

On principal left ideals of βG

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ABSTRACT. Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group written additively, let $G = \bigoplus_{\alpha < \kappa} G_\alpha$, and let $H_\alpha = \{x \in G : x(\gamma) = 0 \text{ for all } \gamma < \alpha\}$. Let βG be the Stone–Čech compactification of G as a discrete semigroup and define a closed subsemigroup $T \subseteq \beta G$ by $T = \bigcap_{\alpha < \kappa} \text{cl}_{\beta G}(H_\alpha \setminus \{0\})$. We show that, for every $p, q \in T$, if $(\beta G + p) \cap (\beta G + q) \neq \emptyset$, then either $p \in \beta G + q$ or $q \in \beta G + p$.

Let S be a discrete semigroup with Stone–Čech compactification βS . The operation on S extends to one on βS in such a way that for each $a \in S$, the left translation

$$\beta S \ni x \mapsto ax \in \beta S$$

is continuous, and for each $q \in \beta S$, the right translation

$$\beta S \ni x \mapsto xq \in \beta S$$

is continuous.

We take the points of βS to be the ultrafilters on S , identifying the principal ultrafilters with the points of S . The topology of βS is generated by taking as a base the subsets of the form

$$\bar{A} = \{p \in \beta S : A \in p\},$$

where $A \subseteq S$. For $p, q \in \beta S$, the ultrafilter pq has a base consisting of subsets of the form

$$\bigcup_{x \in A} xB_x,$$

where $A \in p$ and $B_x \in q$.

The semigroup βS is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. An elementary introduction to βS can be found in [1].

In the study of algebraic structure of βS an important role is played by the following fact.

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Theorem 1 ([1, Corollary 6.20]). *Let G be a countable group. For every $p, q \in \beta G$, if $((\beta G)p) \cap ((\beta G)q) \neq \emptyset$, then either $p \in (\beta G)q$ or $q \in (\beta G)p$.*

Theorem 1 tells us that for any countable group G and for any two principal left ideals of βG , either one of them is contained in another or they are disjoint.

In this note we prove the following extension of Theorem 1.

Theorem 2. *Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group written additively, let $G = \bigoplus_{\alpha < \kappa} G_\alpha$, and let*

$$H_\alpha = \{x \in G : x(\gamma) = 0 \text{ for all } \gamma < \alpha\}.$$

Define a closed subsemigroup $T \subseteq \beta G$ by $T = \bigcap_{\alpha < \kappa} \overline{H_\alpha \setminus \{0\}}$. For every $p, q \in T$, if $(\beta G + p) \cap (\beta G + q) \neq \emptyset$, then either $p \in \beta G + q$ or $q \in \beta G + p$.

Before proving Theorem 2, let us check that T is indeed a subsemigroup. It suffices to show that for every $p, q \in T$ and $\alpha < \kappa$, $p + q \in \overline{H_\alpha \setminus \{0\}}$, equivalently $H_\alpha \setminus \{0\} \in p + q$. Define $A \in p$ and $B_x \in q$ for every $x \in A$ by $A = H_\alpha$ and $B_x = H_\alpha \setminus \{-x\}$. Then $x + B_x \subseteq H_\alpha \setminus \{0\}$, and so $\bigcup_{x \in A} (x + B_x) \subseteq H_\alpha \setminus \{0\}$. But $\bigcup_{x \in A} (x + B_x) \in p + q$. Hence, $H_\alpha \setminus \{0\} \in p + q$.

Proof of Theorem 2. Assume on the contrary that $p \notin \beta G + q$ and $q \notin \beta G + p$ for some $p, q \in T$. We shall show that

$$(\beta G + p) \cap (\beta G + q) = \emptyset,$$

which is a contradiction.

