the map factors as a composite $C_1[x] \sqcup C_1[x] \rightarrow \uparrow x \uparrow \rightarrow X$. The last assertion is obvious. \square

So the functor $\underline{\mathbf{L}}_{\mathcal{S}} : \mathbf{CTS} \rightarrow \mathbf{CTS}$ induces a functor from \mathbf{CTS} to the full reflective subcategory \mathcal{S}^{\perp} of cubical transition systems consisting of \mathcal{S} -orthogonal objects. By [Gau11, Theorem 8.11], the functor $\underline{\mathbf{L}}_{\mathcal{S}}$ is left adjoint to the inclusion functor $\iota_{\mathcal{S}} : \mathcal{S}^{\perp} \subset \mathbf{CTS}$.

8.3. Proposition. *For every cubical transition system X , the cubical transition system $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is \mathcal{S}^{cof} -orthogonal.*

Proof. We want to describe explicitly a cofibrant replacement of

$$p_x : C_1[x] \sqcup C_1[x] \rightarrow \uparrow x \uparrow$$

in \mathbf{CTS} . The functor $V \times -$ is described in [Gau11, Proposition 5.5] and in [Gau11, Proposition 5.8]. The cubical transition system $V \times C_1[x]$ has the same state as $C_1[x]$ (one initial state α and one final state β), has two actions x_1 and x_2 labelled by x and two 1-transitions (α, x_1, β) and

(α, x_2, β) (Figure 4). A cofibrant replacement of p_x can then be obtained by considering the composite map

$$\begin{aligned} p_x^{\text{cof}} : C_1[x] \sqcup C_1[x] &\rightarrow (V \times C_1[x] \sqcup V \times C_1[x]) \\ &\rightarrow (V \times C_1[x] \sqcup V \times C_1[x]) / (x_2 = x_4 = x) \end{aligned}$$

where $\{x_1, x_2\}$ is the set of actions of the left-hand copy of $V \times C_1[x]$ and where $\{x_3, x_4\}$ is the set of actions of the right-hand copy of $V \times C_1[x]$. To completely determine this map, we must say that it induces a bijection on the set of states and it is the inclusion $\{x_1, x_3\} \subset \{x_1, x_3, x\}$ on actions with $\mu(x_1) = \mu(x_2) = \mu(x_3) = \mu(x_4) = x$. So p_x^{cof} is a cofibration. The target of p_x^{cof} is depicted in Figure 5. One has the equalities of cubical transition systems

$$\text{CSA}_1 [(V \times C_1[x] \sqcup V \times C_1[x]) / (x_2 = x_4 = x)] = \uparrow x \uparrow = \text{CSA}_1(\uparrow x \uparrow).$$

Therefore, using Theorem 7.15, the cubical transition systems

$$(V \times C_1[x] \sqcup V \times C_1[x]) / (x_2 = x_4 = x)$$

and $\uparrow x \uparrow$ are weakly equivalent in **CTS**. Thus, p_x^{cof} is a cofibrant replacement of p_x in **CTS**.

We now want to check that $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is p_x^{cof} -orthogonal for all $x \in \Sigma$ to complete the proof. Consider a commutative diagram of solid arrows:

$$\begin{array}{ccc} C_1[x] \sqcup C_1[x] & \xrightarrow{\quad\quad\quad} & \underline{\mathbf{L}}_{\mathcal{S}}(X) \\ \downarrow p_x^{\text{cof}} & \nearrow k & \downarrow \\ (V \times C_1[x] \sqcup V \times C_1[x]) / (x_2 = x_4 = x) & \xrightarrow{\quad\quad\quad} & \mathbf{1} \end{array}$$

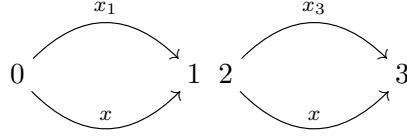
Remember by Proposition 8.2 that the labelling map of $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is one-to-one. The existence and uniqueness of the lift k is then clear on states (because p_x^{cof} is bijective on states) and on actions (k takes all actions of its source to x). \square

8.4. Proposition. *If X is a cubical transition system, then $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is fibrant in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ and in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$.*

Note that the argument given in [Gau11, Theorem 8.11] to prove the fibrancy of $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ is wrong: \mathcal{S} -orthogonality was used instead of \mathcal{S}^{cof} -orthogonality.

