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Lattice points on hyperboloids of one sheet

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Abstract. The problem of counting lattice points on a hyperboloid of two sheets is Gauss' circle problem in hyperbolic geometry. The problem of counting lattice points on a hyperboloid of one sheet does not have the same geometric interpretation, and in general, the solution(s) to Gauss' circle problem gives a lower bound, but not an upper bound. In this paper, we describe an exception. Given an *ample* height, and a lattice on a hyperboloid of one sheet generated by a point in the interior of the *effective cone*, the problem can be reduced to Gauss' circle problem.

Contents

Introduction

Let J be an $(m+1)\times(m+1)$ real symmetric matrix with $m \geq 2$ and signature $(m,1)$ (that is, J has m positive eigenvalues and one negative eigenvalue). Then $\mathbf{x}\circ\mathbf{y} := \mathbf{x}^t J\mathbf{y}$ is a Lorentz product and $\mathbb{R}^{m+1}$ equipped with this product is a Lorentz space, denoted $\mathbb{R}^{m,1}$. The hypersurface

$$\mathcal{V}_k = \{\mathbf{x}\in\mathbb{R}^{m,1} : \mathbf{x}\circ\mathbf{x} = k\}$$

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is a hyperboloid of two sheets if $k<0$, and a hyperboloid of one sheet if $k>0$.

For $k<0$, let us pick one of the sheets and denote it with $\mathcal{H}$. Then $\mathcal{H}\cong\mathbb{H}^m$ has a natural hyperbolic structure, where the distance $|AB|$ between two points on $\mathcal{H}$ is defined via the equation $k\cosh|AB| = A\circ B$. (See the appendix for a short synopsis of this model of hyperbolic geometry.) Let

$$\mathcal{O}=\mathcal{O}(\mathbb{R})=\{T\in M_{(m+1)\times(m+1)}(\mathbb{R}) : T^tJT=J\}$$

and

$$\mathcal{O}^+=\mathcal{O}^+(\mathbb{R})=\{T\in\mathcal{O} : T\mathcal{H}=\mathcal{H}\}.$$

Then $\mathcal{O}^+$ is the group of isometries on $\mathcal{H}$. Let Γ be a discrete subgroup of $\mathcal{O}^+$. That is, for any $P\in\mathcal{H}$, there exists a real M (that may depend on P) such that $|PQ|>M$ for all $Q\neq P$ in the Γ-orbit of P. Suppose further that Γ has the *finite geometric property*, which is to say Γ, as a group acting on $\mathcal{H}$, has a polyhedral fundamental domain $\mathcal{F}\subset\mathcal{H}$ with a finite number of faces. Let us define the quantity

$$N(P,D,\Gamma,t)=\#\{\gamma\in\Gamma : |\gamma P\circ D|<t\},$$

for P and D in $\mathbb{R}^{m,1}$. If there exist P and D on $\mathcal{H}$ such that

$$\limsup_{t\to\infty}\frac{N(P,D,\Gamma,t)}{\log t}>\frac{m-1}{2},$$

then we say Γ is *sufficiently thick*. In that case, the limit

$$\lim_{t\to\infty}\frac{N(P,D,\Gamma,t)}{\log t}$$

exists and depends only on Γ (see Theorem 1.1 below due to Lax and Phillips). Let us call this limit $\alpha(\Gamma)$. It has a variety of interpretations, which are discussed following Theorem 1.1.

Our main result is to extend this classical result to include certain cases when P is on a hyperboloid of one sheet. Let us call $\mathcal{K}\subset\mathbb{R}^{m,1}$ an *ample cone* if $\mathcal{K}$ is convex, open, is a cone with vertex $\mathbf{0}$ (that is, $\lambda\mathbf{x}\in K$ for all $\mathbf{x}\in\mathcal{K}$ and $\lambda>0$), is fixed by Γ (that is, $\gamma\mathbf{x}\in\mathcal{K}$ for all $\mathbf{x}\in\mathcal{K}$ and $\gamma\in\Gamma$), and $\mathbf{x}\circ\mathbf{x}<0$ for all $\mathbf{x}\in\mathcal{K}$. Given an ample cone $\mathcal{K}$, let us define the *effective cone* to be

$$\mathcal{E}=\{\mathbf{x}\in\mathbb{R}^{m,1} : \mathbf{x}\circ\mathbf{y}<0 \text{ for all } \mathbf{y}\in\mathcal{K}\}.$$

Note that $\mathcal{E}$ is closed. We say $\mathbf{x}$ is *ample* if $\mathbf{x}\in\mathcal{K}$, and *effective* if $\mathbf{x}\in\mathcal{E}$. The terms are borrowed from algebraic/arithmetic geometry, wherein a problem motivated discovery of our main result, which is the following:

Theorem 0.1. *Let Γ be a discrete group of isometries with the finite geometric property, and let $\mathcal{K}$ be an ample cone with respect to Γ. Suppose that D is ample and P lies in the interior of the effective cone. If Γ is sufficiently thick, then*

$$\lim_{t\to\infty}\frac{\log N(P,D,\Gamma,t)}{\log t}=\alpha(\Gamma).$$

If Γ *is not sufficiently thick, then*

$$\limsup_{t\to\infty} \frac{\log N(P,D,\Gamma,t)}{\log t} \le \frac{m-1}{2}.$$

Our motivation is discussed in Section 4. Those who are already familiar with the ample and effective cone might have noticed the peculiar direction of the inequality in the definition of $\mathcal{E}$. This is because we chose J to have signature $(m,1)$, rather than the signature $(1,m)$ of the intersection pairing.

1. The Gauss circle problem

1.1. Euclidean case. How many integer lattice points are there in a circle of radius r? More precisely, what is

$$n(r) := \#\{Q \in \mathbb{Z}^2 : |OQ| < r\},$$

where $O = (0,0)$? By thinking of each lattice point as the center of a square tile of unit area, we see that $n(r)$ is bounded below by the area of a disc of radius $r - \sqrt{2}$ and above by the area of a disc with radius $r + \sqrt{2}$. Thus,

$$n(r) = \pi r^2 + E(r),$$

where $|E(r)| < 2\pi(r\sqrt{2}+1)$. The first nontrivial bound on the error term is due to Sierpinski (1906), who showed $|E(r)| = O(r^{2/3})$; the current best known bound $O(r^{46/73+\epsilon})$ is due to Huxley [Hux93]; and the conjectured best possible bound is $O(r^{1/2+\epsilon})$ (where the constants implied by the big O depend on $\epsilon > 0$).

