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Mixing subalgebras of finite von Neumann algebras

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Abstract. Jolissaint and Stalder introduced definitions of mixing and weak mixing for von Neumann subalgebras of finite von Neumann algebras. In this note, we study various algebraic and analytical properties of subalgebras with these mixing properties. We prove some basic results about mixing inclusions of von Neumann algebras and establish a connection between mixing properties and normalizers of von Neumann subalgebras. The special case of mixing subalgebras arising from inclusions of countable discrete groups finds applications to ergodic theory, in particular, a new generalization of a classical theorem of Halmos on the automorphisms of a compact abelian group. For a finite von Neumann algebra M and von Neumann subalgebras A, B of M, we introduce a notion of weak mixing of $B \subset M$ relative to A. We show that weak mixing of $B \subset M$ relative to a subalgebra $A \subset B$ is equivalent to the following property: if $x \in M$ and there exist a finite number of elements $x_1, \ldots, x_n \in M$ such that $Ax \subset \sum_{i=1}^n x_i B$, then $x \in B$. We conclude the paper with an assortment of further examples of mixing subalgebras arising from the amalgamated free product and crossed product constructions.

Contents

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1. Introduction

In [6], Jolissaint and Stalder defined weak mixing and mixing for abelian von Neumann subalgebras of finite von Neumann algebras. These properties arose as natural extensions of corresponding notions in ergodic theory in the following sense: If σ is a measure preserving action of a countable discrete abelian group Γ_0 on a finite measure space (X, μ), then the action is (weakly) mixing in the sense of [1] if and only if the abelian von Neumann subalgebra $L(\Gamma_0)$ is (weakly) mixing in the crossed product finite von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma_0$.

In this note, we extend the definitions of weak mixing and mixing to general von Neumann subalgebras of finite von Neumann algebras, and study various algebraic and analytical properties of these subalgebras. In a forthcoming note, the authors will specialize to the study of mixing properties of maximal abelian von Neumann subalgebras. If B is a von Neumann subalgebra of a finite von Neumann algebra M, and $\mathbb{E}_B$ denotes the usual trace-preserving conditional expectation onto B, we call B a *weakly mixing* subalgebra of M if there exists a sequence of unitary operators $\{u_n\}$ in B such that

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)\|_2 = 0, \quad \forall x, y \in M.$$

We call B a *mixing* subalgebra of M if the above limit is satisfied for all elements x, y in M and all sequences of unitary operators $\{u_n\}$ in B such that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. When B is an abelian algebra, our definition of weak mixing is precisely the *weak asymptotic homomorphism property* introduced by Robertson, Sinclair and Smith [15]. Although our definitions of weak mixing and mixing are slightly different from those of Jolissaint and Stalder, our definitions coincide with theirs in the setting of the action of a countable discrete group on a probability space. Using arguments similar to those in the proofs of Proposition 2.2 and Proposition 3.6 of [6], one can show:

Proposition 1.1. *If σ is a measure preserving action of a countable discrete group Γ_0 on a finite measure space (X, μ), then the action is (weakly) mixing in the sense of* [1] *if and only if the von Neumann subalgebra $L(\Gamma_0)$ is (weakly) mixing in the crossed product finite von Neumann algebra*

$$L^\infty(X, \mu) \rtimes \Gamma_0.$$

For an inclusion of finite von Neumann algebras $B \subset M$, we call a unitary operator $u \in M$ a *normalizer* of B in M if $uBu^* = B$ [3]. Clearly, every unitary in B satisfies this condition; the subalgebra B is said to be *singular* in M if the only normalizers of B in M are elements of B. There is a close relationship between the concepts of weak mixing and singularity. Sinclair and Smith [17] noted one connection in proving that weakly mixing von Neumann subalgebras are singular in their containing algebras. The converse was proved by Sinclair, Smith, White and Wiggins [20] under the assumption that the subalgebra is also masa (maximal abelian self-adjoint subalgebra) in the ambient von Neumann algebra. In other words, the measure preserving action of a countable discrete abelian group Γ_0 on a finite measure space (X, μ) is weakly mixing if and only if the associated von Neumann algebra $L(\Gamma_0)$ is singular in $L^\infty(X, \mu) \rtimes \Gamma_0$. This provides an operator algebraic characterization of weak mixing in the abelian setting, which is the main motivation for the study undertaken here. In contrast to the abelian case, Grossman and Wiggins [4] showed that for general finite von Neumann algebras, weakly mixing is not equivalent to singularity, so techniques beyond those known for singular subalgebras are required. In what follows, we develop basic theory for mixing properties of general subalgebras of finite von Neumann algebras. This leads to a number of new observations about mixing properties of subalgebras and group actions, a characterization of weakly mixing subalgebras in terms of their finite bimodules, and a variety of new examples of inclusions of von Neumann algebras satisfying mixing conditions. The paper is organized as follows.

Section 2 contains some preliminary technical results. We show that if B is a diffuse finite von Neumann algebra, then

$$B^\omega \ominus B = \{x \in B^\omega : \tau_\omega(x^*b) = 0, \forall b \in B\}$$

is the weak operator closure of the linear span of unitary operators in $B^\omega \ominus B$, where B^ω is the ultra-power algebra of B. This result plays an important role in the subsequent sections.

In Section 3, we prove that if B is a mixing von Neumann subalgebra of a finite von Neumann algebra M, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(xb_ny) - \mathbb{E}_B(x)b_n\mathbb{E}_B(y)\|_2 = 0, \quad \forall x, y \in M,$$

if $\{b_n\}$ is a bounded sequence of operators in B such that $\lim_{n\to\infty} b_n = 0$ in the weak operator topology. As applications, we show that if B is mixing in M, k is a positive integer, and $e \in B$ is a projection, then $M_k(\mathbb{C}) \otimes B$ is mixing in $M_k(\mathbb{C}) \otimes M$ and eBe is mixing in eMe. We also show that, in contrast to weakly mixing masas, one cannot distinguish mixing masas by the presence or absence of centralizing sequences in the masa for the containing II_1 factor.

Section 4 concerns the special case of inclusions of group von Neumann algebras. We extend some results of [6] for abelian subgroups to the case of a general inclusion of countable, discrete groups $\Gamma_0 \subset \Gamma$ in showing that $L(\Gamma_0)$ is mixing in $L(\Gamma)$ if and only if $g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite group for every $g \in \Gamma \setminus \Gamma_0$. These two conditions are seen to be equivalent the property that for every diffuse von Neumann subalgebra A of B and every $y \in M$, $yAy^* \subset B$ implies $y \in B$. Some applications to ergodic theory are given. In particular, Theorem 4.3 generalizes results of Kitchens and Schmidt [9] and Halmos [5].

In Section 5, we introduce and study the concept of relative weak mixing for a triple of finite von Neumann algebras, and obtain several characterizations of weakly mixing triples. It turns out that relative weak mixing of an inclusion $B \subset M$ with respect to a von Neumann subalgebra $A \subset B$ is closely related to the bimodule structure between the two subalgebras A and B. In particular, we show that $B \subset M$ is weakly mixing relative to A if and only the following property holds: if $x \in M$ satisfies $Ax \subset \sum_{i=1}^n x_i B$ for a finite number of elements $x_1, \ldots, x_n$ in M, then $x \in B$.

The results in Section 6 show that mixing von Neumann subalgebras have hereditary properties which are notably different from those of general singular subalgebras. We also consider the relationship between mixing and normalizers; in particular, we show that subalgebras of mixing algebras inherit a strong singularity property from the containing algebra. Finally, we provide an assortment of new examples of mixing von Neumann subalgebras which arise from the amalgamated free product and crossed product constructions.

We collect here some basic facts about finite von Neumann algebras, which will be used in the sequel. Throughout this paper, M is a finite von Neumann algebra with a given faithful normal trace τ. Denote by $L^2(M) = L^2(M, \tau)$ the Hilbert space obtained by the GNS-construction of M with respect to τ. The image of $x \in M$ via the GNS-construction is denoted by $\hat{x}$, and the image of a subset L of M is denoted by $\hat{L}$. The trace norm of $x \in M$ is defined by $\|x\|_2 = \|x\|_{2,\tau} = \tau(x^*x)^{1/2}$. Suppose that B is a von Neumann subalgebra of M. Then there exists a unique faithful normal conditional expectation $\mathbb{E}_B$ from M onto B preserving τ. Let e_B be the projection of $L^2(N)$ onto $L^2(B)$. Then the von Neumann algebra $\langle M, e_B \rangle$ generated by M and e_B is called *the basic construction* of M, which plays a crucial role in the study of von Neumann subalgebras of finite von Neumann algebras. There is a unique faithful tracial weight Tr on $\langle M, e_B \rangle$ such that

$$\mathrm{Tr}(x e_B y) = \tau(xy), \quad \forall x, y \in M.$$

For $\xi \in L^2(\langle M, e_B \rangle, \mathrm{Tr})$, define $\|\xi\|_{2,\mathrm{Tr}} = \mathrm{Tr}(\xi^* \xi)^{1/2}$. For more details of the basic construction, we refer to [2, 7, 11, 18]. For a detailed account of finite von Neumann algebras and the theory of masas, we refer the reader to [18].