Proof. A cubical transition system of the form $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is \mathcal{S}^{cof} -orthogonal by Proposition 8.3 and satisfies CSA1 by Proposition 8.2. So it is fibrant in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ by Proposition 7.12 and by Corollary 4.7. The map

$$p_x^{\text{cof}} : C_1[x] \sqcup C_1[x] \rightarrow B$$


 FIGURE 5. $(V \times C_1[x] \sqcup V \times C_1[x]) / (x_2 = x_4 = x)$.

introduced in the proof of Proposition 8.3 is a cofibration of **CTS** such that $\text{CSA}_1(B) = \text{CSA}_1(\uparrow x \uparrow)$. So it is a cofibration of **CTS**⁺ as well since there are more cofibrations in **CTS**⁺ than in **CTS** and B is weakly equivalent to $\uparrow x \uparrow$ in **CTS**⁺ as well by Theorem 7.15. Hence p_x^{cof} is a cofibrant replacement of p_x in **CTS**⁺ as well and $\underline{\mathbf{L}}_{\mathcal{S}}(X)$ is fibrant in **CTS**⁺ as well. \square

8.5. Proposition. *For every cubical transition system X , the unit*

$$X \rightarrow \underline{\mathbf{L}}_{\mathcal{S}}(X)$$

is a weak equivalence of both $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ and $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$.

Proof. The map $X \rightarrow \underline{\mathbf{L}}_{\mathcal{S}}(X)$ is a transfinite composition of pushouts of maps of \mathcal{S} . Then we can use the argument [Gau11, Proposition 8.5]. Let us briefly recall it. Consider a pushout in **CTS** of the form (with $x \in \Sigma$)

$$\begin{array}{ccc} C_1[x] \sqcup C_1[x] & \xrightarrow{\phi} & X \\ \downarrow p_x & & \downarrow f \\ \uparrow x \uparrow & \xrightarrow{\quad} & Y \end{array}$$

The map p_x is *never* a cofibration of course. The point is that ϕ is either a cofibration or it takes the two actions of $C_1[x] \sqcup C_1[x]$ to the same action of X : in the first case, f is a weak equivalence in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ and $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$ because of the left properness; in the second case, f is just an isomorphism. \square

8.6. Theorem. *The two model categories $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ and $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$ have the same class of weak equivalences: the class of maps of cubical transition systems f such that $\underline{\mathbf{L}}_{\mathcal{S}}(f)$ is an isomorphism.*

Proof. By Proposition 8.4 and Proposition 8.5, the functor

$$\underline{\mathbf{L}}_{\mathcal{S}} : \mathbf{CTS} \rightarrow \mathbf{CTS}$$

is a cofibrant-fibrant replacement both in $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$ and $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$. A map f is a weak equivalence in the Bousfield localization if and only if $\underline{\mathbf{L}}_{\mathcal{S}}(f)$ is a weak equivalence of **CTS** (or of **CTS**⁺) by [Hir03, Theorem 3.2.13]. But

the source and target of $\underline{\mathbf{L}}_{\mathcal{S}}(f)$ satisfy CSA1 by Proposition 8.2, hence the conclusion using Theorem 7.15. \square

8.7. Corollary. *The left Quillen functor $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}) \rightarrow \underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$ induced by the identity functor is a Quillen equivalence.*

Proof. The left Quillen functor $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}) \rightarrow \underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$ induces an adjoint equivalence of categories between the homotopy categories. \square

Before stating the next theorem, we need to introduce again a few notations.

8.8. Definition. Let $X \in \mathbf{WHDTS}$. The cubification functor is the functor

$$\underline{\mathbf{Cub}} : \mathbf{WHDTS} \longrightarrow \mathbf{WHDTS}$$

defined by

$$\underline{\mathbf{Cub}}(X) = \varinjlim_{C_n[x_1, \dots, x_n] \rightarrow X} C_n[x_1, \dots, x_n],$$

the colimit being taken in \mathbf{CTS} (or equivalently in \mathbf{WHDTS}).

We can now prove one of the main result of this paper:

8.9. Theorem. *There exists a Bousfield localization of $\square_{\mathcal{S}}^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma$ which is Quillen equivalent to the Bousfield localization of the left determined model category structure of \mathbf{CTS} by the cubification functor.*

Proof. By [Gau11, Corollary 8.7], the Bousfield localization of the left determined model category structure of \mathbf{CTS} by the cubification functor is $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$. By Corollary 8.7 and [Dug01, Proposition 3.2], it suffices to prove that the left Quillen functor $\square_{\mathcal{S}}^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma \rightarrow \underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$ induced by \mathbb{T} is homotopically surjective in the sense of [Dug01, Definition 3.1].

