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Separating pants decompositions in the pants complex

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Abstract. We study the topological types of pants decompositions of a surface by associating to any pants decomposition P, its *pants decomposition graph*, $\Gamma(P)$. This perspective provides a convenient way to analyze the maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a nontrivial separating curve for all surfaces of finite type. We provide an asymptotically sharp approximation of this nontrivial distance in terms of the topology of the surface. In particular, for closed surfaces of genus g we show the maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a separating curve grows asymptotically like the function $\log(g)$. The lower bounds follow from an explicit constructive algorithm for an infinite family of high girth log-length connected graphs, which may be of independent interest.

Contents

1. Introduction

The large scale geometry of Teichmüller space has been an object of interest in recent years, especially within the circles of ideas surrounding Thurston's Ending Lamination Conjecture. In this context, the pants complex, $\mathcal{P}(S)$, associated to a hyperbolic surface, S, becomes relevant, as by a theorem of Brock $\mathcal{P}(S)$ is quasi-isometric to the Teichmüller space of a surface equipped with the Weil–Petersson metric, [Bro]. Accordingly, in order

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to study large scale geometric properties of Teichmüller space with the Weil–Petersson metric, it suffices to study the pants complex of a surface. For instance, significant recent results of [BF, Beh, BeMi, BrM] among others can be viewed from this perspective.

One feature of the coarse geometry of the pants complex in common to many analyses of the subject is the existence of natural quasi-isometrically embedded product regions in the thin part of Teichmüller space. These product regions, which are obstructions to δ-hyperbolicity, correspond to pants decompositions of the surface containing a fixed nontrivially separating (multi)curve. In fact, often in the course of studying the coarse geometry of the pants complex it proves advantageous to pass to the net of pants decompositions that contain a nontrivially separating curve. See for instance [BrM, BDM] in which such methods are used to prove that the certain pants complexes are relatively hyperbolic or thick, respectively. Similarly, work of [MS], uses similar methods to prove the pants complex is one ended.

In this paper, we study the net of pants decompositions of a surface that contain a nontrivially separating curve within the entire pants complex. Specifically, for all surfaces of finite type we approximate the maximum distance in the pants complex of any pants decomposition to a pants decomposition containing a nontrivially separating curve. The following theorem is a highlight of the paper:

Theorem 1.1. *Let* $S = S_{g,n}$ *and set* $D_{g,n} = \max_{P\in\mathcal{P}(S)}(d_{\mathcal{P}(S)}(P, \mathcal{P}_{\mathrm{sep}}(S)))$. *Then, for any fixed number of boundary components (or punctures)* n, $D_{g,n}$ *grows asymptotically like the function* $\log(g)$, *that is* $D_{g,n} = \Theta(\log(g))$. *On the other hand, for any fixed genus* $g \geq 2$, $\forall n \geq 6g-5$, $D_{g,n} = 2$.

There is a sharp contrast between the nets provided by the subcomplexes $\mathcal{C}_{\mathrm{sep}}(S) \subset \mathcal{C}(S)$ and $\mathcal{P}_{\mathrm{sep}}(S) \subset \mathcal{P}(S)$. It is easy to see that $N_1(\mathcal{C}_{\mathrm{sep}}(S)) = \mathcal{C}(S)$. On the other hand, Theorem 1.1 says that the maximal distance from an arbitrary pants decomposition to any pants decompositions containing a nontrivial separating curve is a nontrivial function depending on the topology of the surface. The lower bounds in Theorem 1.1 follow from an original and explicit constructive algorithm for an infinite family of high girth at most cubic graphs with the following expander like property, namely the minimum cardinality of connected cutsets is a logarithmic function with respect to the vertex size of the graphs. This family of graphs may be of independent interest.

The following lemma used in the course of proving the lower bounds in Theorem 1.1 may also be of independent interest. Its proof brings together ideas related to the topology of the surfaces and graph theory in a simple yet elegant manner.

Lemma 1.2 (Key Lemma)**.** *For* $P \in \mathcal{P}(S)$ *and* $\Gamma(P)$ *its pants decomposition graph, let* d *be the cardinality of a minimal nontrivial connected cut-set* $C \subset \Gamma(P)$. *Then* $d_{\mathcal{P}(S)}(P, P') \geq \min\{\mathrm{girth}(\Gamma(P)), d\} - 1$, *for* P' *any pants decomposition containing a separating curve cutting off genus.*

The results of this paper have some overlap with recent results in [CP, RT]. Nonetheless, the results presented are in fact distinct from the aforementioned articles. Specifically, due to the fact that the quasi-isometry constants of [Bro] between the pants complex and Teichmüller space equipped with the Weil–Petersson metric are dependent on the topology of the surface, the results of this paper are more properly related to complex of cubic graphs than to Moduli Space. Conversely, while methods in [CP] do contain lower bounds on the diameter of entire complex of cubic graphs, this paper focuses on the finer question of the density of a natural subset inside the entire space. On the other hand, while methods in [RT] provide an independent and alternative (albeit nonconstructive) proof of the lower bounds achieved in Section 5 of this paper by considering pants decompositions whose pants decomposition graphs are expanders. The explicit and constructive nature of the family of graphs in Section 5 is a novelty of this paper as compared to [RT].

