New York Journal of Mathematics

New York J. Math. **20** (2014) 727–741.

On the AJ conjecture for cables of the figure eight knot

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Abstract. The AJ conjecture relates the A-polynomial and the colored Jones polynomial of a knot in the 3-sphere. It has been verified for some classes of knots, including all torus knots, most double twist knots, $(-2, 3, 6n \pm 1)$-pretzel knots, and most cabled knots over torus knots. In this paper we study the AJ conjecture for $(r, 2)$-cables of a knot, where r is an odd integer. In particular, we show that the $(r, 2)$-cable of the figure eight knot satisfies the AJ conjecture if r is an odd integer satisfying $|r| \geq 9$.

Contents

Received July 20, 2014.

2010 *Mathematics Subject Classification.* Primary 57N10. Secondary 57M25.

Key words and phrases. Colored Jones polynomial, A-polynomial, AJ conjecture, figure eight knot.

ISSN 1076-9803/2014

1. Introduction

1.1. The colored Jones function. For a knot K in the 3-sphere and a positive integer n, let $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ denote the n-colored Jones polynomial of K with framing zero. The polynomial $J_K(n)$ is the quantum link invariant, as defined by Reshetikhin and Turaev [RT], associated to the Lie algebra $sl_2(\mathbb{C})$, with the color n standing for the irreducible $sl_2(\mathbb{C})$-module V_n of dimension n. Here we use the functorial normalization, i.e., the one for which the colored Jones polynomial of the unknot U is

$$J_U(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

For example, the colored Jones polynomial of the figure eight knot E is

$$J_E(n) = [n] \sum_{k=0}^{n-1} \prod_{l=1}^{k} (t^{4n} + t^{-4n} - t^{4l} - t^{-4l}).$$

It is known that $J_K(1) = 1$ and $J_K(2)$ is the usual Jones polynomial [Jo]. The colored Jones polynomials of higher colors are more or less the usual Jones polynomials of parallels of the knot. The color n can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, we have $J_K(0) = 0$.

The colored Jones polynomials are not random. For a fixed knot K, Garoufalidis and Le [GaL] proved that the colored Jones function

$$J_K : \mathbb{Z} \to \mathbb{Z}[t^{\pm 1}]$$

satisfies a nontrivial linear recurrence relation of the form

$$\sum_{k=0}^{d} a_k(t, t^{2n}) J_K(n+k) = 0,$$

where $a_k(u, v) \in \mathbb{C}[u, v]$ are polynomials with greatest common divisor 1.

1.2. Recurrence relations and *q*-holonomicity. Let $\mathcal{R} := \mathbb{C}[t^{\pm 1}]$. Consider a discrete function $f : \mathbb{Z} \to \mathcal{R}$, and define the linear operators L and M acting on such functions by

$$(Lf)(n) := f(n+1), \qquad (Mf)(n) := t^{2n} f(n).$$

It is easy to see that $LM = t^2 ML$. The inverse operators L^{-1}, M^{-1} are well-defined. We can consider L, M as elements of the quantum torus

$$\mathcal{T} := \mathcal{R}\langle L^{\pm 1}, M^{\pm 1}\rangle / (LM - t^2 ML),$$

which is a noncommutative ring.

The recurrence ideal of the discrete function f is the left ideal $\mathcal{A}_f$ in $\mathcal{T}$ that annihilates f:

$$\mathcal{A}_f := \{P \in \mathcal{T} \mid Pf = 0\}.$$

We say that f is q-holonomic, or f satisfies a nontrivial linear recurrence relation, if $\mathcal{A}_f \neq 0$. For example, for a fixed knot K the colored Jones function J_K is q-holonomic.

1.3. The recurrence polynomial of a *q*-holonomic function. Suppose that $f : \mathbb{Z} \to \mathcal{R}$ is a q-holonomic function. Then $\mathcal{A}_f$ is a nonzero left ideal of $\mathcal{T}$. The ring $\mathcal{T}$ is not a principal left ideal domain, i.e., not every left ideal of $\mathcal{T}$ is generated by one element. Garoufalidis [Ga] noticed that by adding all inverses of polynomials in t, M to $\mathcal{T}$ we get a principal left ideal domain $\tilde{\mathcal{T}}$, and hence from the ideal $\mathcal{A}_K$ we can define a polynomial invariant. Formally, we can proceed as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \left\{ \sum_{k \in \mathbb{Z}} a_k(M) L^k \mid a_k(M) \in \mathcal{R}(M),\ a_k = 0 \text{ almost always} \right\},$$

and define the product in $\tilde{\mathcal{T}}$ by $a(M)L^k \cdot b(M)L^l = a(M)b(t^{2k}M)L^{k+l}$.

Then it is known that every left ideal in $\tilde{\mathcal{T}}$ is principal, and $\mathcal{T}$ embeds as a subring of $\tilde{\mathcal{T}}$. The extension $\tilde{\mathcal{A}}_f := \tilde{\mathcal{T}}\mathcal{A}_f$ of $\mathcal{A}_f$ in $\tilde{\mathcal{T}}$ is then generated by a single polynomial

$$\alpha_f(t, M, L) = \sum_{k=0}^{d} \alpha_{f,k}(t, M)\, L^k,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{f,k}(t, M) \in \mathbb{C}[t^{\pm 1}, M]$ are assumed to be co-prime. The polynomial α_f is defined up to a polynomial in $\mathbb{C}[t^{\pm 1}, M]$. We call α_f the recurrence polynomial of the discrete function f.

