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Lattice vertex algebras and combinatorial bases: general case and $\mathcal{W}$-algebras

Antun Milas and Michael Penn

Abstract. We introduce what we call the principal subalgebra of a lattice vertex (super) algebra associated to an arbitrary $\mathbb{Z}$-basis of the lattice. In the first part (to appear), the second author considered the case of positive bases and found a description of the principal subalgebra in terms of generators and relations. Here, in the most general case, we obtain a combinatorial basis of the principal subalgebra W_L and of related modules. In particular, we substantially generalize several results in Georgiev, 1996, covering the case of the root lattice of type A_n, as well as some results from Calinescu, Lepowsky and Milas, 2010. We also discuss principal subalgebras inside certain extensions of affine $\mathcal{W}$-algebras coming from multiples of the root lattice of type A_n.

Contents

1. Introduction

This paper continues our investigation of principal subspaces in [29], where it was shown that (suitably defined) positive basis of an integral lattice L give rise to a supercommutative principal subspace W_L, which can be described explicitly in terms of generators and relations. Our description is useful for purposes of getting graded dimensions of W_L and the related difference equations. All this can be also done for certain W_L-modules, at least those that naturally come from irreducible V_L-modules. Commutative principal subspaces associated to representations of affine Lie algebras were

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also studied in other works such as [17], [24], [25], [6], [31], [32], [33], [19], [13], [14], [10], [11], etc. Moreover, in [16] principal subspaces of positive definite rank one even lattices were examined. But the "full" principal subspace introduced in the pioneering work [18] and investigated further in [4], [5], [8], [9], [12], [23] is generically noncommutative so results from Part I [29] cannot be easily modified.

Motivated by these developments, here we start from an arbitrary $\mathbb{Z}$-basis $\mathcal{B} = \{\alpha_i\}_{i=1}^n$ of the integral lattice L and consider the vertex (super)algebra $W_L(\mathcal{B}) \subset V_L$ generated by the corresponding extremal vectors $e^{\alpha_i} \in \mathbb{C}[L]$, $i = 1, \ldots, n$. Our first main result is a combinatorial basis of $W_L(\mathcal{B})$ (this is the statement of Theorem 4.13). Having an explicit combinatorial basis allows us to easily compute the multi-graded graded dimension directly, without relying on q-difference equations as in [29]. That was obtained in Theorem 5.3. We stress that this result applies even to indefinite, or negative definite lattices. For example, for the rank one lattice $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = -n < 0$, the bi-graded dimension of the principal subspace $W_L = \langle e^{\alpha} \rangle$ equals

$$\chi_{W_L}(x, q) = \sum_{k=0}^{\infty} \frac{q^{-\frac{nk^2}{2}} x^k}{(q)_k},$$

where the x-variable controls the "charge". For $n = 2$, this produces the "opposite" Rogers–Ramanujan series. Clearly, the pure q-dimension is not well-defined so the charge variable x is required here, explaining the need for "multi-graded" dimensions.

These two results conclude our analysis of principal subspaces (and their modules) of lattice vertex superalgebras. In the most interesting case of positive definite lattices, where the charge variables can be omitted, all this can be summarized as:

Theorem 1.1. *Let A be a positive definite symmetric $n \times n$ matrix with integer entries, $B \in \mathbb{Z}^n$, and*

$$f_{A,B}(q) = \sum_{\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n} \frac{q^{\frac{\mathbf{k}A\mathbf{k}^T}{2} + B\cdot\mathbf{k}}}{(q)_{k_1} \cdots (q)_{k_n}}$$

denote the corresponding n-fold q-hypergeometric series (Nahm's sum [34]*). Then there is a principal subspace $W_{L+\beta}$, with an explicit combinatorial basis, whose graded dimension is precisely $f_{A,B}(q)$.*

While principal subalgebras and subspaces are very nice combinatorial objects they are odd species because of lack of conformal structure. One can in theory add the conformal vector and generate a larger subalgebra, but this procedure is somewhat *ad hoc*. Instead, at least in the case of multiple of the root lattice Q, we can generate interesting object via *screening operators*. This method was widely used in the recent investigations of irrational C_2-cofinite $\mathcal{W}$-algebras coming from multiples of root lattices [1], [2], [19].

Although nothing prevents us from studying all simply-laced types, here we only focus on the A-type. We first give construction of a one-parameter family $\mathcal{W}(p)^{\diamond}_{A_n}$ of extensions of the affine $\mathcal{W}$-algebra $\mathcal{W}_p(\mathfrak{sl}_{n+1})$. For $p = 1$ we have a fairly explicit description of this vertex algebra. We prove it is generated by n vectors coming from $\mathcal{W}_1(\mathfrak{sl}_{n+1})$ and certain extremal vectors e^{β_i}, $i = 1, \ldots, k$, parametrized by primitive nonnegative solutions of the Diophantine equation

$$x_1 + 2x_2 + \cdots + nx_n \equiv 0 \pmod{n+1}. \tag{1.1}$$

Let us illustrate this on a low rank example.

Example 1.2. ($Q = A_2$) It is known that $W_1(\mathfrak{sl}_3)$ (Zamolodchikov algebra) has two generators: the conformal vector ω and a primary vector Ω of conformal weight 3. The equation (1.1) has three primitive indecomposable solutions: $(1, 1)$, $(3, 0)$ and $(0, 3)$, corresponding to weights $\omega_1 + \omega_2$, $3\omega_1$, $3\omega_2$, respectively. Thus the relevant extremal vectors are $e^{\alpha_1+\alpha_2}$, $e^{\alpha_1+2\alpha_2}$ and $e^{2\alpha_1+\alpha_2}$, and therefore

$$\mathcal{W}(1)^{\diamond}_{A_2} = \langle \omega, \Omega, e^{\alpha_1+\alpha_2}, e^{\alpha_1+2\alpha_2}, e^{2\alpha_1+\alpha_2} \rangle.$$

However, if the rank is bigger than one, these generators clearly do not generate $W(1)^{\diamond}_{A_n}$ freely. Instead, more subtle relations occur [28]. The vertex algebra $\langle e^{\beta_1}, \ldots, e^{\beta_k} \rangle$ is what we call the *principal subalgebra* of $\mathcal{W}(1)^{\diamond}_{Q}$. For $p \geq 2$, and $Q = A_1$ we were able to completely describe $\mathcal{W}(p)^{\diamond}_{Q}$ (see Section 6).