The lattice can be thought of as the orbit of the point O under the action of a group Γ, and a tile can be thought of as a fundamental domain $\mathcal{F}$. In higher dimensions, we get the elementary asymptotic for $n(r)$ of the volume of the ball $B(r)$ of radius r, divided by the volume of the fundamental domain $\mathcal{F}$, with an error term no larger than a constant multiple of the surface area of the ball. That is,

$$n(r) = \frac{|B(r)|}{|\mathcal{F}|} + O(|\partial B(r)|).$$

1.2. Hyperbolic case. In hyperbolic geometry, we have analogous results, but no geometric proofs. Given a discrete group of isometries Γ with a fundamental domain $\mathcal{F}$ (with natural properties), and points P and O in $\mathbb{H}^m$, we have

$$\#\{\sigma \in \Gamma : |\sigma(P)O| < r\} = \frac{|B(r)|}{|\mathcal{F}|} + o(|B(r)|). \tag{1}$$

Let us call this quantity $n(P,O,\Gamma,r)$. For P inside $\mathcal{F}$, this counts the number of points in the Γ-orbit of P. For P on the boundary of $\mathcal{F}$, it is out by a factor of $|\operatorname{Stab}(P)|$. The area of a disc of radius r in $\mathbb{H}^2$ is $|B(r)| = 2\pi(\cosh r - 1) \sim \pi e^r$, so $|\partial B(r)| = O(|B(r)|)$ (take the derivative), which explains why there are no geometric proofs. For compact $\mathcal{F}$ in $\mathbb{H}^2$, the result in Equation (1) was proved by Huber [Hub56]. It was generalized to $\mathcal{F}$ with finite area by Patterson [Pat75], and the generalization to m dimensions is attributed to Selberg (see [LP82]).

Unlike in Euclidean geometry, the hyperbolic case when $\mathcal{F}$ has infinite volume is very interesting. In particular, we have the following result referred to earlier, which is a simplification of results due to Lax and Phillips [LP82, Theorems 1 and 5.7]. Their results are worded in terms of the eigenvalues for the Laplace–Beltrami operator associated to Γ.

Theorem 1.1 (Lax and Phillips, [LP82])**.** *Suppose Γ is a group of isometries on $\mathbb{H}^m$ that has the finite geometric property. If the discrete part of the spectrum for the associated Laplace–Beltrami operator is not empty, then*

$$\lim_{r\to\infty} \frac{\log n(P,O,\Gamma,r)}{r}$$

exists, depends only on Γ, and is greater than $\frac{m-1}{2}$. If the discrete part of the spectrum is empty, then

$$\limsup_{r\to\infty} \frac{\log n(P,O,\Gamma,r)}{r} \leq \frac{m-1}{2}.$$

For P and D on $\mathcal{H}$, the quantities N and n are related by

$$N(P,D,\Gamma,t) = n(P,D,\Gamma,\cosh^{-1}(-t/k)). \tag{2}$$

Thus, Γ is sufficiently thick if and only if the discrete part of the spectrum is not empty. The discrete part of the spectrum lies in the interval $\big(-(\frac{m-1}{2})^2, 0\big]$, and $\alpha(\Gamma)$ is the largest root of $x^2-(m-1)x+\lambda_0=0$, where λ_0 is the largest eigenvalue in the spectrum.

The quantity $\alpha(\Gamma)$ can also be interpreted as the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ for Γ [Sul82]. Fix P in $\mathbb{H}^m$ and let $\mathbb{S}^{m-1}=\partial\mathbb{H}^m$ be the usual compactification of $\mathbb{H}^m$. The *limit set* $\Lambda(\Gamma)$ is the set of points y on $\mathbb{S}^{m-1}=\partial\mathbb{H}^m$ such that for any hyperplane in $\mathbb{H}^m$ that does not contain y, there exists an $x\in\Gamma(P)$ such that x and y are on the same side of the hyperplane. The limit set is independent of the choice $P\in\mathbb{H}^m$, and the Hausdorff dimension of the limit set is independent of the Poincaré model chosen for $\mathbb{H}^m$. When $\mathcal{F}$ has finite volume, the limit set is all of $\mathbb{S}^{m-1}$ and $\alpha(\Gamma)=m-1$.

2. An instructive example

Consider the Lorentz product $\mathbf{x}\circ\mathbf{y} := x_1y_1+x_2y_2-x_3y_3$ (where $\mathbf{x}=(x_1,x_2,x_3)$, etc.), and the hyperboloids $\mathcal{V}_k$ given by $\mathbf{x}\circ\mathbf{x}=k$ for $k=\pm1$. Let us study the quantity

$$N_k'(t) = \#\big\{\mathbf{x}\in\mathbb{Z}^3 : \mathbf{x}\circ\mathbf{x}=k,\ |x_3|<t\big\}.$$

Given a solution $\mathbf{x}$ (to either equation), it is clear that (x_2,x_1,x_3) and $(-x_1,x_2,x_3)$ are also solutions. So let us define $R_1(\mathbf{x})=(x_2,x_1,x_3)$ and $R_2(\mathbf{x})=(-x_1,x_2,x_3)$. The map R_1, acting on $\mathbb{R}^{2,1}$, is reflection through the plane p_1 given by $x_1 - x_2 = \mathbf{x} \circ (1, -1, 0) = 0$. Let $\mathcal{H}$ be the sheet of $\mathcal{V}_{-1}$ that contains $(0, 0, 1)$. Then, when restricted to $\mathcal{H}$, the map R_1 is reflection through the line $l_1 = p_1 \cap \mathcal{H}$. The map R_2 is reflection through the plane p_2 given by $x_1 = \mathbf{x} \circ (1, 0, 0) = 0$ and has a similar interpretation when restricted to $\mathcal{H}$.

In general, reflection through a hyperplane $p \subset \mathbb{R}^{m,1}$ given by $\mathbf{n} \circ \mathbf{x} = 0$ (so with *normal* vector $\mathbf{n}$) is given by

$$R(\mathbf{x}) = \mathbf{x} - 2\mathrm{proj}_{\mathbf{n}}(\mathbf{x}) = \mathbf{x} - 2\frac{\mathbf{n} \circ \mathbf{x}}{\mathbf{n} \circ \mathbf{n}}\mathbf{n}.$$

If $\mathbf{n} \circ \mathbf{n} = \pm 1$ or ± 2, then the resulting reflection R is in $\mathcal{O}(\mathbb{Z})$. If $\mathbf{n} \circ \mathbf{n} > 0$, then $R \in \mathcal{O}^+(\mathbb{Z})$ and is a reflection in $\mathcal{H}$ through the hyperline $p \cap \mathcal{H}$.