The outline of the paper is as follows. In Section 2 we review relevant background concepts. In Section 3 we introduce a pants decomposition graph. In Section 4 we prove Theorem 1.1 modulo a construction of an infinite family of high girth, log-length connected, at most cubic graphs, which is explicitly described in Section 5.

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2. Preliminaries

2.1. Graph Theory.

Let $\Gamma = \Gamma(V, E)$ be an undirected graph with vertex set V and edge set E. The *degree of a vertex* $v \in V$, is the number of times that the vertex v arises as an endpoint in E. The *degree of a graph* is the maximal degree over all vertices. Γ is called *at most cubic* if the degree of Γ is at most three, and *cubic* if every vertex has degree exactly three. A simple closed path in a graph is called a *cycle*. A cycle of length one is a *loop*. The *girth* of a graph Γ is defined to be the length of a shortest cycle in Γ, unless Γ is acyclic, in which case the girth is infinity.

Given a graph, $\Gamma(V, E)$ for any subset $S \subset V(\Gamma)$, the *complete subgraph of S in* Γ, denoted $\Gamma[S]$, is the subgraph of Γ with vertex set S and edges between any pair of vertices $x, y \in S$ if and only if there is a corresponding edge $e \in E(\Gamma)$. A graph Γ is said to be *connected* if there is a path between any two vertices of the graph, and disconnected otherwise. If a subset of vertices, $C \subset V$, has the property that the *deletion subgraph*, $\Gamma[V \setminus C]$, is disconnected, then C is called a *cut-set* of a graph. If the deletion subgraph $\Gamma[V \setminus C]$, is disconnected and moreover it has at least two connected components each consisting of at least two vertices or a single vertex with a loop, C is said to be a *nontrivial cut-set.* A (nontrivial) [connected] cut-set C is called a *minimal sized (nontrivial) [connected] cut-set* if $|C|$ is minimal over all (nontrivial) [connected] cut-sets of Γ.

We will be interested in families of graphs that are robust with regard to nontrivial disconnection by the removal of connected cut-sets. More formally, we define an infinite family of graphs, $\Gamma_i(V_i, E_i)$, with increasing vertex size to be *log-length connected* if they have the property that the size of minimal nontrivial connected cut-sets of the graphs, asymptotically grows logarithmically in the vertex size of the graphs. Specifically, if we set the function $f(i)$ to be equal to the cardinality of a minimal nontrivial connected cut-set of the graph Γ_i, then $f(i) = \Theta(\log(|V_i|))$. The robust connectivity property of log-length connected graphs is quite different than the connectivity property enjoyed by expander graphs. Informally, a family of graphs are expanders if the graphs are strongly connected in the sense the deletion of small number of arbitrary vertices will not separate the graph. On the other hand, a family of graphs are log-length connected if the graphs are strongly connected in the sense that the deletion of a small locally connected subgraph will not separate the graph. This seems to be a novel type of connectivity property for graphs and may be of independent interest.

2.2. Curve and Pants Complex.

Given any surface of finite type, $S = S_{g,n}$, that is a genus g surface with n boundary components (or punctures), the *complexity* of S, denoted $\xi(S)$, is defined to be $3g-3+n$. For purposes of this paper there is no distinction between boundary components and punctures. Throughout this paper while we will always refer to surfaces with boundary components, the same results hold mutatis mutandis for surfaces with punctures.

A simple closed curve $\gamma \subset S$ is *essential* if it does not bound a disk containing at most one boundary component. For S any surface with positive complexity, the *curve complex* of $S, \mathcal{C}(S)$, is the simplicial complex obtained by associating to each isotopy class of an essential simple closed curve a 0-cell, and more generally a k-cell to each unordered tuple $\{\gamma_0, ..., \gamma_k\}$ of $k+1$ isotopy classes of disjoint homotopically distinct essential simple closed curves, or *multicurves.* A maximal dimensional multicurve is called a *pants decomposition.*

Among simple closed curves on a surface of finite type we differentiate between two types of curves. Specifically, a simple closed curve $\gamma \subset S$ is called a *nontrivially separating curve*, or simply a *separating curve*, if $S \setminus \gamma$ consists of two connected components Y_1 and Y_2 such that $\xi(Y_i) \geq 1$. Any other simple closed curve is *nonseparating.* It should be stressed that a *trivially separating curve*, that is a simple closed curve that cuts off two boundary components of the surface for our purposes is not considered a separating curve. While counterintuitive, this point of view is in fact quite natural in the context of Teichmüller space. Restricting $\mathcal{C}(S)$ to the set of separating curves one obtains the *complex of separating curves*, $\mathcal{C}_{\text{sep}}(S)$.