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2. Unitary operators in $M \ominus B$

Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M. We denote by $M \ominus B$

the orthogonal complement of B in M with respect to the standard inner product on M, that is,

$$M \ominus B = \{x \in M : \tau(x^*b) = 0 \text{ for all } b \in B\}.$$

Then $x \in M \ominus B$ if and only if $\mathbb{E}_B(x) = 0$, where $\mathbb{E}_B$ is the trace-preserving conditional expectation of M onto B. Note that if $x \in M \ominus B$, then $\tau(x) = \tau(\mathbb{E}_B(x)) = 0$, so the unique positive element in $M \ominus B$ is 0. On the other hand, it is easy to see that $M \ominus B$ is the linear span of self-adjoint elements in $M \ominus B$.

In the following section, we will use the fact that a bounded sequence (b_n) in a finite von Neumann algebra B converges to 0 in the weak operator topology if and only if it defines an element of the ultrapower B^ω which is orthogonal to B in the above sense. A key step in the proof of Theorem 3.3 will then be to approximate an arbitrary $z \in B^\omega \ominus B$ by linear combinations of unitary operators in $B^\omega \ominus B$. That such an approximation is possible is the main technical result of this section.

When $B \subset M$ comes from an inclusion of countable discrete groups, there is an obvious dense linear subspace of $M \ominus B$: if G is a subgroup of a discrete group Γ, then $\mathcal{L}(\Gamma) \ominus \mathcal{L}(G)$ is the weak closure of the linear span of unitary operators corresponding to elements in $\Gamma \setminus G$. Although in the case of a general inclusion $B \subseteq M$, such a canonical set of unitaries is not available, we nevertheless obtain a partial answer to the following question: If B is a subalgebra of a diffuse finite von Neumann algebra M such that $eMe \neq eBe$ for every nonzero projection $e \in B$, is $M \ominus B$ the weak closure of the linear span of unitaries in $M \ominus B$?

The assumption that $eMe \neq eBe$ for every nonzero projection $e \in B$ is necessary, as is the assumption that M is diffuse. For instance, if $M = \mathbb{C} \oplus \mathbb{C}$ and $B = \mathbb{C}$ and $\tau(1 \oplus 0) \neq \tau(0 \oplus 1)$, then there are no unitary operators in $M \ominus B$.

Let $(M)_1$ be the operator norm-closed unit ball of M, and let

$$\Lambda = \{x \in (M)_1 : x = x^*, \mathbb{E}_B(x) = 0\}.$$

Then Λ is a convex set which is closed, hence also compact, in the weak operator topology. By the Krein–Milman Theorem, Λ is the weak operator closure of the convex hull of its extreme points. Thus, we need only characterize the extreme points of Λ.

Lemma 2.1. *Suppose that for every nonzero projection $p \in M$, there exists a nonzero element $x_p \in pMp$ satisfying $\mathbb{E}_B(x_p) = 0$. Then the extreme points of Λ are*

$$\left\{2e - 1 : e \in M \text{ a projection, with } \mathbb{E}_B(e) = \frac{1}{2}\right\}.$$

Proof. If $e \in M$ is a projection with $\mathbb{E}_B(e) = \frac{1}{2}$, then it is easy to see that the operator $u = 2e-1 \in \Lambda$ is an extreme point of the unit ball $(M)_1$, hence also an extreme point of Λ. On the other hand, suppose that $a \in \Lambda$ is an extreme point of Λ, but is not of the form $2e-1$, for some projection $e \in M$, as above. By the spectral decomposition theorem, there exists an $\epsilon > 0$ and a nonzero spectral projection e of a such that

$$(-1+\epsilon)e \leq ae \leq (1-\epsilon)e.$$

By assumption, there is a nonzero self-adjoint element $x \in eMe$ such that $\mathbb{E}_B(x) = 0$. By multiplying by a scalar, we may insist that $-\epsilon e \leq x \leq \epsilon e$. Then $a+x, a-x \in \Lambda$ and $a = \frac{1}{2}(a+x) + \frac{1}{2}(a-x)$, so a is not an extreme point of Λ, contradicting our assumption. Therefore, $a = 2e-1$ for some projection $e \in M$. Since $\mathbb{E}_B(a) = 0$, $\mathbb{E}_B(e) = \frac{1}{2}$. This completes the proof. $\square$

The following example shows that the assumptions of the above lemma are essential.

Example 2.2. In the inclusion $\mathbb{C} \subset M_3(\mathbb{C})$, there is no projection $e \in M_3(\mathbb{C})$ satisfing $\tau(e) = \frac{1}{2}$. In this case, the partial isometry

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is an extreme point of Λ.

Corollary 2.3. *Let M be a diffuse finite von Neumann algebra with a faithful normal trace τ. Then $M \ominus \mathbb{C}1$ is the weak operator closure of the linear span of self-adjoint unitary operators in $M \ominus \mathbb{C}1$.*

Proof. For every nonzero projection $p \in M$, pMp is diffuse and hence $pMp \neq \mathbb{C}p$. So there is a nonzero operator $x_p \in pMp$ with $\tau(x_p) = 0$. By Lemma 2.1, $M \ominus \mathbb{C}1$ is the weak operator closure of the linear span of self-adjoint unitary operators in $M \ominus \mathbb{C}1$. $\square$

For the next result, recall that every diffuse finite von Neumann algebra N with faithful trace τ contains a *Haar unitary,* that is, a unitary element $u \in N$ such that $\tau(u^n) = 0$ for all $n \in \mathbb{N}$.

Lemma 2.4. *Suppose B is a diffuse finite von Neumann algebra with a faithful normal trace τ. For $\epsilon > 0$ and $x_1, \ldots, x_n \in B$, there exists a Haar unitary operator $u \in B$ such that*

$$|\tau(x_i u^*)| < \epsilon, \quad 1 \leq i \leq n.$$

Proof. Since B is diffuse, B contains a Haar unitary operator v. Note that $v^n \to 0$ in the weak operator topology. So there exists an N such that

$$|\tau(x_i (v^N)^*)| < \epsilon, \quad 1 \leq i \leq n.$$

Let $u = v^N$. Then u is a Haar unitary operator and the lemma follows. $\square$

Given a separable diffuse von Neumann algebra B with faithful normal trace τ and an ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, denote by B^ω the corresponding ultrapower algebra, and the induced faithful normal trace by τ_ω (see [16]). We again use the standard notation of $(B^\omega \ominus B)_1$ for the norm-closed unit ball of $B^\omega \ominus B$. The following proposition is the main result of this section.

Proposition 2.5. *Suppose B is a separable diffuse finite von Neumann algebra with a faithful normal trace τ. Then $(B^\omega \ominus B)_1$ is the trace norm closure of the convex hull of self-adjoint unitary operators in $B^\omega \ominus B$.*

Proof. We claim that for every nonzero projection $p \in B^\omega$, there exists a nonzero element x_p in $pB^\omega p$ such that $\mathbb{E}_B(x_p) = 0$, where $\mathbb{E}_B$ is the conditional expectation of B^ω onto B preserving τ_ω. Let $p = (p_n) \in B^\omega$, where $p_n \in B$ is a projection with $\tau(p_n) = \tau_\omega(p) > 0$. Since B is separable, there is a sequence $\{y_k\}$ in B which is dense in the trace norm. We may assume that $y_1 = 1$. By Lemma 2.4, for any initial segment $\{y_1, \dots, y_n\}$ of the dense sequence, there is a Haar unitary operator $u_n \in p_nBp_n$ such that

$$|\tau(p_n y_i p_n u_n^*)| < \frac{1}{n}, \quad \forall 1 \le i \le n.$$

Now define an element x_p of B^ω by $x_p = (u_n)$. Then

$$\|x_p\|_2^2 = \lim_{n\to\omega} \|u_n\|_2^2 = \tau(p) > 0.$$

Hence, $x_p \neq 0$ and $x_p \in pB^\omega p$. Note that for each $k \in \mathbb{N}$, we have

$$\tau_\omega(y_k(x_p)^*) = \tau_\omega((py_kp)(x_p)^*) = \lim_{n\to\omega} \tau(p_n y_k p_n u_n^*) = 0.$$

Since $\{y_k\}$ is dense in B in the trace norm topology, $\tau_\omega(y(x_p)^*) = 0$ for all $y \in B$. This implies $\mathbb{E}_B(x_p) = 0$. By Lemma 2.1, $(B^\omega \ominus B)_1$ is the weak operator closure of the convex hull of self-adjoint unitary operators in $B^\omega \ominus B$. Note that $(B^\omega \ominus B)_1$ is a convex set, so its weak operator closure coincides with its closure in the strong operator and trace norm topologies. This proves the result. $\square$

Corollary 2.6. *Suppose B is a separable diffuse finite von Neumann algebra with a faithful normal trace τ. Then $B^\omega \ominus B$ is the weak operator closure of the linear span of self-adjoint unitary operators in $B^\omega \ominus B$.*

Using a similar approach, we can also prove the following result.

