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The universal simplicial bundle is a simplicial group

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Abstract. The universal bundle functor $W : s\mathbf{Grp}(C) \to sC$ for simplicial groups in a category C with finite products lifts to a monad on $s\mathbf{Grp}(C)$ landing in contractible simplicial groups. The construction extends to simplicial algebras for any multisorted Lawvere theory.

Contents

1. Introduction

The present note is motivated by two observations. The first, by Segal [8], is that the total space EG of the universal bundle for a (well-pointed) topological group G can be chosen to be a topological group. The easiest way to see this is to pass through the simplicial construction Segal introduced, which from the group G gives a simplicial topological group. The geometric realisation of this simplicial group is then a topological group (using the product in the category of k-spaces). Additionally, the group G is a closed subgroup of EG and the quotient is one of the standard constructions of the classifying space of a topological group.

The second observation, appearing in [6], is that given a strict 2-group G there is a natural construction of a universal bundle $\mathrm{INN}(G)$ for G which is a grouplike object in $2\mathbf{Gpd}$. Again, there is an injective 'homomorphism' $G \hookrightarrow \mathrm{INN}(G)$. This was proved for 2-groups in **Set**, but also works for strict 2-groups internal to a finitely complete category.

Given a growing interest in higher gauge theory, derived geometry and higher topos theory, it is natural to consider a generalisation of these results

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to ∞-groups, at least in the first instance as presented by simplicial groups. Because all the constructions involved are very simple, we can work internal to an arbitrary category C with finite products.

The functor $W\colon s\mathbf{Grp} \to s\mathbf{Set}$, introduced in [2], plays the rôle in the simplicial world analogous to that which $E\colon \mathbf{Grp}(\mathbf{Top}) \to \mathbf{Top}$ does for universal bundles for topological groups. One can easily see that the construction of W works for simplicial groups in any category C with finite products. The main result here is that W lifts through the forgetful functor $s\mathbf{Grp}(C) \to sC$, up to isomorphism. Furthermore, not only is this an endofunctor on $s\mathbf{Grp}(C)$, it is a monad, with the unit of the monad being the subgroup inclusion. The result that WG is a group, at least under the assumption that C has all finite limits, is proved in [7] in a more conceptual manner.

We can say even more about the status of WG as a universal bundle in categories other than $s\mathbf{Set}$, by recent joint work of Nikolaus, Schreiber and Stevenson [5]. If $\mathrm{Sh}_\infty(S)$ is the ∞-topos of ∞-sheaves on a site S with a terminal object, then any ∞-group object $\mathcal{G}$ in $\mathrm{Sh}_\infty(S)$ is presented by a simplicial group G in $C = \mathrm{Sh}(S)$, and moreover every principal ∞-bundle is presented by a pullback (in $\mathrm{Sh}(S)$) of the universal bundle $WG \to \overline{W}G$ described here.

For background on simplicial objects and simplicial groups the reader may consult the classic [4]. We shall describe simplicial objects in $\mathbf{Set}$, using elements, but all constructions here are possible in a category with finite products, if we take the definition as using generalised elements.

A remark is perhaps necessary for the history of this result. Theorem 3 was proved around the time [6] was written, but the original version of the notes languished, being referred to in one or two places, themselves until recently unpublished work. Thanks are due to Jim Stasheff for encouraging a broader distribution. Urs Schreiber and Danny Stevenson made useful suggestions on a draft and the referee made many helpful, detailed suggestions about how to improve this article.

2. Main construction

We must first present the definitions of the objects we are considering. To start with, we have the classical universal bundle for a simplicial group G.

Definition 1. The *universal G-bundle WG* has as its set of n-simplices

$$(WG)_n = G_n \times \cdots \times G_0$$

and face and degeneracy maps given by

$$d_i(g_n, \ldots, g_0)$$
$$= \begin{cases} (d_0 g_n g_{n-1}, g_{n-2}, \ldots, g_0), & \text{if } i = 0; \\ (d_i g_n, \ldots, d_1 g_{n-i+1}, d_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \ldots, g_0), & \text{if } i = 1, \ldots, n-1; \\ (d_n g_n, \ldots, d_1 g_1), & \text{if } i = n, \end{cases}$$

and

$$s_i(g_n, \dots, g_0) = (s_i g_n, \dots, s_0 g_{n-i}, \mathrm{id}_{G_{n-i}}, g_{n-i-1}, \dots, g_0), \quad \text{for } i = 0, \dots, n.$$

The simplicial group G acts (on the left) on WG by multiplication on the first factor, and the quotient $(WG)/G$ is denoted $\overline{W}G$. The set of n-simplices of $(\overline{W}G)_n$ is $G_{n-1} \times \cdots \times G_0$. We will not need a description of the face and degeneracy maps for the present purposes (the reader may find a description in [4, §20]), we only need to note that this quotient of simplicial objects exists even if we consider simplicial objects internal to other categories C without assuming existence of colimits.