Another related natural complex associated to any surface of finite type with positive complexity is the *pants complex*, first developed in [HT]. The 1-skeleton of the pants complex, the *pants graph*, $\mathcal{P}(S)$, has vertices corresponding to pants decompositions, and edges between two vertices when the two corresponding pants decompositions differ by an *elementary move*. Specifically, two pants decompositions differ by an elementary move, if the pants decompositions differ in exactly one curve and those differing curves intersect minimally inside the unique complexity one component of the surface in the complement of all the other agreeing curves of the pants decompositions. Restricting $\mathcal{P}(S)$ to the set of pants decompositions containing a separating curve we have the *pants complex of separating curves*, $\mathcal{P}_{\text{sep}}(S)$. This paper analyzes the net of $\mathcal{P}_{\text{sep}}(S)$ in $\mathcal{P}(S)$.

3. Pants decomposition graph

By topological considerations, for $P \in P(S_{g,n})$,

$$|P| = \xi(S) = 3g - 3 + n,$$

while the number of connected components, or "pairs of pants," in the complement $S \setminus P$ is equal to $|\chi(S)| = 2(g-1)+n$. Given $P \in \mathcal{P}(S)$ we define its *pants decomposition graph*, $\Gamma(P)$, as follows: $\Gamma(P)$ is a graph with vertices corresponding the connected components of $S \setminus P$, and edges between vertices corresponding to connected components that share a common boundary curve. See Figure 1 for an example.

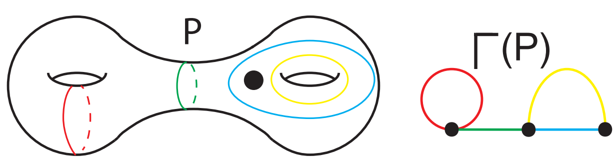

FIGURE 1. $\Gamma(P)$ for $P \in P(S_{2,1})$.

Remark 3.1. The notion of pants decomposition graphs is considered in [Bus] as well as in [Par]. Moreover, replacing the vertices in $\Gamma(P)$ with edges and vice versa yields the *adjacency graph* in [BeMa].

The following self evident lemma organizes elementary properties of $\Gamma(P)$ and gives a one to one correspondence between certain graphs and pants decomposition graphs:

Lemma 3.2. *For $P \in \mathcal{P}(S_{g,n})$, and $\Gamma(P)$ its pants decomposition graph:*

(1) *$\Gamma(P)$ is a connected graph with $2(g-1)+n$ vertices and $3(g-1)+n$ edges. In particular, $\pi_1(\Gamma(P))$ is the free group of rank g.*

(2) $\Gamma(P)$ *is at most cubic.*

Moreover, for all $q, p \in \mathbb{N}$, *given any connected, at most cubic graph* $\Gamma = \Gamma(V, E)$ *with* $|V| = 2(p-1) + q$ *and* $|E| = 3(p-1) + q$, *there exists a pants decomposition* $P \in \mathcal{P}(S_{p,q})$ *with* $\Gamma(P) \cong \Gamma$.

3.1. Elementary moves and pants decomposition graphs. Recall the two types of elementary moves:

E1: Inside a $S_{1,1}$ component of the surface in the complement of all of the pants curves except α, the curve α is replaced with β where α and β intersect once.

E2: Inside a $S_{0,4}$ component of the surface in the complement of all of the pants curves except α, the curve α is replaced with β where α and β intersect twice.

Elementary move E1 has a trivial action on $\Gamma(P)$, while the impact of the elementary move E2 can be described as follows: identify any two adjacent vertices, v_1, v_2 in the pants decomposition graph connected by an edge e, then the action of an elementary move E2 on the pants decomposition graph has the effect of interchanging any edge other than e impacting v_1, or possibly the empty set, with any edge other than e, impacting v_2, or possibly the empty set. The one stipulation is that in the event that the empty set is being interchanged with an edge, the result of the action must yield a connected at most cubic graph.

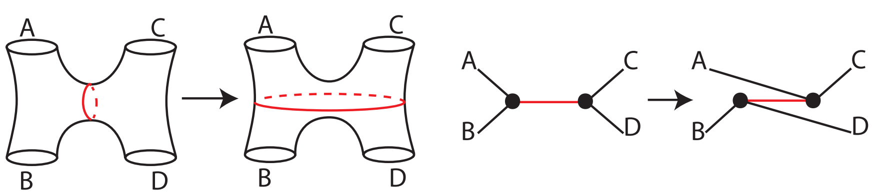

FIGURE 2. An example of an elementary pants move action on $\Gamma(P)$.