The right adjoint $\mathbb{R} : \mathbf{CTS}^+ \rightarrow \square_{\mathcal{S}}^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma$ of $\mathbb{T} : \square_{\mathcal{S}}^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma \rightarrow \mathbf{CTS}^+$ may be defined as follows. Let X be a cubical transition system. The set $\mathbb{R}(X)$ of n -cubes labelled by $(a_1, \dots, a_n) \in \Sigma^n$ is the set of maps of cubical transition systems $C_n[a_1, \dots, a_n] \rightarrow X$ (so for $n = 0$, it is the set of states) with an obvious definition of the face maps and symmetry maps. Using the isomorphism of categories $\text{CUBE}(\square_{\mathcal{S}}^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma) \cong \text{CUBE}(\mathbf{WHDTS})$ by Theorem 7.3, one deduces that $\mathbb{T}(\mathbb{R}(X)) \cong \underline{\mathbf{Cub}}(X)$. Hence by [Gau11, Proposition 8.4] and by [Gau11, Proposition 8.5], for any cubical transition system X , the counit $\mathbb{T}(\mathbb{R}(X)) \rightarrow X$ is a weak equivalence of $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS})$, and therefore of $\underline{\mathbf{L}}_{\mathcal{S}}(\mathbf{CTS}^+)$ by Theorem 8.6.⁷ Since all cubical transition systems are cofibrant, we see that the last assertion is nothing else but the definition of homotopically surjective. \square

⁷[Gau11, Proposition 8.4] actually proves that the counit $\underline{\mathbf{Cub}}(X) \rightarrow X$ is a transfinite composition of pushouts of the maps $p_x : C_1[x] \sqcup C_1[x] \rightarrow \uparrow x \uparrow$ for $x \in \Sigma$; so $\underline{\mathbf{Cub}}(X) \rightarrow X$ is a weak equivalence in the Bousfield localizations by the same argument as for proving Proposition 8.5.

9. The higher-dimensional automata paradigm

If $K = \square_S[x_1, \dots, x_n] \sqcup_{\partial \square_S[x_1, \dots, x_n]} \square_S[x_1, \dots, x_n]$ with $x_1, \dots, x_n \in \Sigma$ and $n \geq 2$, then $\mathbb{R}(\underline{\mathbf{L}}_S(\mathbb{T}(K))) = \square_n[x_1, \dots, x_n]$ by Proposition 8.2 and [Gau10b, Proposition 9.3]. This suggests to recall now the HDA paradigm for a slight improvement of Theorem 8.9.

9.1. Definition ([Gau10b, Definition 7.1]). A labelled symmetric precubical set $(K//\Sigma)$ satisfies the paradigm of higher-dimensional automata (HDA paradigm) if for every $p \geq 2$, every commutative square of solid arrows (called a labelled p -shell or labelled p -dimensional shell)

$$\begin{array}{ccc}
 \partial \square_S[p] & \xrightarrow{\quad} & K \\
 \downarrow & \nearrow k & \downarrow \\
 \square_S[p] & \xrightarrow{\quad} & !^S \Sigma
 \end{array}$$

admits at most one lift k (i.e., a map k making the two triangles commutative).

The interest of the HDA paradigm in computer science is that it is satisfied by all real examples (see for example [Gau08, Theorem 5.2] and [Gau08, Corollary 5.3]). A full n -cube with $n \geq 2$ models the concurrent execution of n actions. An empty n -cube with $n \geq 2$ models the concurrent execution of $n - 1$ actions maximum among a set of n actions [Gau08]. It is impossible to have two n -cubes (for $n \geq 2$) with the same boundary. Either it is possible for the n actions to run concurrently (full), or there is an obstruction (empty).

Note that the HDA paradigm is automatically satisfied by higher dimensional transition systems because for $n \geq 2$, there is the isomorphism

$$C_n[x_1, \dots, x_n] \sqcup_{\partial C_n[x_1, \dots, x_n]} C_n[x_1, \dots, x_n] \cong C_n[x_1, \dots, x_n]$$

for all $x_1, \dots, x_n \in \Sigma$ by [Gau10b, Proposition 9.3].

By [Gau10b, Proposition 7.3], the HDA paradigm is equivalent to being orthogonal to the set of maps

$$\{\square_S[a_1, \dots, a_p] \sqcup_{\partial \square_S[a_1, \dots, a_p]} \square_S[a_1, \dots, a_p] \rightarrow \square_S[a_1, \dots, a_p]\}$$

for $p \geq 2$ and $a_1, \dots, a_p \in \Sigma$. So (see [Gau10b, Corollary 7.4]), the full subcategory, denoted by \mathbf{HDA}^Σ , of $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$ containing the objects satisfying the HDA paradigm is a full reflective locally presentable category of the category $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$ of labelled symmetric precubical sets. In other terms, the inclusion functor $i_\Sigma : \mathbf{HDA}^\Sigma \subset \square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$ has a left adjoint $\text{Sh}_\Sigma : \square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma \rightarrow \mathbf{HDA}^\Sigma$.

