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Nonreciprocal units in a number field with an application to Oeljeklaus–Toma manifolds

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Abstract. In this paper we show that if a number field K contains a nonreciprocal unit u of degree $s+2t$ with s positive conjugates and $2t$ complex conjugates of equal moduli, where $t \geqslant 2$, then $s = (2t+2m)q-2t$ for some integers $m \geqslant 0$ and $q \geqslant 2$. On the other hand, for any s and $t \geqslant 2$ related as above we construct a number field K with s real and $2t$ complex embeddings that contains a nonreciprocal unit u of degree $s+2t$ with s positive conjugates and $2t$ complex conjugates of equal moduli. From this, for any pair of integers $s \geqslant 1$, $t \geqslant 2$ satisfying $s \neq (2t+2m)q-2t$ we deduce that the rank of the subgroup of units U whose $2t$ complex conjugates have equal moduli is smaller than s and, therefore, for any choice of an admissible subgroup A of K the corresponding Oeljeklaus–Toma manifold $X(K,A)$ admits no locally conformal Kähler metric.

Contents

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1. Introduction

Let throughout K be a number field of degree $d = [K : \mathbb{Q}]$ with $s \geqslant 0$ real embeddings $\sigma_1, \dots, \sigma_s$ into $\mathbb{C}$ and $2t$ complex embeddings

$$\sigma_{s+1}, \overline{\sigma}_{s+1}, \dots, \sigma_{s+t}, \overline{\sigma}_{s+t}$$

into $\mathbb{C}$, so that $d = s + 2t$. Here, for each $i = 1, \dots, t$ the embedding $\overline{\sigma}_{s+i} : K \to \mathbb{C}$ is defined as $\overline{\sigma}_{s+i}(a) = \overline{\sigma_{s+i}(a)}$ for every $a \in K$, where $\overline{z}$ stands for the complex conjugate of $z \in \mathbb{C}$. Let $\mathcal{O}_K^*$ be the group of units in the ring of integers of K. Put

$$\mathcal{U}_K := \{u \in \mathcal{O}_K^* : \sigma_i(u) > 0 \text{ for every } i = 1, \dots, s\}$$

for a subgroup of $\mathcal{O}_K^*$ consisting of units whose real conjugates are all positive.

Consider the logarithmic representation of units $l : \mathcal{O}_K^* \to \mathbb{R}^{s+t}$ given by

$$l(u) := (\log|\sigma_1(u)|, \dots, \log|\sigma_s(u)|, 2\log|\sigma_{s+1}(u)|, \dots, 2\log|\sigma_{s+t}(u)|).$$

By the Dirichlet's unit theorem, $l(\mathcal{U}_K)$ is a full discrete lattice in the subspace

$$\mathcal{S} := \left\{(x_1, \dots, x_{s+t}) \in \mathbb{R}^{s+t} : \sum_{i=1}^{s+t} x_i = 0\right\}$$

of $\mathbb{R}^{s+t}$. Equivalently (see, e.g., [13] and [20]), one can choose $s + t - 1$ multiplicatively independent units in $\mathcal{U}_K$, say $u_1, \dots, u_{s+t-1}$, such that every other unit in $\mathcal{U}_K$ can be written as $wu_1^{k_1} \dots u_{s+t-1}^{k_{s+t-1}}$ with a root of unity $w \in K$ and some $k_1, \dots, k_{s+t-1} \in \mathbb{Z}$. From now on, assume that $s \geqslant 1$. Then the projection $\mathcal{P} : \mathcal{S} \to \mathbb{R}^s$ given by the first s coordinates is surjective. Thus, there are subgroups A of rank s in $\mathcal{U}_K$ such that $\mathcal{P}(l(A))$ is a full discrete lattice in $\mathbb{R}^s$. Throughout, such a subgroup A will be called *admissible* for K. An admissible subgroup A for K is generated by s units $u_1, \dots, u_s \in \mathcal{U}_K$ such that the matrix

$$M(u_1, \dots, u_s) := (\log|\sigma_j(u_i)|)_{1 \leqslant i,j \leqslant s}$$

has rank s, so that these units are multiplicatively independent.

The results of this paper are motivated by some applications to the so-called locally conformal Kähler complex manifolds X (according to Vaisman [29], such a manifold is defined as a Hermitian manifold whose metric is conformal to a Kähler metric in some neighborhood of every point) and the corresponding study of locally conformal Kähler metrics (LCK metrics) on X (see, e.g., [3], [4], [17], [19], [28], [29], [30]). In [16], Oeljeklaus and Toma introduced some compact complex manifold $X(K, A)$ associated to a number field K and to an admissible subgroup A for K. These manifolds were named as *Oeljeklaus–Toma manifolds* and have many interesting properties (see the recent papers of Battisti and Oeljeklaus [1], Kasuya [14], Ornea and Verbitsky [18], Verbitsky [32], Vuletescu [33], etc.). In particular, it is known that if an Oeljeklaus–Toma manifold $X(K, A)$ admits an LCK metric then for all $u \in A$ we have

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)|$$

(see the proof of Proposition 2.9 in [16]). Since the numbers $\sigma_{s+i}(u)$ and $\overline{\sigma}_{s+i}(u)$ are complex conjugate, the previous condition can be written in the form

$$|\sigma_{s+1}(u)| = |\overline{\sigma}_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| = |\overline{\sigma}_{s+t}(u)|. \tag{1}$$

In the appendix of this paper (written by Laurent Battisti), it is shown (see Theorem 8) that the Oeljeklaus–Toma manifold $X(K, A)$ admits an LCK metric if and only if for all $u \in A$ the condition (1) holds (which is stronger than just the 'only if' condition that was proved in the previous result in [16]).

This raises the following natural question: are there s multiplicatively independent units $u_1, \dots, u_s$ in $\mathcal{U}_K$ such that (1) holds for each $u = u_1, \dots, u_s$? Such units would generate an admissible subgroup A for K and a corresponding manifold $X(K, A)$ with LCK metric. The answer is 'yes' for $s \geqslant 1$ and $t = 1$ (so far this is the only known case with a positive answer) and 'no' for $s = 1$ and $t \geqslant 2$ (see Proposition 2.9 in [16]). It is not clear whether or not there are some cases with $s, t \geqslant 2$ when the answer is positive. Vuletescu [33] has shown recently that the answer is 'no' for $1 < s < t$. Below, we will show that the answer is 'no' for any s that is not of the form (2) below. In particular, for $t \geqslant 2$ this implies a negative answer for $1 < s < 2t$ and also for s odd. Unfortunately, the second statement of Theorem 1 shows that for all other s the field K may contain a nonreciprocal unit. This leaves the problem open for some special pairs $s, t \geqslant 2$ satisfying (2), since our construction gives only one nonreciprocal unit in $\mathcal{U}_K$ instead of s multiplicatively independent units.