The vector $\mathbf{n} = (1, 1, 1)$ satisfies $\mathbf{n} \circ \mathbf{n} = 1$, so yields another reflection in $\mathcal{O}^+(\mathbb{Z})$, namely

$$R_3 = \begin{bmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{bmatrix}.$$

Stereographic projection of $\mathcal{H}$ through the point $(0, 0, -1)$ and onto the plane $x_3 = 0$ sends $\mathcal{H}$ to the Poincaré disc model of $\mathbb{H}^2$. Under this projection, R_1 becomes reflection through the line l_1 given by $x_2 = x_1$; R_2 becomes reflection through the line l_2 given by $x_1 = 0$; and R_3 becomes reflection through the (hyperbolic) line l_3 with endpoints $(1, 0)$ and $(0, 1)$ (suppressing the x_3 component, which is zero), as shown in Figure 1. This model can also be thought of as a perspective view of $\mathcal{H}$, where our eye is at the point $(0, 0, -1)$, so we will identify it with $\mathcal{H}$. Note that the group $\Gamma = \langle R_1, R_2, R_3 \rangle$ has a fundamental domain $\mathcal{F}$ bounded by these three lines, which is a region with finite area (in fact, $|\mathcal{F}| = \pi/4$). Thus, every integer lattice point on $\mathcal{H}$ is in the Γ-orbit of a point $P \in \mathcal{H}$ that lies in $\mathcal{F}$.

Let $\mathbf{n}_i$ be normal vectors associated to each of the reflections R_i. If $\mathbf{x} \in \mathcal{F}$, then $\mathbf{n}_i \circ \mathbf{x}$ has the same sign as $\mathbf{n}_i \circ P$ for any $P \in \mathcal{F}$. Thus, if we let $\mathbf{n}_1 = (-1, 1, 0)$, $\mathbf{n}_2 = (1, 0, 0)$, and $\mathbf{n}_3 = (-1, -1, -1)$ so that they point into $\mathcal{F}$, a point $\mathbf{x} \in \overline{\mathcal{F}}$ (so allowing boundary points too) if and only if $\mathbf{n}_i \circ P \geq 0$ for all i. This gives us the conditions $0 \leq x_1 \leq x_2$, $x_1 + x_2 \leq x_3$, $\mathbf{x} \circ \mathbf{x} = -1$, and $\mathbf{x} \in \mathbb{Z}^3$. Solving, we get $\mathbf{x} = (0, 0, 1)$, which we call O. Thus, every integer solution in $\mathcal{H}$ is in the Γ-orbit of $O = (0, 0, 1)$.

Because O is on a vertex of $\mathcal{F}$, we can also describe the set of integer solutions as the orbit of O under the action of Γ' generated by reflection in the four lines with normal vectors $(\pm 1, \pm 1, 1)$. The tiling induced by Γ' is shown in Figure 1.

Thus,

$$N'_{-1}(t) = 2N(O, O, \Gamma', t) = 2n(O, O, \Gamma', \cosh^{-1}(t)) \sim 2t.$$

The factor of 2 in the first equality arises because $N'_{-1}(t)$ counts the lattice points on both sheets and because the stabilizer of O in Γ' is trivial. The second equality follows from Equation (2), and the third equality follows

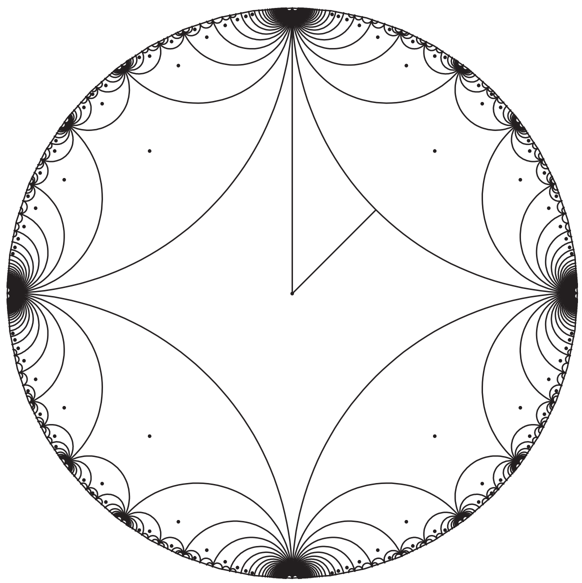

FIGURE 1. The tiling induced by Γ', with centers the Γ'-orbit of O. Also pictured is the fundamental domain $\mathcal{F}$ for Γ.

from Equation (1) and the observation that the fundamental domain for Γ' has area 2π.

Points on the hyperboloid of one sheet $\mathcal{V}_1$ also have geometric interpretations. A point $Q \in \mathcal{V}_1$ represents the line on $\mathcal{H}$ given by the intersection of the plane $Q \circ \mathbf{x} = 0$ with $\mathcal{H}$. Thus, an integer lattice point is in $\mathcal{V}_1(\mathbb{Z})$ if and only if it is in the Γ-orbit of a point $P \in \mathcal{V}_1(\mathbb{Z})$ such that the line l given by $P \circ \mathbf{x} = 0$ intersects $\mathcal{F}$. In general, if two lines given by $A \circ \mathbf{x} = 0$ and $B \circ \mathbf{x} = 0$ intersect, then

$$|A \circ B| = ||A||||B|| \cos\theta, \tag{3}$$

where θ is the acute angle between the lines. When $\cos\theta = 1$, (so $\theta = 0$), the intersection is at infinity. Using this, we solve for all $P \in \mathcal{V}_1(\mathbb{Z})$ such that the line l given by $P \circ \mathbf{x} = 0$ intersects $\mathcal{F}$. This can be solved algebraically, but it might be easier to see it geometrically.