In fact the category \mathbf{HDA}^Σ is locally finitely presentable; indeed, the labelled n -cubes for $n \geq 0$ are in \mathbf{HDA}^Σ by [Gau10b, Proposition 7.2], and one can prove that they form a dense set of generators.

9.2. Notation. When Σ is the singleton $\{\tau\}$, let $i := i_\Sigma$ and $\text{Sh} := \text{Sh}_\Sigma$.

One has

$$i_\Sigma(K//\Sigma) \cong (i(K)//\Sigma) = (K//\Sigma)$$

and

$$\text{Sh}_\Sigma(K//\Sigma) \cong (\text{Sh}(K) \rightarrow \text{Sh}(!^S\Sigma) \cong !^S\Sigma)$$

because the symmetric precubical set $!^S\Sigma$ belongs to \mathbf{HDA} .

We want to restrict the homotopy theory of labelled symmetric precubical sets to the full reflective subcategory \mathbf{HDA}^Σ . So we must explain how to restrict the Olschok construction to a full reflective subcategory, at least within our particular setting.

9.3. Theorem ([Ols09a, Lemma 5.2] with some additional remarks). (*Restriction to a full reflective subcategory*) Let \mathcal{K} , I , S and V be as in Theorem 2.6. Let \mathcal{A} be a full reflective locally presentable subcategory with $V \in \mathcal{A}$ and $I \subset \text{Mor}(\mathcal{A})$. Let $R : \mathcal{K} \rightarrow \mathcal{A}$ be the reflection. Suppose that $S \subset \mathbf{cof}_{\mathcal{A}}(I)$ and that for every object $X \in \mathcal{A}$, $X^V \in \mathcal{A}$ where $(-)^V$ is the right adjoint in \mathcal{K} to the cylinder functor $\text{Cyl}(-) = V \times -$. Then there exists a unique combinatorial model category structure such that the class of cofibrations is generated by I and such that an object of \mathcal{A} is fibrant if and only if it is $\Lambda_{\mathcal{K}}(V, S, I)$ -injective. In particular, the fibrant objects of \mathcal{A} are the fibrant objects of \mathcal{K} belonging to \mathcal{A} . The reflection $R : \mathcal{K} \rightarrow \mathcal{A}$ is a homotopically surjective left Quillen adjoint. All objects are cofibrant.

Proof. The inclusion functor $\mathcal{A} \subset \mathcal{K}$ is a right adjoint, therefore it preserves binary products. And $V \in \mathcal{A}$ by hypothesis. Hence $V \times - : \mathcal{K} \rightarrow \mathcal{K}$ restricts to an endofunctor of \mathcal{A} .

Note that $I \subset \mathcal{A}$ implies that $I = R(I)$. One has

$$\mathbf{inj}_{\mathcal{A}}(I) = \text{Mor}(\mathcal{A}) \cap \mathbf{inj}_{\mathcal{K}}(I)$$

because \mathcal{A} is a full subcategory. So for all $f \in \mathbf{inj}_{\mathcal{A}}(I)$, $(\emptyset \rightarrow A) \square f$ for every object A of \mathcal{A} because $(\mathbf{cof}_{\mathcal{K}}(I), \mathbf{inj}_{\mathcal{K}}(I))$ is a cofibrant small weak factorization system, hence $(\mathbf{cof}_{\mathcal{A}}(I), \mathbf{inj}_{\mathcal{A}}(I))$ is a cofibrant small weak factorization system of \mathcal{A} .

Since R is colimit-preserving, $R(\mathbf{cell}_{\mathcal{K}}(I)) \subset \mathbf{cell}_{\mathcal{A}}(I)$. Every map of $\mathbf{cof}_{\mathcal{K}}(I)$ is a retract of a map of $\mathbf{cell}_{\mathcal{K}}(I)$, therefore $R(\mathbf{cof}_{\mathcal{K}}(I)) \subset \mathbf{cof}_{\mathcal{A}}(I)$. Let $f \in \text{Mor}(\mathcal{A}) \cap \mathbf{cof}_{\mathcal{K}}(I)$. Then $R(f) = f \in \mathbf{cof}_{\mathcal{A}}(I)$. One obtains the inclusion $\text{Mor}(\mathcal{A}) \cap \mathbf{cof}_{\mathcal{K}}(I) \subset \mathbf{cof}_{\mathcal{A}}(I)$. Let $A \in \mathcal{A}$. Then $A \sqcup A \rightarrow V \times A$ is a map of \mathcal{A} belonging to $\mathbf{cof}_{\mathcal{K}}(I)$ because V is good in \mathcal{K} . So V is good in \mathcal{A} as well.