The previous discussion give rise to the following problem: Find a combinatorial basis of the vertex algebra $\langle e^{\beta_1}, \ldots, e^{\beta_k} \rangle$, where $\{\beta_1, \ldots, \beta_k\}$ is an arbitrary *set* in L, and not just a $\mathbb{Z}$-basis. This will be pursued in a forthcoming paper [28].

N.B. Sections 1–5 of this work are essentially included in the Ph.D. thesis of the second author [30], written under the advisement of the first author.

2. The setting

Similar to Part I [29] and [12], consider the rank n integral lattice L with a $\mathbb{Z}$-basis $\{\alpha_i\}_{i=1}^n$

$$L = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i, \tag{2.1}$$

equipped with a nondegenerate symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$ such that the matrix A defined by $A_{(i,j)} = \langle \alpha_i, \alpha_j \rangle$ is nonsingular. We centrally extend the lattice L and L° (the dual lattice of L) as in [12], and consider the corresponding lattice vertex superalgebra

$$V_L \cong M(1) \otimes \mathbb{C}[L]$$

[7, 27, 22, 26]. As in [12] [29], we make use of the vertex operators [27]

$$Y(e^{\beta},x)=\sum_{m\in\mathbb{Z}}(e^{\beta})_m x^{-m-1}=E^-(-\beta,x)E^+(-\beta,x)e_{\beta}x^{\beta},$$

where

$$e_{\beta}\cdot(h\otimes e^{\gamma})=\epsilon(\beta,\gamma)h\otimes e^{\beta+\gamma},\quad e^{\gamma}\in\mathbb{C}[L],\ h\in M(1).$$

Also recall from Part I [29], the *principal subalgebra*[1] associated to $\mathcal{B}$

$$W_L(\mathcal{B})=\langle e^{\alpha_1},\ldots,e^{\alpha_n}\rangle,$$

the smallest vertex subalgebra of V_L containing $\{e^{\alpha_i}\}_{i=1}^n$. Once a basis is fixed, we shall drop $\mathcal{B}$ in the parenthesis and write W_L for brevity. For every $\beta\in L^{\circ}$ we define the cyclic W_L-module

$$W_{L+\beta}:=W_L\cdot e^{\beta}\subset V_{L^{\circ}}.$$

We refer to this space as a *principal subspace* (again see [29] for more details). We also denote by $\{\omega_j\}_{j=1}^n$ the dual basis of $\mathcal{B}$ such that $\langle\alpha_i,\omega_j\rangle=\delta_{i,j}$. From Part I (or [23], [12]), we recall the intertwining operators $\mathcal{Y}(e^{\omega_j},x)$ acting among appropriate triples of V_L-modules. We also use $\mathcal{Y}_c(\cdot,x)$ to denote the constant term of the intertwining operator $\mathcal{Y}(\cdot,x)$ [12, 29]. Recall also the following result ([29], Proposition 3.0.3):

Lemma 2.1. *With α_i and ω_j as above,*

$$[Y(e^{\alpha_i},x_1),\mathcal{Y}(e^{\omega_j},x_2)]=0.$$

Our goal is to obtain a monomial basis of W_L and of $W_{L+\omega_i}$, where monomials are of the form $(e^{\alpha_{i_k}})_{m_k^{(1)}}\ldots(e^{\alpha_{i_1}})_{m_1^{(1)}}\mathbf{1}$. We consider the following partial ordering on the monomials in W_L and $W_{L+\beta}$.

Definition 2.2. Let

$$v_1=(e^{\alpha_{i_k}})_{m_k^{(1)}}\ldots(e^{\alpha_{i_1}})_{m_1^{(1)}}\mathbf{1},\quad v_2=(e^{\alpha_{i_k}})_{m_k^{(2)}}\ldots(e^{\alpha_{i_1}})_{m_1^{(2)}}\mathbf{1}.$$

We say

$$v_1\prec v_2, \tag{2.2}$$

if $m_r^{(1)}=m_r^{(2)}$ for $1\le r\le s$ and $m_{s+1}^{(1)}<m_{s+1}^{(2)}$. We extend this definition to arbitrary nonzero multiples of monomials, and to monomials in $W_{L+\beta}$, where $\mathbf{1}$ is replaced by e^{β}.

For example,

$$e_{-3}^{\alpha_i}e_{-4}^{\alpha_i}e_{-2}^{\alpha_i}\mathbf{1}\prec e_{-5}^{\alpha_i}e_{-3}^{\alpha_i}e_{-2}^{\alpha_i}\mathbf{1}.$$

Notice that $\prec$ is a total ordering on the set of monomials of W_L of the same charge (or color in this case). Observe also that every chain with respect to this ordering does not have an upper bound (for instance there are infinitely many monomials which are “bigger” than $e_{-3}^{\alpha_i}e_{-4}^{\alpha_i}e_{-2}^{\alpha_i}\mathbf{1}$). However,

[1] We use “principal subalgebra” for the principal subspace inside V_L.

if we consider monomials of the same color-charge and the same degree (or weight), every chain will have an upper bound due to lower truncation property of the vertex algebra.

3. Rank one subspaces

Consider the sublattice $\mathbb{Z}\alpha_i \subset \tilde{L}$ and the rank one subalgebra generated by e^{α_i},

$$W_{L_i} = \langle e^{\alpha_i} \rangle . \tag{3.1}$$

We will first find a combinatorial basis of W_{L_i}. The proof of the spanning of this set will be used directly to find a spanning set for W_L in general. The proof of the linear independence will serve as a template for the higher rank case (cf. Section 4).

To simplify notation, we write $\alpha_i = \alpha$ and $L_i = L$ for the remainder of the section.