3.2. Adding boundary components. Any pants decomposition of $S_{g,n+1}$ can be obtained by beginning with a suitable pants decomposition of $S_{g,n}$, adding a boundary component appropriately, and then appropriately completing the resulting multicurve into a pants decomposition of $S_{g,n+1}$. The effect that this process of adding a boundary component has on the pants decomposition graph has two forms as depicted in Figure 3.

3.3. Separating curves and pants decomposition graphs. Examining a pants decomposition graph $\Gamma(P)$ provides an easy way to determine if a pants decomposition P contains a separating curve. Specifically, a curve in a pants decomposition is a separating curve of the surface if and only if the effect of removing the corresponding edge in $\Gamma(P)$ nontrivially separates the

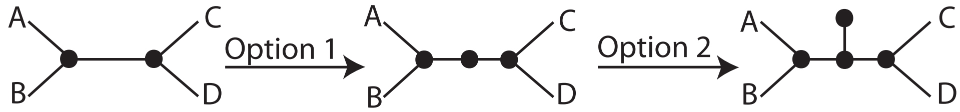

FIGURE 3. Adding a boundary component to a pants decomposition graph.

graph into two connected components. Recall that a nontrivial separation of a graph is a separation such that there are at least two connected components each consisting of at least two vertices or a single vertex and a loop.

We differentiate two categories of separating curves, separating curves that *cut off genus* and separating curves that *cut off boundary components*. By the former, we refer to separating curves on the surface whose removal separates that surface into two nontrivial subsurfaces each with genus at least one. By the latter, we refer to to separating curves on the surface whose removal separates that surface into two nontrivial subsurfaces at least one of which is a topological sphere with boundary components. Equivalently, a separating curve cuts off genus if the removal of the edge in $\Gamma(P)$ corresponding to the curve disconnects the graph into two cyclic components, otherwise if at least one of the connected components is acyclic, then the separating curve cuts off boundary components. Separating curves that cut off genus only exist on surfaces $S_{g,n}$ with $g \geq 2$,, while separating curves that cut off boundary components only exist on surfaces with $n \geq 3$.

4. Proof of Theorem 1.1

In this section we will prove the following theorem which in particular implies the statement of Theorem 1.1. The proof will from directly from the combination of the Lemmas and Corollaries in this section. To simplify the exposition we will first deal with the case of closed surfaces, and then we will explain how boundary components affect the arguments.

Theorem 4.1 (Main Theorem). *Let $S = S_{g,n}$ and set*

$$D_{g,n} = \max_{P \in \mathcal{P}(S)} (d_{\mathcal{P}(S)}(P, \mathcal{P}_{\text{sep}}(S))).$$

Then, for any fixed number of boundary components (or punctures) n, $D_{g,n}$ grows asymptotically like the function $\log(g)$. *that is $D_{g,n} = \Theta(\log(g))$. In particular, for closed surfaces of sufficiently large genus,*

$$\frac{\log_2(2g+2)}{2} - 2 \leq D_{g,0} \leq \lfloor 2\log_2(g-1) + 1 \rfloor.$$

On the other hand, for any fixed genus $g \geq 2$, $\forall n \geq 6g-5$, $D_{g,n} = 2$.

Remark 4.2. It is not hard to see by direct consideration that $D_{0,6} = 1$. More generally, for $n \geq 7$, $D_{0,n} = 0$, and $\forall n \geq 3$, $D_{1,n} = 2$. The exact terms in the upper and lower bounds on $D_{g,0}$ are not believed to be sharp but instead come from the technical details in the proofs.

4.1. Upper bounds for closed surfaces using girth.

Lemma 4.3. *For $P \in \mathcal{P}(S)$ and $\Gamma(P)$ its pants decomposition graph*

$$d_{\mathcal{P}}(P, \mathcal{P}_{\text{sep}}) \leq \text{girth}(\Gamma(P)) - 1.$$

In particular, $D_{g,0} \leq \lfloor 2\log_2(g-1) + 1 \rfloor$.

Proof. By valence considerations, a loop in $\Gamma(P)$ implies P contains a separating curve. Hence, in order to prove the first statement it suffices to show that given any cycle of length $n \geq 2$, there exists an elementary move decreasing the length of the cycle by one. See Figure 4.

Regarding the second statement, it is known that a girth h cubic graph must have at least $2^{h/2}$ vertices, [Big]. It follows that any cubic graph Γ with $2(g-1)$ vertices, has $\text{girth}(\Gamma) \leq 2\log_2(g-1) + 2\rfloor$. The second statement now follows from the first one. □

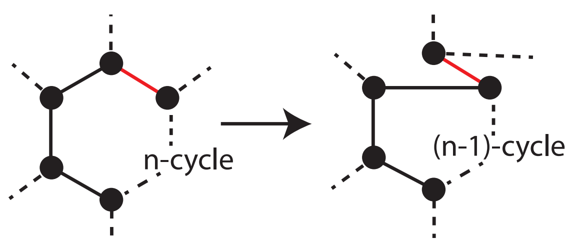

FIGURE 4. Elementary pants move decreasing the length of a cycle in Γ.