When f is the colored Jones function J_K of a knot K, we let $\mathcal{A}_K$ and α_K denote the recurrence ideal $\mathcal{A}_{J_K}$ and the recurrence polynomial α_{J_K} of J_K respectively. We also say that $\mathcal{A}_K$ and α_K are the recurrence ideal and the recurrence polynomial of the knot K. Since $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$, we can assume that $\alpha_K(t, M, L) = \sum_{k=0}^{d} \alpha_{K,k}(t, M)L^k$ where all the coefficients $\alpha_{K,k} \in \mathbb{Z}[t^{\pm 1}, M]$ are co-prime.

1.4. The AJ conjecture. The colored Jones polynomials are powerful invariants of knots, but little is known about their relationship with classical topology invariants like the fundamental group. Inspired by the theory of noncommutative A-ideals of Frohman, Gelca and Lofaro [FGL, Ge] and the theory of q-holonomicity of quantum invariants of Garoufalidis and Le [GaL], Garoufalidis [Ga] formulated the following conjecture that relates the A-polynomial and the colored Jones polynomial of a knot in the 3-sphere.

Conjecture 1 (AJ conjecture)**.** *For every knot K, $\alpha_K|_{t=-1}$ is equal to the A-polynomial, up to a factor depending on M only.*

The A-polynomial of a knot was introduced by Cooper et al. [CoCGLS]; it describes the $SL_2(\mathbb{C})$-character variety of the knot complement as viewed from the boundary torus. The A-polynomial carries important information about the geometry and topology of the knot. For example, it distinguishes the unknot from other knots [DG, BZ], and the sides of its Newton polygon give rise to incompressible surfaces in the knot complement [CoCGLS]. Here in the definition of the A-polynomial, we also allow the factor $L-1$ coming from the abelian component of the character variety of the knot group. Hence the A-polynomial in this paper is equal to $L-1$ times the A-polynomial defined in [CoCGLS].

The AJ conjecture has been verified for the trefoil knot, the figure eight knot (by Garoufalidis [Ga]), all torus knots (by Hikami [Hi], Tran [Tr13]), some classes of two-bridge knots and pretzel knots including most double twist knots and $(-2,3,6n\pm 1)$-pretzel knots (by Le [Le], Le and Tran [LeT12]), the knot 7_4 (by Garoufalidis and Koutschan [GaK]), and most cabled knots over torus knots (by Ruppe and Zhang [RZ]).

Note that there is a stronger version of the AJ conjecture, formulated by Sikora [Si], which relates the recurrence ideal and the A-ideal of a knot. The A-ideal determines the A-polynomial of a knot. This conjecture has been verified for the trefoil knot (by Sikora [Si]), all torus knots [Tr13] and most cabled knots over torus knots [Tr14].

1.5. Main result. Suppose K is a knot with framing zero, and r, s are two integers with c their greatest common divisor. The (r,s)-cable $K^{(r,s)}$ of K is the link consisting of c parallel copies of the $(\frac{r}{c},\frac{s}{c})$-curve on the torus boundary of a tubular neighborhood of K. Here an $(\frac{r}{c},\frac{s}{c})$-curve is a curve that is homologically equal to $\frac{r}{c}$ times the meridian and $\frac{s}{c}$ times the longitude on the torus boundary. The cable $K^{(r,s)}$ inherits an orientation from K, and we assume that each component of $K^{(r,s)}$ has framing zero. Note that if r and s are co-prime, then $K^{(r,s)}$ is again a knot.

In [LeT10], we studied the volume conjecture [Ka, MuM] for $(r,2)$-cables of a knot and especially $(r,2)$-cables of the figure eight knot, where r is an integer. In this paper we study the AJ conjecture for $(r,2)$-cables of a knot, where r is an odd integer. In particular, we will show the following.

Theorem 1. *The $(r,2)$-cable of the figure eight knot satisfies the AJ conjecture if r is an odd integer satisfying $|r|\geq 9$.*

1.6. Plan of the paper. In Section 2 we prove some properties of the colored Jones polynomial of cables of a knot. In Section 3 we study the AJ conjecture for $(r,2)$-cables of the figure eight knot and prove Theorem 1.

1.7. Acknowledgments. I would like to thank Thang T.Q. Le and Xingru Zhang for helpful discussions. I would also like to thank the referee for comments and suggestions. Dennis Ruppe [Ru] has independently obtained a similar result to Theorem 1.

2. The colored Jones polynomial of cables of a knot

Recall from the introduction that for each positive integer n, there is a unique irreducible $sl_2(\mathbb{C})$-module V_n of dimension n.