Consider the following set:

$$\mathcal{B}_i = \left\{ (e^{\alpha})_{m_1} \dots (e^{\alpha})_{m_k} \;\middle|\; m_{j-1} \leq m_j - \langle \alpha, \alpha \rangle , m_k < -i, k \geq 0 \right\},$$

viewed as elements in $\operatorname{End}(W_{L+\beta})$, where $\beta \in L^{\circ}$. The set of "monomials"

$$\mathcal{B}^{(i)} = \mathcal{B}_i \cdot e^{\frac{i\alpha}{\langle \alpha, \alpha \rangle}}$$

will be shown to be a basis of $W_{L+\frac{i\alpha}{\langle \alpha,\alpha\rangle}}$. Notice that throughout these computations $i = 0$ corresponds to the case of W_L itself.

Our investigation into the structure of the rank one subspaces will begin with the construction of some relations involving quadratic elements. It will be important to separate into cases when $\langle \alpha, \alpha \rangle$ is negative, positive or zero. We define a set of quadratic elements of $\operatorname{End}(W_{L+\beta})$

$$\mathcal{B}(2) = \{ (e^{\alpha})_r (e^{\alpha})_t \mid r \leq t - \langle \alpha, \alpha \rangle \}. \tag{3.2}$$

We will start with the case when $\langle \alpha, \alpha \rangle > 0$. From [27] we have the following:

Proposition 3.1. *For $1 \leq k \leq \langle \alpha, \alpha \rangle$, $k \in \mathbb{N}$,*

$$(e^{\alpha})_{-k} e^{\alpha} = 0. \tag{3.3}$$

Proof. We will utilize the notation found in [27], as well as the following result

$$Y(e^{\alpha}, x) e^{\alpha} = \epsilon(\alpha, \alpha) x^{\langle \alpha, \alpha \rangle} \exp \left(\sum_{n \in \mathbb{Z}_-} \frac{-\alpha(n)}{n} x^{-n} \right) e^{2\alpha}.$$

This provides us with

$$(e^{\alpha})_{-k} e^{\alpha} = \frac{1}{\epsilon(\alpha, \alpha)} \operatorname{Coeff}_{x^{k-1}} \left(Y(e^{\alpha}, x) e^{\alpha} \right) = 0,$$

for $1 \leq k \leq \langle \alpha, \alpha \rangle$, because the smallest power of x in the above is $\langle \alpha, \alpha \rangle$. $\square$

From the previous proposition we build some useful relations. For $1 \leq k \leq \langle\alpha,\alpha\rangle$, we have the following:

$$\begin{aligned} A_+(k,x) &= Y((e^\alpha)_{-k}e^\alpha, x) \\ &= \frac{1}{(k-1)!}\left(\left(\frac{d}{dx}\right)^{(k-1)} Y(e^\alpha,x)\right)Y(e^\alpha,x) = 0, \end{aligned} \tag{3.4}$$

where there is no need for normal ordering because when $\langle\alpha,\alpha\rangle \geq 0$, the algebra W_L is supercommutative. Taking coefficients of the above equation will give us relations within W_L, so define

$$R_+(k,n) := \mathrm{Coeff}_{x^{-n-1}}\left(A_+(k,x)\right).$$

The relations $R_+(k,n)$ can be written as follows:

$$\begin{aligned} R_+(k,n) &= \sum_{m\leq -k}\binom{m+k-1}{k-1}(e^\alpha)_{-k-m}(e^\alpha)_{n+m} \\ &\quad + \sum_{m\geq 0}\binom{m+k-1}{k-1}(e^\alpha)_{-k-m}(e^\alpha)_{n+m}. \end{aligned} \tag{3.5}$$

Now we will decompose $R_+(k,n)$ into terms $(e^\alpha)_r(e^\alpha)_s$ for which $|s-r| \geq \langle\alpha,\alpha\rangle$ and those for which $|s-r| < \langle\alpha,\alpha\rangle$. If $\langle\alpha,\alpha\rangle - n - k$ is even, we have $\langle\alpha,\alpha\rangle$ terms of $R_+(k,n)$ for which $|s-r| < \langle\alpha,\alpha\rangle$. These are

$$(e^\alpha)_{-k-m_1}(e^\alpha)_{n+m_1}, \ldots, (e^\alpha)_{-k-m_{\langle\alpha,\alpha\rangle-1}}(e^\alpha)_{n+m_{\langle\alpha,\alpha\rangle-1}}$$

where $m_1 = \frac{1}{2}(\langle\alpha,\alpha\rangle - n - k) + 1$ and $m_i = m_1 + i - 1$ for $2 \leq i \leq \langle\alpha,\alpha\rangle - 1$. If $\langle\alpha,\alpha\rangle - n - k$ is odd, we have $\langle\alpha,\alpha\rangle$ terms of $R_+(k,n)$ for which $|s-r| < \langle\alpha,\alpha\rangle$:

$$(e^\alpha)_{-k-m_1}(e^\alpha)_{n+m_1}, \ldots, (e^\alpha)_{-k-m_{\langle\alpha,\alpha\rangle}}(e^\alpha)_{n+m_{\langle\alpha,\alpha\rangle}}$$

where $m_1 = \frac{1}{2}(-\langle\alpha,\alpha\rangle - n - k + 1)$ and $m_i = m_1 + i - 1$ for $2 \leq i \leq \langle\alpha,\alpha\rangle$.

For $\langle\alpha,\alpha\rangle - n - k$ even (resp. odd) consider the $(\langle\alpha,\alpha\rangle - 1 \times \langle\alpha,\alpha\rangle - 1)$ (resp. $\langle\alpha,\alpha\rangle \times \langle\alpha,\alpha\rangle$) matrix $\mathcal{P}$ defined by

$$(\mathcal{P})_{i,j} = \mathrm{Coeff}_{(e^\alpha)_{-1-m_j}(e^\alpha)_{n+m_j}} R_+(i, n+i-1) = \binom{m_j}{i-1} \tag{3.6}$$

for $1 \leq i,j \leq \langle\alpha,\alpha\rangle - 1$ (resp. $1 \leq i,j \leq \langle\alpha,\alpha\rangle$) where the m_j are defined as above.

We will make use of the following lemma involving matrices.