4.2. Lower bounds for closed surfaces. Recall that a separating curve $\gamma \in \mathcal{C}_{\text{sep}}(S)$ is said to *cut off genus* if $S \setminus \gamma$ consists of two connected complexity at least one subsurfaces neither of which is topologically a sphere with boundary components. Also recall that for a graph $\Gamma(V, E)$, a subset $C \subset V$ is called a *nontrivial connected cut-set* of Γ if $\Gamma[C]$ is a connected graph and $\Gamma[V \setminus C]$ has at least two connected components each consisting of at least two vertices or a vertex and a loop. The following lemma gives a lower bound on the distance of a pants decomposition to a pants decomposition which cuts off genus.

Lemma 4.4 (Key Lemma)**.** *For $P \in \mathcal{P}(S)$ and $\Gamma(P)$ its pants decomposition graph, let d be the cardinality of a minimal nontrivial connected cut-set $C \subset \Gamma(P)$. Then*

$$d_{\mathcal{P}(S)}(P, P') \geq \min\{\text{girth}(\Gamma(P)) - 1, d - 1\}$$

for P' any pants decomposition containing a separating curve cutting off genus.

Proof. Let γ be any curve in the pants decomposition P, and let α be any separating curve of the surface S that cuts off genus. It suffices to show that the number of elementary moves needed to take the curve γ to α is at least $\min\{\text{girth}(\Gamma(P)) - 1, d - 1\}$. In fact, considering the effect of an elementary move, it suffices to show that α nontrivially intersects at least $\min\{\text{girth}(\Gamma(P)), d\}$ different connected components of $S \setminus P$.

Corresponding to α consider the subgraph $[\alpha] \subset \Gamma(P)$ consisting of all vertices in $\Gamma(P)$ corresponding to connected components of $S \setminus P$ nontrivially intersected by α, as well as all edges in $\Gamma(P)$ corresponding to curves of the pants decomposition P nontrivially intersected by α. By construction, the subgraph $[\alpha]$ is connected. Note that the subgraph $[\alpha]$ need not be equal to the induced subgraph $\Gamma[\alpha]$, but may be a proper subgraph of it. Nonetheless, $V(\Gamma[\alpha]) = V([\alpha])$. (See Figure 5 for an example of a subgraph [a] $\subset \Gamma(P)$.)

As noted, it suffices to show $|V(\Gamma[\alpha])| \geq \min\{\text{girth}(\Gamma(P)),\ d\}$. Assuming not, by the girth condition it follows that $\Gamma[\alpha]$ is acyclic. However, this implies that α is entirely contained in a union of connected components of $S \setminus P$ such that in the ambient surface S, the connected components glue together to yield an essential subsurface Y, which is topologically a sphere with boundary components. Moreover, by the cardinality of the minimal nontrivial connected cut-set condition, it follows that the removal of the essential subsurface Y, or any essential subsurface thereof, from the ambient surface S does not, nontrivially separate S. In particular, for all $U \subset Y$, $S \setminus U$ consists of a disjoint union of at most one nontrivial essential subsurface as well as some number of pairs of pants. It follows that α cannot be a separating curve cutting off genus. □

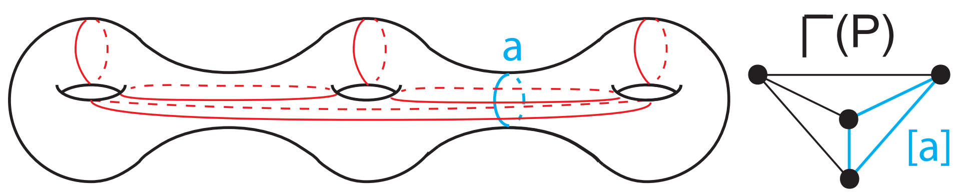

FIGURE 5. An example of a subgraph $[a] \subset \Gamma(P)$ corresponding to a separating curve $a \subset S_{3,0}$, cutting off genus. In this example, the girth of $\Gamma(P)$ is three and there are no nontrivial connected cut-sets of $\Gamma(P)$. By Lemma 4.4, the distance from P to any pants decomposition with a separating curve cutting off genus is at least (and in fact exactly) two.