2. Main results

Recall that an algebraic number α is called *reciprocal* if α^{-1} is its conjugate over $\mathbb{Q}$ and *nonreciprocal* otherwise. The main result of this paper is the following:

Theorem 1. *If a number field K of degree $d = s + 2t$ over $\mathbb{Q}$ with s real and $2t$ complex embeddings, where $t \geqslant 2$, contains a nonreciprocal unit $u \in \mathcal{U}_K$ of degree d whose $2t$ nonreal algebraic conjugates satisfy* (1) *then for some integers $m \geqslant 0$ and $q \geqslant 2$ we have*

$$s = (2t + 2m)q - 2t. \tag{2}$$

On the other hand, if s and $t \geqslant 2$ satisfy (2) *with some integers $m \geqslant 0$ and $q \geqslant 2$ then there is a number field K with s real and $2t$ complex embeddings that contains a nonreciprocal unit $u \in \mathcal{U}_K$ of degree $d = s + 2t$ satisfying* (1).

In general, the situation when a number field K contains a nonreciprocal unit $u \in \mathcal{U}_K$ as described in Theorem 1 happens very rarely. If, for instance, the Galois group $\mathrm{Gal}(F/\mathbb{Q})$, where F is the normal closure of K over $\mathbb{Q}$, is 'large' (say the group $\mathrm{Gal}(F/\mathbb{Q})$ acts on d conjugates of $\alpha \in K$ as a full symmetric group S_d which is the 'generic' situation, by an old result of van der Waerden ([31]), then the equality (5) below cannot hold (see, e.g., [27]). Hence, such fields K do not contain units with the required properties.

From Theorem 1 we shall derive the following:

Theorem 2. *Let K be a number field of degree $d = s + 2t$ over $\mathbb{Q}$ with s real and $2t$ complex embeddings, where $s \geqslant 1$ and $t \geqslant 2$ are not of the form* (2)*. Then the rank of the subgroup U of $\mathcal{U}_K$ of units satisfying* (1) *is smaller than s and, therefore, for any choice of an admissible subgroup A for K the Oeljeklaus–Toma manifold $X(K, A)$ has no LCK metric.*

This implies the main result of [33], where the same conclusion as that of Theorem 2 has been proved under the assumption $1 < s < t$.

In the next section we shall give some auxiliary results. The proof of Theorem 1 is then given in Sections 4 and 5. In Section 5 one can also find an explicit example corresponding to the case $s = 4$, $t = 2$, $m = 0$ and $q = 2$ of Theorem 1. In Section 6 we shall give the proof of Theorem 2. Finally, in an appendix Laurent Battisti gives the proof of his Theorem 8 and using an alternative (geometrical) approach derives Theorem 2 from Theorem 1 as well.

3. Auxiliary results

An algebraic integer $\alpha > 1$ is called a *Perron number* if all of its conjugates over $\mathbb{Q}$ are less than α in absolute value. In particular, a Perron number is a *Pisot number* if its conjugates over $\mathbb{Q}$ (if any) are less than 1 in absolute value. We shall use *totally positive Pisot units* (Pisot numbers that are units whose algebraic conjugates are all positive) in Lemma 5 and subsequently in the proof of Theorem 1.

A version of the next lemma appears in [26]. Its proof is based on the argument of applying an automorphism of the Galois group that maps an algebraic number to its maximal (or minimal) conjugate which leads to a contradiction. This simple argument also plays a crucial role in the papers [7], [9], [27]. Below, we shall give a proof of the next lemma, since a similar argument appears several times in this paper as well.

Lemma 3. *Let α or α^{-1} be a Perron number of degree $d \geqslant 3$, and let $\alpha_1, \alpha_2, \alpha_3$ be any three distinct conjugates of α. Then $\alpha_1^2 \neq \alpha_2\alpha_3$.*

Proof. Assume that $\alpha_1^2 = \alpha_2\alpha_3$. Let F be the normal closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Take an automorphism of the Galois group of $F/\mathbb{Q}$ which maps α_1 to α. It maps α_2, α_3 to some conjugates $\alpha_i \neq \alpha_j$ of α, different from α, and so it maps the equality $\alpha_1^2 = \alpha_2\alpha_3$ into the equality $\alpha^2 = \alpha_i\alpha_j$. However, the modulus of its left hand side is greater than the modulus of its right hand side if α is Perron number (resp. smaller if α^{-1} is a Perron number), a contradiction. □

Corollary 4. *Suppose that a unit $u \in \mathcal{U}_K$ satisfying* (1) *has $2t' > 0$ distinct complex conjugates and s distinct real ones. Then none of its real conjugates has the same modulus as the complex (nonreal) one.*

Proof. There is nothing to prove if $s = 0$. Assume that $s > 0$. If one of the real conjugates of u, say, $\alpha_1 > 0$ has the same modulus as the complex (nonreal) conjugate α_2 then $\alpha_1^2 = \alpha_2\overline{\alpha_2}$. Here, $\alpha_1, \alpha_2, \overline{\alpha_2}$ are distinct. By Lemma 3, none of the conjugates of u (and of u^{-1}) is a Perron number. This happens only if there are no other positive conjugates of u except for α_1. Thus, $s = 1$ and all the conjugates of u lie on the circle $|z| = \alpha_1$. Since u is a unit, the radius of the circle must be 1, so that $u = \alpha_1 = 1$. Hence, $\deg u = 1$, contrary to $\deg u = s + 2t' \geqslant 2$. □

We remark that an alternative proof of Corollary 4 can be given by applying the results of Boyd [2] and Ferguson [12].

A standard construction of Pisot numbers in a real field gives Pisot numbers but not Pisot units [23]; see also [11] for a construction of a dense set of Pisot numbers in a given field with very small conjugates and [6] for a contruction of Pisot units. In the proof of Theorem 1 we shall need the following result (which is also of independent interest):

Lemma 5. *For each number field L, each constant $c > 1$ and each integer $q \geqslant 2$ there is a totally positive Pisot unit $\beta > c$ of degree q whose minimal polynomial over $\mathbb{Q}$ is irreducible in the ring $L[x]$.*

Proof. Consider the polynomial

$$H(x) := (-x)^q + k_{q-1}(-x)^{q-1} + \cdots + k_{q-1}k_{q-2}\dots k_1(-x) + 1, \tag{3}$$

where $k_1 < k_2 < \cdots < k_{q-1}$ is a rapidly increasing sequence of positive integers; for example, $k_{j+1} > k_j^2$ for $j = 1, \dots, q-2$ and k_1 is large enough. (This construction is similar in spirit to that of Lemma 3 in [6].) Then $H(0) = 1 > 0$ and $H(1/k_1) < 0$. Also, it is easy to see that the sign of $H(2k_j)$ is the same as that of $(-1)^{j+1}$. Indeed, inserting $x = 2k_j$ into $H(x)$ we see that among the two largest terms $k_{q-1}\dots k_{j+1}(-2k_j)^{j+1}$ and $k_{q-1}\dots k_{j+1}k_j(-2k_j)^j$ the first one is greater in absolute value. Similarly, the sign of $H(k_j/2)$ is the same as that of $(-1)^j$. So in each of the q intervals $(0, 1/k_1)$ and $(k_j/2, 2k_j)$, where $j = 1, \dots, q-1$, there is a root of the polynomial H. Consequently, the polynomial $G(x) = x^q H(1/x)$ reciprocal to H has $q-1$ roots in the interval $(0, 1)$ and one root β in (k_1, ∞). As $\deg G = q$, this polynomial must be irreducible in $\mathbb{Z}[x]$, since the product of any number of its roots without β is of modulus smaller than 1. Therefore, $\beta > k_1$ is a totally positive Pisot unit of degree q.