We now define a simplicial group $W_{\mathrm{gr}}G$ for any simplicial group G.

Definition 2. The set of n-simplices of $W_{\mathrm{gr}}G$ is given by

$$(W_{\mathrm{gr}}G)_n = G_n \times \cdots \times G_0.$$

The face maps are

$$d_i(g_n, \dots, g_0) = \begin{cases} (g_{n-1}, g_{n-2}, \dots, g_0), & \text{if } i = 0; \\ (d_i g_n, \dots, d_1 g_{n-i+1}, g_{n-i-1}, \dots, g_0), & \text{if } i = 1, \dots, n-1; \\ (d_n g_n, \dots, d_1 g_1), & \text{if } i = n, \end{cases}$$

and the degeneracy maps are

$$s_i(g_n, \dots, g_0) = (s_i g_n, \dots, s_0 g_{n-i}, g_{n-i}, g_{n-i-1}, \dots, g_0), \text{ for } i = 0, \dots, n.$$

If we let the product on $(W_{\mathrm{gr}}G)_n$ be componentwise, these face and degeneracy maps are homomorphisms, because those of G are. $W_{\mathrm{gr}}G$ is then a simplicial group. The construction is clearly functorial.

Our result is that this simplicial group *is* the universal G-bundle.

Theorem 3. *The endofunctor $W_{\mathrm{gr}} \colon s\boldsymbol{Grp}(C) \to s\boldsymbol{Grp}(C)$ is a lift, up to isomorphism, of the universal bundle functor W through the forgetful functor $s\boldsymbol{Grp}(C) \to sC$. Moreover, W_{gr} is a monad.*

After proving this in the next section, the last section contains some observations on generalisations which are more open ended.

3. Proof of Theorem 3

There is an isomorphism between $(WG)_n$ and (the underlying set of) $(W_{\mathrm{gr}}G)_n$, given by

$$\begin{aligned} \Phi_n \colon (WG)_n &\to (W_{\mathrm{gr}}G)_n \\ (g_n, \dots, g_0) &\mapsto (k_n, k_{n-1}, \dots, k_0), \end{aligned}$$

where the k_j are defined recursively as

$$k_n = g_n, \qquad k_{j-1} = d_0 k_j g_{j-1} \ \ (j < n).$$

One can see that the maps Φ_n define a map Φ of simplicial sets by the use of the standard identities for the boundary and degeneracy maps for G. One can check the inverse map is

$$\begin{aligned}\Phi^{-1}\colon W_{\mathrm{gr}}G &\to WG\\ (h_n,\dots,h_0) &\mapsto (h_n, d_0h_n^{-1}h_{n-1},\dots,d_0h_1^{-1}h_0).\end{aligned}$$

Thus W_{gr} is a lift of W, up to isomorphism. Note that since $W_{\mathrm{gr}}G$ is isomorphic to WG its underlying simplicial set is contractible.

We use the isomorphism Φ to see how G includes into $W_{\mathrm{gr}}G$:

$$\begin{aligned} G_n \hookrightarrow (WG)_n &\xrightarrow{\Phi} (W_{\mathrm{gr}}G)_n \\ g_n \mapsto (g_n,1,\dots,1) &\mapsto (g_n, d_0g_n, d_0^2g_n,\dots,d_0^ng_n)\end{aligned} \tag{1}$$

Call this homomorphism ι_G. Since the (left) action of G on $W_{\mathrm{gr}}G$ is defined via Φ, it is trivial to see that Φ is a G-equivariant isomorphism between free G-spaces. This means that $(W_{\mathrm{gr}}G)/G \simeq \overline{W}G$; this quotient therefore exists in all categories C with finite products.

Thus far we have an endofunctor

$$W_{\mathrm{gr}}\colon s\mathbf{Grp}(C)\to s\mathbf{Grp}(C),$$

and a natural transformation

$$\iota\colon 1_{s\mathbf{Grp}}\to W_{\mathrm{gr}}$$

whose component at G is given by the inclusion $\iota_G\colon G\hookrightarrow W_{\mathrm{gr}}G$.