Lemma 3.2. *For all $r, s \in \mathbb{N}$, the matrix*

$$\begin{pmatrix} \binom{r}{0} & \binom{r+1}{0} & \cdots & \binom{r+s}{0} \\ \binom{r}{1} & \binom{r+1}{1} & \cdots & \binom{r+s}{1} \\ \vdots & \vdots & & \vdots \\ \binom{r}{s} & \binom{r+1}{s} & \cdots & \binom{r+s}{s} \end{pmatrix}$$

is invertible.

The proof of the lemma is trivial (simply subtract s-th column from the $(s+1)$-th column, $(s-1)$-th from the s-th, etc. It is easy to see that the determinant of the matrix is one).

Now for $1 \leq i \leq \langle\alpha,\alpha\rangle - 1$ (resp. $1 \leq i \leq \langle\alpha,\alpha\rangle$) define new expressions $\mathcal{R}_+(i,n)$ to be the linear combinations of of $R_+(i, n+i-1)$ corresponding to the row reduction of $\mathcal{P}$ to the identity matrix. These new expressions are of the form

$$\mathcal{R}_+(i,n) = (e^\alpha)_{-1-m_i}(e^\alpha)_{n+m_i} + \sum_{|s-r|\geq\langle\alpha,\alpha\rangle} c_{r,s}(e^\alpha)_r(e^\alpha)_s \tag{3.7}$$

and since $\langle\alpha,\alpha\rangle \geq 0$, W_L is supercommutative so we can write

$$\begin{aligned}\mathcal{R}_+(i,n) = \\ &(e^\alpha)_{-1-m_i}(e^\alpha)_{n+m_i} + \sum_{r\leq s-\langle\alpha,\alpha\rangle} (c_{r,s} + (-1)^{\langle\alpha,\alpha\rangle}c_{s,r})(e^\alpha)_r(e^\alpha)_s.\end{aligned} \tag{3.8}$$

Notice that $\mathcal{R}_+(i,n)v = 0$ for any $v \in W_{L+\beta}$ for $\beta \in L^\circ$, so we may consider these expressions as a new family of relations of these quadratic elements.

Lemma 3.3. *For $\langle\alpha,\alpha\rangle > 0$, every quadratic element*

$$(e^\alpha)_m(e^\alpha)_n \in \mathrm{End}(W_{L+\beta})$$

can be written as a (possibly infinite) linear combination of elements of $\mathcal{B}(2)$. This reduces to a finite linear combination when applied on a vector v in $W_{L+\beta}$, where $\beta \in L^\circ$.

Proof. Suppose we have a quadratic element $(e^\alpha)_u(e^\alpha)_v \in \mathrm{End}(W_{L+\beta})$ such that $v - \langle\alpha,\alpha\rangle < u < v + \langle\alpha,\alpha\rangle$. In other words $(e^\alpha)_u(e^\alpha)_v$ is not an element of $\mathcal{B}(2)$ and cannot be written as an element of $\mathcal{B}(2)$ simply by invoking the supercommutativity of the space. We can find $n \in \mathbb{Z}$, $1 \leq k \leq \langle\alpha,\alpha\rangle$, and $1 \leq i \leq \langle\alpha,\alpha\rangle$ so that $u = -1 - m_i$ and $v = n + m_i$ where m_i is defined in terms of n, k, and i as above. Observe that (3.8) allows us to write

$$(e^\alpha)_u(e^\alpha)_v = -\sum_{r\leq s-\langle\alpha,\alpha\rangle} (c_{r,s} + (-1)^{\langle\alpha,\alpha\rangle}c_{s,r})(e^\alpha)_r(e^\alpha)_s \in \mathcal{B}(2),$$

thus finishing the proof. $\square$

Now we prove the spanning in the case when $\langle\alpha,\alpha\rangle < 0$. In order to do this we first need a set of relations within the quadratic elements of W_L. From [27] we have the following

$$A_-(x_1,x_2) = (x_1 - x_2)^{-\langle\alpha,\alpha\rangle}[Y(e^\alpha,x_1), Y(e^\alpha,x_2)] = 0, \tag{3.9}$$

where the bracket stands for a commutator or anti-commutator depending on the parity. Now taking strategic coefficients we form useful relations.

$$\begin{aligned}(3.10)\qquad R_-(m,n) &= \mathrm{Coeff}_{x_1^{-m-1}x_2^{-n-1}}\, A_-(x_1,x_2)\\ &= \sum_{k=0}^{-\langle\alpha,\alpha\rangle} (-1)^k \binom{-\langle\alpha,\alpha\rangle}{k} [(e^{\alpha})_{m-\langle\alpha,\alpha\rangle-k}, (e^{\alpha})_{n+k}].\end{aligned}$$

Lemma 3.4. *For $\langle\alpha,\alpha\rangle < 0$, every quadratic element*

$$(e^{\alpha})_m(e^{\alpha})_n \in \mathrm{End}(W_{L+\beta})$$

can be written as a linear combination of elements of $\mathcal{B}(2)$, where $\beta \in L^{\circ}$.

Proof. For a monomial $(e^{\alpha})_m(e^{\alpha})_n$ we define the index sum as $-m-n$. Fix an index sum S, and define

$$(3.11)\qquad r = \begin{cases} \frac{S-\langle\alpha,\alpha\rangle+1}{2} & \text{if } S \text{ is odd,}\\ \frac{S-\langle\alpha,\alpha\rangle}{2}+1 & \text{if } S \text{ is even,}\end{cases}$$

and

$$(3.12)\qquad t = \begin{cases} \frac{S-\langle\alpha,\alpha\rangle-1}{2} & \text{if } S \text{ is odd,}\\ \frac{S-\langle\alpha,\alpha\rangle}{2}-1 & \text{if } S \text{ is even.}\end{cases}$$

Notice $(e^{\alpha})_{-t-\langle\alpha,\alpha\rangle}(e^{\alpha})_{-r}$ is the largest quadratic element in the $\prec$ ordering with index sum S which is not an element of $\mathcal{B}(2)$, so we shall begin by showing that it is in $\mathrm{span}(\mathcal{B}(2))$.