Let $A, B \in \mathcal{A}$. Then $\mathcal{A}(V \times A, B) \cong \mathcal{K}(V \times A, B) \cong \mathcal{K}(A, B^V) \cong \mathcal{A}(A, B^V)$ because $B^V \in \mathcal{A}$ and because \mathcal{A} is a full subcategory of \mathcal{K} . So the interval

$$\begin{array}{ccc}
 \partial \square_S[a_1, \dots, a_p] & \xrightarrow{f} & (K//\Sigma)^{(!^S(\Sigma \times \{0,1\})//\Sigma)} \\
 \downarrow & \nearrow \bar{f} & \\
 \square_S[a_1, \dots, a_p] & &
 \end{array}$$

FIGURE 6. $(K//\Sigma)^{(!^S(\Sigma \times \{0,1\})//\Sigma)}$ must satisfy the HDA paradigm.

$$\begin{array}{ccc}
 \text{Cyl}(\partial \square_S[a_1, \dots, a_p]) & \xrightarrow{f} & K \\
 \downarrow & \nearrow \bar{f} & \\
 \text{Cyl}(\square_S[a_1, \dots, a_p]) & &
 \end{array}$$

FIGURE 7. HDA paradigm for $(K//\Sigma)^{(!^S(\Sigma \times \{0,1\})//\Sigma)}$ and adjunction.

object V is exponential in \mathcal{A} . Since R is colimit-preserving, one has

$$I \star \gamma \in R(\mathbf{cof}_{\mathcal{K}}(I)) \subset \mathbf{cof}_{\mathcal{A}}(I)$$

and $I \star \gamma^k \in R(\mathbf{cof}_{\mathcal{K}}(I)) \subset \mathbf{cof}_{\mathcal{A}}(I)$, hence V is cartesian in \mathcal{A} as well.

Let T be an object of \mathcal{A} . Then T is $\Lambda_{\mathcal{K}}(V, S, I)$ -injective, if and only if it is $\Lambda_{\mathcal{A}}(V, S, I)$ -injective because $R(\Lambda_{\mathcal{K}}(V, S, I)) = \Lambda_{\mathcal{A}}(V, S, I)$. So the fibrant objects of \mathcal{A} are the fibrant objects of \mathcal{K} belonging to \mathcal{A} . Hence the existence of the model category structure on \mathcal{A} by Theorem 2.6.

Since $R(\mathbf{cof}_{\mathcal{K}}(I)) \subset \mathbf{cof}_{\mathcal{A}}(I)$, the reflection R takes cofibrations to cofibrations. Let f be a weak equivalence of \mathcal{K} . Let T be a fibrant object of \mathcal{A} . Then $\mathcal{K}(f, T)/\simeq$ is a bijection. By adjunction, one gets that $\mathcal{A}(R(f), T)/\simeq$ is a bijection as well, i.e., that $R(f)$ is a weak equivalence of \mathcal{A} . Therefore the reflection $R : \mathcal{K} \rightarrow \mathcal{A}$ is a left Quillen adjoint. It is homotopically surjective because for any fibrant object $A \in \mathcal{A}$, A is cofibrant in \mathcal{K} and there is the isomorphism $R(A) \cong A$. \square

Denote by $(K//\Sigma) \mapsto (K//\Sigma)^{(!^S(\Sigma \times \{0,1\})//\Sigma)}$ the right adjoint to the cylinder functor of $\square_S^{\text{op}} \mathbf{Set} \downarrow !^S \Sigma$ which exists by Proposition 6.2.

9.4. Proposition. *For every labelled symmetric precubical set $(K//\Sigma)$, if $(K//\Sigma)$ satisfies the HDA paradigm, then $(K//\Sigma)^{(!^S(\Sigma \times \{0,1\})//\Sigma)}$ satisfies the HDA paradigm as well.*

Proof. We have to prove that there exists at most one lift k for every commutative diagram of the form of Figure 6 with $p \geq 2$ and $a_1, \dots, a_p \in \Sigma$. By adjunction, that means that we have to prove that there exists at most one list \bar{f} for every commutative diagram of the form of Figure 7.