If l intersects $\mathcal{F}$, it intersects two of the three lines l_1, l_2, l_3 that bound $\mathcal{F}$. Since $\mathbf{n}_1 \circ \mathbf{n}_1 = 2$ and $\mathbf{n}_2 \circ \mathbf{n}_2 = \mathbf{n}_3 \circ \mathbf{n}_3 = 1$, the intersection with l_1 is either perpendicular or at an angle of $\pi/4$, while the intersection with the other two sides is either perpendicular or at the cusp. A line cannot be perpendicular to two sides of a triangle, as the sum of angles in a triangle is always less than π. It also cannot be perpendicular to one side and intersect l_1 at an angle of $\pi/4$, as this gives either a triangle with two right angles, or a right angle and two angles of $\pi/4$, neither of which are possible. If l goes through the cusp, then the two possibilities (intersecting l_1 perpendicularly or at an angle of $\pi/4$) are the lines l_2 and l_3. Finally, l cannot be l_1 as the normal $\mathbf{n}_1$ is not on $\mathcal{V}_1$. This gives us two possibilities for P, namely $P = \mathbf{n}_2$ or $\mathbf{n}_3$. Their orbits are disjoint. Their stabilizers both have order 2. To see this, note that the line l_2 is a boundary line for four copies of the fundamental domain $\mathcal{F}$. Of the four relevant isometries, two send $\mathbf{n}_2$ to $\mathbf{n}_2$, while the other two send $\mathbf{n}_2$ to $-\mathbf{n}_2$. The same argument works for $\mathbf{n}_3$. Thus,

$$N_1'(t) = \frac{1}{2}\left(N(\mathbf{n}_2, O, \Gamma, t) + N(\mathbf{n}_3, O, \Gamma, t)\right).$$

The quantity $|Q \circ O| < t$ has a geometric interpretation too. Note that $|Q \circ O| = \sinh|OQ'|$, where Q' is the point closest to O on the line given by $Q \circ \mathbf{x} = 0$. Thus the quantity $N_1'(t)$ can be interpreted as counting the number of lines in the orbit of the lines given by $\mathbf{n}_1$ and $\mathbf{n}_2$ that intersect the disc centered at O with radius $\sinh^{-1}(t)$. If we choose an arbitrary point A on one of these lines, then an element of the Γ-orbit of A is in the disc only if the corresponding line intersects the disc, from which we get the lower bound

$$N_1'(t) \gg t.$$

An upper bound is not so obvious.

We can also interpret $N(P, O, \Gamma, t)$ in a dual fashion as follows:

$$\begin{aligned} N(P, O, \Gamma, t) &= \#\{\gamma \in \Gamma : |\gamma P \circ O| < t\} \\ &= \#\{\gamma \in \Gamma : |P \circ \gamma O| < t\}. \end{aligned}$$

Thus, for $P \in \mathcal{V}_1$, we seek to count the number of points in the Γ-orbit of O that lie in the region bounded by $P \circ \mathbf{x} = \pm t$, as shown in Figure 2. As noted before, the curve $P \circ \mathbf{x} = t$ is the locus of points a distance $\sinh^{-1}(t)$ away from the line $P \circ \mathbf{x} = 0$. We can certainly fit a disc of radius $\sinh^{-1}(t)$ between the two curves, giving us the lower bound we found before. That we can fit infinitely many disjoint discs of this radius between these curves suggests that the lower bound is not sharp, and in fact Babillot [Bab02], using results of Duke, Rudnick, and Sarnak [DRS93], show

$$N_1'(t) \gg\ll t \log t.$$

Remark 1. The vector $\mathbf{n}_1$ lies on the hyperboloid of one sheet $\mathcal{V}_2$. Since the portion of l_1 on the boundary of $\mathcal{F}$ is bounded, the stabilizer in Γ for $\mathbf{n}_1$ must be infinite. The definition of n, which counts elements of the group rather than elements in the orbit, is historical, and was likely made so that the stabilizer would not be part of the asymptotic formula. Note that the stabilizer of a point in a discrete group of isometries on $\mathbb{H}^n$ is always finite, so the definition made sense.

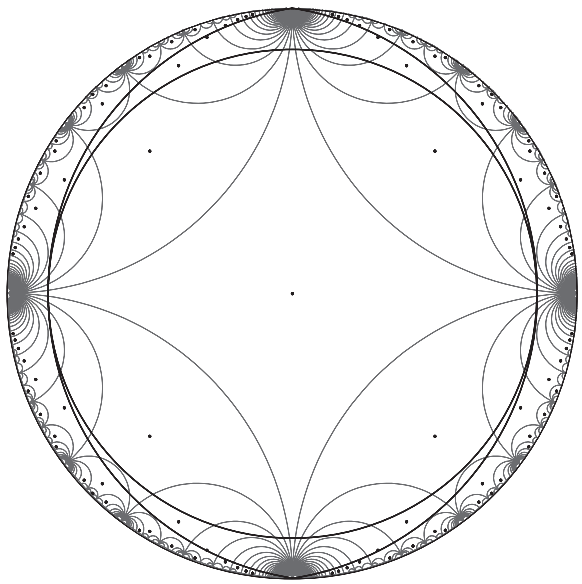

FIGURE 2. The region bounded by the curves $P \circ \mathbf{x} = \pm t$ and a disc between them.

3. The main result

The example in Section 2 illustrates how counting lattice points on hyperboloids of one sheet is different from counting lattice points on hyperboloids of two sheets. Since $\alpha(\Gamma) = 1$ for both, it satisfies the conclusion of our main result. However, the main result does not include this case, and in general has no new content when $\mathcal{F}$ is finite, since in that case, the effective cone is the same as the ample cone and does not intersect any hyperboloid of one sheet so it is covered by Selberg's result (as quoted in Section 1.2).

Before presenting the proof, let us look at an example that illustrates the main idea. A group Γ acting on $\mathbb{R}^{2,1}$ is generated by two reflections and a rotation that, when viewed in the Poincaré disc model of $\mathcal{H}$, are the reflections through the lines AE and $A'E'$ in Figure 3, and rotation by π about C. Its fundamental domain $\mathcal{F} \subset \mathcal{H}$ is the region bounded by the lines AA', AE, and $A'E'$. The line FF' intersects AE and $A'E'$ perpendicularly at B and B'. Let $\mathcal{F}'$ be the closed region bounded by $ABB'A'$. We take for the ample cone $\mathcal{K}$ the cone which, when intersected with $\mathcal{H}$, is the largest open subset of $\Gamma(\mathcal{F}')$. This cone is open, convex, and is fixed by Γ. (It is, in fact, the ample cone for a class of K3 surfaces, as described in [Bar11].)