Since $p \notin \beta G + q$, there are $P \in p$ and $Q_x \in q$ for every $x \in G$ such that

$$P \cap \bigcup_{x \in G} (x + Q_x) = \emptyset.$$

And since $q \notin \beta G + p$, there are $Q \in q$ and $P_x \in p$ for every $x \in G$ such that

$$Q \cap \bigcup_{x \in G} (x + P_x) = \emptyset.$$

For every $x \in G \setminus \{0\}$, let

$$\phi(x) = \max \text{supp}(x) \text{ and } \theta(x) = \min \text{supp}(x).$$

As usual, $\text{supp}(x) = \{\alpha < \kappa : x(\alpha) \neq 0\}$. Also let

$$\phi(0) = -1 \text{ and } \theta(0) = \kappa.$$

Define partial orders \leq_L and \leq_R on G by

$$\begin{aligned} x \leq_L y & \text{ if and only if } x(\alpha) = y(\alpha) \text{ for each } \alpha \leq \phi(x), \\ x \leq_R y & \text{ if and only if } x(\alpha) = y(\alpha) \text{ for each } \alpha \geq \theta(x). \end{aligned}$$

Now for every $x \in G$, define $A_x \in p$ and $B_x \in q$ by

$$A_x = \left(\bigcap_{y \leq_R x} P_y \right) \cap P \cap H_{\phi(x)+1},$$

$$B_x = \left(\bigcap_{y \leq_R x} Q_y \right) \cap Q \cap H_{\phi(x)+1}.$$

(Notice that $\{y \in G : y \leq_R x\}$ is finite.) It then follows that:

- (1) $(\bigcup_{x \in G} (x + A_x)) \cap B_0 = \emptyset$ and $(\bigcup_{x \in G} (x + B_x)) \cap A_0 = \emptyset$.
- (2) For every $x \in G$ and for every $y \leq_R x$, one has $A_x \subseteq A_y$ and $B_x \subseteq B_y$, in particular, $A_x \subseteq A_0$ and $B_x \subseteq B_0$.
- (3) For every $x \in G$, one has $A_x, B_x \subseteq H_{\phi(x)+1}$.

We claim that

$$\left(\bigcup_{x \in G} (x + A_x) \right) \cap \left(\bigcup_{x \in G} (x + B_x) \right) = \emptyset.$$

To see this, let $x, y \in G$. We have to show that $(x + A_x) \cap (y + B_y) = \emptyset$. Consider two cases.

Case 1: neither $x \leq_L y$ nor $y \leq_L x$. Then $(x + H_{\phi(x)+1}) \cap (y + H_{\phi(y)+1}) = \emptyset$. Consequently by (3), $(x + A_x) \cap (y + B_y) = \emptyset$.

Case 2: either $x \leq_L y$ or $y \leq_L x$. Let $y \leq_L x$. By (1), $(x - y + A_{x-y}) \cap B_0 = \emptyset$, so $(x + A_{x-y}) \cap (y + B_0) = \emptyset$. But $x - y \leq_R x$ and $0 \leq_R y$. Consequently by (2), again $(x + A_x) \cap (y + B_y) = \emptyset$.

Since the subsets

$$U = \bigcup_{x \in G} (x + A_x) \quad \text{and} \quad V = \bigcup_{x \in G} (x + B_x)$$

of G are disjoint, the subsets \bar{U} and \bar{V} of βG are also disjoint. But $\beta G + p \subseteq \bar{U}$ and $\beta G + q \subseteq \bar{V}$. Hence, $(\beta G + p) \cap (\beta G + q) = \emptyset$. \square

Remark 1. Theorem 2 was inspired by [2, Proposition 3.4].

Remark 2. The semigroup T from Theorem 2 depends only on two cardinals: κ and $\lambda = \min\{|\bigoplus_{\gamma \leq \alpha < \kappa} G_\alpha| : \gamma < \kappa\}$ [3].

We conclude this note with the following question.

Question. Is it true that for any (Abelian) group G and for any two principal left ideals of βG , either one of them is contained in another or they are disjoint?

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