One has

$$\begin{aligned} (\text{Cyl}(\partial \square_S[a_1, \dots, a_p]))_0 &= (\partial \square_S[a_1, \dots, a_p])_0 \cong (\square_S[a_1, \dots, a_p])_0 \\ &= (\text{Cyl}(\square_S[a_1, \dots, a_p]))_0 \end{aligned}$$

and

$$\begin{aligned} (\text{Cyl}(\partial \square_S[a_1, \dots, a_p]))_1 &= (\partial \square_S[a_1, \dots, a_p])_1 \times \{0, 1\} \\ &\cong (\square_S[a_1, \dots, a_p])_1 \times \{0, 1\} = (\text{Cyl}(\square_S[a_1, \dots, a_p]))_1. \end{aligned}$$

Therefore $\bar{f}_0 = f_0$ and $\bar{f}_1 = f_1$ exist and are unique.

The fact that $(K//\Sigma)$ satisfies the HDA paradigm is going to be used only now. Let $x \in (\text{Cyl}(\square_S[a_1, \dots, a_p]))_p$ with $p \geq 2$. Since

$$\text{Cyl}(\square_S[a_1, \dots, a_p])_p = (\square_S[a_1, \dots, a_p])_p \times \{0, 1\}^p$$

one sees that all $\bar{f}(\partial_i^\alpha x) = f(\partial_i^\alpha x)$ are determined and the proof is complete. \square

Hence the theorem:

9.5. Theorem. *There exists a unique combinatorial model category structure on \mathbf{HDA}^Σ such that the class of cofibrations is generated by \mathcal{I} and such that a fibrant object is a $\Lambda_{\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma}((!^S(\Sigma \times \{0, 1\})//\Sigma), \emptyset, \mathcal{I})$ -injective object of \mathbf{HDA}^Σ . All objects are cofibrant.*

Now we can state the last theorem:

9.6. Theorem. *There exists a Bousfield localization of \mathbf{HDA}^Σ which is Quillen equivalent to the Bousfield localization of the left determined model category structure of \mathbf{CTS} by the cubification functor.*

Proof. By [Gau10b, Theorem 9.5], the functor

$$\mathbb{T} : \square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma \rightarrow \mathbf{WHDTs}$$

factors uniquely (up to isomorphism of functors) as a composite

$$\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma \xrightarrow{\text{Sh}_\Sigma} \mathbf{HDA}^\Sigma \xrightarrow{\bar{\mathbb{T}}} \mathbf{WHDTs}.$$

Moreover, the functor $\bar{\mathbb{T}}$ is a left adjoint. And it is not a left Quillen adjoint with exactly the same proof as for Proposition 7.5. We work with the left Quillen functor $\bar{\mathbb{T}} : \mathbf{HDA}^\Sigma \rightarrow \mathbf{L}_S(\mathbf{CTS}^+)$ like in the proof of Theorem 8.9. We then just have to check that the functor $\bar{\mathbb{T}}$ is, like the functor \mathbb{T} , homotopically surjective.

Let X be a cubical transition system. Let $n \geq 2$ and $x_1, \dots, x_n \in \Sigma$. Then one has $(\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma)(\square_n[x_1, \dots, x_n], \mathbb{R}(X)) \cong \mathbf{CTS}(\mathbb{T}(\square_n[x_1, \dots, x_n]), X)$ by the adjunction $\mathbb{T} \dashv \mathbb{R}$, and

$$\begin{aligned} & \mathbf{CTS}(\mathbb{T}(\square_S[x_1, \dots, x_n]), X) \\ &= \mathbf{CTS}(\mathbb{T}(\square_S[x_1, \dots, x_n] \sqcup_{\partial \square_S[x_1, \dots, x_n]} \square_S[x_1, \dots, x_n]), X) \\ &\cong \mathbf{CTS}(\mathbb{T}(\square_S[x_1, \dots, x_n]), X) \sqcup_{\mathbf{CTS}(\mathbb{T}(\partial \square_S[x_1, \dots, x_n]), X)} \mathbf{CTS}(\mathbb{T}(\square_S[x_1, \dots, x_n]), X) \end{aligned}$$

by [Gau10b, Proposition 9.3], because the contravariant functor $\mathbf{CTS}(-, X)$ is limit-preserving. So by the adjunction $\mathbb{T} \dashv \mathbb{R}$ again and because the contravariant functor

$$(\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma)(-, X)$$

is limit-preserving, one obtains the bijection

$$\begin{aligned} & (\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma)(\square_n[x_1, \dots, x_n], \mathbb{R}(X)) \\ & \cong (\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma)(\square_S[x_1, \dots, x_n] \sqcup_{\partial \square_S[x_1, \dots, x_n]} \square_S[x_1, \dots, x_n], \mathbb{R}(X)), \end{aligned}$$

which means that $\mathbb{R}(X) \in \mathbf{HDA}^\Sigma$. Therefore

$$\overline{\mathbb{T}}(\mathbb{R}(X)) \cong \mathbb{T}(\mathbb{R}(X)) \cong \underline{\mathbf{Cub}}(X). \quad \square$$