In Section 5 for any even number $2m$, such that h is the largest integer satisfying $\left(\left\lceil \frac{2^h-4}{h-4} \right\rceil\right) \cdot h \leq 2m$, we construct a graph, Γ_{2m}, such that $|V(\Gamma_{2m})| = 2m$, $\text{girth}(\Gamma_{2m}) = h$, and any connected cut-set of the graph contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. By Lemma 4.4, the pants decomposition corresponding to Γ_{2m} is distance at least $\frac{h}{2} - 2$ from a pants decomposition containing a separating curve. Because the pants decomposition graph Γ_{2m} corresponds to a pants decomposition of a closed surfaces of genus $m-1$, it follows that $\frac{h}{2} - 2 < D_{m-1,0}$. Since for large enough values of h,

$$2m < \left(\left\lceil \frac{2^{h+1}-4}{h-3} \right\rceil \right) \cdot (h+1) < 2^{h+2},$$

after algebraic manipulation one obtains $\frac{\log_2(2(m-1)+2)}{2} - 2 < \frac{h}{2} - 2$. In conjunction with Lemma 4.3 we have proven the following:

Corollary 4.5. *For large enough values of g, we have the bounds on $D_{g,0}$ recorded in Theorem 4.1. In particular, $D_{g,0} = \Theta(\log(g))$.*

4.3. Adding boundary components. In this section we modify the previously described arguments to allow for the case that our surface S has boundary components. We begin with a lemma describing a local situation in $\Gamma(P)$ which can be manipulated to generate a pants decomposition containing a separating curve.

Lemma 4.6. *For $P \in \mathcal{P}(S)$ and $\Gamma(P)$ its pants decomposition graph. If $\Gamma(P)$ has three consecutive vertices of degree at most two, then*

$$d_{\mathcal{P}}(P, \mathcal{P}_{\text{sep}}) \leq 2.$$

Proof. See Figure 6. □

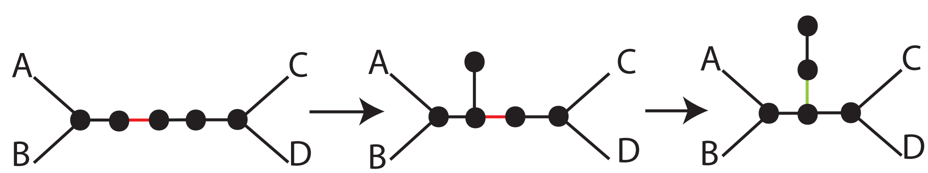

FIGURE 6. Two elementary moves creating a separating curve cutting off boundary components.

Using Lemma 4.6, presently we generalize Lemma 4.3 to surfaces with boundary.

Corollary 4.7. $\forall g \geq 2,\ \ D_{g,n} \leq \lfloor 2\log_2(g-1) + 3 \rfloor$.

Proof. Recall that Lemma 4.3 gives an upper bound of $\lfloor 2\log_2(g-1)+1 \rfloor$ on $D_{g,0}$. Specifically, this upper bound was obtained by taking the smallest cycle C in the graph $\Gamma(P)$ which had length at most $\lfloor 2\log_2(g-1)+2 \rfloor$ and then successively decreasing the length of cycle C by elementary pants moves as in the proof of Lemma 4.3. Consider what can happen to this cyclic subgraph C as we add boundary components to our surface as in Subsection 3.2. If the added boundary components do not affect the length of cycle C, the upper bound is unaffected. On the other hand, if the added boundary components increase the length of the cycle C by adding one (two) degree two vertex (vertices) to the cycle C, then the distance to a separating curve increases by at most one (two). However, once at least three degree two vertices have been added to the cycle C, instead of reducing the cycle to a loop, we can instead use elementary moves to gather together three consecutive vertices of degree two and then create a separating curve locally, as in Lemma 4.6. The statement of the corollary follows. □

Again using Lemma 4.6 we have the following corollary, also proving a special case of Theorem 4.1.

Corollary 4.8. *For all* $g \geq 2$, $n \geq 6g-5 \implies D_{g,n} = 2$.

Proof. By Lemma 3.2 for $P \in \mathcal{P}(S_{g,n})$, $\Gamma(P)$ is a connected at most cubic graph with $2(g-1)+n$ vertices and $3(g-1)+n$ edges. Since $n \geq 6g-5$, by pigeon hole considerations it follows that $\Gamma(P)$ has three consecutive vertices of degree at most two. By Lemma 4.6, $D_{g,n} \leq 2$. Then to see that $D_{g,n} = 2$ it suffices to explicitly exhibit connected at most cubic graphs with $2(g-1)+n$ vertices and $3(g-1)+n$ edges for all $g \geq 2, n \geq 6g-5$ such that the graphs neither contain nontrivial cut edges nor are one elementary move away from a graph with a nontrivial cut edge. See Figure 7 for an explicit construction of such a family of graphs. □

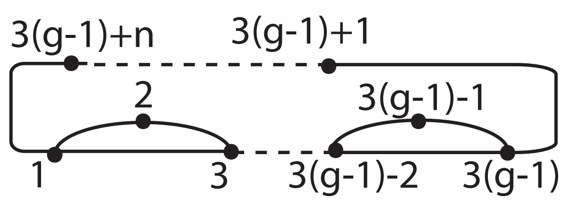

FIGURE 7. Pants decompositions graphs of pants decompositions which are distance two from a pants decomposition containing a separating curve.