From now on we assume that r is an odd integer. Then the $(r, 2)$-cable $K^{(r,2)}$ of a knot K is a knot. The calculation of the colored Jones polynomial of $K^{(r,2)}$ is standard: we decompose $V_n \otimes V_n$ into irreducible components

$$V_n \otimes V_n = \bigoplus_{k=1}^{n} V_{2k-1}.$$

Since the R-matrix commutes with the actions of the quantized algebra, it acts on each component V_{2k-1} as a scalar μ_k times the identity. The value of μ_k is well-known:

$$\mu_k = (-1)^{n-k} t^{-2(n^2-1)} t^{2k(k-1)}.$$

Hence from the theory of quantum invariants (see, e.g., [Oh]), we have

$$\begin{aligned} J_{K^{(r,2)}}(n) &= \sum_{k=1}^{n} \mu_k^r J_K(2k-1) \\ &= t^{-2r(n^2-1)} \sum_{k=1}^{n} (-1)^{r(n-k)} t^{2rk(k-1)} J_K(2k-1). \end{aligned} \tag{1}$$

Note that t in this paper is equal to $q^{1/4}$ in [LeT10].

Lemma 2.1. *We have*

$$J_{K^{(r,2)}}(n+1) = -t^{-2r(2n+1)} J_{K^{(r,2)}}(n) + t^{-2rn} J_K(2n+1).$$

Proof. From Equation (1) we have

$$\begin{aligned} &J_{K^{(r,2)}}(n+1) \\ &= t^{-2r(n^2+2n)} \sum_{k=1}^{n+1} (-1)^{r(n+1-k)} t^{2rk(k-1)} J_K(2k-1) \\ &= t^{-2rn} J_K(2n+1) + (-1)^r t^{-2r(n^2+2n)} \sum_{k=1}^{n} (-1)^{r(n-k)} t^{2rk(k-1)} J_K(2k-1) \\ &= t^{-2rn} J_K(2n+1) + (-1)^r t^{-2r(2n+1)} J_{K^{(r,2)}}(n). \end{aligned}$$

The lemma follows, since $(-1)^r = -1$. □

Let $\mathbb{J}_K(n) := J_K(2n+1)$. Note that q-holonomicity is preserved under taking subsequences of the form $kn + l$, see, e.g., [KK]. Since J_K is q-holonomic, we have the following.

Proposition 2.2. *For a fixed knot K, the function $\mathbb{J}_K$ is q-holonomic.*

Note that $\mathbb{J}_K(n-1) + \mathbb{J}_K(-n) = 0$. Recall that $\mathcal{A}_{\mathbb{J}_K}$ and $\alpha_{\mathbb{J}_K}$ denote the recurrence ideal and the recurrence polynomial of $\mathbb{J}_K$ respectively.

Lemma 2.3. *If* $P(t,M,L) \in \mathcal{A}_{\mathbb{J}_K}$ *then* $P(t,(t^2M)^{-1},L^{-1}) \in \mathcal{A}_{\mathbb{J}_K}$.

Proof. Suppose that $P(t,M,L) = \sum \lambda_{k,l} M^k L^l$, where $\lambda_{k,l} \in \mathcal{R} = \mathbb{C}[t^{\pm 1}]$, annihilates $\mathbb{J}_K$. Since $\mathbb{J}_K(n-1) + \mathbb{J}_K(-n) = 0$ for all integers n, we have

$$\begin{aligned} 0 &= P\mathbb{J}_K(-n-1) \\ &= \sum \lambda_{k,l}\, t^{-2(n+1)k} \mathbb{J}_K(-n-1+l) \\ &= -\sum \lambda_{k,l}\, t^{-2(n+1)k} \mathbb{J}_K(n-l) \\ &= -\sum \lambda_{k,l} (t^2M)^{-k} L^{-l} \mathbb{J}_K(n). \end{aligned}$$

Hence $P(t,(t^2M)^{-1},L^{-1})\mathbb{J}_K = 0$. □

For a Laurent polynomial $f(t) \in \mathcal{R}$, let $d_+[f]$ and $d_-[f]$ be respectively the maximal and minimal degree of t in f. The difference $\mathrm{br}[f] := d_+[f] - d_-[f]$ is called the breadth of f.

Lemma 2.4. *Suppose K is a nontrivial alternating knot. Then* $\mathrm{br}[\mathbb{J}_K(n)]$ *is a quadratic polynomial in n.*

Proof. Since K is a nontrivial alternating knot, [Le, Proposition 2.1] implies that $\mathrm{br}[J_K(n)]$ is a quadratic polynomial in n. Since

$$\mathrm{br}[\mathbb{J}_K(n)] = \mathrm{br}[J_K(2n+1)],$$

the lemma follows. □

Proposition 2.5. *Suppose K is a nontrivial alternating knot. Then the recurrence polynomial $\alpha_{\mathbb{J}_K}$ of $\mathbb{J}_K$ has L-degree > 1.*