We now have to prove that W_{gr} is a monad. For background, see for example [3, chapter VI]. Notice that

$$\begin{aligned}(W_{\mathrm{gr}}^2G)_n &= (W_{\mathrm{gr}}G)_n\times(W_{\mathrm{gr}}G)_{n-1}\times\dots(W_{\mathrm{gr}}G)_0\\ &= (G_n\times\dots\times G_0)\times(G_{n-1}\times\dots\times G_0)\times\dots\times(G_0).\end{aligned}$$

If $\mathrm{pr}_1\colon (W_{\mathrm{gr}}G)_j\to G_j$ denotes projection on the first factor, define the maps

$$(\mu_G)_n = \mathrm{pr}_1\times\dots\times\mathrm{pr}_1\colon (W_{\mathrm{gr}}^2G)_n\to G_n\times\dots\times G_0 = (W_{\mathrm{gr}}G)_n,$$

which assemble into a map of simplicial groups

$$\mu_G\colon W_{\mathrm{gr}}^2G\to W_{\mathrm{gr}}G,$$

and these clearly form the components of a natural transformation

$$\mu : W_{\mathrm{gr}}^2\to W_{\mathrm{gr}}.$$

Now to show that W_{gr} is a monad we need to check that the diagrams

$$\begin{tikzcd} W_{\mathrm{gr}}^3 G \arrow[r, "W_{\mathrm{gr}}(\mu_G)"] \arrow[d, "\mu_{W_{\mathrm{gr}}G}"'] & W_{\mathrm{gr}}^2 G \arrow[d, "\mu"] \\ W_{\mathrm{gr}}^2 G \arrow[r, "\mu"'] & W_{\mathrm{gr}} G \end{tikzcd}$$

and

$$\begin{tikzcd} W_{\mathrm{gr}} G \arrow[r, "W_{\mathrm{gr}}(\iota_G)"] \arrow[dr, "="'] & W_{\mathrm{gr}}^2 G \arrow[d, "\mu"] & W_{\mathrm{gr}} G \arrow[l, "\iota_{W_{\mathrm{gr}}G}"'] \arrow[dl, "="] \\ & W_{\mathrm{gr}} G & \end{tikzcd}$$

commute, which can be done levelwise and is a fairly easy if tedious exercise in indices. This completes the proof. □

4. Extensions

First recall that our original motivating example was the construction by Segal [8] of a group model for the universal bundle for a well-pointed topological group, and that the original group became a closed subgroup. We have a similar statement for the case when we have a faithful functor $C \to$ **CGHaus** into the category of compactly generated Hausdorff spaces, in which case we will say that objects in C *have an underlying space*. As an example, one could take some category of manifolds. The following result will also hold for the various extensions covered below.

Proposition 4. *If G is a simplicial group object with an underlying space, then it is levelwise a closed subgroup of $W_{\mathrm{gr}}G$.*

Proof. Recall that G_n is the subgroup of $(W_{\mathrm{gr}}G)_n$ given by the conjunction of the collection of equations $g_{n-i} = d_0^i g_n$ for $i = 0, \dots, n$, so it is the intersection of a finite number of closed subspaces. □

Now notice that the definition of $W_{\mathrm{gr}}G$ does not depend in any way on the fact that G is a simplicial *group*; only the map Φ^{-1} requires the inversion map. Thus we can define the endofunctor $W_{\mathrm{mon}} \colon s\mathbf{Mon}(C) \to s\mathbf{Mon}(C)$ on the category of simplicial monoids in C. It lifts the functor $W \colon s\mathbf{Mon}(C) \to sC$ up to a natural *transformation* $W \Rightarrow W_{\mathrm{mon}}$ (that the functor W is defined on the whole category of simplicial monoids has been known since the earliest constructions).

Proposition 5. *For any simplicial monoid M in C, $W_{\mathrm{mon}}M$ is contractible in $s\boldsymbol{Mon}(C)$.*

Proof. The extra degeneracies

$$\begin{aligned} s_{-1}\colon (W_{\mathrm{mon}}M)_n &\to (W_{\mathrm{mon}}M)_{n+1} \\ (g_n,\dots,g_0) &\mapsto (1,g_n,g_{n-1},\dots,g_0), \end{aligned}$$

give rise to a contracting homotopy. □

This result is stronger than the contractibility statement in the previous section, which is only at the level of underlying simplicial objects. It clearly extends to the contractibility of $W_{\mathrm{gr}}G$ in $s\mathbf{Grp}(C)$. However, we do not have a simple construction and interpretation of $W_{\mathrm{mon}}M/M$ as before. Indeed, under the minimal assumptions here—the presence of finite products—we do not necessarily have this quotient. One could perhaps consider a homotopy quotient in sC if necessary, such as the diagonal of the bisimplicial object associated to the action groupoid of M on $W_{\mathrm{mon}}M$.