Generalizing the aforementioned family of cubic graphs, Γ_{2m}, in Section 5 we show that for any fixed $n \in \mathbb{N}$ we can add n boundary components to the our graphs, Γ_{2m}, creating a family of pants decomposition graphs Γ_{2m}^n, whose corresponding pants decompositions have girth, minimum nontrivial cut-set size, and distance between valence less than three vertices growing logarithmically in the vertex size of the graph. By Lemma 4.4, the fact that girth and minimum nontrivial connected cut-set size grow logarithmically in the vertex size of the graph implies that the distance between pants decompositions corresponding to the constructed graphs to any pants decompositions containing a separating curve cutting off genus grows logarithmically in the vertex size of the graph. Moreover, the fact that the distance between valence less than three vertices grows logarithmically in the vertex size of the graphs, implies that the distance between pants decompositions corresponding to the constructed graphs to any pants decompositions containing a separating curve cutting off boundary components also grows logarithmically in the vertex size of the graphs. As a corollary, we have:

Corollary 4.9. *For any fixed* $n \in \mathbb{N}$, $D_{g,n} = \Theta(\log(g))$.

5. Construction of large girth, log-length connected graphs

We first describe a construction for a family, Γ_h, of cubic girth $h \geq 5$ graphs with

$$\left(\left\lceil \frac{2^h - 4}{h - 4} \right\rceil\right) \cdot h + \{0, 1\}$$

vertices (where the final term is simply to ensure the total number of vertices is even), which have the property that any connected cut-set of Γ_h contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. Afterward, we generalize our construction, interpolating between the family of graphs Γ_h. Specifically, for all $m \in \mathbb{N}$, such that $h \geq 5$ is the largest integer satisfying $2m \geq \left(\left\lceil \frac{2^h-4}{h-4} \right\rceil\right) \cdot h$, there exists a cubic girth h graph Γ_{2m} with $2m$ vertices and the property that any connected cut-set of the graph contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. Finally, we demonstrate that for any fixed number of boundary components n, we can add n boundary components to our graphs Γ_{2m} yielding a family of graphs Γ^n_{2m} with the same desired properties.

5.1. Construction of $\boldsymbol{\Gamma_h}$. Begin with $\left(\left\lceil \frac{2^h-4}{h-4} \right\rceil\right)$ disjoint cycles each of length h (possibly one of length $h+1$ if necessary to make the total number of vertices even). Then, chain together these disjoint cycles into an at most cubic connected tower T_h, connecting each cycle to its neighboring cycle(s) by adding two edges between pairs of vertices, one from each cycle, such that each of the two vertices from the same cycle, to which edges are being attached, are of distance at least $\lfloor \frac{h}{2} \rfloor$. See Figure 8 for an example of T_8.

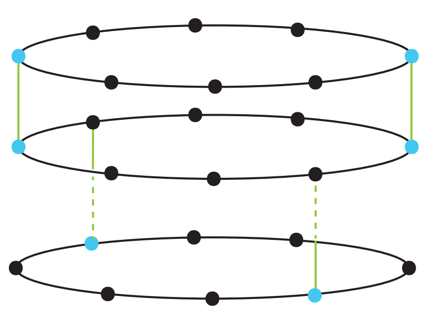

FIGURE 8. T_8, an at most cubic girth eight tower graph.

By construction, the tower graphs, T_h, have the following properties:

T1: T_h has $\left(\left\lceil \frac{2^h-4}{h-4} \right\rceil\right) \cdot h + \{0, 1\}$ vertices.

T2: T_h is an at most 3-regular and at least 2-regular graph with girth h.

T3: If we denote the subset of vertices of T_h of valence two by $V_2^{T_h}$, then $|V_2^{T_h}| \geq 2^h$.

T4: Any connected cut-set of T_h has at least $\lfloor \frac{h}{2} \rfloor$ vertices.

5.2. Algorithm completing T_h to a 3-regular graph Γ_h. Motivated by ideas in [Big], presently we describe a constructive algorithm to add edges to the tower T_h completing it to a 3-regular graph $\Gamma = \Gamma_h$ which also has girth h, and retains the property that any connected cut-set of Γ_h has at least $\lfloor \frac{h}{2} \rfloor$ vertices. By abuse of notation, we will always refer to the graph that has been constructed up to the current point as Γ. In terms of ensuring the girth condition, the main observation being used implicitly throughout is that removing edges from a graph never decreases girth, while adding an edge connecting vertices which were previously at least distance $h-1$ apart, in a girth at least h graph, yields a girth at least h graph.