Proof. Suppose that $\alpha_{\mathbb{J}_K}(t,M,L) = P_1(t,M)L + P_0(t,M)$, where $P_1, P_0 \in \mathbb{Z}[t^{\pm 1}, M]$ are co-prime. Note that the polynomial

$$\alpha_{\mathbb{J}_K}(t,(t^2M)^{-1},L^{-1}) = P_1(t,t^{-2}M^{-1})L^{-1} + P_0(t,t^{-2}M^{-1})$$

is in the recurrence ideal $\mathcal{A}_{\mathbb{J}_K}$ of $\mathbb{J}_K$, by Lemma 2.3. Since $\alpha_{\mathbb{J}_K}$ is the generator of $\tilde{\mathcal{A}}_{\mathbb{J}_K} = \tilde{\mathcal{T}}\mathcal{A}_{\mathbb{J}_K}$ in $\tilde{\mathcal{T}}$, there exists $\gamma(t,M) \in \mathcal{R}(M)$ such that

$$\gamma(t,M)L\left(P_1(t,t^{-2}M^{-1})L^{-1} + P_0(t,t^{-2}M^{-1})\right) = P_1(t,M)L + P_0(t,M).$$

This is equivalent to

$$\begin{aligned} P_0(t,M) &= \gamma(t,M)P_1(t,t^{-4}M^{-1}), \\ P_1(t,M) &= \gamma(t,M)P_0(t,t^{-4}M^{-1}). \end{aligned}$$

Since P_0 and P_1 are coprime in $\mathbb{Z}[t^{\pm 1}, M]$, it follows from the above equations that $\gamma(t,M)$ is a unit element in $\mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$, i.e., $\gamma(t,M) = \pm t^k M^l$. Hence $P_0(t,M) = \pm t^k M^l P_1(t,t^{-4}M^{-1})$.

The equation $\alpha_{\mathbb{J}_K}\mathbb{J}_K = 0$ can now be written as

$$\mathbb{J}_K(n+1) = \pm \frac{t^{2nl+k} P_1(t,t^{-4-2n})}{P_1(t,t^{2n})}\, \mathbb{J}_K(n).$$

This implies that

$$\operatorname{br}[\mathbb{J}_K(n+1)] - \operatorname{br}[\mathbb{J}_K(n)] = \operatorname{br}(t^{2nl+k}P_1(t, t^{-4-2n})) - \operatorname{br}(P_1(t, t^{2n}).$$

It is easy to see that for n big enough, $\operatorname{br}(t^{2nl+k}P_1(t, t^{-4-2n})) - \operatorname{br}(P_1(t, t^{2n}))$ is a constant independent of n. Hence the breadth of $\mathbb{J}_K(n)$, for n big enough, is a linear function on n. This contradicts Lemma 2.4, since K is a nontrivial alternating knot. □

Let ε be the map reducing $t = -1$.

Proposition 2.6. *For any $P \in \mathcal{A}_{\mathbb{J}_K}$, $\varepsilon(P)$ is divisible by $L - 1$.*

Proof. The proof of Proposition 2.6 is similar to that of [Le, Proposition 2.3], which makes use of the Melvin–Morton conjecture proved by Bar-Natan and Garoufalidis [BG].

It is known that for any knot K (with framing zero), $J_K(n)/[n]$ is a Laurent polynomial in t^4. Moreover, the Melvin–Morton conjecture [MeM] says that for any $z \in \mathbb{C}^*$ we have

$$\lim_{n\to\infty}\left(\frac{J_K(n)}{[n]}\ |_{t^2=z^{1/n}}\right) = \frac{1}{\Delta_K(z)},$$

where $\Delta_K(z)$ is the Alexander polynomial of K.

For $l \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0, \pm 1\}$, we let

$$\begin{aligned}\widehat{\mathbb{J}}_K(l, z) &:= \lim_{n\to\infty}\left(\frac{J_K(2n+2l+1)}{[2n+2l+1]}\ |_{t^2=z^{1/(2n+1)}}\right)\\ &= \lim_{n\to\infty}\left(\frac{t^2-t^{-2}}{z-z^{-1}}\,\mathbb{J}_K(n+l)\ |_{t^2=z^{1/(2n+1)}}\right).\end{aligned}$$

Then

$$\widehat{\mathbb{J}}_K(0, z) = \lim_{n\to\infty}\left(\frac{J_K(2n+1)}{[2n+1]}\ |_{t^2=z^{1/(2n+1)}}\right) = \frac{1}{\Delta_K(z)}.$$

In particular, we have $\widehat{\mathbb{J}}_K(0, z) \neq 0$.

Claim 1. *For any $l \in \mathbb{Z}$, we have $\widehat{\mathbb{J}}_K(l, z) = \widehat{\mathbb{J}}_K(0, z)$.*

Proof of Claim 1. For any knot K, by [MeM] we have

$$\frac{J_K(n)}{[n]}|_{t^4=e^h} = \sum_{k=0}^{\infty} P_k(n)h^k,$$

where $P_k(n)$ is a polynomial in n of degree at most k:

$$P_k(n) = P_{k,k}n^k + P_{k,k-1}n^{k-1} + \cdots + P_{k,1}n + P_{k,0}.$$

Then

$$\widehat{\mathbb{J}}_K(l,z) = \lim_{n\to\infty}\left(\frac{J_K(2n+2l+1)}{[2n+2l+1]}\ |_{t^2=z^{1/(2n+1)}}\right)$$
$$= \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{k} P_{k,j}\,(2n+2l+1)^j h^k\ |_{h=\frac{2\ln z}{2n+1}}\right).$$

We have

$$\lim_{n\to\infty}(2n+2l+1)^j\left(\frac{2\ln z}{2n+1}\right)^k = \begin{cases} 0 & \text{if } j<k \\ (2\ln z)^k & \text{if } j=k, \end{cases}$$

which is independent of l. Claim 1 follows. □

We now complete the proof of Proposition 2.6. Suppose $P = \sum \lambda_{k,l} M^k L^l$, where $\lambda_{k,l} \in \mathcal{R}$. Then $\sum \lambda_{k,l}\, t^{2kn}\mathbb{J}_K(n+l) = 0$ for all integers n.