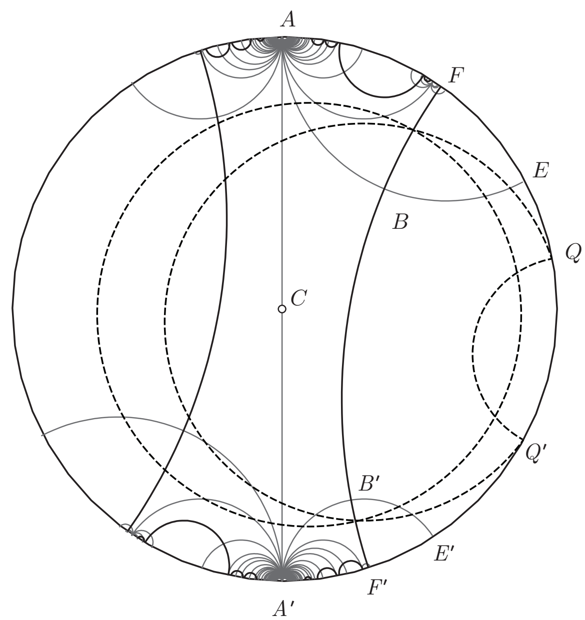

FIGURE 3. An ample cone $\mathcal{K}$ intersected with $\mathcal{H}$, the region bounded by the dark lines. The gray lines represent the symmetries of $\mathcal{K} \cap \mathcal{H}$, or the tiling of $\mathcal{K} \cap \mathcal{H}$ with $\mathcal{F}'$. The dotted curves represent the line $P \circ \mathbf{x} = 0$, the curve $-P \circ \mathbf{x} = t$, and a disc that contains the intersection of $\mathcal{K} \cap \mathcal{H}$ with the region $-P \circ \mathbf{x} \leq t$.

Its intersection with $\mathcal{H}$ is the region bounded by the dark lines in Figure 3. Given P inside the effective cone, the line $P \circ \mathbf{x} = 0$ does not intersect $\mathcal{K} \cap \mathcal{H}$ (since $P \circ \mathbf{x} < 0$ for all $\mathbf{x} \in \mathcal{K}$). Suppose, for example, that the line $P \circ \mathbf{x} = 0$ is the line QQ' in Figure 3. The idea is to find a line, like FF', that separates the ample cone from the line $P \circ \mathbf{x} = 0$. For large enough t, the region $-P \circ \mathbf{x} \leq t$ intersects this line, forming a line segment with midpoint O. We consider the disc centered at O that contains the intersection of this region with the ample cone. This disc gives a suitable upper bound. Lower bounds are easier to find.

Proof of Theorem 0.1. The argument given here is for any dimension $m \geq 2$. We use terminology appropriate for $m = 3$ (so $\mathcal{H} \cong \mathbb{H}^3$). Let $k = D \circ D < 0$ and let $\mathcal{H}$ be the sheet of $\mathbf{x} \circ \mathbf{x} = k$ that contains D.

When $P \circ P < 0$, the result follows directly from Theorem 1.1 (using Equation (1) and $N(P, D, \Gamma, t) = N(\lambda P, D, \Gamma, \lambda t)$, where λ is chosen so that

$\lambda P \in \mathcal{H}$). For the other cases, we use the dual interpretation of $N(P, D, \Gamma, t)$. That is, we look at the intersection of the Γ-orbit of D with the region bounded by $P \circ \mathbf{x} = \pm t$. Note that, since D is ample, the Γ-orbit of D is contained in $\mathcal{K} \cap \mathcal{H}$ and $P \circ \gamma D < t$ (so we are interested in a subset of the region $0 \leq -P \circ x \leq t$). The idea is to find two balls: A ball in $\mathcal{H}$ that contains the intersection of $\mathcal{K} \cap \mathcal{H}$ with the region in $\mathcal{H}$ bounded by the curves $P \circ \mathbf{x} = \pm t$, which gives us an upper bound; and a ball that is contained in the region bounded by $P \circ \mathbf{x} = \pm t$, which gives us a lower bound.

We begin with the lower bound, which is relevant only when Γ is sufficiently thick. When $P \circ P > 0$ we pick any point $O \in \mathcal{H}$ on the (hyper)plane $P \circ \mathbf{x} = 0$. The ball of radius $\sinh^{-1}\left(\frac{t}{\sqrt{-kP \circ P}}\right)$ centered at O lies between the surfaces $P \circ \mathbf{x} = \pm t$, so by Theorem 1.1,

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log N(P, D, \Gamma, t)}{\log t} &\geq \lim_{t\to\infty} \frac{n\left(O, D, \Gamma, \sinh^{-1}(ct)\right)}{\sinh^{-1}(ct)} \frac{\sinh^{-1}(ct)}{\log t} \\ &\geq \alpha(\Gamma) \lim_{t\to\infty} \frac{\sinh^{-1}(ct)}{\log t} = \alpha(\Gamma),\end{aligned}$$

where $c = \frac{1}{\sqrt{-kP \circ P}}$.

In $\mathbb{R}^{m,1}$, we can identify $\mathcal{H}$ with the open cone $\{\mathbf{x} \in \mathbb{R}^{m,1} : \mathbf{x} \circ \mathbf{x} < 0\}$ modulo the action of $\mathbb{R}^*$. Then the boundary at infinity of $\mathcal{H}$ is the cone $\{\mathbf{x} \in \mathbb{R}^{m,1} : \mathbf{x} \circ \mathbf{x} = 0, \mathbf{x} \neq \mathbf{0}\}$ modulo $\mathbb{R}^*$. This is the usual compactification of $\mathbb{H}^m$ with $\mathbb{S}^{m-1}$. If $P \circ P = 0$, then P is on the cone and represents a point at infinity. We will say $P \in \partial\mathcal{H}$. The hyperplane $P \circ \mathbf{x} = -t$ (in $\mathbb{R}^{m,1}$) is parallel to the edge of the cone (along the line spanned by P), so its intersection with $\mathcal{H}$ is a horoball that is tangent to $\partial\mathcal{H}$ at P.

Let $O \in \mathcal{H}$ be an arbitrary point in the horoball. Then the point Q on the horoball closest to O lies on the (hyperbolic) line through O and P. Thus, Q lies in the subspace spanned by O and P. That is, $Q = \lambda O + \mu P$. Solving for $O \circ Q$ (using $Q \circ Q = O \circ O = k$ and $Q \circ P = -t$), we get

$$k \cosh r = -O \circ Q = \frac{k}{2}\left(\frac{t}{c} + \frac{c}{t}\right),$$

where $c = -P \circ O$ and r is the radius of the largest ball centered at O that lies in the horoball. Again, by Theorem 1.1, we get

$$\liminf_{t\to\infty} \frac{\log N(P, D, \Gamma, t)}{\log t} \geq \lim_{t\to\infty} \frac{n(O, D, \Gamma, r)}{r} \frac{\cosh^{-1}\left(\frac{1}{2}\left(\frac{t}{c} + \frac{c}{t}\right)\right)}{\log t} = \alpha(\Gamma).$$