Step One (An easy opportunity to add an edge). If Γ is 3-regular, we're done. If not, fix a vertex $v \in V_2^{T_h}$ of valence two. If there exists some other vertex $x \in V_2^{T_h}$ with $d_\Gamma(v,x) \geq h-1$, add an edge between x and v.

Step Two (Exhaust easy opportunities). Iterate Step One until all possibilities to add edges to Γ are exhausted.

Step Three (One step backward, two steps forward). If Γ is 3-regular, we're done. If not, since the total number of vertices is even, there must exist at least two vertices, x and y, of valence two. Consider the sets $U = N^\Gamma_{h-2}(x) \cup N^\Gamma_{h-2}(y)$ and $I = N^\Gamma_{h-2}(x) \cap N^\Gamma_{h-2}(y)$. Due to the valence considerations, since x, y are valence TWO vertices in an at most cubic graph it follows that $|N^\Gamma_{h-2}(x)| \leq 1 + 2 + ... + 2^{h-2} = 2^{h-1} - 1$, and similarly for $N^\Gamma_{h-2}(y)$. Note that $|U| = |N^\Gamma_{h-2}(x)| + |N^\Gamma_{h-2}(y)| - |I| \leq 2^h - 2 - |I|$. Then consider the set $W = V_2^{T_h} \setminus U$. Since $|V_2^{T_h}| \geq 2^h$, it follows that $|W| \geq 2 + |I|$. In particular, the set W is nonempty. Furthermore, considering that Step Two was completed to exhaustion, it follows that $\forall w \in W$, w is of valence three in Γ. Moreover, by definition, the vertex w is of valence two in T_h. Denote the vertex that is connected to w in Γ but not in T_h by w'. Perforce, w' is distance at least $h-2$ from both x and y. In fact, we can assume that w' is not exactly distance $h-2$ from both x and y because $|W| > |I|$. For concreteness, we can assume that $d_\Gamma(x, w') \geq h-1$.

Remove from Γ the edge e connecting w to w', and in its place include two edges: e_1 between x and w', and e_2 between w and y. Adding the two edges e_1 and e_2 does not decrease girth to less than h as they each connect vertices that were distance at least $h-1$ apart: After removing e, the vertices w and w' are distance at least $h-1$ because Γ was girth at least h. Hence, even after adding edge e_1 we can still be sure that the vertices y and w remain distance at least $h-1$ apart, thereby allowing us to add edge e_2 without decreasing girth to less than h.

Step Four (Repeat)*.* If Γ is not yet 3-regular, return to Step Three.

The algorithm terminates as Step Three can always be performed if the graph is not yet 3-regular, and the net effect of Step Three increases the number of edges in the at most 3-regular graph by one. By construction the graph Γ_h has girth h. Moreover, as that the algorithm never removes edges from the tower T_h, and hence the resulting graph Γ_h includes the tower T_h as a subgraph. Using the girth condition in conjunction with the fact that any connected cut-set of Γ_h is a cut-set of T_h, it is not hard to see that any connected cut-set of Γ_h has at least $\lfloor \frac{h}{2} \rfloor$ vertices.

5.3. Construction of $\boldsymbol{\Gamma_{2m}}$. For any even number of vertices $2m$ such that $2m \geq |V(\Gamma_h)|$, for some h, we can construct a 3-regular girth h graph on $2m$ vertices, which we denote Γ_{2m}, with the property that any connected cut-set of Γ_{2m} contains at least $\lfloor \frac{h}{2} \rfloor$ vertices. In fact, we can construct the graphs Γ_{2m} using the exact same process as in the construction of Γ_h with the exception that we now start with additional cycles in the building our initial tower which is subsequently completed to a cubic graph. Specifically, to construct Γ_{2m}, we begin with $\lfloor \frac{2m}{h} \rfloor$ cycles of length h and $(h+1)$ as necessary.

5.3.1. Adding a fixed number $\boldsymbol{n}$ of boundary components to $\boldsymbol{\Gamma_{2m}}$. For any fixed number $n \in \mathbb{N}$, we can add n boundary components to the graphs Γ_{2m}, to obtain graphs Γ^n_{2m}. Moreover, we can easily ensure that no two added boundary components are within distance $\lfloor \frac{h}{2} \rfloor$ from each other, past some minimal threshold for $2m$. This is because for x, an added boundary component in Γ_{2m}, $|N_{\lfloor \frac{h}{2} \rfloor}(x)| \leq 2^{\lfloor \frac{h}{2} \rfloor + 1}$, while $|V(\Gamma_{2m})| \geq 2^h$. It follows that that for any fixed number of boundary components n, we have a family of graphs Γ^n_{2m} with girth, nontrivial minimum cut-set size, and the distance between valence less than three vertices all growing logarithmically in the vertex size of the graph.

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