Note that the polynomial (3) is linear in k_{q-1}, so the polynomial (3), as a polynomial in two variables $H(x, k_{q-1})$, is irreducible in the ring $L[x, a_{d-1}]$. Indeed, otherwise the polynomials $(-x)^q + 1$ and

$$H_1(x) := \frac{H(x) - (-x)^q - 1}{k_{q-1}} = (-x)^{q-1} + k_{q-2}(-x)^{q-2} + \dots + k_{q-2} \dots k_1(-x)$$

must have a common factor in $L[x]$. Hence, $(-x)^q + 1$ and $H_1(x)$ must have a common root. However, this is not the case, since the roots of $(-x)^q + 1$ are roots of unity whereas the modulus of lowest term in H_1 is greater than the sum of the moduli of the other terms for any x of modulus 1. Thus, by Hilbert's irreducibility theorem (see Theorem 46 on p. 298 in [24]), for some positive integer $k_{q-1}^* \geqslant k_{q-1}$ the polynomial $H(x, k_{q-1}^*)$ is irreducible in the ring $L[x]$, and so the polynomial $G(x, k_{q-1}^*) = x^q H(1/x, k_{q-1}^*)$ is irreducible in $L[x]$ too. □

For any $u_1, \dots, u_s \in \mathcal{U}_K$ we write

$$S(u_1, \dots, u_s) := \{u_1^{k_1} \dots u_s^{k_s} \ : \ k_1, \dots, k_s \in \mathbb{N} \cup \{0\}\}$$

for the multiplicative semigroup generated by $u_1, \dots, u_s$.

Lemma 6. *Let $u_1, \dots, u_s$ be some $s \geqslant 1$ multiplicatively independent units in $\mathcal{U}_K$ satisfying* (1). *Then either $\mathbb{Q}(u_1, \dots, u_s)$ is a proper subfield of K or $\mathcal{U}_K$ contains s multiplicatively independent units $v_1, \dots, v_s \in S(u_1, \dots, u_s)$, each of degree $d = [K : \mathbb{Q}]$.*

Proof. Assume that $\mathbb{Q}(u_1, \dots, u_s) = K$ (otherwise there is nothing to prove). Suppose $S(u_1, \dots, u_s)$ does not contain s multiplicatively independent units of degree d each. Choose multiplicatively independent units $v_1, \dots, v_s \in S(u_1, \dots, u_s)$ satisfying $\mathbb{Q}(v_1, \dots, v_s) = K$ for which the sum $D := \deg v_1 + \dots + \deg v_s$ is the largest possible. If $D = sd$ we are done. In case $D < sd$, we will show that D can be increased, and so arrive to a contradiction.

Without restriction of generality we may assume that $h := \deg v_1 < d$. Then $s > 1$ and for some v_j with $j \geqslant 2$, say for v_2, we have $v_2 \notin \mathbb{Q}(v_1)$, since otherwise $\mathbb{Q}(v_1, \dots, v_s) = \mathbb{Q}(v_1)$ is a proper subfield of K. Now, replace the set $v_1, v_2, \dots, v_s$ by the set $v_1 v_2^p, v_2, \dots, v_s$, where p is an integer that will be chosen later. The latter set is also multiplicatively independent, all of its elements belong to $S(u_1, \dots, u_s)$ and also

$$\mathbb{Q}(v_1 v_2^p, v_2, \dots, v_s) = \mathbb{Q}(v_1, v_2, \dots, v_s) = K.$$

In order to complete the proof it remains to show that

$$\deg v_1 v_2^p > h = \deg v_1 \tag{4}$$

for some large positive integer p.

It is clear that $v_1, v_2 > 0$, since $v_1, v_2 \in S(u_1, \dots, u_s)$ and $u_1, \dots, u_s > 0$. If all the conjugates of v_2 are of equal moduli then, as v_2 is a unit, they all lie on the circle $|z| = 1$. Hence, $v_2 = 1$, which is a contradiction to

$v_2 \notin \mathbb{Q}(v_1)$. It follows that not all conjugates of v_2 have the same modulus. Since $v_2 \in S(u_1, \dots, u_s)$, and the units $u_1, \dots, u_s$ satisfy the condition (1), the unit v_2 satisfies (1) as well. Consequently, either the largest positive conjugate of v_2 (it can be v_2 itself) is a Perron number or a reciprocal of the smallest positive conjugate of v_2 is a Perron number.

Select the smallest $\ell \in \mathbb{N}$ for which $v_2^\ell \in \mathbb{Q}(v_1)$, if such an ℓ exists. Take p of the form $\ell k+1$ with large $k \in \mathbb{N}$ if $\ell \in \mathbb{N}$ as above exists and take any large p otherwise. For such p we have $v_2^p \notin \mathbb{Q}(v_1)$. Thus, v_2^p has a conjugate over the field $\mathbb{Q}(v_1)$ distinct from v_2^p. Assume that w_2^p is such a conjugate, where $w_2 \neq v_2$ are conjugate over $\mathbb{Q}$. It follows that the numbers $v_1 v_2^p \neq v_1 w_2^p$ are conjugate over $\mathbb{Q}$. Now, consider some h automorphisms of the Galois group of $\mathbb{Q}(v_1, v_2)/\mathbb{Q}$ that map v_1 into its h conjugates over $\mathbb{Q}$. These map v_2 and w_2 to some of their conjugates over $\mathbb{Q}$ and the pair $v_1 v_2^p, v_1 w_2^p$ into some h pairs of two distinct conjugates of $v_1 v_2^p$. Therefore, either $\deg v_1 v_2^p \geqslant 2h$ (which finishes the proof of (4)) or the list of $2h$ conjugates contains some equal elements. This means that for some two distinct conjugates of v_1, say for v_1 itself and $w_1 \neq v_1$, we have $v_1 v_2^p = w_1 (w_2^*)^p$, where w_2^* is a conjugate of v_2 over $\mathbb{Q}$. Then $w_2^* \neq v_2$.