Consider

$$\begin{aligned}(3.13)\quad R_-(-r,-t) &= \sum_{k=0}^{-\langle\alpha,\alpha\rangle} (-1)^k \binom{-\langle\alpha,\alpha\rangle}{k} [(e^{\alpha})_{-r-\langle\alpha,\alpha\rangle-k}, (e^{\alpha})_{-t+k}]\\ &= \sum_{k=0}^{-\langle\alpha,\alpha\rangle-1} (-1)^k \binom{-\langle\alpha,\alpha\rangle}{k} [(e^{\alpha})_{-r-\langle\alpha,\alpha\rangle-k}, (e^{\alpha})_{-t+k}]\\ &\quad + (-1)^{-\langle\alpha,\alpha\rangle}[(e^{\alpha})_{-r}, (e^{\alpha})_{-t-\langle\alpha,\alpha\rangle}] = 0.\end{aligned}$$

Notice that this can be rewritten

$$\begin{aligned}(3.14)\qquad & (e^{\alpha})_{-t-\langle\alpha,\alpha\rangle}(e^{\alpha})_{-r}\\ &= (e^{\alpha})_{-r}(e^{\alpha})_{-t-\langle\alpha,\alpha\rangle}\\ &\quad + \sum_{k=0}^{-\langle\alpha,\alpha\rangle-1} (-1)^{k+\langle\alpha,\alpha\rangle} \binom{-\langle\alpha,\alpha\rangle}{k} [(e^{\alpha})_{-r-\langle\alpha,\alpha\rangle-k}, (e^{\alpha})_{-t+k}],\end{aligned}$$

proving that $(e^{\alpha})_{-t-\langle\alpha,\alpha\rangle}(e^{\alpha})_{-r} \in \mathrm{span}(\mathcal{B}(2))$.

Before we continue, notice that if $(e^{\alpha})_u(e^{\alpha})_v$ is such that

$$(e^{\alpha})_u(e^{\alpha})_v \prec (e^{\alpha})_{-t-\langle\alpha,\alpha\rangle}(e^{\alpha})_{-r}$$

then for some $i > 0$ we have $u = -t - \langle \alpha, \alpha \rangle + i$ and $v = -r - i$. Now we will use this version of the $\prec$ ordering to inductively finish the proof.

Suppose for $i < l$ we have $(e^{\alpha})_{-t-\langle \alpha,\alpha \rangle+i}(e^{\alpha})_{-r-i} \in \operatorname{span}(\mathcal{B}(2))$, and, similarly to the base case, we can take a suitable relation and rewrite it to finish the argument.

$$
\begin{aligned}
(3.15)\qquad & R_{-}(-r-l,-t+l) \\
&= \sum_{k=0}^{-\langle \alpha,\alpha \rangle} (-1)^k \binom{-\langle \alpha, \alpha \rangle}{k} [(e^{\alpha})_{-r-l-\langle \alpha,\alpha \rangle-k}, (e^{\alpha})_{-t+l+k}] \\
&= \sum_{k=0}^{-\langle \alpha,\alpha \rangle-1} (-1)^k \binom{-\langle \alpha, \alpha \rangle}{k} [(e^{\alpha})_{-r-l-\langle \alpha,\alpha \rangle-k}, (e^{\alpha})_{-t+l+k}] \\
&\quad + (-1)^{-\langle \alpha,\alpha \rangle} [(e^{\alpha})_{-r-l}, (e^{\alpha})_{-t-\langle \alpha,\alpha \rangle+l}] = 0,
\end{aligned}
$$

which, recalling that $[\cdot,\cdot]$ is a supercommutator in this case, we can rewrite as

$$
\begin{aligned}
(3.16)\quad & (e^{\alpha})_{-t-\langle \alpha,\alpha \rangle+l}(e^{\alpha})_{-r-l} \\
&= (-1)^{\langle \alpha,\alpha \rangle}(e^{\alpha})_{-r-l}(e^{\alpha})_{-t-\langle \alpha,\alpha \rangle+l} \\
&\quad + \sum_{k=0}^{-\langle \alpha,\alpha \rangle-1} (-1)^{k+\langle \alpha,\alpha \rangle} \binom{-\langle \alpha, \alpha \rangle}{k} [(e^{\alpha})_{-r-l-\langle \alpha,\alpha \rangle-k}, (e^{\alpha})_{-t+l+k}] \\
&\in \operatorname{span}(\mathcal{B}(2)),
\end{aligned}
$$

finishing the proof. □

Now we are ready to show that we have a spanning set for each rank one subspace.

Theorem 3.5. *The set $\mathcal{B}^{(0)}$ spans W_L .*

Proof. For the special case $\langle \alpha, \alpha \rangle = 0$, we easily get $W_L \cong \mathbb{C}[x_{-1}, x_{-2}, \dots]$, where the variable x_{-i} corresponds to e^{α}_{-i}. In this case $\mathcal{B}^{(0)}$ is clearly a spanning set (and a basis) of W_L. The remaining cases can be handled simultaneously in light of Lemmas 3.3 and 3.4. Suppose $\mathcal{B}^{(0)}$ does not span W_L. Since $\prec$ is a total ordering of elements with the same charge and a partial ordering for all of W_L we choose $a \in \mathcal{B}^{(0)}$ to be the maximal element in the ordering $\prec$ such that $a \notin \operatorname{span}(\mathcal{B}^{(0)})$. Take

$$
(3.17)\qquad a = (e^{\alpha})_{m_l}(e^{\alpha})_{m_{l-1}} \dots (e^{\alpha})_{m_2}(e^{\alpha})_{m_1}\mathbf{1}.
$$

Find n with $1 \le n \le l$ such that

$$
n = \min\{s \mid m_s > m_{s-1} - \langle \alpha, \alpha \rangle\}.
$$

Use Lemmas 3.3 and 3.4 to write

$$
(3.18)\qquad (e^{\alpha})_{m_n}(e^{\alpha})_{m_{n-1}} = \sum c_{r,s}(e^{\alpha})_{-r}(e^{\alpha})_{-s},
$$

where the sum is taken with $s+r=m_n+m_{n-1}$ and $-r \leq -s-\langle\alpha,\alpha\rangle$. Let

$$b_{r,s}=(e^{\alpha})_{m_l}\cdots(e^{\alpha})_{m_{n+1}}(e^{\alpha})_{-r}(e^{\alpha})_{-s}(e^{\alpha})_{m_{n-2}}\cdots(e^{\alpha})_{m_1}\mathbf{1}. \tag{3.19}$$

So we have

$$a=\sum_{r>s} b_{r,s}. \tag{3.20}$$

Where r and s are in the sum as before. Notice we have $a \prec b_{r,s}$ for each pair (r,s), also notice that since $a \notin \mathrm{Span}(\mathcal{B}^{(0)})$ at least one $b_{r,s}\cdot\mathbf{1} \notin \mathcal{B}^{(0)}$, which contradicts the maximality of a. $\square$

Recall the linear isomorphisms $e_\lambda : V_{L^\circ} \to V_{L^\circ}$, where $\lambda \in L^\circ$ as in [12].