For $z \in \mathbb{C}\setminus\{0,\pm1\}$, by Claim 1 we have

$$0 = \lim_{n\to\infty}\left(\sum \lambda_{k,l}\, t^{2kn}\frac{t^2-t^{-2}}{z-z^{-1}}\,\mathbb{J}_K(n+l)\ |_{t^2=z^{1/(2n+1)}}\right)$$
$$= \sum(\lambda_{k,l}\ |_{t^2=1})z^{k/2}\,\widehat{\mathbb{J}}_K(l,z)$$
$$= (P\ |_{t^2=1,M=z^{1/2},L=1})\widehat{\mathbb{J}}_K(0,z).$$

Since $\widehat{\mathbb{J}}_K(0,z)\neq 0$, we have $P\ |_{t^2=1,M=z^{1/2},L=1}= 0$ for all $z\in\mathbb{C}\setminus\{0,\pm1\}$. This implies that $P\ |_{t^2=1}$ is divisible by $L-1$. Proposition 2.6 follows. □

Proposition 2.7. *$\varepsilon(\alpha_{\mathbb{J}_K})$ has L-degree 1 if and only if $\alpha_{\mathbb{J}_K}$ has L-degree 1.*

Proof. The backward direction is obvious since $\varepsilon(\alpha_{\mathbb{J}_K})$ is always divisible by $L-1$, by Proposition 2.6. Suppose that $\varepsilon(\alpha_{\mathbb{J}_K}) = g(M)(L-1)$ for some $g(M)\in\mathbb{C}[M^{\pm1}]\setminus\{0\}$. Then

$$\alpha_{\mathbb{J}_K} = g(M)(L-1) + (1+t)\sum_{k=0}^{d} a_k(M)L^k, \tag{2}$$

where $a_k(M)\in\mathcal{R}[M^{\pm1}]$ and d is the L-degree of $\alpha_{\mathbb{J}_K}$.

Since $\alpha_{\mathbb{J}_K}(t,(t^2M)^{-1},L^{-1})$ is also in the recurrence ideal of $\mathbb{J}_K$,

$$\alpha_{\mathbb{J}_K}(t,M,L) = h(M)\alpha_{\mathbb{J}_K}(t,(t^2M)^{-1},L^{-1})L^d$$

for some $h(M)\in\mathcal{R}(M)$. Equation (2) then becomes

$$g(M)(L-1) + (1+t)\sum_{k=0}^{d} a_k(M)L^k$$
$$= h(M)g(t^{-2}M^{-1})(L^{-1}-1)L^d + (1+t)\sum_{k=0}^{d} h(M)a_k(t^{-2}M^{-1})L^{d-k}.$$

Suppose that $d > 1$. By comparing the coefficients of L^0 in both sides of the above equation, we get $-g(M)+(1+t)a_0(M) = (1+t)h(M)a_d(t^{-2}M^{-1})$. This is equivalent to

$$g(M) = (1+t)\left(a_0(M) - h(M)a_d(t^{-2}M^{-1})\right). \tag{3}$$

Since $g(M)$ is a Laurent polynomial in M with coefficients in $\mathbb{C}$, Equation (3) implies that $g(M) = 0$. This is a contradiction. Hence $d = 1$. □

3. Proof of Theorem 1

Let E be the figure eight knot. By [Ha] we have

$$J_E(n) = [n]\sum_{k=0}^{n-1}\prod_{l=1}^{k}(t^{4n} + t^{-4n} - t^{4l} - t^{-4l}). \tag{4}$$

Recall that $E^{(r,2)}$ is the $(r,2)$-cable of E and $\mathbb{J}_E(n) = J_E(2n+1)$. By Lemma 2.1, we have

$$M^r(L + t^{-2r}M^{-2r})J_{E^{(r,2)}} = \mathbb{J}_E. \tag{5}$$

For nonzero $f, g \in \mathbb{C}[M^{\pm 1}, L]$, we write $f \overset{M}{=} g$ if the quotient f/g does not depend on L. Proving Theorem 1 is then equivalent to proving that $\varepsilon(\alpha_{E^{(r,2)}}) \overset{M}{=} A_{E^{(r,2)}}$, where

$$\begin{aligned} &A_{E^{(r,2)}} \\ &= (L-1)\left\{L^2 - ((M^8 + M^{-8} - M^4 - M^{-4} - 2)^2 - 2)L + 1\right\}(L + M^{-2r}) \end{aligned}$$

is the A-polynomial of $E^{(r,2)}$ cf. [NZ].