Now, take an automorphism σ of the Galois group of $\mathbb{Q}(v_1, v_2)/\mathbb{Q}$ that maps w_2^* into v. (Recall that v is a conjugate of v_2 which is a Perron number, or v^{-1} is a Perron number.) This maps the equality $v_1 v_2^p = w_1 (w_2^*)^p$ into $\sigma(v_1)\sigma(v_2)^p = \sigma(w_1) v^p$, where $\sigma(v_1) \neq \sigma(w_1)$ and $\sigma(v_2) \neq v$. However, this is impossible, since $|\sigma(v_2)| \neq v$ and so the modulus of the right hand side, $\sigma(w_1)v^p$, is greater (resp. smaller) than that of the left hand side, $\sigma(v_1)\sigma(v_2)^p$, if v is a Perron number (resp. v^{-1} is a Perron number) and p is large enough. ☐

Finally, in the proof of Theorem 2 we shall use the next lemma (which is Lemma 1.6 in [16]):

Lemma 7. *Let K' be a proper subfield of K and a proper extension of $\mathbb{Q}$, i.e., $\mathbb{Q} \subset K' \subset K$, and let $A \subset \mathcal{U}_{K'}$ be an admissible subgroup for K. Suppose that s' and $2t'$ are the numbers of distinct real and complex embeddings of K', respectively. Then t' is positive, $s = s'$ and A is admissible for K'.*

4. The restriction on the number of real embeddings in Theorem 1

The aim of this section is to prove (2). Take a nonreciprocal unit in $\mathcal{U}_K$ of degree d with conjugates

$$\alpha_1, \dots, \alpha_s, \alpha_{s+1}, \overline{\alpha_{s+1}}, \dots, \alpha_{s+t}, \overline{\alpha_{s+t}},$$

satisfying (1). Then

$$\alpha_{s+1}\overline{\alpha_{s+1}} = \dots = \alpha_{s+t}\overline{\alpha_{s+t}} = \beta \tag{5}$$

for some $\beta > 0$. Here, $\beta \neq 1$, since otherwise the unit α_1 is reciprocal. Thus, $s > 0$. We claim that the set $\{\alpha_1, \dots, \alpha_s, \alpha_1^{-1}, \dots, \alpha_s^{-1}\}$ contains a Perron number. As the conjugates $\alpha_1, \dots, \alpha_s$ are positive, at most one of them can lie on the circle $|z| = \sqrt{\beta}$. If $s = 1$ then we are done, unless all the conjugates of α_1 lie on the circle $|z| = \sqrt{\beta}$. However, then the norm of α_1 is $\beta^{d/2}$. In view of $\alpha_1 \in \mathcal{U}_K$ we obtain $\beta = 1$, a contradiction. In the alternative case, $s \geqslant 2$, we take α to be the largest number in the set $\{\alpha_1, \dots, \alpha_s\}$ if it is greater than $\sqrt{\beta}$, and the smallest one if all the conjugates of α_1 lie in $|z| \leqslant \sqrt{\beta}$. Then α (resp. α^{-1}) is a Perron number.

Obviously, β cannot be written as a product of two complex conjugates of α other than given in (5), and it is not a product of a real conjugate and a complex (nonreal) conjugate. Assume that among the real conjugates of α there are $m \geqslant 0$ pairs of conjugates that multiply to β, where $m = 0$ if there are no such pairs. Then, without restriction of generality (5) can be extended to

$$\beta = \alpha_{s-2m+1}\alpha_{s-2m+2} = \dots = \alpha_{s-1}\alpha_s = \alpha_{s+1}\overline{\alpha_{s+1}} = \dots = \alpha_{s+t}\overline{\alpha_{s+t}}. \tag{6}$$

Note that $s > 2m$, since otherwise the norm of α is equal to $\beta^{d/2} \neq 1$ and α is not a unit.

Assume that the degree of β over $\mathbb{Q}$ is q, and the conjugates of β are $\beta_1 = \beta, \beta_2, \dots, \beta_q$. If $q = 1$ then mapping α_{s-2m+1} to α_1, we find that $\alpha_1\alpha' = \beta$ for some conjugate $\alpha' \neq \alpha_1$ of α, since $\beta \mapsto \beta$. But the pair α_1, α' does not appear in (6), a contradiction. Hence, $q \geqslant 2$.

Take any automorphism $\sigma = \sigma_j$ of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$, where F is the normal closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, that maps β to β_j. Then (6) (which corresponds to $\sigma = \mathrm{id}$) maps to

$$\begin{aligned} \beta_j &= \sigma(\alpha_{s-2m+1})\sigma(\alpha_{s-2m+2}) = \dots = \sigma(\alpha_{s-1})\sigma(\alpha_s) \\ &= \sigma(\alpha_{s+1})\sigma(\overline{\alpha_{s+1}}) = \dots = \sigma(\alpha_{s+t})\sigma(\overline{\alpha_{s+t}}). \end{aligned} \tag{7}$$

Here, σ acts as a permutation of the set $\mathcal{C} := \{\alpha_1, \alpha_2, \dots, \alpha_{s+t}, \overline{\alpha_{s+t}}\}$. Consider q multiple equalities (7), where $j = 1, \dots, q$. Evidently, each of the q sets

$$\begin{aligned} \mathcal{C}_\sigma := \{&\sigma(\alpha_{s-2m+1}), \sigma(\alpha_{s-2m+2}), \dots, \sigma(\alpha_{s-1}), \sigma(\alpha_s), \\ &\sigma(\alpha_{s+1}), \sigma(\overline{\alpha_{s+1}}), \dots, \sigma(\alpha_{s+t}), \sigma(\overline{\alpha_{s+t}})\} \end{aligned}$$

contains $2t + 2m$ distinct elements. We will show that they are disjoint, so that $\cup_\sigma \mathcal{C}_\sigma = \mathcal{C}$.

Suppose first that some set $\mathcal{C}_\sigma$, where $\sigma \neq \mathrm{id}$, contains a complex (nonreal) number. Then $\alpha_i\alpha_j = \alpha_k\alpha_l = \beta_J \neq \beta$, where the indices i, j, k, l are distinct and

$$\mathcal{I} := \{\alpha_i, \alpha_j, \alpha_k, \alpha_l\} \cap \{\alpha_{s+1}, \overline{\alpha_{s+1}}, \dots, \alpha_{s+t}, \overline{\alpha_{s+t}}\} \neq \emptyset.$$