The reflection $\text{Sh}_\Sigma : \square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma \rightarrow \mathbf{HDA}^\Sigma$ is a homotopically surjective left Quillen functor by Theorem 9.3. So, by [Dug01, Proposition 3.2] there exists a set of maps \mathcal{Y} and a Quillen equivalence

$$\text{Sh}_\Sigma : \underline{\mathbf{L}}_{\mathcal{Y}}(\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma) \simeq \mathbf{HDA}^\Sigma.$$

From Theorem 9.6, there exists a set of maps \mathcal{X} and a Quillen equivalence

$$\overline{\mathbb{T}} : \underline{\mathbf{L}}_{\mathcal{X}} \mathbf{HDA}^\Sigma \simeq \underline{\mathbf{L}}_{\underline{\mathbf{Cub}}}(\mathbf{CTS}).$$

One obtains the Quillen equivalence $\underline{\mathbf{L}}_{\mathcal{X}} \underline{\mathbf{L}}_{\mathcal{Y}}(\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma) \simeq \underline{\mathbf{L}}_{\underline{\mathbf{Cub}}}(\mathbf{CTS})$. Hence

$$\mathbb{T} : \underline{\mathbf{L}}_{\mathcal{X} \cup \mathcal{Y}}(\square_S^{\text{op}} \mathbf{Set} \downarrow!^S \Sigma) \simeq \underline{\mathbf{L}}_{\underline{\mathbf{Cub}}}(\mathbf{CTS}).$$

References

- [AHS06] ADÁMEK, J.; HERRLICH, H.; STRECKER, G. E. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, 17 (2006), 1–507. MR2240597, Zbl 1113.18001. Reprint of the 1990 original [Wiley, New York; MR1051419 (91h:18001)].
- [AR94] ADÁMEK, J.; ROSICKÝ, J. Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, 189. *Cambridge University Press, Cambridge*, 1994. xiv+316 pp. ISBN: 0-521-42261-2. MR1294136 (95j:18001), Zbl 0795.18007.
- [Cis02] CISINSKI, D-C. Théories homotopiques dans les topos. *J. Pure Appl. Algebra* **174** (2002), no. 1, 43–82. MR1924082 (2003i:18021), Zbl 1015.18009, doi: 10.1016/S0022-4049(01)00176-1.
- [CS96] CATTANI, G. L.; SASSONE, V. Higher-dimensional transition systems. 11th Annual IEEE Symposium on Logic in Computer Science (New Brunswick, NJ, 1996), 55–62. *IEEE Comput. Soc. Press, Los Alamitos, CA*, 1996. MR1461821, doi: 10.1109/LICS.1996.561303.