The proof of $\varepsilon(\alpha_{E^{(r,2)}}) \overset{M}{=} A_{E^{(r,2)}}$ is divided into 4 steps.

3.1. Degree formulas for the colored Jones polynomials.

The following lemma will be used later in the proof of Theorem 1.

Lemma 3.1. *For $n > 0$ we have*

$$\begin{aligned} d_+[J_E(n)] &= 4n^2 - 2n - 2, \\ d_-[J_E(n)] &= -4n^2 + 2n + 2, \end{aligned}$$

$$d_+[J_{E^{(r,2)}}(n)] = \begin{cases} 16n^2 - (2r+20)n + 2r + 4 & \text{if } r \geq -7 \\ -2rn^2 + 2r & \text{if } r \leq -9, \end{cases}$$

$$d_-[J_{E^{(r,2)}}(n)] = \begin{cases} -2rn^2 + 2r & \text{if } r \geq 9 \\ -16n^2 - (2r-20)n + 2r - 4 & \text{if } r \leq 7. \end{cases}$$

Proof. The first two formulas follow directly from Equation (4). We now prove the formula for $d_+[J_{E^{(r,2)}}(n)]$. The one for $d_-[J_{E^{(r,2)}}(n)]$ is proved similarly.

From Equation (1), we have

$$\begin{aligned} d_+[J_{E^{(r,2)}}(n)] &= -2r(n^2-1) + \max_{1\le k\le n}\{2rk(k-1) + d_+[J_E(2k-1)]\} \\ &= -2r(n^2-1) + \max_{1\le k\le n}\{(2r+16)k^2 - (2r+20)k + 4\}. \end{aligned}$$

Let $f(k) := (2r+16)k^2 - (2r+20)k + 4$, where $1 \le k \le n$. If $r \ge -7$, $f(k)$ attains its maximum at $k = n$. If $r \le -9$, $f(k)$ attains its maximum at $k = 1$. The lemma follows. □

3.2. An inhomogeneous recurrence relation for $\mathbb{J}_E$. Let

$$\begin{aligned} P_1(t,M) &:= t^{-2}M^2 - t^2M^{-2}, \\ P_{-1}(t,M) &:= t^2M^2 - t^{-2}M^{-2}, \\ P_0(t,M) &:= (M^2 - M^{-2})(-M^4 - M^{-4} + M^2 + M^{-2} + t^4 + t^{-4}). \end{aligned}$$

From [ChM, Proposition 4.4] (see also [GaS]) we have

$$(P_1L + P_{-1}L^{-1} + P_0)J_E \in \mathcal{R}[M^{\pm 1}]. \tag{6}$$

Let

$$\begin{aligned} Q_1(t,M) &:= P_1(t,M)P_1(t,t^2M)P_0(t,t^{-2}M), \\ Q_{-1}(t,M) &:= P_{-1}(t,M)P_{-1}(t,t^{-2}M)P_0(t,t^2M), \\ Q_0(t,M) &:= P_1(t,M)P_{-1}(t,t^2M)P_0(t,t^{-2}M) \\ &\quad + P_{-1}(t,M)P_1(t,t^{-2}M)P_0(t,t^2M) \\ &\quad - P_0(t,M)P_0(t,t^2M)P_0(t,t^{-2}M). \end{aligned}$$

Proposition 3.2. *We have*

$$\left\{Q_1(t,t^2M^2)L + Q_{-1}(t,t^2M^2)L^{-1} + Q_0(t,t^2M^2)\right\}\mathbb{J}_E \in \mathcal{R}[M^{\pm 1}].$$

Proof. We first note that

$$\begin{aligned} &Q_1(t,M)L^2 + Q_{-1}(t,M)L^{-2} + Q_0(t,M) \\ &= P_1(t,M)P_1(t,t^2M)P_0(t,t^{-2}M)L^2 \\ &\quad + P_{-1}(t,M)P_{-1}(t,t^{-2}M)P_0(t,t^2M)L^{-2} \\ &\quad + P_1(t,M)P_{-1}(t,t^2M)P_0(t,t^{-2}M) \\ &\quad + P_{-1}(t,M)P_1(t,t^{-2}M)P_0(t,t^2M) \\ &\quad - P_0(t,M)P_0(t,t^2M)P_0(t,t^{-2}M) \\ &= \big\{P_1(t,M)P_0(t,t^{-2}M)L + P_{-1}(t,M)P_0(t,t^2M)L^{-1} \\ &\qquad - P_0(t,t^2M)P_0(t,t^{-2}M)\big\} \\ &\quad \times \left\{P_1(t,M)L + P_{-1}(t,M)L^{-1} + P_0(t,M)\right\}. \end{aligned}$$