For the upper bound, we first find a plane in $\mathcal{H}$ that separates the ample cone $\mathcal{K}$ from the plane given by $P \circ \mathbf{x} = 0$ (or the point P on $\partial\mathcal{H}$, if $P \circ P = 0$). We consider $\mathbf{n} = P - \lambda D$. Note that if $\lambda > 0$ is small enough then $\mathbf{n} \circ D < 0$ and $\mathbf{n} \circ \mathbf{n} > 0$. Since $\mathbf{n} \circ \mathbf{n} > 0$, the hyperplane $\mathbf{n} \circ \mathbf{x} = 0$ intersects $\mathcal{H}$, so defines a plane in $\mathcal{H}$. Suppose $Q \in \mathcal{H}$ and $\mathbf{n} \circ Q = 0$. Then

$$P \circ Q = (\mathbf{n} + \lambda D) \circ Q = \lambda D \circ Q.$$

Since Q and D are in $\mathcal{H}$, $Q \circ D < 0$. Since $P \circ D < 0$ (P is effective), Q and D are on the same side of $P \circ \mathbf{x} = 0$. Because P is in the interior of $\mathcal{E}$, a small enough (but positive) choice for λ will ensure $\mathbf{n} \in \mathcal{E}$, so $\mathbf{n} \circ \mathbf{x} = 0$ does not intersect $\mathcal{K}$. When $P \circ P = 0$, it is clear $\mathbf{n} \circ P \neq 0$, so $\mathbf{n} \circ \mathbf{x} = 0$ separates $\mathcal{K}$ and P. So let us assume $P \circ P > 0$. Note

$$\begin{aligned} (\mathbf{n} \circ P)^2 &= (P \circ P)^2 - 2\lambda(P \circ P)(P \circ D) + \lambda^2 (P \circ D)^2 \\ &> (P \circ P)(P \circ P - 2\lambda(P \circ D) + \lambda^2 (D \circ D)) \\ &= (P \circ P)(\mathbf{n} \circ \mathbf{n}), \end{aligned} \tag{4}$$

since $P \circ P > 0$ and $D \circ D < 0$. Thus the angle θ between the planes $\mathbf{n} \circ \mathbf{x} = 0$ and $P \circ \mathbf{x} = 0$ does not exist (as $|\cos\theta| > 1$; see Equation (3)), so the planes do not intersect on $\mathcal{H}$ nor on $\partial\mathcal{H}$. Thus, $\mathbf{n} \circ \mathbf{x} = 0$ separates the ample cone $\mathcal{K}$ and the hyperplane $P \circ \mathbf{x} = 0$.

Consider the point

$$O = P - \frac{P \circ \mathbf{n}}{\mathbf{n} \circ \mathbf{n}} \mathbf{n}.$$

Note that

$$O \circ O = P \circ P - \frac{(P \circ \mathbf{n})^2}{\mathbf{n} \circ \mathbf{n}} < 0,$$

since either $P \circ P = 0$, or when $P \circ P > 0$, the planes $P \circ \mathbf{x} = 0$ and $\mathbf{n} \circ \mathbf{x} = 0$ do not intersect in $\mathcal{H}$ (see Equation (4)). Thus $\mu O \in \mathcal{H}$ for $\mu = \sqrt{\frac{k}{O \circ O}}$ and the set $\{\mathbf{x} \in \mathcal{H} : -\mu O \circ x < t\}$ is a ball in $\mathcal{H}$. Now suppose $Q = \gamma D$ for some $\gamma \in \Gamma$, so $Q \in \mathcal{K}$ and hence $\mathbf{n} \circ Q < 0$ (Q and D are on the same side of $\mathbf{n} \circ \mathbf{x} = 0$). Suppose also that $|P \circ Q| < t$. Since $P \circ Q < 0$, this means $-P \circ Q < t$. Hence

$$-O \circ Q = -P \circ Q + \frac{P \circ \mathbf{n}}{\mathbf{n} \circ \mathbf{n}}(\mathbf{n} \circ Q) < t + \frac{P \circ \mathbf{n}}{\mathbf{n} \circ \mathbf{n}}(\mathbf{n} \circ Q) < t,$$

since $P \circ \mathbf{n} > 0$, $\mathbf{n} \circ \mathbf{n} > 0$, and $\mathbf{n} \circ Q < 0$. Thus,

$$N(P, D, \Gamma, t) \leq N(O, D, \Gamma, t) = N(\mu O, D, \Gamma, \mu t).$$

Hence, if Γ is sufficiently thick, then

$$\limsup_{t \to \infty} \frac{\log N(P, D, \Gamma, t)}{\log t} \leq \limsup_{t \to \infty} \frac{n(\mu O, D, \Gamma, r)}{r} \frac{r}{\log t} = \alpha(\Gamma), \tag{5}$$

where $r = \cosh^{-1}\left(\frac{t}{\sqrt{k O \circ O}}\right)$. If Γ is not sufficiently thick, then we replace the conclusion '$= \alpha(\Gamma)$' with '$\leq \frac{m-1}{2}$' in Equation (5). □

Remark 2. We note that the condition that P be in the interior of the effective cone cannot be relaxed to $P \in \mathcal{E}$. In the example of Figure 3, let the line FF' be given by $P \circ \mathbf{x} = 0$. Since D is ample, and $\mathcal{K}$ is open, $D \circ P \neq 0$, so we may choose P so that $P \circ D < 0$. Note that $P \in \mathcal{E}$, since all of $\mathcal{K}$ lies on one side of $P \circ \mathbf{x} = 0$. However, the line FF', and hence P, is fixed by the reflections through AE and $A'E'$. Hence the stabilizer of P in Γ' is infinite, so $N(P, D, \Gamma', t)$ is infinite whenever it is nonempty.