Proposition 2.7. *If M is a separable type II_1 factor and B is an abelian von Neumann subalgebra of M, then $M \ominus B$ is the weak operator closure of the linear span of unitary operators in $M \ominus B$.*

It is not clear whether Proposition 2.7 holds for nonabelian subalgebras. We are unable, for instance, to establish the conclusion of the result when B is a hyperfinite subfactor of a nonhyperfinite type II_1 factor M, e.g. $L\mathbb{F}_2$.

3. Mixing von Neumann subalgebras

Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M.

Definition 3.1. An algebra B is a *mixing von Neumann subalgebra* of M if

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)\|_2 = 0$$

holds for all $x, y \in M$ and every sequence of unitary operators $\{u_n\}$ in B such that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. If B is a mixing von Neumann subalgebra of M, then we say $B \subseteq M$ a *mixing inclusion* of finite von Neumann algebras.

It is easy to see that B is a mixing von Neumann subalgebra of M if and only if for all elements x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny)\|_2 = 0$$

whenever $\{u_n\}$ is a sequence of unitary operators in B such that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology.

Remark 3.2. By the Kaplansky density theorem, we may assume that x and y are in a subset F of M such that M is the von Neumann algebra generated by F in Definition 3.1.

The following theorem, which is the main result of this section, provides a useful equivalent condition for mixing inclusions of finite von Neumann algebras.

Theorem 3.3. *If B is a mixing von Neumann subalgebra of M and $x, y \in M$ with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, then*

$$\lim_{n\to\infty} \|\mathbb{E}_B(xb_ny)\|_2 = 0$$

whenever $\{b_n\}$ is a bounded sequence of operators in B such that $\lim_{n\to\infty} b_n = 0$ in the weak operator topology.

Proof. Let ω be a free ultrafilter of the set of natural numbers and let M^ω be the ultrapower algebra of M. Then M^ω is a finite von Neumann algebra with a faithful normal trace τ_ω. We can identify B^ω with a von Neumann subalgebra of M^ω in the natural way. Every bounded sequence (b_n) in B defines an element z of B^ω. We may assume that $\|z\| \leq 1$. It is easy to see that $\lim_{n\to\omega} b_n = 0$ in the weak operator topology if and only if

$$\tau_\omega(zb) = 0, \quad \forall b \in B.$$

Recall that $M \ominus B = \{x \in M : \tau(x^*b) = 0 \text{ for all } b \in B\}$. It is easy to see that Definition 3.1 is equivalent to the following: For any x, y in $M \ominus B$, and any unitary operator $u \in B^\omega \ominus B$, one has $\mathbb{E}_{B^\omega}(xuy) = 0$.

Note that B is a diffuse subalgebra of M. Indeed, suppose $p \in B$ is a minimal projection. Since B is mixing, then in particular we have that $B' \cap M \subset B$, so Theorem 12.2.4 of [18] implies that there exists a masa A of M such that $p \in A \subset B$. But then p is a minimal projection of A, a contradiction. Thus, Proposition 2.5 applies, and $(B^\omega \ominus B)_1$ is the trace norm closure of the convex hull of unitary operators in $B^\omega \ominus B$. Let $\epsilon > 0$. Then there exist unitary operators $u_1, \dots, u_n$ in $B^\omega \ominus B$ and positive numbers $\alpha_1, \dots, \alpha_n$ with $\alpha_1 + \cdots + \alpha_n = 1$ such that

$$\left\| z - \sum_{k=1}^{n} \alpha_k u_k \right\|_{2,\tau_\omega} < \epsilon.$$

For any elements x and y of $M \ominus B$,

$$\begin{aligned} \|\mathbb{E}_{B^\omega}(xzy)\|_{2,\tau_\omega} &= \left\| \mathbb{E}_{B^\omega}\left(x \left(z - \sum_{k=1}^{n} \alpha_k u_k \right) y \right) \right\|_{2,\tau_\omega} \\ &\le \left\| x \left(z - \sum_{k=1}^{n} \alpha_k u_k \right) y \right\|_{2,\tau_\omega} \\ &\le \|x\| \cdot \left\| z - \sum_{k=1}^{n} \alpha_k u_k \right\|_{2,\tau_\omega} \cdot \|y\| \\ &\le \epsilon \|x\| \|y\|. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mathbb{E}_{B^\omega}(xzy) = 0$, which is equivalent to

$$\lim_{n\to\infty} \|\mathbb{E}_B(xb_n y)\|_2 = 0. \qquad \square$$

Two applications of the above theorem are the following.

Corollary 3.4. *If B is a mixing von Neumann subalgebra of M and k is a positive integer, then $M_k(\mathbb{C}) \otimes B$ is mixing in $M_k(\mathbb{C}) \otimes M$.*

Proof. Note that $x = (x_{ij}) \in (M_k(\mathbb{C}) \otimes M) \ominus (M_k(\mathbb{C}) \otimes B)$ if and only if $x_{ij} \in M \ominus B$ for all $1 \le i, j \le k$. Moreover, $b_n = (b_{ij}^n) \in M_k(\mathbb{C}) \otimes B$ converges to 0 in the weak operator topology if and only if b_{ij}^n converges to 0 in the weak operator topology for all $1 \le i, j \le k$. Now the corollary follows from Theorem 3.3. $\square$

Corollary 3.5. *If B is a mixing von Neumann subalgebra of M and e is a projection of B, then eBe is mixing in eMe.*

Proof. Let (b_n) be a bounded sequence in eBe which converges to 0 in the weak operator topology. For $x, y \in eMe \ominus eBe$, we have $x, y \in M \ominus B$. By Theorem 3.3,

$$\lim_{n\to\infty} \|\mathbb{E}_{eBe}(xb_n y)\|_2 = \lim_{n\to\infty} \|\mathbb{E}_B(xb_n y)\|_2 = 0. \qquad \square$$

It is well-known that the presence of centralizing sequences in a masa for its containing II_1 factor is a conjugacy invariant for the masa. More generally, it is possible to build nonconjugate masas of a II_1 factor by controlling the existence of centralizing sequences in various cutdowns of each masa. Sinclair and White [19] developed this technique to produce uncountably many nonconjugate weakly mixing masas in the hyperfinite II_1 factor with the same Pukánszky invariant. The final result of this section implies that, in contrast to the larger class of weakly mixing masas, there is no hope of distinguishing mixing masas along these lines. Following the notation of [19], for a von Neumann subalgebra B of a II_1 factor M, we denote by $\Gamma(B)$ the maximal trace of a projection $e \in B$ for which eBe contains a nontrivial centralizing sequences for eMe.

Proposition 3.6. *If B is a mixing subalgebra of a type II_1 factor M and $eBe \neq eMe$ for each nonzero projection $e \in B$, then $\Gamma(B) = 0$.*

Proof. By Corollary 3.5, we need only show that there is no nontrivial sequence $\{b_n\}$ in B which is centralizing for M. Suppose $\{b_n\} \subset B$ is such a centralizing sequence for M. We may assume that $\tau(b_n) = 0$ for each n. Suppose that $\lim_{n\to\omega} b_n = z \in B$ in the weak operator topology. Then for all $x \in M$,

$$zx = \lim_{n\to\omega} b_n x = \lim_{n\to\omega} x b_n = xz.$$

Since M is a type II_1 factor, $z = \tau(z)1 = 0$. Hence $\lim_{n\to\omega} b_n = 0$ in the weak operator topology. Choose a nonzero element $x \in M$ such that $\tau(xb) = 0$ for all $b \in B$. Note that

$$\begin{aligned}\|xb_n - b_n x\|_2^2 &= \|xb_n\|_2^2 + \|b_n x\|_2^2 - 2\,\mathrm{Re}\,\tau(b_n^* x^* b_n x)\\ &\geq \tau(b_n^* x^* x b_n) - 2\,\mathrm{Re}\,\tau(b_n^* \mathbb{E}_B(x^* b_n x))\\ &= \tau(x^* x b_n b_n^*) - 2\,\mathrm{Re}\,\tau(b_n^* \mathbb{E}_B(x^* b_n x)).\end{aligned}$$