If $|\mathcal{I}| = 1$, then one side of the equality

$$\alpha_i\alpha_j = \alpha_k\alpha_l \tag{8}$$

is real and the other side is nonreal, a contradiction. Suppose next that $|\mathcal{I}| = 2$. If both complex numbers are on one side of (8), say on its right hand side, then $|\alpha_i\alpha_j| = |\alpha_k| \cdot |\alpha_l| = \sqrt{\beta} \cdot \sqrt{\beta} = \beta$, so $\alpha_i\alpha_j = \beta$, contrary to $\alpha_i\alpha_j = \beta_J \neq \beta$. If the two complex numbers are on different sides of (8), say α_j and α_l, then $|\alpha_i/\alpha_k| = |\alpha_l|/|\alpha_j| = \sqrt{\beta}/\sqrt{\beta} = 1$. Thus, $\alpha_i = \pm\alpha_k$, which is impossible in view of $i \neq k$ and $\alpha_i, \alpha_k > 0$. Next, if $|\mathcal{I}| = 3$ then, assuming that the remaining real conjugate in (8) is α_i, we obtain $|\alpha_i| = |\alpha_k||\alpha_l|/|\alpha_j| = \sqrt{\beta}$.Thus, $\alpha_i = \sqrt{\beta}$. Then $\beta = \alpha_i^2 = \alpha_{s+1}\overline{\alpha_{s+1}}$, which contradicts Lemma 3. Finally, if $|\mathcal{I}| = 4$, then all 4 conjugates of α in (8) are complex, $\alpha_j \neq \overline{\alpha_i}$ and $\alpha_l \neq \overline{\alpha_k}$. We have already proved that the product of such α_i and α_j cannot be the product of two real conjugates or a real and a complex conjugate. Hence, the set $\mathcal{C}_\sigma$ corresponding to β_J (which is equal to $\alpha_i\alpha_j$) consists entirely of complex (nonreal) numbers. As $|\mathcal{C}_\sigma| = 2t + 2m \geqslant 2t$, all the complex conjugates of α must belong to $\mathcal{C}_\sigma$. Thus, $\overline{\alpha_i} \in \mathcal{C}_\sigma$, and so $\alpha_i\alpha_j = \overline{\alpha_i}\alpha_\ell$ with some complex (nonreal) conjugate α_ℓ. Multiplying both sides by $\alpha_i\overline{\alpha_j}/\beta$, we deduce that

$$\alpha_i^2 = \alpha_i^2\alpha_j\overline{\alpha_j}/\beta = \overline{\alpha_i}\alpha_\ell\alpha_i\overline{\alpha_j}/\beta = \alpha_\ell\overline{\alpha_j}.$$

Now, if $\overline{\alpha_j} = \alpha_\ell$ then $\alpha_i = \alpha_\ell$, which is not the case. If otherwise $\overline{\alpha_j} \neq \alpha_\ell$ then the conjugates α_i, α_ℓ and $\overline{\alpha_j}$ are distinct. Then equality $\alpha_i^2 = \alpha_\ell\overline{\alpha_j}$ is impossible, by Lemma 3. Hence, the set $\mathcal{I}$ is empty.

We have thus proved that all the numbers in (8) are distinct positive numbers. Hence, for each $\sigma \neq \mathrm{id}$ the set $\mathcal{C}_\sigma$ consists of $2t + 2m$ positive conjugates of α. Assume that some positive conjugate, say α_1, appears in $b > 1$ sets $\mathcal{C}_\sigma$. Then an automorphism of $\mathrm{Gal}(F/\mathbb{Q})$ that maps α_1 to α_{s+1} acts as a permutation of the set $\mathcal{C}$ and as a permutation of the set $\{\beta_1, \dots, \beta_q\}$. In this way we will obtain q equalities of the type (7), where the complex conjugate α_{s+1} appears b times. As $b > 1$, this contradicts the fact that $\beta_J = \alpha_i\alpha_j$ with complex (nonreal) α_i, α_j only happens once when $J = 1$ and $\alpha_j = \overline{\alpha_i}$. By the same argument, a conjugate of α cannot appear $b = 0$ times in the sets $\mathcal{C}_\sigma$. Consequently, every conjugate of α appears exactly once in the union of q sets $\mathcal{C}_\sigma$. Hence,

$$s + 2t = d = |\mathcal{C}| = |\mathcal{C}_{\mathrm{id}}|q = (2t + 2m)q.$$

As $q \geqslant 2$, this finishes the proof of (2).

5. The construction of a nontrivial unit in Theorem 1

Let $I(n)$ be the infimum among all positive numbers I_n with the following property: any closed real interval of length at least I_n contains a full set of conjugates of an algebraic integer of degree n. By a result of Robinson [21], every interval of length greater than 4 contains infinitely many full sets of conjugates of algebraic integers (see also [22]). Later, Ennola [10] proved that such an interval contains full sets of conjugates of algebraic integers of degree n for all n sufficiently large. Hence, for each positive ε we have $I(n) <$

$4+\varepsilon$ for every $n > n(\varepsilon)$. From [5] we know that $I(2) = (1+\sqrt{5})/2+\sqrt{2}$ and it is evident that $I(1) = 1$. It seems very likely that every interval of length, say 5, or even smaller (although greater than 4, by an old result of Schur [25]) contains a full set of conjugates of an algebraic integer of degree n for every $n \in \mathbb{N}$. However, since no result of such type is given explicitly in the literature, we simply put

$$I := \max\{5, \sup_{n\in\mathbb{N}} I(n)\}.$$

Therefore, every interval of length I contains a full set of conjugates of an algebraic integer of degree n for every $n \in \mathbb{N}$.

We claim that for any integers $t \geqslant 1$ and $m \geqslant 0$ there is an algebraic integer γ of degree $t+m$ with t conjugates in the interval $[1, I+1]$ and m conjugates in the interval $(2I, \infty)$, say

$$(9) \qquad 1 \leqslant \gamma = \gamma_1 < \dots < \gamma_t \leqslant I+1 < 2I < \gamma_{t+1} < \dots < \gamma_{t+m}.$$

Indeed, by the definition of I, such an algebraic integer γ exists for $m = 0$. To show the existence of such γ for $m \geqslant 1$ we can use a theorem of Motzkin [15]. Let us take, for instance, arbitrary t points $\lambda_1 < \dots < \lambda_t$ in the interval $(1, I+1)$ and arbitrary $m-1$ points $\lambda_{t+1} < \dots < \lambda_{t+m-1}$ in the interval $(2I, 2I+1)$. Then, by the main result of [8], for each $\varepsilon > 0$ there is a constant $c(\varepsilon, \lambda_1, \dots, \lambda_{t+m-1})$ and a totally positive algebraic integer

$$\gamma_{t+m} > \max\{c(\varepsilon, \lambda_1, \dots, \lambda_{t+m-1}), 2I+1\}$$

of degree $t+m$ such that the numbers $\gamma_1, \dots, \gamma_{t+m-1}$ conjugate to γ_{t+m} over $\mathbb{Q}$ lie in the ε-neighborhoods of the points $\lambda_1, \dots, \lambda_{t+m-1}$, respectively. By taking a sufficiently small $\varepsilon > 0$, we see that this algebraic integer γ_{t+m} of degree $t+m$ with conjugates $\gamma_1 = \gamma, \dots, \gamma_{t+m}$ satisfies (9).

By Lemma 5, we can take a totally positive Pisot unit β of degree $q \geqslant 2$ such that, firstly,

$$(10) \qquad \beta = \beta_1 > \frac{I(I+1)}{I-1} > 1 > \beta_2 > \dots > \beta_q$$

and, secondly, the minimal polynomial of β over $\mathbb{Q}$ is irreducible in the ring $\mathbb{Q}(\gamma_1, \dots, \gamma_{t+m})[x]$.