4. Motivation

Let X be a K3 surface defined over a number field K, and let $\mathcal{P}$ be a point on $X(K)$. Let $\mathcal{A} = \mathrm{Aut}(X/K)$ be the group of K-rational automorphisms on X. Given a Weil height h_D associated to an ample divisor D, let us consider the following counting function:

$$\mathcal{N}(\mathcal{P}, X, h_D, t) = \#\{\mathcal{Q} \in \mathcal{A}(\mathcal{P}) : h_D(\mathcal{Q}) < t\}.$$

As with Gauss' lattice point problem, it is natural to ask for the asymptotic behavior of $\mathcal{N}$. The pull back map sends $\mathcal{A}$ to Γ', a discrete subgroup of isometries of the Picard lattice. It is therefore natural to believe the asymptotic behavior of $\mathcal{N}$ is related to $\alpha(\Gamma')$ when Γ' has the finite geometric property. In fact, for an X in a particular class of K3 surfaces, we know [Bar11]

$$\lim_{t\to\infty} \frac{\log(\mathcal{N}(\mathcal{P}, X, h_D, t))}{\log t} = \alpha(\Gamma').$$

This is the same group Γ' of the example illustrated in Figure 3. The limit $\alpha(\Gamma')$ is computed in [Bar03a], where it is shown that $\alpha(\Gamma') = .6527 \pm .0012$. As remarked earlier, this is the Hausdorff dimension of the limit set $\Lambda(\Gamma')$, the Cantor like subset of the boundary $\mathbb{S} = \partial\mathbb{H}^2$ formed by removing the open arcs outside $\mathcal{K}$ for each boundary line denoted by a dark curve in Figure 3.

We connect $\mathcal{N}(\mathcal{P}, X, h_D, t)$ with $N(P, D, \Gamma', t)$, via vector heights [Bar03b]. A vector height $\mathbf{h}$ is a map from $X(K)$ to $\mathrm{Pic}(X) \otimes \mathbb{R}$ that has several nice properties. For any Weil height h_D,

$$h_D(\mathcal{P}) = \mathbf{h}(\mathcal{P}) \cdot D + O(1), \tag{6}$$

where $\cdot$ is the intersection pairing, which has signature $(1, m)$, and the error term is bounded independent of $\mathcal{P}$ but may depend on D. Furthermore, for any $\sigma \in \mathcal{A}$,

$$\mathbf{h}(\sigma\mathcal{P}) = \sigma_* \mathbf{h}(\mathcal{P}) + O(1),$$

where $\sigma_* \in \Gamma'$ is its push forward (the pull back of σ^{-1}), and the bound in the error term is independent of $\mathcal{P}$, but may depend on σ.

We observe that, if D is ample, then $h_D(\mathcal{P}) > 0$ for all but finitely many points $\mathcal{P}$, so modulo the error term, $\mathbf{h}(\mathcal{P})$ is in the effective cone (see Equation (6)). Thus, one might suspect that Theorem 0.1 is a crucial step in the generalization of the result in [Bar11].

We close with a final observation.

If $[C]$ is a divisor class that represents a smooth curve C on X, then $[C] \cdot [C] = 2g - 2$ (by the adjunction formula). Thus, for $g > 1$, the number of divisor classes represented by elements in the $\mathcal{A}$-orbit of C and satisfying

$[C] \cdot D < t$ for fixed ample D is exactly the lattice point problem. If $g = 1$ (an elliptic curve) or 0 (a smooth rational curve, also known as a -2 curve), then $[C]$ lies on the boundary of the effective cone. For example, the line FF' in Figure 3 represents a -2 curve on X. That is, there is a -2 curve C on X so that $[C]$ is a normal to the plane that represents FF'. Thus, $N([C], D, \Gamma', t)$ is infinite for some fixed values of t. (The stabilizer of $[C]$ is the infinite group generated by the reflections in AE and $A'E'$.) However, there is a natural modification of N, as noted in Remark 1:

$$N'(P, D, \Gamma', t) = \#\{Q \in \Gamma'(P) : |Q \circ D| < t\}.$$

In the limited case where C is an elliptic curve or -2 curve on a K3 surface X in the class of K3 surfaces considered in [Bar03a], one has (see Theorem 3.9 and the remarks following it in [Bar03a])

$$\lim_{t\to\infty} \frac{\log(N'([C], D, \Gamma', t))}{\log t} = \alpha(\Gamma').$$

Thus, one might wonder whether this is true in general. Note that the conclusion drawn in [Bar03a] is a consequence of the dimension calculation, a complicated computation that is specific to that example.

Remark 3. After initial submission of this paper, the author became aware of the work by Kontorovich and Oh. In their paper [KoO11, Theorem 1.5], among other results, the authors prove that if Γ is sufficiently thick, $m = 3$, $P \circ P = 0$, and $D \circ D > 0$, then

$$N'(P, D, \Gamma, t) \sim ct^{\alpha(\Gamma)},$$

and explicitly give c. Their techniques look promising.

Appendix: The pseudosphere in Lorentz space

The pseudospherical model of hyperbolic geometry may not be familiar to many readers, so here is a quick description. More detailed treatments appear in [Bar01, Chapter 12] and [Rat94, Chapter 3]. We assume the reader is familiar with the Poincaré disc model of $\mathbb{H}^2$. Let us begin in $\mathbb{R}^{2,1}$ with the standard Lorentz product $\mathbf{x} \circ \mathbf{y} = x_1y_1 + x_2y_2 - x_3y_3$. The pseudosphere $\mathcal{H}$ is the set of points $\mathbf{x}$ in $\mathbb{R}^{2,1}$ with $\mathbf{x} \circ \mathbf{x} = -1$ and $x_3 > 0$. The distance function on $\mathcal{H}$ induced by the arclength element $ds^2 = -d\mathbf{x} \circ d\mathbf{x}$ is $|AB|$ where $\cosh |AB| = -A \circ B$. This surface can be projected onto the Poincaré disc, as described in Section 2. It is not too difficult to verify that the arclength elements in both models coincide, so $\mathcal{H}$ is a model of hyperbolic geometry.

For the sake of intuition, it is useful to think of the pseudosphere as a sphere of radius i embedded in Lorentz space, and while the analogy is not quite exact, it is very close. As a consequence, results in hyperbolic geometry can be thought of as analogs of similar results in spherical geometry and with similar proofs.

A sphere of radius r is the surface $\mathbb{S}$ in $\mathbb{R}^3$ given by $\mathbf{x}\cdot\mathbf{x} = r^2$. The distance $|AB|$ between two points on $\mathbb{S}$ is given by $r^2\cos(|AB|/r) = A\cdot B$. Replacing r with i and the dot product with a Lorentz product, we get the surface given by $\mathbf{x}\circ\mathbf{x} = -1$. However, as this is a hyperboloid of two sheets and we want a connected geometry, we take $\mathcal{H}$ to be only one sheet. Alternatively, we can identify antipodal points on the hyperboloid, and in this respect, the pseudosphere is more closely an analog of elliptic geometry, which is the geometry of the sphere modulo the relation of antipodal points (or projective geometry with a metric). Distance on $\mathcal{H}$ is given by $-\cos(|AB|/i) = A\circ B$, and noting that $\cos(\theta/i) = \cosh\theta$ (Euler's formula), we get $-\cosh|AB| = A\circ B$.