By Equation (6) we have $(P_1L + P_{-1}L^{-1} + P_0)J_E \in \mathcal{R}[M^{\pm 1}]$. Hence

$$(Q_1L^2 + Q_{-1}L^{-2} + Q_0)J_E \in \mathcal{R}[M^{\pm 1}]. \tag{7}$$

We have $(M^k L^{2l} J_E)(2n+1) = ((t^2M^2)^k L^l \mathbb{J}_E)(n)$. It follows that

$$(P(t,M)L^{2l}J_E)(2n+1) = (P(t,t^2M^2)L^l\mathbb{J}_E)(n)$$

for any $P(t,M) \in \mathcal{R}[M^{\pm 1}]$. Hence Equation (7) implies that

$$\left\{Q_1(t,t^2M^2)L + Q_{-1}(t,t^2M^2)L^{-1} + Q_0(t,t^2M^2)\right\}\mathbb{J}_E \in \mathcal{R}[M^{\pm 1}].$$

This proves Proposition 3.2. □

3.3. A recurrence relation for $J_{E(r,2)}$. Let

$$Q(t,M,L) := Q_1(t,t^2M^2)L + Q_{-1}(t,t^2M^2)L^{-1} + Q_0(t,t^2M^2).$$

By Proposition 3.2, we have $Q\mathbb{J}_E \in \mathcal{R}[M^{\pm 1}]$. Equation (5) then implies that

$$QM^r(L + t^{-2r}M^{-2r})J_{E^{(r,2)}} \in \mathcal{R}[M^{\pm 1}]. \tag{8}$$

Let $Q'(t,M) := LQ(t,M)M^r(L+t^{-2r}M^{-2r})$. From Equation (8) we have $Q'J_{E^{(r,2)}} \in \mathcal{R}[M^{\pm 1}]$.

Let $R := Q'J_{E^{(r,2)}} \in \mathcal{R}[M^{\pm 1}]$. We claim that $R \neq 0$, which means that $Q'J_{E^{(r,2)}} = R$ is an inhomogeneous recurrence relation for $J_{E^{(r,2)}}$. Indeed, assume that $R = 0$. Then Q' annihilates the colored Jones function $J_{E^{(r,2)}}$. By [Le, Proposition 2.3], $\varepsilon(Q')$ is divisible by $L-1$. However this cannot occur, since

$$\varepsilon(Q') \overset{M}{=} \left\{L^2 - \left((M^8 + M^{-8} - M^4 - M^{-4} - 2)^2 - 2\right)L + 1\right\}(L + M^{-2r})$$

is not divisible by $L-1$. Hence $R \neq 0$ in $\mathcal{R}[M^{\pm 1}]$.

Write $R(t,M) = (1+t)^m R'(t,M)$, where $m \geq 0$ and $R'(-1,M) \neq 0$ in $\mathbb{C}[M^{\pm 1}]$. Let

$$S(t,M,L) := (R'(t,M)L - R'(t,t^2M))Q'(t,M).$$

Since $Q'J_{E^{(r,2)}} = (1+t)^m R' \in \mathcal{R}[M^{\pm 1}]$ is an inhomogeneous recurrence relation for $J_{E^{(r,2)}}$, we have the following.

Proposition 3.3. *The polynomial $S \in \mathcal{T}$ annihilates the colored Jones function $J_{E^{(r,2)}}$ and has L-degree 4.*

3.4. Completing the proof of Theorem 1. Note that S has L-degree 4 and $\varepsilon(S) \overset{M}{=} A_{E^{(r,2)}}$. Hence to complete the proof of Theorem 1, we only need to show that if $|r| \geq 9$ then S is equal to the recurrence polynomial $\alpha_{E^{(r,2)}}$ in $\tilde{\mathcal{T}}$, up to a rational function in $\mathcal{R}(M)$. This is achieved by showing that there does not exist a nonzero polynomial $P \in \mathcal{R}[M^{\pm 1}][L]$ of degree ≤ 3 that annihilates the colored Jones function $J_{E^{(r,2)}}$. We will make use of the degree formulas in Subsection 3.1.

From now on we assume that r is an odd integer satisfying $|r| \geq 9$. Suppose that $P = P_3L^3 + P_2L^2 + P_1L + P_0$, where $P_k \in \mathcal{R}[M^{\pm 1}]$, annihilates $J_{E^{(r,2)}}$. We want to show that $P_k = 0$ for $0 \leq k \leq 3$.

Indeed, by applying Lemma 2.1 we have

$$\begin{aligned}0 &= P_3 J_{E^{(r,2)}}(n+3) + P_2 J_{E^{(r,2)}}(n+2) + P_1 J_{E^{(r,2)}}(n+1) + P_0 J_{E^{(r,2)}}(n)\\ &= \left(-t^{-2r(6n+9)} P_3 + t^{-2r(4n+4)} P_2 - t^{-2r(2n+1)} P_1 + P_0\right) J_{E^{(r,2)}}(n)\\ &\quad + \left(t^{-2r(5n+8)} P_3 - t^{-2r(3n+3)} P_2 + t^{-2rn} P_1\right) J_E(2n+1)\\ &\quad + \left(-t^{-2r(3n+6)} P_3 + t^{-2r(n+1)} P_2\right) J_E(2n+3) + t^{-2r(n+2)} P_3 J_E(2n+5)\\ &= P_3' J_{E^{(r,2)}}(n) + P_2' J_E(2n+5) + P_1' J_E(2n+3) + P_0' J_E(2n+1).\end{aligned}$$

It is easy to see that $P_k = 0$ for $0 \le k \le 3$ if and only if $P_k' = 0$ for $0 \le k \le 3$.