Set

$$(11) \qquad k := \lceil (1+I^{-1})\beta \rceil \geqslant (1+I^{-1})\beta > (1+I^{-1})\frac{I(I+1)}{I-1} > I+1 \geqslant 6.$$

Consider the polynomial

$$H(x, \beta_j) := \prod_{i=1}^{t+m} (x^2 - \gamma_i(k-\beta_j)x + \beta_j^2).$$

If $j > 1$ then the discriminant of each quadratic factor in $H(x, \beta_j)$ is positive. Indeed, using (9), (10) and (11), we obtain

$$(\gamma_i(k-\beta_j))^2 - 4\beta_j^2 \geqslant (k-\beta_j)^2 - 4\beta_j^2 > (6-1)^2 - 4 > 0.$$

Since $\gamma_i(k-\beta_j) > 0$, the factor $x^2 - \gamma_i(k-\beta_j)x + \beta_j^2$ has two positive roots. Hence, the polynomial $H(x,\beta_j)$ has $2t+2m$ positive roots.

We claim that for $j=1$ the polynomial

$$H(x,\beta_1) = \prod_{i=1}^{t+m} (x^2 - \gamma_i(k-\beta)x + \beta^2)$$

has $2m$ positive roots and $2t$ complex roots lying on the circle $|z| = \beta$. Indeed, this time the discriminant

$$\Delta_i := (\gamma_i(k-\beta))^2 - 4\beta^2$$

is positive for $i = t+1, \dots, t+m$. To see this, we use $k - \beta \geqslant \beta/I$ and $\gamma_i > 2I$ which gives

$$(\gamma_i(k-\beta))^2 - 4\beta^2 \geqslant \gamma_i^2(\beta/I)^2 - 4\beta^2 > 4I^2(\beta/I)^2 - 4\beta^2 = 0.$$

Thus, the quadratic polynomial $x^2 - \gamma_i(k-\beta)x + \beta^2$ has two positive roots for every $i = t+1, \dots, t+m$.

Similarly, we may check that Δ_i is negative for $i = 1, \dots, t$. Indeed, by (9), (10) and (11),

$$\begin{aligned}(\gamma_i(k-\beta))^2 - 4\beta^2 &\leqslant (I+1)^2(\lceil(1+I^{-1})\beta\rceil - \beta)^2 - 4\beta^2 \\ &< (I+1)^2(\beta/I+1)^2 - 4\beta^2 \\ &< (I+1)^2(2\beta/(I+1))^2 - 4\beta^2 = 4\beta^2 - 4\beta^2 = 0,\end{aligned}$$

where the inequality $\beta/I + 1 < 2\beta/(I+1)$ follows from (10). Consequently, for each $i = 1, \dots, t$ the roots of $x^2 - \gamma_i(k-\beta)x + \beta^2$ are

$$\frac{\gamma_i(k-\beta) \pm \sqrt{(\gamma_i(k-\beta))^2 - 4\beta^2}}{2}.$$

These are complex conjugate numbers lying on the circle $|z| = \beta$. Thus, $H(x,\beta_1)$ has $2m$ positive roots and $2t$ complex roots all lying on the circle $|z| = \beta$.

Summarizing, we conclude that the polynomial

$$P(x) := \prod_{j=1}^{q} H(x,\beta_j) = \prod_{j=1}^{q}\prod_{i=1}^{t+m} (x^2 - \gamma_i(k-\beta_j)x + \beta_j^2) \in \mathbb{Z}[x] \tag{12}$$

has $(2t+2m)(q-1) + 2m = (2t+2m)q - 2t$ positive roots and $2t$ complex roots.

We next show that the polynomial $P(x)$ of (12), of degree $d = s + 2t$, where $s = (2t+2m)q - 2t$, is irreducible in $\mathbb{Z}[x]$. Let α be one of its complex roots, say

$$\alpha = \frac{\gamma(k-\beta) + i\sqrt{4\beta^2 - (\gamma(k-\beta))^2}}{2},$$

where $i = \sqrt{-1}$, $\beta = \beta_1$ and $\gamma = \gamma_1$. Assume that $\ell := \deg \alpha < 2(t+m)q$ and consider the set of conjugates of α, say $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_\ell$. Evidently,

$$\overline{\alpha} = \frac{\gamma(k-\beta) - i\sqrt{4\beta^2 - (\gamma(k-\beta))^2}}{2}$$

is a conjugate of α over $\mathbb{Q}$, so both α and $\overline{\alpha}$ belong to the set $\{\alpha_1, \dots, \alpha_\ell\}$.

Let F be the Galois closure of $\mathbb{Q}(\beta, \gamma)$ over $\mathbb{Q}$ and $G := \mathrm{Gal}(F/\mathbb{Q})$. Note that the numbers β_j, $j = 1, \dots, q$, are conjugate over $\mathbb{Q}(\gamma)$, since, by the choice of β, the minimal polynomial of β is irreducible in the ring $\mathbb{Q}(\gamma)[x]$. Thus, given any j in the range $1 \leqslant j \leqslant q$, there is an automorphism $\sigma \in G$ that fixes γ and maps $\beta \mapsto \beta_j$. This automorphism maps the factor $x^2 - \gamma(k-\beta)x + \beta^2$ to the factor $x^2 - \gamma(k-\beta_j)x + \beta_j^2$, so it takes the pair of roots of the first quadratic polynomial, $\alpha = \alpha_1(\gamma, \beta), \overline{\alpha} = \alpha_2(\gamma, \beta)$, to the pair of roots $\alpha_1(\gamma, \beta_j), \alpha_2(\gamma, \beta_j)$ of the second quadratic polynomial. In particular, this implies that these four roots are conjugate over $\mathbb{Q}$ for each $j = 1, \dots, q$.

Next, map γ to γ_i, where i is one of the indices $1, \dots, t+m$. This automorphism of G takes β to some β_J and $x^2 - \gamma(k-\beta)x + \beta^2$ to $x^2 - \gamma_i(k-\beta_J)x + \beta_J^2$. Hence, their roots $\alpha_1(\gamma, \beta), \alpha_2(\gamma, \beta)$ and $\alpha_1(\gamma_i, \beta_J), \alpha_2(\gamma_i, \beta_J)$ are conjugate over $\mathbb{Q}$. Furthermore, by the same argument as above and the fact that the minimal polynomial of β_J is irreducible in $\mathbb{Q}(\gamma_i)[x]$, the roots of $x^2 - \gamma_i(k-\beta_J)x + \beta_J^2$ (namely $\alpha_1(\gamma_i, \beta_J), \alpha_2(\gamma_i, \beta_J)$) and the roots of $x^2 - \gamma_i(k-\beta_r)x + \beta_r^2$ (say $\alpha_1(\gamma_i, \beta_r), \alpha_2(\gamma, \beta_r)$) are conjugate over $\mathbb{Q}$ for any indices J, r in the range $1 \leq J, r \leqslant q$. Thus, we conclude that all $(2t+2m)q$ roots of the polynomial P defined in (12) are conjugate over $\mathbb{Q}$.