Lines on $\mathbb{S}$ are the intersection of $\mathbb{S}$ with planes through the origin, so can be described by equations of the form $\mathbf{n}\cdot\mathbf{x} = 0$. Lines on $\mathcal{H}$ are also the (nonempty) intersection of $\mathcal{H}$ with planes through the origin, so are described by equations of the form $\mathbf{n}\circ\mathbf{x} = 0$. The plane $\mathbf{n}\circ\mathbf{x} = 0$ intersects $\mathcal{H}$ if and only if $\mathbf{n}\circ\mathbf{n} > 0$, and as in the spherical case, we can normalize $\mathbf{n}$ so that $\mathbf{n}\circ\mathbf{n} = 1$.

The surface $\mathcal{H}$ admits a group of isometries that allow us to do the usual things. Namely, we can translate any point to any other point, rotate by any angle about any point, and reflect through any line. Since distance is defined by the Lorentz product, isometries must preserve it. We can use the existence of isometries to establish a number of results. For example, any line l can be moved to the line given by $y = 0$, which has normal vector $(0, 1, 0)$. By inverting this map, we conclude that l can be described by the equation $\mathbf{n}\circ\mathbf{x} = 0$ where $\mathbf{n}\circ\mathbf{n} = 1$.

An explicit description of the group of isometries is again inspired by our knowledge of the sphere. The isometries on $\mathbb{S}$ are generated by rotations about the z-axis, rotations about the y-axis, and reflection through the plane $y = 0$. The isometries on $\mathcal{H}$ are generated by rotations about the z-axis, "rotations" by imaginary angles about the y-axis (which are translations), and reflection through $y = 0$.

The reflection through the plane $\mathbf{n}\circ\mathbf{x}$ in $\mathbb{R}^3$ is a reflection on $\mathbb{S}$, and is given by

$$R(\mathbf{x}) = \mathbf{x} - 2(\mathrm{proj}_{\mathbf{n}}\mathbf{x})\mathbf{n} = \mathbf{x} - 2(\mathbf{n}\cdot\mathbf{x})\mathbf{n},$$

(where $\mathbf{n}\cdot\mathbf{n} = 1$). A reflection in $\mathbb{R}^{2,1}$ has the same formula, with the dot product replaced by the Lorentz product, and if $\mathbf{n}$ has length one (so $\mathbf{n}\circ\mathbf{x} = 0$ intersects $\mathcal{H}$), then when restricted to $\mathcal{H}$ it is the reflection through the line $\mathbf{n}\circ\mathbf{x} = 0$.

The angle θ between two lines $\mathbf{n}\cdot\mathbf{x} = 0$ and $\mathbf{m}\cdot\mathbf{x} = 0$ on $\mathbb{S}$ is the angle between their normal vectors. Thus, with $\mathbf{n}$ and $\mathbf{m}$ vectors of length one, θ is given by $\cos\theta = \pm\mathbf{n}\cdot\mathbf{m}$. The ambiguity of $\pm$ corresponds to the ambiguity of the direction of the normal vectors, and the ambiguity of whether the angle between the lines is θ or its supplement. The angle between two intersecting lines on $\mathcal{H}$ satisfies the same formula, after replacing the dot product with the Lorentz product. Since θ is an angle and not a distance, the radius plays no role in the formula, so passing to the pseudosphere, there is no i in the formula. That is, the angle between two lines on $\mathcal{H}$ given by the equations $\mathbf{n} \circ \mathbf{x} = 0$ and $\mathbf{m} \circ \mathbf{x} = 0$ (and with $\mathbf{n}$ and $\mathbf{m}$ vectors of length one), is given by $\cos\theta = \pm\mathbf{n} \circ \mathbf{m}$. Unlike on the sphere, it is possible for $|\mathbf{n} \circ \mathbf{m}|$ to be greater than one, in which case θ does not exist (or is not real), and the lines do not intersect. A pair of such lines are called *ultraparallel*. If $|\mathbf{n} \circ \mathbf{m}| = 1$, then the lines intersect at infinity and are called *parallel*.

The intersection of $\mathbb{S}$ with the plane $\mathbf{n} \cdot \mathbf{x} = t$ (for $t < r$) is a circle of radius $\cos^{-1}(t/r)$ centered at $r\mathbf{n}$. It is also the curve a constant distance $\sin^{-1}(t/r)$ away from the line $\mathbf{n} \cdot \mathbf{x} = 0$. On $\mathcal{H}$, the curve $\mathbf{n} \circ \mathbf{x} = t$ is a circle of radius $\cosh^{-1} t$ centered at $\mathbf{n}$ if $\mathbf{n} \circ \mathbf{n} = -1$; and a curve a constant distance $\sinh^{-1} t$ away from the line $\mathbf{n} \circ \mathbf{x} = 0$ if $\mathbf{n} \circ \mathbf{n} = 1$ (see [Rat94, Theorem 3.2.12]). If $\mathbf{n} \circ \mathbf{n} = 0$, then it is a horocycle.

The area of a triangle ΔABC on $\mathbb{S}$ is $r^2(A + B + C - \pi)$. Again, letting $r = i$, we get that the area of a triangle ΔABC on $\mathcal{H}$ is $\pi - A - B - C$. The area of a disc of radius ρ on the sphere of radius r is $2\pi r^2(1 - \cos(\rho/r))$, which can be derived using calculus. Thus, the area of a disc of radius ρ on $\mathcal{H}$ is $2\pi(\cosh(\rho) - 1)$, which can also be derived using calculus.

The distance $|AB|$ on the sphere of radius r defined by $r^2 \cos(|AB|/r) = A \cdot B$ is the one induced by the arclength element in $\mathbb{R}^3$. The canonical choice of distance on a sphere is to measure it in radians, which is to instead define $|AB|$ by $r^2 \cos|AB| = A \cdot B$. Using these units (radians), the sphere has curvature 1. In a similar fashion, the distance $|AB|$ defined in the introduction, where $\mathcal{H}$ is one of the sheets in $\mathcal{V}_k$, is the canonical distance chosen so that $\mathcal{H}$ has curvature -1.

Extending either the spherical or hyperbolic geometry to higher dimensions is of comparable difficulty.

An inner product $\mathbf{x} \cdot \mathbf{y} := \mathbf{x}^t J\mathbf{y}$ defined by a symmetric positive definite J can be thought of as the standard dot product after a suitable change of basis. In the same way, a Lorentz product defined by a J with signature $(n, 1)$ can be expressed in terms of the standard Lorentz product after a suitable change of basis.

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