Let $g(n) = P_2' J_E(2n+5) + P_1' J_E(2n+3) + P_0' J_E(2n+1)$. Then

$$P_3' J_{E^{(r,2)}}(n) + g(n) = 0. \tag{9}$$

We first show that $P_3' = 0$. Indeed, assume that $P_3' \neq 0$ in $\mathcal{R}[M^{\pm 1}]$. If $r \ge 9$ then, by Lemma 3.1, we have

$$d_-[P_3' J_{E^{(r,2)}}(n)] = d_-[J_{E^{(r,2)}}(n)] + O(n) = -2rn^2 + O(n).$$

Similarly, we have $d_-[P_k' J_E(2n+2k+1)] = -16n^2 + O(n)$ if $P_k' \neq 0$, where $k = 0, 1, 2$. It follows that, for n big enough,

$$\begin{aligned}&d_-[P_3' J_{E^{(r,2)}}(n)]\\ &< \min\{d_-[P_2' J_E(2n+5)], d_-[P_1' J_E(2n+3)], d_-[P_0' J_E(2n+1)]\}\\ &\le d_-[g(n)].\end{aligned}$$

Hence $d_-[P_3' J_{E^{(r,2)}}(n)] < d_-[g(n)]$. This contradicts Equation (9).

If $r \le -9$ then, by similar arguments as above, we have

$$\begin{aligned}&d_+[P_3' J_{E^{(r,2)}}(n)]\\ &> \max\{d_+[P_2' J_E(2n+5)], d_+[P_1' J_E(2n+3)], d_+[P_0' J_E(2n+1)]\}\\ &\ge d_+[g(n)].\end{aligned}$$

for n big enough. This also contradicts Equation (9). Hence $P_3' = 0$.

Since $g(n) = 0$, we have $(P_2' L^2 + P_1' L + P_0')\mathbb{J}_E = 0$. This means that $\mathbb{J}_E$ is annihilated by $P' := P_2' L^2 + P_1' L + P_0'$. We claim that $P' = 0$ in $\mathcal{R}[M^{\pm 1}][L]$. Indeed, assume that $P' \neq 0$. Since P' annihilates $\mathbb{J}_E$, it is divisible by the recurrence polynomial $\alpha_{\mathbb{J}_E}$ in $\tilde{\mathcal{T}}$. It follows that $\alpha_{\mathbb{J}_E}$, and hence $\varepsilon(\alpha_{\mathbb{J}_E})$, has L-degree ≤ 2.

Since E is a nontrivial alternating knot, Propositions 2.5, 2.6 and 2.7 imply that $\varepsilon(\alpha_{\mathbb{J}_E})$ is divisible by $L-1$ and has L-degree ≥ 2. Hence we conclude that $\varepsilon(\alpha_{\mathbb{J}_E})$ is divisible by $L-1$ and has L-degree exactly 2.

By Proposition 3.2, we have $Q\mathbb{J}_E \in \mathcal{R}[M^{\pm 1}]$. Let $Q'' := Q\mathbb{J}_E$. Then $Q'' \neq 0$ (otherwise, Q annihilates $\mathbb{J}_E$. However, this contradicts Proposition 2.6 since $\varepsilon(Q) \overset{M}{=} L^2 - \left((M^8 + M^{-8} - M^4 - M^{-4} - 2)^2 - 2\right) L + 1$ is not divisible by $L-1$). This means that $Q\mathbb{J}_E = Q'' \in \mathcal{R}[M^{\pm 1}]$ is an inhomogeneous recurrence relation for $\mathbb{J}_E$

Write $Q''(t,M) = (1+t)^m Q'''(t,M)$, where $m \geq 0$ and $Q'''(-1,M) \neq 0$ in $\mathbb{C}[M^{\pm 1}]$. Then $(Q'''(t,M)L - Q'''(t,t^2M))Q$ annihilates $\mathbb{J}_E$ and hence is divisible by $\alpha_{\mathbb{J}_E}$ in $\tilde{\mathcal{T}}$. Consequently, $(L-1)\varepsilon(Q)$ is divisible by $\varepsilon(\alpha_{\mathbb{J}_E})$ in $\mathbb{C}(M)[L]$. This means $\frac{\varepsilon(\alpha_{\mathbb{J}_E})}{L-1}$ divides $\varepsilon(Q)$ in $\mathbb{C}(M)[L]$. However this cannot occur, since $\frac{\varepsilon(\alpha_{\mathbb{J}_E})}{L-1}$ has L-degree exactly 1 and $\varepsilon(Q)$ is an irreducible polynomial in $\mathbb{C}[M^{\pm 1}, L]$ of L-degree 2.

Hence $P' = 0$, which means that $P'_k = 0$ for $0 \leq k \leq 2$. Consequently, $P_k = 0$ for $0 \leq k \leq 3$. This completes the proof of Theorem 1.

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This paper is available via http://nyjm.albany.edu/j/2014/20-36.html.