Since $\{b_n\}$ is a central sequence of M, $\{b_n b_n^*\}$ is also a central sequence of M. The uniqueness of the trace on M implies that

$$\lim_{n\to\omega} \tau(x^* x b_n b_n^*) = \lim_{n\to\omega} \tau(x^* x)\tau(b_n b_n^*) = \lim_{n\to\omega} \|x\|_2^2 \cdot \|b_n\|_2^2.$$

By Theorem 3.3,

$$0 = \lim_{n\to\infty} \|xb_n - b_n x\|_2 \geq \|x\|_2 \lim_{n\to\infty} \|b_n\|_2,$$

which implies that $\lim_{n\to\omega} \|b_n\|_2 = 0$. This completes the proof. □

Corollary 3.7. *If B is a mixing masa of a type II_1 factor M, then $\Gamma(B) = 0$.*

4. Mixing inclusions of group von Neumann algebras

In this section, we apply our operator-algebraic machinery to the special case of mixing inclusions of von Neumann algebras that arise from actions of countable, discrete groups. This direction was taken up in [6], where it was shown that, for an infinite abelian subgroup Γ_0 of a countable group Γ, the inclusion $L(\Gamma_0) \subset L(\Gamma)$ is mixing if and only if the following condition (called (ST)) is satisfied:

For every finite subset C of $\Gamma \setminus \Gamma_0$, there exists a finite exceptional set $E \subset \Gamma_0$ such that $g\gamma h \notin \Gamma_0$ for all $\gamma \in \Gamma_0 \setminus E$ and $g, h \in C$.

Theorem 4.3 of this section supplies a similar characterization for the case in which Γ_0 is not abelian, and also establishes a connection between the group normalizer of the subgroup Γ_0 and the "analytic" normalizer of its associated group von Neumann algebra. The key observation required is the following, which shows that mixing subalgebras satisfy a much stronger form of singularity.

Theorem 4.1. *Let B be a mixing von Neumann subalgebra of M, and suppose that A is a diffuse von Neumann subalgebra of B. If $y \in M$ satisfies $yAy^* \subseteq B$, then $y \in B$.*

Proof. We may assume that A is a diffuse abelian von Neumann algebra. Then A is generated by a Haar unitary operator w. In particular, $\lim_{n\to\infty} w^n = 0$ in the weak operator topology. Let $x \in M$ and $\mathbb{E}_B(x) = 0$. Then

$$|\tau(xy)|^2 \le \|\mathbb{E}_{A'\cap M}(xy)\|_2^2.$$

Note that

$$\mathbb{E}_{A'\cap M}(xy) = \lim_{n\to\omega} \frac{\sum_{k=1}^n w^k(xy)(w^*)^k}{n}$$

in the weak operator topology. Hence,

$$\begin{aligned}
|\tau(xy)|^2 &\le \|\mathbb{E}_{A'\cap M}(xy)\|_2^2 \\
&\le \lim_{n\to\omega} \left\| \frac{\sum_{k=1}^n w^k(xy)(w^*)^k}{n} \right\|_2^2 \\
&= \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^n \tau(w^i(xy)(w^*)^i w^j(y^*x^*)(w^*)^j) \\
&\le \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^n |\tau(x(yw^{j-i}y^*)x^*(w^*)^{j-i})| \\
&\le \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^n \|\mathbb{E}_B(x(yw^{j-i}y^*)x^*(w^*)^{j-i})\|_2 \\
&= \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^n \|\mathbb{E}_B(x(yw^{j-i}y^*)x^*)\|_2.
\end{aligned}$$

By hypothesis, $yw^ny^* \in B$. Note that $\lim_{n\to\infty} yw^ny^* = 0$ in the weak operator topology. By Theorem 3.3,

$$\lim_{n\to\infty} \|\mathbb{E}_B(x(yw^ny^*)x^*)\|_2 = 0.$$

So

$$|\tau(xy)|^2 \leq \lim_{n\to\omega} \frac{1}{n^2} \sum_{i,j=1}^{n} \|\mathbb{E}_B(x(yw^{j-i}y^*)x^*)\|_2 = 0.$$

Therefore, $\tau(xy) = 0$ for all $y \in M \ominus B$. This implies that $y \in B$. □

Remark 4.2. In Theorem 4.1, it is not necessary that the unit of A be the same as the unit of B.

Theorem 4.3. *Let $M = L(\Gamma)$ and $B = L(\Gamma_0)$. Then the following conditions are equivalent:*

(1) *$B = L(\Gamma_0)$ is mixing in $M = L(\Gamma)$.*
(2) *$g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite group for every $g \in \Gamma \setminus \Gamma_0$.*
(3) *For every diffuse von Neumann subalgebra A of B and every unitary operator $v \in M$, if $vAv^* \subseteq B$, then $v \in B$.*
(4) *For every diffuse von Neumann subalgebra A of B and every operator $y \in M$, if $yAy^* \subseteq B$, then $y \in B$.*

Proof. (1) $\Rightarrow$ (4) follows from Theorem 4.1 and (4) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2) Suppose $M = L(\Gamma)$ and $B = L(\Gamma_0)$. Suppose for some $g \in \Gamma\setminus\Gamma_0$, $g\Gamma_0 g^{-1}\cap\Gamma_0$ is an infinite group. Let $\Gamma_1 = \Gamma_0\cap g^{-1}\Gamma_0 g = g^{-1}(g\Gamma_0 g^{-1}\cap\Gamma_0)g$. Then Γ_1 is an infinite group, and $g\Gamma_1 g^{-1} \subseteq \Gamma_0$. So $\lambda(g)L(\Gamma_1)\lambda(g^{-1}) \subseteq L(\Gamma_0)$. By the third statement, $\lambda(g) \in L(\Gamma_0)$ and $g \in \Gamma_0$. This is a contradiction.

(2) $\Rightarrow$ (1) First, we show that if $g_1, g_2 \in \Gamma \setminus \Gamma_0$, then $g_1\Gamma_0 g_2 \cap \Gamma_0$ is a finite set. Suppose $h_1, h_2 \in \Gamma_0$ and $g_1h_1g_2, g_1h_2g_2 \in \Gamma_0$. Then

$$g_1h_1h_2^{-1}g_1^{-1} = g_1h_1g_2(g_1h_2g_2)^{-1} \in \Gamma_0 \cap g_1\Gamma_0 g_1^{-1}.$$

Since $\Gamma_0 \cap g_1\Gamma_0 g_1^{-1}$ is a finite group,

$$\{h_1h_2^{-1} : h_1, h_2 \in \Gamma_0 \,\text{and}\, g_1h_1g_2, g_1h_2g_2 \in \Gamma_0\}$$

is a finite set. Hence, $g_1\Gamma_0 g_2 \cap \Gamma_0$ is a finite set.

Let $\{v_n\}$ be a sequence of unitary operators in B such that $\lim\limits_{n\to\infty} v_n = 0$ in the weak operator topology. Write $v_n = \sum_{k=1}^{\infty} \alpha_{n,k}\lambda(h_k)$. Then for each k, $\lim_{n\to\infty} \alpha_{n,k} = 0$. Suppose $g_1, g_2 \in \Gamma \setminus \Gamma_0$. There exists an N such that for all $m \geq N$, $g_1h_mg_2 \notin \Gamma_0$. Hence,

$$\|\mathbb{E}_B(g_1v_ng_2)\|_2 = \sum_{i=1}^{N} \|\alpha_{n,i}\mathbb{E}_B(g_1\lambda(h_i)g_2)\|_2 \leq \sum_{i=1}^{N} |\alpha_{n,i}| \to 0$$

when $n \to \infty$. By Remark 3.2, M is mixing relative to B. □

We now apply Theorem 4.3 to the group-theoretic situation arising from a semidirect product $\Gamma = G \rtimes \Gamma_0$, where Γ_0 is an infinite group. Let $\sigma_h(g) = hgh^{-1}$ for $h \in \Gamma_0$ and $g \in G$. Then σ_h is an automorphism of G. Note that $hg = hgh^{-1}h = \sigma_h(g)h$ for $h \in \Gamma_0$ and $g \in G$.

Proposition 4.4. *Let $M = L(G \rtimes \Gamma_0)$ and $B = L(\Gamma_0)$. Then B is mixing in M if and only if for each $g \in G$, $g \neq e$, the group*

$$\{h \in \Gamma_0 : \sigma_h(g) = g\}$$

is finite.