Hence, $\ell = \deg \alpha$ can be smaller than $\deg P = 2(t+m)q$ only if α is a multiple root of P. However, if α is equal to another complex root α' of P corresponding, say to $\gamma_i \neq \gamma$ and β (which is the only possibility to get a complex root), then

$$\alpha + \overline{\alpha} = \gamma(k-\beta) = \gamma_i(k-\beta) = \alpha' + \overline{\alpha'}.$$

This yields $\gamma = \gamma_i$, a contradiction. The proof of Theorem 1 is now completed.

We conclude this section with an example which shows that the unit

$$\begin{aligned} \alpha &:= 15 + 5\sqrt{2} + 6\sqrt{3} + 2\sqrt{6} + \sqrt{310 + 222\sqrt{2} + 276\sqrt{3} + 120\sqrt{6}} \\ &= 74.724635\dots \end{aligned} \tag{13}$$

is a nonreciprocal unit of degree 8 with 4 real conjugates and two pairs of complex conjugates of equal moduli. This corresponds to the case $K = \mathbb{Q}(\alpha)$ and $s = 4$, $t = 2$, $m = 0$, $q = 2$ in equality (2) of Theorem 1.

Take a quadratic algebraic integer $\gamma = 3 - \sqrt{2}$ with conjugate $\gamma' = 3 + \sqrt{2}$ and a quadratic Pisot unit $\beta := 7 + 4\sqrt{3}$ with conjugate $\beta' = 7 - 4\sqrt{3}$. Then the conditions (9) and (10) are satisfied with $I = 5$. Evidently, the minimal polynomial of β is irreducible in the ring $\mathbb{Q}(\gamma,\gamma')[x] = \mathbb{Q}(\sqrt{2})[x]$. By (11), we obtain $k = 17$. Hence, $H(x,\beta)$ is the product of the polynomials

$$x^2 - (3-\sqrt{2})(10-4\sqrt{3})x + 97 + 56\sqrt{3}$$

and

$$x^2 - (3+\sqrt{2})(10-4\sqrt{3})x + 97 + 56\sqrt{3}.$$

Thus, $H(x,\beta)$ is equal to

$$x^4-(60-24\sqrt{3})x^3+(1230-448\sqrt{3})x^2-(1788+1032\sqrt{3})x+18817+10864\sqrt{3}.$$

Similarly, $H(x,\beta')$ equals

$$x^4-(60+24\sqrt{3})x^3+(1230+448\sqrt{3})x^2-(1788-1032\sqrt{3})x+18817-10864\sqrt{3}.$$

Now, calculating the product $H(x,\beta)H(x,\beta')$ we find the polynomial (12)

$$\begin{aligned} P(x) =& x^8 - 120x^7 + 4332x^6 - 86664x^5 + 1311590x^4 - 10994952x^3 \\ &+ 75494124x^2 - 19704x + 1, \end{aligned}$$

which is irreducible in $\mathbb{Z}[x]$. It has four positive roots

$$0.000068\ldots, 0.000192\ldots, 26.844323\ldots, 74.724635, \ldots$$

where the last one is the root α defined in (13), which is the larger of the roots of the quadratic factor $x^2 - (3+\sqrt{2})(10+4\sqrt{3})x + (7-4\sqrt{3})^2$ and is the largest positive root of P, and two pairs of complex conjugate roots

$$6.779783\ldots \pm i12.166732\ldots, 2.435606\ldots \pm i13.713594\ldots$$

on the circle $|z| = \beta = 7 + 4\sqrt{3} = 13.928203\ldots$.

6. Proof of Theorem 2

Consider the subgroup U of $\mathcal{U}_K$ of units satisfying (1). If U has rank at least s then it contains s multiplicatively independent units $u_1, \ldots, u_s$. In particular, $u_1 \notin \mathbb{Q}$. Suppose first that $K' := \mathbb{Q}(u_1, \ldots, u_s)$ is a proper subfield of K. Note that K' is a proper extension of $\mathbb{Q}$, since $u_1 \notin \mathbb{Q}$. Applying Lemma 7 we find that K' has s real and $2t' > 0$ complex embeddings. By Corollary 4, the conjugates of u_1 have at least $s+1$ distinct moduli. Note that the restrictions of the embeddings $\sigma_1, \ldots, \sigma_s$ of K to K' are the real embeddings of K'. Hence, the list

$$\sigma_1(u_1), \ldots, \sigma_s(u_1), \sigma_{s+1}(u_1), \overline{\sigma}_{s+1}(u_1), \ldots, \sigma_{s+t}(u_1), \overline{\sigma}_{s+t}(u_1)$$

contains at least $s+1$ numbers with distinct moduli. Since the last $2t$ numbers in this list have the same modulus, the first s must have distinct moduli. Now, as $\sigma_1(u_1)$ appears in the list exactly $k = (s+2t)/(s+2t') > 1$ times and $k \in \mathbb{N}$, it must appear at least once among the last $2t$ numbers of the list. However, then the number of distinct moduli in the list is at most s, a contradiction.

It remains to consider the alternative case when $\mathbb{Q}(u_1, \dots, u_s) = K$. Then, by Lemma 6, the semigroup $S(u_1, \dots, u_s)$ contains s multiplicatively independent units $v_1, \dots, v_s$ of degree d each. Since $v_1, \dots, v_s \in S(u_1, \dots, u_s)$ and the units $u_1, \dots, u_s$ satisfy the condition (1), the units $v_1, \dots, v_s$ must satisfy (1) as well. In particular, this implies that the matrix

$$M = M(v_1, \dots, v_s) := (\log |\sigma_j(v_i)|)_{1 \leqslant i,j \leqslant s}$$

has rank s. However, by Theorem 1, the units $v_1, \dots, v_s$ of degree d must be reciprocal. Hence, for each $i = 1, \dots, s$ the product over real embeddings $\prod_{j=1}^{s} \sigma_j(v_i)$ is equal to 1. Thus, the columns of the matrix M are linearly dependent, which implies that the rank of M is smaller than s. (This is also true for $s = 1$ when M is the 1×1 matrix with entry 0.) Therefore, the rank of U is smaller than s. This completes the proof of Theorem 2, by the result of Oeljeklaus and Toma [16] stated in Section 1. (See also a stronger result given in Theorem 8 of the Appendix.)

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Appendix (by Laurent Battisti[1])

This appendix has two objectives. First, we prove a criterion for detecting OT-manifolds admitting a locally conformally Kähler metric and in the second part we give an alternative proof of Theorem 2 by using a geometric property of OT-manifolds (namely, their non-Kählerianity). In what follows, we keep the notation defined in the introduction of the main article.