- [Dij68] DIJKSTRA, E. W. Cooperating Sequential Processes. Programming Languages: NATO Advanced Study Institute, 3–112. *Academic Press*, 1968. <http://www.cs.utexas.edu/users/EWD/ewd01xx/EWD123.PDF>.
- [Dug01] DUGGER, D. Combinatorial model categories have presentations. *Adv. Math.* **164** (2001), no. 1, 177–201. MR1870516 (2002k:18022), Zbl 1001.18001, arXiv:math/0007068, doi: 10.1006/aima.2001.2015.
- [Gau03] GAUCHER, P. A model category for the homotopy theory of concurrency. *Homology Homotopy Appl.* **5** (2003), no. 1, 549–599. MR2072345 (2005g:55029), Zbl 1069.55008, arXiv:math/0308054.
- [Gau08] GAUCHER, P. Towards a homotopy theory of process algebra. *Homology Homotopy Appl.* **10** (2008), no. 1, 353–388 (electronic). MR2426108 (2009d:68115), Zbl 1151.68037, arXiv:math/0701552.
- [Gau09] GAUCHER, P. Homotopical interpretation of globular complex by multipointed d -space. *Theory Appl. Categ.* **22** (2009), 588–621 (electronic). MR2591950 (2011b:55018), Zbl 1191.55013, arXiv:0710.3553.
- [Gau10a] GAUCHER, P. Combinatorics of labelling in higher-dimensional automata. *Theoret. Comput. Sci.* **411** (2010), no. 11–13, 1452–1483. MR2642826 (2011c:18007), Zbl 1191.68384, arXiv:0808.1667, doi: 10.1016/j.tcs.2009.11.013.
- [Gau10b] GAUCHER, P. Directed algebraic topology and higher dimensional transition systems. *New York J. Math.* **16** (2010), 409–461 (electronic). MR2740585 (2011k:18002), Zbl 1250.18004.
- [Gau11] GAUCHER, P. Towards a homotopy theory of higher dimensional transition systems. *Theory Appl. Categ.* **25** (2011), No. 12, 295–341 (electronic). MR2861110, Zbl 1235.18004, arXiv:1011.0918.
- [GM03] GRANDIS, M.; MAURI, L. Cubical sets and their site. *Theory Appl. Categ.* **11** (2003), No. 8, 185–211 (electronic). MR1988396 (2004e:18018), Zbl 1022.18009.
- [Gou02] GOUBAULT, E. Labelled cubical sets and asynchronous transition systems: an adjunction. Presented at CMCIM’02, 2002.
- [Gou03] GOUBAULT, E. Some geometric perspectives in concurrency theory. Algebraic topological methods in computer science (Stanford, CA, 2001). *Homology Homotopy Appl.* **5** (2003), no. 2, 95–136. MR1994942 (2005f:68106), Zbl 1034.68059.
- [Gun94] GUNAWARDENA, J. Homotopy and concurrency. *Bull. EATCS* **54** (1994), 184–193. Zbl 0938.68605.
- [Hir03] HIRSCHHORN, P. S. Model categories and their localizations. Mathematical Surveys and Monographs, 99. *American Mathematical Society, Providence, RI*, 2003. xvi+457 pp. ISBN: 0-8218-3279-4. MR1944041 (2003j:18018), Zbl 1017.55001.
- [Hov99] HOVEY, M. Model categories. Mathematical Surveys and Monographs, 63. *American Mathematical Society, Providence, RI*, 1999. xii+209 pp. ISBN: 0-8218-1359-5. MR1650134 (99h:55031), Zbl 0909.55001.
- [JNW96] JOYAL, A.; NIELSEN, M.; WINSKEL, G. Bisimulation from open maps. *Inform. Comput.* **127** (1996), no. 2, 164–185. MR1407994 (98f:68082), Zbl 0856.68067, doi: 10.1006/inco.1996.0057.
- [KR05] KURZ, A.; ROSICKÝ, J. Weak factorizations, fractions and homotopies. *Appl. Categ. Structures* **13** (2005) no. 2, 141–160. MR2141595 (2006c:18001), Zbl 1077.18002, doi: 10.1007/s10485-004-6730-z.
- [ML98] MAC LANE, S. Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, 5. *Springer-Verlag, New York*, 1998. xii+314 pp. ISBN: 0-387-98403-8. MR1712872 (2001j:18001), Zbl 0906.18001.
- [Ols09a] OLSCHOK, M. Left determined model structures for locally presentable categories. *Appl. Categ. Structures* **19** (2011), no. 6, 901–938. MR2861071

- (2012k:18008), Zbl 1251.18003, arXiv:0901.1627, doi: 10.1007/s10485-009-9207-2.
- [Ols09b] OLSCHOK, M. On constructions of left determined model structures. PhD thesis, Masaryk University, Faculty of Science, 2009.
- [Pra91] PRATT, V. Modeling concurrency with geometry. Proceedings of the 18th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, 311–322. *ACM Press, New York, NY*, 1991. doi: 10.1145/99583.99625.
- [Ros09] ROSICKÝ, J. On combinatorial model categories. *Appl. Categ. Structures* **17** (2009), no. 3, 303–316. MR2506258 (2010f:55023), Zbl 1175.55013, arXiv:0708.2185, doi: 10.1007/s10485-008-9171-2.
- [RT03] ROSICKÝ, J.; THOLEN, W. Left-determined model categories and universal homotopy theories. *Trans. Amer. Math. Soc.* **355** (2003), no. 9, 3611–3623 (electronic). MR1990164 (2004e:55023), Zbl 1030.55015, doi: 10.1090/S0002-9947-03-03322-1.
- [VG06] VAN GLABBEEK, R. J. Erratum to “On the expressiveness of higher dimensional automata” [Theoret. Comput. Sci. 356 (2006), no. 3, 265–290; MR2223695]. *Theoret. Comput. Sci.* **368** (2006), no. 1–2, 168–194. MR2274947, Zbl 1171.68582, doi: 10.1016/j.tcs.2006.06.024.
- [WN95] WINSKEL, G.; NIELSEN, M. Models for concurrency. *Handbook of logic in computer science*, 1–48. Handb. Log. Comput. Sci., 4, *Oxford Univ. Press, New York*, 1995. MR1365754 (97g:68072).

LABORATOIRE PPS (CNRS UMR 7126), CASE 7014, UNIV PARIS DIDEROT. SORBONNE PARIS CITÉ, F-75205 PARIS, FRANCE

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