Proof. Let $g \in G$ and $h \in \Gamma_0$. Suppose $h \in g\Gamma_0 g^{-1} \cap \Gamma_0$. Then $ghg^{-1} \in \Gamma_0$. Note that $ghg^{-1} = hh^{-1}ghg^{-1} = h(\sigma_{h^{-1}}(g)g^{-1})$. So $ghg^{-1} \in \Gamma_0$ implies that $\sigma_{h^{-1}}(g)g^{-1} \in \Gamma_0 \cap G = \{e\}$, i.e., $\sigma_{h^{-1}}(g) = g$ and hence $\sigma_h(g) = g$. Conversely, suppose $\sigma_h(g) = g$. Then $\sigma_{h^{-1}}(g) = g$ and hence $ghg^{-1} = h\sigma_{h^{-1}}(g)g^{-1} = h \in \Gamma_0 \cap g\Gamma_0 g^{-1}$. This proves

$$\{h \in \Gamma_0 : \sigma_h(g) = g\} = \{h \in \Gamma_0 : h \in g\Gamma_0 g^{-1} \cap \Gamma_0\}.$$

Suppose B is mixing in M. By (2) of Theorem 4.3, $g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite group for every $g \in G$ with $g \neq e$. So the group $\{h \in H : \sigma_h(g) = g\}$ is finite. Conversely, suppose that for each $g \in G$, $g \neq e$, the group $\{h \in \Gamma_0 : \sigma_h(g) = g\}$ is finite. Our previous observations then imply that the group $g\Gamma_0 g^{-1} \cap \Gamma_0$ is finite. A group element of $\Gamma \setminus \Gamma_0$ can be written as gh, $g \in G$, $g \neq e$, $h \in \Gamma_0$. Note that

$$gh\Gamma_0 h^{-1} g^{-1} \cap \Gamma_0 = g\Gamma_0 g^{-1} \cap \Gamma_0$$

is finite. So B is mixing in M by 2 of Theorem 4.3. $\square$

Recall that the action σ of a group H on a finite von Neumann algebra N is called *ergodic* if $\sigma_h(x) = x$ for all $h \in H$ implies that $x = \lambda 1$. The following result extends Theorem 2.4 of [9] to the noncommutative setting.

Corollary 4.5. *Let $M = L(G \rtimes \Gamma_0)$ and $B = L(\Gamma_0)$. Suppose Γ_0 is a finitely generated, infinite, abelian group or Γ_0 is a torsion free group. Then B is mixing in M if and only if every element $h \in \Gamma_0$ of infinite order is ergodic on $L(G)$.*

Proof. If B is mixing in M, then clearly every element $h \in \Gamma_0$ of infinite order is ergodic on $L(G)$. Now suppose every element $h \in \Gamma_0$ of infinite order is ergodic on $L(G)$. If B is not mixing in M, then there is a $g \in G$, $g \neq e$, such that $\{h \in \Gamma_0 : \sigma_h(g) = g\}$ is an infinite group. Under the above hypotheses on Γ_0, there exists an element h_0 of infinite order such that $\sigma_{h_0}(g) = g$. This implies that the action of h_0 on $L(G)$ is not ergodic, which is a contradiction. $\square$

Corollary 4.6. *Let $M = L(G \rtimes \mathbb{Z})$ and $B = L(\mathbb{Z})$. Then the following conditions are equivalent:*

(1) *The action of $\mathbb{Z}$ on $L(G)$ is mixing, i.e., B is mixing in M.*
(2) *The action of $\mathbb{Z}$ on $L(G)$ is weakly mixing, i.e., B is weakly mixing in M.*
(3) *The action of $\mathbb{Z}$ on $L(G)$ is ergodic.*
(4) *For every $g \in G$, $g \neq e$, the orbit $\{\sigma_h(g)\}$ is infinite.*
(5) *For every $g \in G$, $g \neq e$, $\{h \in \mathbb{Z} : \sigma_h(g) = g\} = \{e\}$.*

Proof. Let γ be a generator of $\mathbb{Z}$. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) Suppose $\sigma_{\gamma^n}(g) = g$ and n is the minimal positive integer satisfies this condition. Let $x = L_g + L_{\sigma_\gamma(g)} + \cdots + L_{\sigma_{\gamma^{n-1}}(g)}$. Then $x \in L(G)$, $x \neq \lambda 1$, and $\sigma_h(x) = x$ for all $h \in \mathbb{Z}$. This implies that the action of $\mathbb{Z}$ on $L(G)$ is not ergodic.

(4) $\Rightarrow$ (5) Suppose $\sigma_{\gamma^n}(g) = g$ for some positive integer n. Then the orbit $\{\sigma_h(g)\}$ has at most n elements.

(5) $\Rightarrow$ (1) follows from Proposition 4.4. □

A special case of Corollary 4.6 implies the following classical result of Halmos [5].

Corollary 4.7 (Halmos's Theorem)**.** *Let X be a compact abelian group, and $T : X \to X$ a continuous automorphism. Then T is mixing if and only if T is ergodic.*

Proof. By the Pontryagin duality theorem, the dual group G of X is a discrete abelian group. Furthermore, there is an induced action of $\mathbb{Z}$ on G, and the action is unitarily conjugate to the action of T on X. Now the corollary follows from Corollary 4.6. □

5. Relative weak mixing

Suppose M is a finite von Neumann algebra with a faithful normal trace τ, and A, B are von Neumann subalgebras of M. We say $B \subset M$ is *weakly mixing relative to* A if there exists a sequence of unitary operators $u_n \in A$ such that

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)\|_2 = 0, \quad \forall x, y \in M.$$

So B is weakly mixing in M if and only if $B \subset M$ is weakly mixing relative to B. Since every diffuse von Neumann algebra contains a sequence of unitary operators converging to 0 in the weak operator topology, B is mixing in M implies that $B \subset M$ is weakly mixing relative to A for all diffuse von Neumann subalgebras A of B.

It is easy to see that $B \subset M$ is weakly mixing relative to A if and only if there exists a sequence of unitary operators $u_n \in A$ such that for all elements x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, one has

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny)\|_2 = 0.$$

The main result of this section is the following, which is inspired by [14].

Theorem 5.1. *Let M be a finite von Neumann algebra with a faithful normal trace τ, and let A, B be von Neumann subalgebras of M with $A \subset B$. Then the following conditions are equivalent:*

(1) *$B \subset M$ is weakly mixing relative to A, i.e., there exists a sequence of unitary operators $\{u_k\}$ in A such that*
$$\lim_{k\to\infty} \|\mathbb{E}_B(xu_ky)\|_2 = 0, \quad \forall x, y \in M \ominus B.$$
(2) *If $z \in A' \cap \langle M, e_B\rangle$ satisfies $\mathrm{Tr}(z^*z) < \infty$, then $e_Bze_B = z$.*
(3) *If $p \in A' \cap \langle M, e_B\rangle$ satisfies $\mathrm{Tr}(p) < \infty$, then $e_Bpe_B = p$.*
(4) *If $x \in M$ satisfies $Ax \subset \sum_{i=1}^n x_iB$ for a finite number of elements $x_1, \ldots, x_n \in M$, then $x \in B$.*

Before we prove Theorem 5.1, we state some corollaries of the theorem.

Corollary 5.2. *Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a von Neumann subalgebra of M. Then the following conditions are equivalent:*

(1) *B is a weakly mixing von Neumann subalgebra of M.*
(2) *If $x \in M$ satisfies $Bx \subset \sum_{i=1}^n x_iB$ for a finite number of elements $x_1, \ldots, x_n \in M$, then $x \in B$.*

The following corollary gives an operator algebraic characterization of weakly mixing actions of countable discrete groups.

Corollary 5.3. *If σ is a measure preserving action of a countable discrete group Γ_0 on a finite measure space (X, μ), then weak mixing of σ is equivalent to the following property: if $x \in L^\infty(X, \mu) \rtimes \Gamma_0$ and $L(\Gamma_0)x \subset \sum_{i=1}^n x_iL(\Gamma_0)$ for a finite number of elements $x_1, \ldots, x_n$ in $L^\infty(X, \mu) \rtimes \Gamma_0$, then $x \in L(\Gamma_0)$.*

Corollary 5.4. *Let M be a finite von Neumann algebra with a faithful normal trace τ, and let B be a mixing von Neumann subalgebra of M. If $A \subset B$ is a diffuse von Neumann subalgebra and $x \in M$ satisfies $Ax \subset \sum_{i=1}^n x_iB$ for a finite number of elements $x_1, \ldots, x_n \in M$, then $x \in B$.*

To prove Theorem 5.1, we need the following lemmas.