Corollary 3.6. *For $\langle\alpha,\alpha\rangle \neq 0$, the set $\mathcal{B}^{(i)}$ spans $W_{L+\frac{i\alpha}{\langle\alpha,\alpha\rangle}}$.*

Proof. The difference in $\mathcal{B}^{(i)}$ when $i \neq 0$ and when $i=0$ comes down to the "initial condition", the right most term of any monomial. As in [10], observe that the simple current operator

$$e_{\lambda_i} : W_L \to W_{L+\lambda_i}$$

is a linear isomorphism, thus sending bases (resp. spanning sets) to bases (resp. spanning sets). Now, we specialize $\lambda_i=\frac{i\alpha}{\langle\alpha,\alpha\rangle}$ and apply the formula (cf. [12])

$$e_{\lambda_i}(e^{\alpha})_m=c(\alpha,-\lambda_i)(e^{\alpha})_{m-i}e_{\lambda_i}. \tag{3.21}$$

Then, from the definition, it is easy to see that $e_{\lambda_i}(\mathcal{B}^{(0)})$ gives all nonzero multiples of $\mathcal{B}^{(i)}$. $\square$

Now we shall look at the linear independence of $\mathcal{B}^{(i)}$.

Theorem 3.7. *The set $\mathcal{B}^{(i)}$ is linearly independent.*

Proof. The idea of the proof is similar to the one used in [23]. From (3.21) we have

$$e_{\frac{i\alpha}{\langle\alpha,\alpha\rangle}}(\mathcal{B}^{(0)})=\mathcal{B}^{(i)},$$

which holds up to a nonzero scalar of elements, meaning that some elements in $\mathcal{B}^{(0)}$ are sent to nonzero multiples in $\mathcal{B}^{(i)}$. Thus it is sufficient to prove the statement for $i=0$. On the contrary, assume

$$\sum_{j=1}^{n}\lambda_j a_j=0 \tag{3.22}$$

with $a_j \in \mathcal{B}^{(0)}$ and $\lambda_j \in \mathbb{C}$ all nonzero, ordered so that $a_n \prec a_{n-1} \prec \cdots \prec a_1$. Without loss of generality, we may assume that all a_i are of the same degree and same charge. Pick an arbitrary monomial $a \in \mathcal{B}^{(0)}$. We consider three type of maps on W_L:

- $A=\mathcal{Y}_c(e^{\frac{\alpha}{\langle\alpha,\alpha\rangle}},x)$,
- $B=e_{-\alpha}$,

- $C = e_{\frac{-\alpha}{\langle\alpha,\alpha\rangle}}$.

Now we associate an endomorphism X_a to $a \in \mathcal{B}^{(0)}$ by the following procedure:

Step 0. Let $b = a$.

Step 1. If b admits factorization $b = b'e^{\alpha}$, where b' is a "shorter" monomial, we compute $B(b)$. If the resulting vector admits the same factorization we compute $B(B(b))$, etc. until no such factorization is possible.

Step 2. Apply the map $(C \circ A)^m$ on the vector computed in Step 1, where m is smallest possible such that that resulting vector can be again written as $b''e^{\alpha}$. This is always possible because $(C \circ A)(e^{\alpha})_m$ is a multiple of $(e^{\alpha})_{m+1}$. Let b be the resulting vector. Return to Step 1.

Observe that all intermediate monomials (denoted b) computed through this procedure always stay within the set $\mathcal{B}^{(0)}$ (up to scalar). Also, because the map $(C \circ A)$ reduces the overall degree, while B decreases the charge, this process will eventually halt when we reach a (nonzero) multiple of $\mathbf{1}$ of charge zero. The operator X_a is defined as composition of B's and $(C \circ A)$'s given by the procedure such that $X_a(a)$ is a nonzero multiple of $\mathbf{1}$. In other words, there are unique nonnegative integers $n_1, \ldots, n_k$ such that

$$X_a = \left[e_{-\alpha}(e_{\frac{-\alpha}{\langle\alpha,\alpha\rangle}}\mathcal{Y}_c(e^{\frac{\alpha}{\langle\alpha,\alpha\rangle}}, x)^{n_1}\right] \circ \cdots \circ \left[e_{-\alpha}(e_{\frac{-\alpha}{\langle\alpha,\alpha\rangle}}\mathcal{Y}_c(e^{\frac{\alpha}{\langle\alpha,\alpha\rangle}}, x)^{n_k}\right]. \tag{3.23}$$

Clearly, $X_a = 1$ if and only if $a = \mathbf{1}$.

Claim: Let $a \prec b$, where $a, b \in \mathcal{B}^{(0)}$ then, $X_a(b) = 0$.

The claim follows immediately from

$$B(a) \prec B(b) \quad \text{and} \quad (C \circ A)(a) \prec (C \circ A)(b),$$

which implies $\mathbf{1} \prec X_a(b)$.