In fact if we are willing to give up any comparison to existing functors, then the only structure one uses to define the functor W_{mon} with values in contractible monoids is the basepoint 1, and the fact projections and diagonal maps are monoid homomorphisms. This gives us great freedom in extending the results above to other algebraic structures. In particular we can mimic the definitions for what are known as *algebras for a Lawvere theory*, and our results extend to this much more general case. First, we give the definition.

Definition 6. A *multisorted Lawvere theory* is a category T with finite products such that there is a set S of objects, known as *sorts*, such that every object of T is isomorphic to a finite product of finite powers of objects in S. An *algebra* for a Lawvere theory T, or T-*algebra*, in a category C with finite products is a finite-product-preserving functor $\mathsf{T} \to C$, and a morphism of T-algebras is a natural transformation.

A multisorted Lawvere theory with only a single sort is simply called a Lawvere theory [1]. In examples the set of sorts is usually finite, but this is not a necessary assumption. One can also consider *infinitary* Lawvere theories, which use products of an arbitrary size.

While this definition appears very abstract, it is in fact very concrete. If we write down the axioms for a group, say, in diagrammatic form, then we have maps $1 \to G$, $G \to G$ and $G \times G \to G$ giving the identity element, inverses and multiplication respectively, and some commuting diagrams for associativity etc. These freely generate a category with objects $\{G^n\}_{n\geq 0}$ and those morphisms which arise as products and composites of various projections, diagonals and structure maps of the group. This is then the Lawvere theory Group for groups. A finite-product-preserving functor $\mathsf{Group} \to \mathbf{Set}$ is exactly a group: a set with the correct finitary operations such that the necessary diagrams commute.

Another example is the Lawvere theory Ring for rings, which has $\mathsf{AbGroup}$, the Lawvere theory for *Abelian groups* as a subcategory, encoding the Abelian group underlying the ring. $\mathsf{AbGroup}$ is a quotient of Group, since there is an additional commuting diagram in the definition of an Abelian group. Similarly, we can consider the two-sorted Lawvere theory Module for modules, the sorts being the coefficient ring and the module itself. It contains Ring, using one sort, and $\mathsf{AbGroup}$, using the other sort. To contrast, given a fixed ring R, there is a Lawvere theory (with a single sort) for R-modules, with a unary operation $r \cdot - \colon M \to M$ for each $r \in R$. In the other direction, there is the Lawvere theory P for pointed objects, which has a single map $1 \to x$, where x is the unique sort, and all other maps generated from this and projections and diagonals; algebras are simply pointed sets, and morphisms are pointed maps. In general maps $1 \to x$ in a (multisorted) Lawvere theory can be called *constants*.

Algebras for a (multisorted) Lawvere theory in **Set** are just groups, rings, modules, algebras and so on, but we can also consider for example *topological* groups, rings etc., and likewise for any category C with finite products, like that of smooth manifolds. We have the category $\mathsf{T}\mathrm{Alg}(C)$ of T-algebras which is just the category $\mathrm{Cat}_\times(\mathsf{T}, C)$ of finite-product-preserving functors and arbitrary natural transformations. This is functorial in C for functors $C \to D$ preserving finite products. Given a category C with finite products, the category sC of simplicial objects also has finite products, and it is examples of this sort which we shall focus on.

Given a multisorted Lawvere theory, there are two further structures we consider them to be equipped with. We can consider a multisorted Lawvere theory T with a specified map from the theory of pointed objects, $1\colon \mathsf{P} \to \mathsf{T}$ — in other words, a specified constant for a given sort. We can also consider the case of a specified constant for each sort of T. Alternatively we can consider T equipped with a specified map from the theory of monoids, $m\colon \mathsf{Monoid} \to \mathsf{T}$, which picks out a sort and a monoid operation on that sort. Given such a monoid operation m there is a functor

$$s\mathsf{T}\mathrm{Alg}(C) \xrightarrow{m^*} s\mathbf{Mon}(C) \xrightarrow{W} sC \tag{1}$$

which we shall call $W^{\mathsf{T},m}$. Note that this forgets all the sorts of T except the one on which m is defined. We also define the forgetful functor $U^m\colon s\mathsf{T}\mathrm{Alg}(C) \to sC$ forgetting everything but the sort with the specified monoid operation, including the monoid operation on it.

Let T be a multisorted Lawvere theory and A a simplicial T-algebra in C. We define another simplicial T-algebra $W_{\mathsf{T}}A$, reducing to the case of W_{gr} for $\mathsf{T} = \mathsf{Group}$.