Lemma 5.5. *Let $p \in \langle M, e_B\rangle$ be a finite projection, $p \leq 1 - e_B$, and $\epsilon > 0$. Then there exist $x_1, \ldots, x_n \in M \ominus B$, and projections $f_1, \ldots, f_n \in B$ such that $\mathbb{E}_B(x_j^*x_i) = \delta_{ij}f_i$, and*
$$\left\| p - \sum_{i=1}^n x_ie_Bx_i^* \right\|_{2,\mathrm{Tr}} < \epsilon.$$

Proof. Let $q = e_B + p$. Then q is a finite projection in $\langle M, e_B\rangle$. By Lemma 1.8 of [12], there are $x_0, x_1, \ldots, x_n \in M$, $x_0 = 1$, such that $\mathbb{E}_B(x_j^*x_i) = \delta_{ij}f_i$ for $0 \leq i, j \leq n$ and
$$\left\| q - \sum_{i=0}^n x_ie_Bx_i^* \right\|_{2,\mathrm{Tr}} < \epsilon.$$
Clearly,
$$\left\| p - \sum_{i=1}^n x_ie_Bx_i^* \right\|_{2,\mathrm{Tr}} < \epsilon. \qquad \square$$

Suppose that $\mathcal{H} \subset L^2(M)$ is a right B-module. Let $\mathcal{L}_B(L^2(B), \mathcal{H})$ be the set of bounded right B-modular operators from $L^2(B)$ into $\mathcal{H}$. The dimension of $\mathcal{H}$ over B is defined as

$$\dim_B(\mathcal{H}) = \mathrm{Tr}(1),$$

where Tr is the unique tracial weight on B' satisfying the following condition

$$\mathrm{Tr}(x^*x) = \tau(xx^*), \quad \forall x \in \mathcal{L}_B(L^2(B), \mathcal{H}).$$

We say $\mathcal{H}$ is a *finite right B-module* if $\mathrm{Tr}(1) < \infty$. For details on finite modules, we refer the reader to appendix A of [21].

Suppose that $\mathcal{H} \subset L^2(M)$ is a right B-module. We say that $\mathcal{H}$ is *finitely generated* if there exist finitely many elements $\xi_1, \dots, \xi_n \in \mathcal{H}$ such that $\mathcal{H}$ is the closure of $\sum_{i=1}^n \xi_i B$. A set $\{\xi_i\}_{i=1}^n$ is called an orthonormal basis of $\mathcal{H}$ if $\mathbb{E}_B(\xi_i^*\xi_j) = \delta_{ij}p_i \in B$, $p_i^2 = p_i$, and for every $\xi \in \mathcal{H}$ we have

$$\xi = \sum_i \xi_i E_B(\xi_i^*\xi).$$

Let p be the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Then $p = \sum_{i=1}^n \xi_i e_B \xi_i$, where $\xi_i \in L^2(M)$ is viewed as an unbounded operator affilated with M. Every finitely generated right B module has an orthonormal basis. For finitely generated right B-modules, we refer to 1.4.1 of [13].

The following lemma is proved by Vaes in [21] (see Lemma A.1).

Lemma 5.6. *Suppose $\mathcal{H}$ is a finite right B-module. Then there exists a sequence of projections z_n of $Z(B) = B' \cap B$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and, for each n, there exists a projection $p_n \in M_{k_n}(B)$ such that $\mathcal{H}z_n$ is unitarily equivalent to the $p_n M_{k_n}(B) p_n$-B-bimodule $p_n(L^2(B)^{(n)})$. In particular, $\mathcal{H}z_n$ is a finitely-generated right B-module.*

The following lemma is motivated by Lemma 1.4.1 of [13].

Lemma 5.7. *Suppose $\mathcal{H} \subset L^2(M)$ is an A-B-bimodule, which is finitely generated as a right B-module. Let p denote the orthogonal projection of $L^2(M)$ onto $\mathcal{H}$. Then there exists a sequence of projections z_n in $A' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and for each n, there exist a finite number of elements $x_{n,1}, \dots, x_{n,k} \in M$ such that*

$$z_n p z_n(\hat{x}) = \sum_{i=1}^k x_{n,i} \widehat{\mathbb{E}_B(x_{n,i}^* x)}, \quad \forall x \in M.$$

Proof. Let $\{\xi_i\}_{i=1}^k \subset \mathcal{H} \subset L^2(M, \tau)$ be an orthonormal basis for $\mathcal{H}$, i.e., $\mathcal{H} = \oplus_{i=1}^k [\xi_i B]$. As in 1.4.1 of [13], the projection p from $L^2(M)$ onto $\mathcal{H}$ has the form $p = \sum_{i=1}^k \xi_i e_B \xi_i^*$, where $\xi_i \in L^2(M)$ is viewed as an unbounded operator affilated with M. Since $\mathcal{H}$ is a left A-submodule of $L^2(M)$, in particular it is an invariant subspace for the von Neumann algebra A, so the projection $p : L^2(M) \to \mathcal{H}$ commutes with A. Thus, $p \in A' \cap \langle M, e_B\rangle$. For $a \in A$, we have

$$a\left(\sum_{i=1}^{n} \xi_i e_B \xi_i^*\right) = \left(\sum_{i=1}^{n} \xi_i e_B \xi_i^*\right) a$$

and, applying the pull down map to both sides, we obtain

$$a\left(\sum_{i=1}^{n} \xi_i \xi_i^*\right) = \left(\sum_{i=1}^{n} \xi_i \xi_i^*\right) a.$$

Hence $aq = qa$ for all spectral projections q of $\xi_i\xi_i^*$. Since $\sum_{i=1}^{n} \xi_i\xi_i^*$ is a densely defined operator affilated with M, $q \in A' \cap M$. We thus obtain a sequence of projections $z_n \in A' \cap M$ such that $\lim_{n\to\infty} z_n = 1$ in the strong operator topology and $\sum_{i=1}^{k} z_n\xi_i\xi_i^* z_n$ is a bounded operator for each n. Let $x_{n,i} = z_n\xi_i$, $1 \le i \le k$. Then $x_{n,i} \in M$ and

$$z_n p z_n(\hat{x}) = \sum_{i=1}^{k} z_n\xi_i e_B \xi_i^* z_n(\hat{x}) = \sum_{i=1}^{k} x_{n,i} e_B x_{n,i}^*(\hat{x}) = \sum_{i=1}^{k} x_{n,i}\widehat{\mathbb{E}_B(x_{n,i}^* x)}$$

for all $x \in M$. □

Proof of Theorem 5.1. (1) $\Rightarrow$ (2) Suppose $e_B z e_B = z$ is not true. We may assume that $(1-e_B)z \neq 0$ (otherwise, consider $z(1-e_B)$). Replacing z by a nonzero spectral projection of $(1-e_B)zz^*(1-e_B)$ corresponding to an interval $[c, 1]$ with $c > 0$, we may assume that $z = p \neq 0$ is a subprojection of $1 - e_B$.

Let $\epsilon > 0$. By Lemma 5.5, there is a natural number n and $x_1, \dots, x_n \in M \ominus B$ such that $\mathbb{E}_B(x_j^* x_i) = \delta_{ij} f_i$, where f_i is a projection in B, and

$$\left\| p - \sum_{i=1}^{n} x_i e_B x_i^* \right\|_{2,\mathrm{Tr}} < \epsilon/2.$$

Let $p_0 = \sum_{i=1}^{n} x_i e_B x_i^*$. Then p_0 is a projection. Note that $u_k p u_k^* = p$. So

$$\|u_k p_0 u_k^* - p_0\|_{2,\mathrm{Tr}} \le \|u_k(p_0 - p)u_k^*\|_{2,\mathrm{Tr}} + \|p_0 - p\|_{2,\mathrm{Tr}} < \epsilon.$$

Therefore,

$$\begin{aligned} 2\|p_0\|_{2,\mathrm{Tr}}^2 &= \|u_k p_0 u_k^* - p_0\|_{2,\mathrm{Tr}}^2 + 2\mathrm{Tr}(u_k p_0 u_k^* p_0) \\ &= \|u_k p_0 u_k^* - p_0\|_{2,\mathrm{Tr}}^2 + 2\sum_{1\le i,j\le n} \mathrm{Tr}(u_k x_i e_B x_i^* u_k^* x_j e_B x_j^*) \\ &\le \epsilon^2 + 2\sum_{1\le i,j\le n} \tau(\mathbb{E}_B(x_i^* u_k^* x_j) x_j^* u_k x_i) \\ &\le \epsilon^2 + 2\sum_{1\le i,j\le n} \|\mathbb{E}_B(x_j^* u_k x_i)\|_{2,\tau}^2. \end{aligned}$$

By the assumption of the lemma, $2\sum_{1\leq i,j\leq n}\|\mathbb{E}_B(x_j^*u_kx_i)\|_{2,\tau}^2 \to 0$ when $k\to\infty$. Hence, $\|p_0\|_{2,\mathrm{Tr}} \leq \epsilon$. Since $\epsilon>0$ was arbitrary, this says $p=0$. This is a contradiction.