Now we invoke the claim, and apply the operator X_{a_1} on (3.22) and get

$$\lambda_1 = 0, \tag{3.24}$$

contradicting the assumption $\lambda_1 \neq 0$. □

Corollary 3.8. *The set $\mathcal{B}^{(0)}$ is a basis of W_L. In addition, if $\langle\alpha,\alpha\rangle \neq 0$, then $\mathcal{B}^{(i)}$ is a basis of $W_{L+\frac{i\alpha}{\langle\alpha,\alpha\rangle}}$.*

4. Higher rank subspaces

Now we switch to the general case as in Section 2. The lattice L is of rank n. Similar to the relations for each individual particles of the same color, we have two sets of relations that connect particles. For $\langle\alpha_i,\alpha_j\rangle \geq k > 0$ we have

$$A_+(k,x,i,j) = Y((e^{\alpha_i})_{-k}e^{\alpha_j},x) = \frac{1}{(k-1)!}\left(\left(\frac{d}{dx}\right)^{(k-1)} Y(e^{\alpha_i},x)\right)Y(e^{\alpha_j},x) = 0, \tag{4.1}$$

while for $\langle\alpha_i,\alpha_j\rangle \leq 0$ we have

$$A_-(i,j,x_1,x_2) = (x_1-x_2)^{-\langle\alpha_i,\alpha_j\rangle}[Y(e^{\alpha_i},x_1),Y(e^{\alpha_j},x_2)] = 0. \tag{4.2}$$

After taking appropriate coefficients in $A_+(k,x,i,j)$ and $A_-(i,j,x_1,x_2)$ we are left with the following sets of relations:

$$R_+^{(0)}(i,j,k,n) = \sum_{m\leq -k}\frac{(m+k-1)!}{(m)!}(e^{\alpha_i})_{-k-m}(e^{\alpha_j})_{n+m} + \sum_{m\geq 0}\frac{(m+k-1)!}{m!}(e^{\alpha_i})_{-k-m}(e^{\alpha_j})_{n+m} \tag{4.3}$$

and

$$R_-^{(0)}(i,j,m,n) = \sum_{k=0}^{-\langle\alpha_i,\alpha_j\rangle}(-1)^k\binom{-\langle\alpha_i,\alpha_j\rangle}{k}[(e^{\alpha_i})_{m-k},(e^{\alpha_j})_{n+k}], \tag{4.4}$$

respectively.

Definition 4.1. We say an element $(e^{\alpha_i})_k \in \mathrm{End}(W_L)$ has color i. The colors are ordered from smallest to largest as $1 < 2 < \cdots < n$.

Definition 4.2. We say a monomial is written in decreasing color order (from the left) if it is written in the form,

$$(e^{\alpha_{m_k}})_{i_k}(e^{\alpha_{m_{k-1}}})_{i_{k-1}}\dots(e^{\alpha_{m_2}})_{i_2}(e^{\alpha_{m_1}})_{i_1}\mathbf{1}, \tag{4.5}$$

with $m_k \geq m_{k-1} \geq \cdots \geq m_2 \geq m_1$.

We shall employ the following notation,

$$\varepsilon^{\alpha_i}_{(m_k,\dots,m_1)} = (e^{\alpha_i})_{m_k}\dots(e^{\alpha_i})_{m_1},$$

where for simplicity we shall often write $\mu_i = (m_k,\dots,m_1)$. The following technical result will be used to write monomials in decreasing color order.

Proposition 4.3. *For fixed $m,n \in \mathbb{Z}$, every $k \in \mathbb{N}$, and $1 \leq i \leq j \leq \mathrm{rank}(L)$, there are integers m_l,n_l,r_l,s_l, and numbers c_l, d_l so that we can write*

$$(e^{\alpha_i})_m(e^{\alpha_j})_n = \sum_{l=0}^{-\langle\alpha_i,\alpha_j\rangle} c_l(e^{\alpha_j})_{m_l}(e^{\alpha_i})_{n_l} + \sum_{l=1}^{-\langle\alpha_i,\alpha_j\rangle} d_l(e^{\alpha_i})_{r_l}(e^{\alpha_j})_{s_l}, \tag{4.6}$$

with $s_l \geq n+k$.

Proof. The focus of this proof will be for the case when $\langle \alpha_i, \alpha_j \rangle < 0$. The reason is that when $\langle \alpha_i, \alpha_j \rangle \geq 0$ we have

$$[(e^{\alpha_i})_m, (e^{\alpha_j})_m] = 0.$$

By using the relation

$$\begin{aligned} (4.7) \quad & R_-(i, j, m + \langle \alpha_i, \alpha_j \rangle, n) \\ &= \sum_{l=0}^{-\langle \alpha_i, \alpha_j \rangle} (-1)^l \binom{-\langle \alpha_i, \alpha_j \rangle}{l} [(e^{\alpha_i})_{m-l}, (e^{\alpha_j})_{n+l}] = 0 \end{aligned}$$

we can write

$$\begin{aligned} (e^{\alpha_i})_m (e^{\alpha_j})_n &= (-1)^{\langle \alpha_i, \alpha_j \rangle} (e^{\alpha_j})_n (e^{\alpha_i})_m \\ &- \sum_{l=1}^{-\langle \alpha_i, \alpha_j \rangle} (-1)^l \binom{-\langle \alpha_i, \alpha_j \rangle}{l} [(e^{\alpha_i})_{m-l}, (e^{\alpha_j})_{n+l}], \end{aligned}$$

since we are in the setting of a vertex superalgebra. If we use this identity k times, each time rewriting the term $(e^{\alpha_i})_{m-l}(e^{\alpha_j})_{n+l}$ where l is smallest, we have

$$\begin{aligned} (e^{\alpha_i})_m (e^{\alpha_j})_n &= (-1)^{\langle \alpha_i, \alpha_j \rangle} (e^{\alpha_j})_n (e^{\alpha_i})_m \\ &- \sum_{l=k}^{-\langle \alpha_i, \alpha_j \rangle + k - 1} (-1)^l R_l [(e^{\alpha_i})_{m-l}, (e^{\alpha_j})_{n+l}], \end{aligned}$$

where

$$R_l = \sum_{l'=0}^{k} (-1)^{l'} \binom{-\langle \alpha_i, \alpha_j \rangle}{l - l'}.$$