A complex manifold X is said to *admit an LCK metric* if there exists a closed positive $(1,1)$-form $\widetilde{\omega}$ on the universal cover $\widetilde{X}$ of X and a representation $\rho : \pi_1(X) \to \mathbb{R}_{>0}$ such that for all $g \in \pi_1(X)$, one has $g^*\widetilde{\omega} = \rho(g)\widetilde{\omega}$. This notion was introduced by Vaisman in [28]. See also the introduction of the main article for further references on the subject.

In the case of an OT-manifold $X(K, A)$, its fundamental group is (up to isomorphism) the semi-direct product $A \ltimes \mathcal{O}_K$ and its universal cover is $\mathbb{H}^s \times \mathbb{C}^t$. In [16] (proof of Proposition 2.9) and in [33] the authors prove that if an OT-manifold $X(K, A)$ admits an LCK metric then one has $\rho(g) = |\sigma_{s+1}(u)|^2 = ... = |\sigma_{s+t}(u)|^2$ for all $g = (u, a) \in A \ltimes \mathcal{O}_K$. It turns out that this relation between the absolute values of the complex embeddings of the elements of A is in fact a characterization:

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Theorem 8. *An OT-manifold $X(K,A)$ admits an LCK metric if and only if the following holds:*

$$\text{for all } u \in A, \quad |\sigma_{s+1}(u)| = ... = |\sigma_{s+t}(u)|. \tag{14}$$

Proof. We only need to check that this condition is sufficient. Let $X(K,A)$ be an OT-manifold with A satisfying condition (14) and define the following real function on $\mathbb{H}^s \times \mathbb{C}^t$:

$$\varphi(z) := \left(\prod_{j=1}^{s} \frac{i}{z_j - \overline{z_j}}\right)^{\frac{1}{t}} + \sum_{k=1}^{t} |z_{s+k}|^2.$$

This definition of φ is very natural: when $t = 1$, this function is the same as the function F defined in [16], example p. 169.

It is enough to prove that it is a Kähler potential on $\mathbb{H}^s \times \mathbb{C}^t$. For this, we will see that the matrix $(\partial_{z_p}\partial_{\overline{z_q}}\varphi_1)$ is positive definite, where we set $\varphi_1(z) = \left(\prod_{j=1}^{s} \frac{i}{z_j - \overline{z_j}}\right)^{\frac{1}{t}}$. For all $q \in \{1, ..., s\}$, one has:

$$\partial_{\overline{z_q}}\varphi_1(z) = \frac{1}{t}\frac{1}{z_q - \overline{z_q}}\varphi_1,$$

and for all $p \in \{1, ..., s\}$, one has:

$$\partial_{z_p}\partial_{\overline{z_q}}\varphi_1(z) = \begin{cases} \dfrac{1}{t^2}\dfrac{-1}{(z_p - \overline{z_p})(z_q - \overline{z_q})}\varphi_1 \text{ if } p \neq q \\[2ex] \dfrac{1}{t^2}(1+t)\dfrac{-1}{(z_p - \overline{z_p})^2}\varphi_1 \text{ if } p = q. \end{cases}$$

Hence, $(\partial_{z_p}\partial_{\overline{z_q}}\varphi_1) = \dfrac{1}{t^2}\varphi_1 B$ where the matrix B is

$$B = \begin{pmatrix} \frac{(1+t)}{4y_1^2} & \frac{1}{4y_1y_2} & \cdots & \frac{1}{4y_1y_s} \\ \frac{1}{4y_2y_1} & \frac{(1+t)}{4y_2^2} & \cdots & \frac{1}{4y_2y_s} \\ \vdots & & & \\ \frac{1}{4y_sy_1} & \frac{1}{4y_sy_2} & \cdots & \frac{(1+t)}{4y_s^2} \end{pmatrix},$$

and $z_j = x_j + iy_j$ for all $j \in \{1, ..., s+t\}$. As in [19], we notice that B is the sum of a diagonal positive definite matrix and a positive semidefinite one. Hence, B is positive definite.

Now, let $\omega_0 := i\partial\overline{\partial}\varphi$ and for all $g = (u, a) \in A \ltimes \mathcal{O}_K$ set $\rho(g) := |\sigma_{s+1}(u)|^2$. First, notice that because u is a unit we have

$$(\sigma_1(u)...\sigma_s(u))(|\sigma_{s+1}(u)|^2...|\sigma_{s+t}(u)|^2) = 1.$$

Then, write

$$\begin{aligned}\partial\overline{\partial}(\varphi\circ g)(z) &= \frac{1}{(\sigma_1(u)...\sigma_s(u))^{\frac{1}{t}}}\partial\overline{\partial}\varphi_1(z)\\ &\quad + \partial\overline{\partial}\sum_{k=1}^{t}|\sigma_{s+k}(u)z_{s+k}+\sigma_{s+k}(a)|^2\\ &= \rho(g)\partial\overline{\partial}\varphi_1(z)+\rho(g)\partial\overline{\partial}\sum_{k=1}^{t}|z_{s+k}|^2\\ &= \rho(g)\partial\overline{\partial}\varphi(z).\end{aligned}$$

We now obtain the following equalities:

$$g^*\omega_0 = g^*(i\partial\overline{\partial}\varphi) = i\partial\overline{\partial}(\varphi\circ g) = i\rho(g)\partial\overline{\partial}(\varphi) = \rho(g)\omega_0.$$

This concludes the proof. □

Recall that in [16], the authors showed that no OT-manifold admits a Kähler structure (this is Proposition 2.5, loc. cit.). Using this fact, we now see how to prove Theorem 2.

Assume that K is a number field of degree $d = s + 2t$ with $t \geqslant 2$ and with s not being of the form (2). We now suppose that the rank of the subgroup U of $\mathcal{U}_K$ of units satisfying equation (1) is at least (therefore, equal to) s and we want to show that this leads to a contradiction.

First, notice that $l(U)$ has a trivial intersection with the kernel of the projecting map $\mathcal{P} : \mathcal{S} \to \mathbb{R}^s$, where l and $\mathcal{P}$ are defined in the introduction of the main article. Thus, U is an admissible subgroup of $\mathcal{U}_K$. Now, consider the OT-manifold $X(K,U)$; it admits an LCK metric by Theorem 8. As a consequence of Theorem 1, all the elements of U are reciprocal. In particular, $|\sigma_{s+j}(u)| = 1$ for all $u \in U$ and for all $j \in \{1,...,t\}$.

Let ω be a Kähler form on $\mathbb{H}^s \times \mathbb{C}^t$ giving rise to an LCK metric on $X(K,U)$. For all $g = (u,a) \in U \ltimes \mathcal{O}_K$, one has $g^*\omega = |\sigma_{s+1}(u)|^2\omega$ (see the paragraph before Theorem 8), which simplifies as $g^*\omega = \omega$. The form ω being invariant under the action of $A \ltimes \mathcal{O}_K$, it descends to a Kähler form on $X(K,U)$. This implies that $X(K,U)$ is a Kähler manifold, which is the desired contradiction.

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