(2) $\Rightarrow$ (1) Suppose (1) is false. Then there exist $\epsilon_0>0$ and $x_1,\ldots,x_n\in N\ominus B$ such that $\sum_{1\leq i,j\leq n}\|\mathbb{E}_B(x_iux_j^*)\|_{2,\tau}^2 \geq \epsilon_0$ for all $u\in\mathcal{U}(A)$. Let $z=\sum_{i=1}^n x_i^*e_Bx_i$. Then $z\perp e_B$, $\mathrm{Tr}(z)<\infty$, and

$$\begin{aligned}\mathrm{Tr}(zuzu^*) &= \sum_{i,j=1}^n \mathrm{Tr}(x_i^*e_Bx_iux_j^*e_Bx_ju^*) = \sum_{i,j=1}^n \mathrm{Tr}(E_B(x_iux_j^*)e_Bx_ju^*x_i^*)\\ &= \sum_{i,j=1}^n \tau(E_B(x_iux_j^*)x_ju^*x_i^*) = \sum_{i,j=1}^n \|E_B(x_iux_j^*)\|_2^2 \geq \epsilon,\end{aligned}$$

for all $u\in\mathcal{U}(A)$. Let Γ_z be the weak operator closure of the convex hull of $\{uzu^* : u\in\mathcal{U}(A)\}$. Then there exists a unique element $y\in\Gamma_z$ such that $\|y\|_{2,\mathrm{Tr}}=\min\{\|x\|_{2,\mathrm{Tr}} : x\in\Gamma_z\}$. The uniqueness implies that $uyu^*=y$ for all $u\in\mathcal{U}(A)$ and hence $y\in A'\cap\langle N,e_B\rangle$. Since $\mathrm{Tr}(zuzu^*)\geq\epsilon_0$, $\mathrm{Tr}(zy)\geq\epsilon_0>0$. So $y>0$ and $y\perp e_B$. Note that

$$\mathrm{Tr}(y^2)\leq\|y\|\mathrm{Tr}(y)\leq\|y\|\mathrm{Tr}(z)<\infty.$$

This contradicts the assumption of (2).

(2) $\Leftrightarrow$ (3) is easy to see.

(3) $\Rightarrow$ (4) Suppose $Ax\subset\sum_{i=1}^n x_iB$. Let $\mathcal{H}$ be the closure of $\widehat{AxB}$ in $L^2(N,\tau)$. Then $\mathcal{H}$ is a left A finitely generated right B bimodule. Let p be the projection of $L^2(N,\tau)$ onto $\mathcal{H}$. Then $p\in A'\cap\langle N,e_B\rangle$ is a finite projection of $\langle N,e_B\rangle$. By the assumption of (3), $p\leq e_B$. So $\hat{x}=p(\hat{x})=e_B(\hat{x})\in\hat{B}$ and $x\in B$.

(4) $\Rightarrow$ (3) Suppose $p\in A'\cap\langle M,e_B\rangle$ satisfies $\mathrm{Tr}(z^*z)<\infty$. Then $\mathcal{H}=pL^2(M)$ is a left A finite right B bimodule. By Lemma 5.6, we may assume that $\mathcal{H}$ is a left A finitely generated right B bimodule. By Lemma 5.7, there exists a sequence of projections z_n in $A'\cap M$ such that $\lim_{n\to\infty}z_n=1$ in the strong operator topology and for each n, there exist $x_{n,1},\ldots,x_{n,k}\in M$ such that

$$z_npz_n(\hat{x})=\sum_{i=1}^k x_{n,i}\widehat{\mathbb{E}_B(x_{n,i}^*}x), \quad \text{for all } x\in M.$$

Note that $z_npz_n\in A'\cap\langle M,e_N\rangle$, and for every $x\in M$,

$$A\left(z_npz_n(\hat{x})\right)=(z_npz_n)(\widehat{Ax})\subset\sum_{i=1}^n\widehat{x_{n,i}B}.$$

By the assumption of (4), $z_npz_n(\hat{x})\in\hat{B}\subset L^2(B)$ for every $x\in M$. Hence, for each $\xi\in L^2(M)$, $z_npz_n(\xi)\in L^2(B)$. Since $\lim_{n\to\infty}z_n=1$ in the strong operator topology, $p(\xi)=\lim_{n\to\infty}z_npz_n(\xi)\in L^2(B)$, i.e., $p\leq e_B$. $\square$

6. Further results and examples

In this section, we explore the hereditary properties of mixing subalgebras of finite von Neumann algebras; that is, we show that if $B \subset M$ is a mixing inclusion, then the properties of an inclusion $B_1 \subset B$ can force certain mixing properties on the inclusion $B_1 \subset M$. In particular, Proposition 6.1 below allows us to construct examples of weakly mixing subalgebras which are not mixing. We also use the crossed product and amalgamated free product constructions to produce further examples of mixing inclusions.

6.1. Hereditary properties of mixing algebras.

Proposition 6.1. *Let B be a mixing von Neumann subalgebra of M, and let B_1 be a diffuse von Neumann subalgebra of B. We have the following:*

(1) $B_1' \cap M = B_1' \cap B$.
(2) *If B_1 is singular in B, then B_1 is singular in M.*
(3) $\mathcal{N}_M(B_1)'' \subseteq B$, *where* $\mathcal{N}_M(B_1) = \{u \in \mathcal{U}(M) : uB_1u^* = B_1\}$.
(4) *If B_1 is weakly mixing in B, then B_1 is weakly mixing in M.*
(5) *If B_1 is mixing in B, then B_1 is mixing in M.*

Proof. (1)–(3) follow from Theorem 4.1.

(4) By Corollary 5.2, we need to show that if $x \in M$ satisfies $B_1x \subset \sum_{i=1}^n x_iB_1$ for a finite number of elements $x_1, \ldots, x_n \in M$, then $x \in B_1$. Note that B is mixing in M. By Corollary 5.4, $x \in B$. Let $b_i = \mathbb{E}_B(x_i)$ for $1 \leq i \leq n$. Applying $\mathbb{E}_B$ to both sides of the inclusion $B_1x \subset \sum_{i=1}^n x_iB_1$, we have $B_1x \subset \sum_{i=1}^n b_iB_1$. Since B_1 is weakly mixing in B, $x \in B_1$ by Corollary 5.2.

(5) Suppose B_1 is mixing in B and u_n is a sequence of unitary operators in B_1 with $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. For $x, y \in M$, we have

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)\|_2 = 0$$

since B is mixing in M. Applying $\mathbb{E}_{B_1}$ to $\mathbb{E}_B(xu_ny) - \mathbb{E}_B(x)u_n\mathbb{E}_B(y)$, we have

$$\lim_{n\to\infty} \|\mathbb{E}_{B_1}(xu_ny) - \mathbb{E}_{B_1}(\mathbb{E}_B(x)u_n\mathbb{E}_B(y))\|_2 = 0. \tag{6.1}$$

Since B_1 is mixing in B,

$$\lim_{n\to\infty} \|\mathbb{E}_{B_1}(\mathbb{E}_B(x)u_n\mathbb{E}_B(y)) - \mathbb{E}_{B_1}(x)u_n\mathbb{E}_{B_1}(u)\|_2 = 0. \tag{6.2}$$

Combining (6.1) and (6.2), we have

$$\lim_{n\to\infty} \|\mathbb{E}_{B_1}(xu_ny) - \mathbb{E}_{B_1}(x)u_n\mathbb{E}_{B_1}(u)\|_2 = 0,$$

which implies that B_1 is mixing in M. □

Remark 6.2. Suppose B_i is a diffuse von Neumann subalgebra of M_i for $i = 1, 2$. If $B_1 \neq M_1$ or $B_2 \neq M_2$, then $B_1 \bar{\otimes} B_2$ is not a mixing von Neumann subalgebra of $M_1 \bar{\otimes} M_2$ by Proposition 6.1. On the other hand, it is easy to check that $B_1 \bar{\otimes} B_2$ is weakly mixing in $M_1 \bar{\otimes} M_2$ if B_1 and B_2 are weakly mixing in M_1 and M_2, respectively. This gives examples of weakly mixing but not mixing subalgebras.

Note that in the proof of statement (4) of Proposition 6.1, we use an equivalent condition of weak mixing (Corollary 5.2) instead of the definition. The essential difficulty is that in the definition of weak mixing, we do not assume that $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. However, we have the following result.

Proposition 6.3. *Let M be a type* II_1 *factor with the faithful normal trace τ, and let B be a proper subfactor of M. If $\{u_n\}$ is a sequence of unitary operators in B such that for all elements x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$, one has*

$$\lim_{n\to\infty} \|\mathbb{E}_B(xu_ny)\|_2 = 0,$$

then $\lim\limits_{n\to\infty} u_n = 0$ in the weak operator topology.