Definition 7. For each sort x of T, the set of n-simplices of $(W_{\mathsf{T}}A)(x)$ is given by

$$(W_{\mathsf{T}}A)(x)_n = A(x)_n \times \cdots \times A(x)_0.$$

The face maps are

$$d_i(g_n,\dots,g_0) = \begin{cases} (g_{n-1}, g_{n-2}, \dots, g_0), & \text{if } i = 0; \\ (d_i g_n, \dots, d_1 g_{n-i+1}, g_{n-i-1}, \dots, g_0), & \text{if } i = 1, \dots, n-1; \\ (d_n g_n, \dots, d_1 g_1), & \text{if } i = n, \end{cases}$$

and the degeneracy maps are

$$s_i(g_n,\dots,g_0) = (s_i g_n, \dots, s_0 g_{n-i}, g_{n-i}, g_{n-i-1}, \dots, g_0), \quad \text{for } i = 0, \dots, n.$$

Note that in the case we do not have any underlying monoid operations we cannot define $W^{\mathsf{T},m}$, whereas we can always define W_{T}. The following is our omnibus theorem for multisorted Lawvere theories.

Theorem 8. *Given a multisorted Lawvere theory* T*,* W_{T} *is a monad on* $s\mathsf{T}\mathrm{Alg}(C)$*. Moreover:*

- *If* T *is equipped with a specified monoid operation* m*, then there is a natural transformation* $\Phi^m \colon W^{\mathsf{T},m} \Rightarrow U^m \circ W_{\mathsf{T}}$*.*
- *If* T *is equipped with a specified* group *operation* m*, then* Φ^m *is an isomorphism.*
- *If a sort* x *of* T *has a constant* $1 \to x$ *then* $(W_{\mathsf{T}}A)(x)$ *is contractible.*

Proof. The proof is essentially that in Section 3 and Proposition 5. □

In the case that there are no constants for any of the sorts, for example the theory of a semigroup acting on a set from automata theory, then there seems little one can say about W_{T} apart from its monad structure. Uses for such examples would be interesting to see.

Example 9. Consider the theory Module for modules. For any simplicial ring R and R-module M we have a contractible $(W_{\mathsf{Ring}}R)$-module $W_{\mathsf{Module}}M$, where $W_{\mathsf{Ring}}R$ is a contractible simplicial ring. Moreover, $W_{\mathsf{Module}}M$ is isomorphic as a simplicial object to WM calculated for the underlying simplicial Abelian group of M.

It is sometimes the case that there is a finite-product-preserving homotopy colimit functor hocolim: $sC \to C$, for example when C is the category of k-spaces and hocolim is geometric realisation.

Proposition 10. *If* hocolim *preserves finite products then* $\mathrm{hocolim}\, W_{\mathsf{T}}A$ *is a* T*-algebra in* C*.*

This brings us full circle back to Segal's result that the total space EG of the universal bundle is a topological group (we have only shown above that the simplicial model was a simplicial group). To continue the above example, $|W_{\mathsf{Module}}M|$ is a topological space which is a module for the topological ring $|W_{\mathsf{Ring}}R|$ (both of which are contractible).

The following example is an amusing reality check.

Example 11 (Trimble)**.** Consider the ring $\mathbb{Z}/(2)$ as a simplicial ring concentrated in the degree zero. The simplicial ring $W_{\mathsf{Ring}}(\mathbb{Z}/(2))$ is contractible. There is a finite-product-preserving functor $\Theta\colon s\mathbf{Set} \to \mathbf{CGHaus}$ defined analogously to geometric realisation, but with the unit interval replaced by the end-compactified long line: a connected, non-path-connected compact Hausdorff space which is an interval object. We thus have a topological ring $\Theta W_{\mathsf{Ring}}(\mathbb{Z}/(2))$ which can be shown to be connected but *not* contractible [9].

Finally, one omission which might be glaring to those who are familiar with the constructions W and $\overline{W}$ from homotopy theory is the case of simplicial *groupoids*. Groupoids are not algebras for any multisorted Lawvere theory, rather they are models for a *finite limit* (or *essentially algebraic*) theory, a concept that subsumes multisorted Lawvere theories and allows for arbitrary finite limits in the definition of operations. Any generalisation of our constructions to finite limit theories would need to give simplicial T-algebras isomorphic to the ones given here. However the natural construction that generalises W_{gr} from simplicial groups to simplicial groupoids does not result in a simplicial groupoid, although this does not rule out more complicated algebraic structures arising. A result in this direction analogous to the one presented here would be far reaching, as any locally finitely presentable category is the category of algebras for *some* finite limit theory.

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