Proof. Note that B is weakly mixing in M and hence singular in M. In particular $B' \cap M = \mathbb{C}1$. Let ω be a non principal ultrafilter of $\mathbb{N}$ and suppose $\lim_{n\to\omega} u_n = b$ in the weak operator topology. For x, y in M with $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$,

$$\mathbb{E}_B(xby) = \lim_{n\to\omega} \mathbb{E}_B(xu_ny) = 0.$$

Let $b = u|b|$ be the polar decomposition of b. Note that

$$\mathbb{E}_B(xu^*) = \mathbb{E}_B(x)u^* = 0.$$

Hence,

$$\mathbb{E}_B(x|b|y) = \mathbb{E}_B(xu^*u|b|y) = \mathbb{E}_B(xu^*by) = 0.$$

Let $x = y^*$. Then $\mathbb{E}_B(y^*|b|y) = 0$ and hence $y^*|b|y = 0$. This implies that $|b|y = 0$ for all $y \in M$ with $\mathbb{E}_B(y) = 0$. For $b' \in B$, $\mathbb{E}_B(b'y) = b'\mathbb{E}_B(y) = 0$. Hence, $|b|b'y = 0$. This implies that $|b|R(b'y) = 0$, where $R(b'y)$ is the range projection of $b'y$. Let $p = \vee_{b'\in B} R(b'y)$. Then $|b|p = 0$. On the other hand, $0 \neq p \in B' \cap M$, so $p = 1$. We then have $|b| = 0$, and $b = 0$. Therefore, $\lim_{n\to\omega} u_n = 0$ in the weak operator topology. Since ω is an arbitrary non principal ultrafilter of $\mathbb{N}$, $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. $\square$

6.2. Further examples of mixing subalgebras.

Lemma 6.4. *Let B be a von Neumann subalgebra of M. Then the following conditions are equivalent:*

1. *B is atomic type* I.
2. *For every* bounded *sequence $\{x_n\}$ in M with $\lim_{n\to\infty} x_n = 0$ in the weak operator topology, $\lim_{n\to\infty} \|\mathbb{E}_B(x_n)\|_2 = 0$.*

Proof. (1) $\Rightarrow$ (2) Since B is a finite atomic type I von Neumann algebra, $B = \oplus_{k=1}^{N} M_{n_k}(\mathbb{C})$, where $1 \leq N \leq \infty$. So there exists a sequence of finite rank central projections $p_n \in B$ such that $p_n \to 1$ in the strong operator topology. Therefore, $\tau(p_n) \to 1$. Let $\{x_n\}$ be a bounded sequence in M with $x_n \to 0$ in the weak operator topology, and let $\epsilon > 0$. We may assume that $\|x_n\| \leq 1$. Choose p_k such that $\tau(1 - p_k) < \epsilon^2/4$. Note that the map $x \in M \to p_k \mathbb{E}_B(x)$ is a finite rank operator. There is an $m > 0$ such that for all $n \geq m$, $\|p_k \mathbb{E}_B(x_n)\|_2 < \epsilon/2$. Then

$$\|\mathbb{E}_B(x_n)\|_2 \leq \|p_k \mathbb{E}_B(x_n)\|_2 + \|(1 - p_k)\mathbb{E}_B(x_n)\|_2 \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This proves that $\|\mathbb{E}_B(x_n)\|_2 \to 0$.

(2) $\Rightarrow$ (1) If M is not atomic type I, then there is a nonzero central projection $p \in M$ such that pM is diffuse. Thus, there is a Haar unitary operator $v \in pM$. Note that $v^n \to 0$ in the weak operator topology. But $\|\mathbb{E}_B(v^n)\|_2 = \|v^n\|_2 = \tau(p)^{1/2}$ does not converge to 0. This contradicts (2). $\square$

Proposition 6.5. *Let $M = M_1 *_A M_2$ be the amalgamated free product of diffuse finite von Neumann algebras (M_1, τ_1) and (M_2, τ_2) over an atomic finite von Neumann algebra A. Then M_1 is a mixing von Neumann subalgebra of M.*

Proof. The following spaces are mutually orthogonal with respect to the unique trace τ on M: $M_2 \ominus A$, $(M_1 \ominus A) \otimes (M_2 \ominus A)$, $(M_2 \ominus A) \otimes (M_1 \ominus A)$, $(M_1 \ominus A) \otimes (M_2 \ominus A) \otimes (M_1 \ominus A)$, $\cdots$. Furthermore, the trace-norm closure of the linear span of the above spaces is $L^2(M, \tau) \ominus L^2(M_1, \tau)$. Suppose $\{u_n\}$ is a sequence of unitary operators in M_1 satisfying $\lim_{n\to\infty} u_n = 0$ in the weak operator topology. To prove M_1 is a mixing von Neumann subalgebra of M, we need only to show for x in each of the above spaces, we have

$$\lim_{n\to\infty} \|\mathbb{E}_{M_1}(x u_n x^*)\|_2 = 0.$$

We will give the proof for x in one of the following spaces:

$$(M_1 \ominus A) \otimes (M_2 \ominus A) \quad \text{and} \quad (M_2 \ominus A) \otimes (M_1 \ominus A).$$

The other cases can be proved similarly.

Suppose $x = x_1 y_1$, where $x_1 \in M_1 \ominus A$ and $y_1 \in M_2 \ominus A$. Then

$$x u_n x^* = x_1 y_1 (u_n - \mathbb{E}_A(u_n)) y_1^* x_1 + x_1 y_1 \mathbb{E}_A(u_n) y_1^* x_1^*.$$

Note that $\mathbb{E}_{M_1}(x_1 y_1 (u_n - \mathbb{E}_A(u_n)) y_1^* x_1) = 0$ and $\lim_{n\to\infty} \|E_A(u_n)\|_2 = 0$ by Lemma 6.4. So

$$\lim_{n\to\infty} \|\mathbb{E}_{M_1}(x u_n x^*)\|_2 = 0.$$

Suppose $x = y_1 x_1$, where $x_1 \in M_1 \ominus A$ and $y_1 \in M_2 \ominus A$. Then

$$x u_n x^* = y_1 x_1 u_n x_1^* y_1 = y_1(x_1 u_n x_1^* - \mathbb{E}_A(x_1 u_n x_1^*)) y_1^* - y_1 \mathbb{E}_A(x_1 u_n x_1^*) y_1^*.$$

Note that

$$\mathbb{E}_{M_1}(y_1(x_1u_nx_1^* - \mathbb{E}_A(x_1u_nx_1^*))y_1^*) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\mathbb{E}_A(x_1u_nx_1^*)\|_2 = 0$$

by Lemma 6.4. So

$$\lim_{n\to\infty} \|\mathbb{E}_{M_1}(xu_nx^*)\|_2 = 0. \qquad \square$$

Note, in particular, that Proposition 6.5 implies that if A is a diffuse mixing masa in a finite von Neumann algebra M_1, and M_2 is also diffuse, then A is mixing in the free product $M_1 * M_2$.

Now let B be a diffuse finite von Neumann algebra with a faithful normal trace τ, and let G be a countable discrete group. Let $*_{g\in G}B_g$ be the free product von Neumann algebra, where B_g is a copy of B for each g. The shift transformation $\sigma(g)((x_h)) = (x_{g^{-1}h})$ defines an action of G on $*_{g\in G}B_g$. Let $M = *_{g\in G}B_g \rtimes G$. Then M is a type II_1 factor and we can identify B with B_e.

Proposition 6.6. *The above algebra B is a mixing von Neumann subalgebra of M.*

Proof. Suppose v_g is the classical unitary operator corresponding to the action g in M. Then for every (x_h) in $*_{g\in G}B_g$,

$$v_g(x_h)v_g^{-1} = (\sigma_g(x_h)) = (x_{g^{-1}h}).$$

Suppose $b_n \in B = B_e$, $b_n \to 0$ in the weak operator topology, $g \neq e$, and $x_h \in B_h$. We may assume $\tau(b_n) = 0$ for each n. Note that

$$x_hv_gv_nv_g^*x_h^* = x_h\sigma_g(b_n)x_h^*.$$

If $h \neq e$, it is clear that $x_h\sigma_g(b_n)x_h^*$ is free with $B = B_e$ and hence orthogonal to B. If $h = e$, direct computations show that $x_e\sigma_g(b_n)x_e^*$ is orthogonal to $B = B_e$. So we have

$$\begin{aligned}\mathbb{E}_B(x_hv_gb_nv_g^*x_h^*) = \mathbb{E}_B(x_h\sigma_g(b_n)x_h^*) = \tau(x_h\sigma_g(b_n)x_h^*) &= \tau(\sigma_g(b_n)x_h^*x_h) \\ &= \tau(b_n\sigma_{g^{-1}}(x_h^*x_h)),\end{aligned}$$

and this last expression above converges to zero. Note that the linear span of the above elements x_hv_g is dense in $M \ominus B$ in the weak operator topology. This proves that B is mixing in M. $\square$

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