Reindexing the sum allows us to write

$$(e^{\alpha_i})_m (e^{\alpha_j})_n = \sum_{l=0}^{-\langle \alpha_i, \alpha_j \rangle} c_l (e^{\alpha_j})_{m_l} (e^{\alpha_i})_{n_l} + \sum_{l=1}^{-\langle \alpha_i, \alpha_j \rangle} d_l (e^{\alpha_i})_{r_l} (e^{\alpha_j})_{s_l},$$

with $s_l \geq n + k$, for all $1 \leq l \leq -\langle \alpha_i, \alpha_j \rangle$. □

Proposition 4.4. *Every monomial $w \in W_L$ can be written as a linear combination of monomials of decreasing color.*

Proof. Let

$$w = (e^{\alpha_{n_m}})_{i_m} \dots (e^{\alpha_{n_1}})_{i_1} v$$

be an arbitrary monomials where v is in decreasing color order and $n_2 < n_1$ (so the color condition is invalid). Pick k as in Proposition 4.3 so that

$$(e^{\alpha_{n_1}})_{k'} v = 0$$

for all $k' \geq i_1 + k$, and write

$$(e^{\alpha_{n_2}})_{i_2}(e^{\alpha_{n_1}})_{i_1} = \sum_l c_l (e^{\alpha_{n_1}})_{r_l}(e^{\alpha_{n_2}})_{s_l} + \sum_l d_l (e^{\alpha_{n_2}})_{t_l}(e^{\alpha_{n_1}})_{u_l},$$

where $u_l \geq i_1 + k$. So we have,

$$w = \sum_l (e^{\alpha_{n_m}})_{i_m} \dots (e^{\alpha_{n_3}})_{i_3}(e^{\alpha_{n_1}})_{r_l}(e^{\alpha_{n_2}})_{s_l} v.$$

Finitely many repetitions of the above calculation (using Proposition 4.3) will result in w written as a linear combination of monomials in decreasing color order. □

In light of the bases we found for rank one subspaces of W_L (Corollary 3.8) in conjunction with the color ordering given by Proposition 4.4 the following set spans W_L:

$$\mathcal{B}_0^{(0)} = \left\{ \varepsilon_{\mu_n}^{\alpha_n} \dots \varepsilon_{\mu_1}^{\alpha_1} \mathbf{1} \;\middle|\; m_{j+1}^i \leq m_j^i - \langle \alpha_i, \alpha_i \rangle \text{ for } 1 \leq j \leq k_i - 1, \right.$$
$$\left. \text{with } 1 \leq j \leq n \right\}.$$

But this set is too large to be a basis as it does not take into account the transition between the particles. We will now present results that will allow us to add a "transition condition" to the elements of $\mathcal{B}_0^{(0)}$.

Proposition 4.5. *For any $\beta_1, \beta_2 \in L$, we have*

$$E^+(\beta_1, x_1)E^-(\beta_2, x_2) = \left(1 - \frac{x_2}{x_1}\right)^{\langle \beta_1, \beta_2 \rangle} E^-(\beta_1, x_2)E^+(\beta_1, x_1) \in (\operatorname{End} W_L)[[x_1^{-1}, x_2]], \tag{4.8}$$

where $E^{\pm}(\cdot, x)$ are as in [27].

This proposition together with the definition of the vertex operator

$$Y(e^\beta, x) = E^-(-\beta, x)E^+(-\beta, x)e_\beta x^\beta,$$

and the action $x^{\beta_1} e_{\beta_2} = x^{\langle \beta_1, \beta_2 \rangle} e_{\beta_2} x^{\beta_1}$ gives us the following result.

Proposition 4.6. *Let $\beta_1, \dots, \beta_k \in L$. Then*

$$Y(e^{\beta_1}, x_1) \dots Y(e^{\beta_k}, x_k) = c \prod_{i<j} (x_i - x_j)^{\langle \beta_i, \beta_j \rangle} E^-(\beta_1, x_1) \dots E^-(\beta_k, x_k)$$
$$\cdot E^+(\beta_1, x_1) \dots E^+(\beta_k, x_k) e_{\sum_i \beta_i},$$

where $c = \prod_{r=1}^k \epsilon(\beta_{k-r}, \sum_{s=0}^{r-1} \beta_{k-s})$ is the contribution from the 2-cocyle associated to the central extension (cf. also [29]*).*

After applying this proposition to the vacuum we have:

Lemma 4.7.

$$Y(e^{\beta_1},x_1)\dots Y(e^{\beta_k},x_k)\mathbf{1} = c\prod_{i<j}(x_i-x_j)^{\langle\beta_i,\beta_j\rangle}E^-(\beta_1,x_1)\dots E^-(\beta_k,x_k)e^{\sum_i\beta_i}. \tag{4.9}$$

If we write

$$(x_i-x_j)^{\langle\beta_i,\beta_j\rangle} = x_i^{\langle\beta_i,\beta_j\rangle}\left(1-\frac{x_j}{x_i}\right)^{\langle\beta_i,\beta_j\rangle}, \tag{4.10}$$

then we have the following relation

$$\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)^{-\langle\beta_i,\beta_j\rangle} Y(e^{\beta_1},x_1)\dots Y(e^{\beta_k},x_k)\mathbf{1} \in \prod_{i=1}^{k} x_i^{\sum_{j=1}^{k}\langle\beta_i,\beta_j\rangle}(W_L)[[x_1,\dots,x_k]]. \tag{4.11}$$

This relation will be used to add a "transition condition" to the set $\mathcal{B}_0^{(0)}$, which we will do after defining a few necessary tools.

By Proposition 4.4 we only need to need to consider monomials of the form $w=\varepsilon^{\alpha_n}_{\mu_n}\dots\varepsilon^{\alpha_1}_{\mu_1}\mathbf{1}$ where $\mu_i=(m^i_{k_i},\dots,m^i_1)$. The grading on this monomial is as follows: charge, total charge, and weight respecitively

$$\mathrm{ch}(w)=(k_1,k_2,\dots,k_n) \tag{4.12}$$

$$\mathrm{Ch}(w)=k_1+k_2+\dots+k_n \tag{4.13}$$

$$\mathrm{wt}(w)=-\left(m^n_{k_n}+\dots+m^1_1\right)+\sum_{l=1}^{n}k_l\left(\frac{\langle\alpha_l,\alpha_l\rangle}{2}-1\right). \tag{4.14}$$

Consider the following subset of W_L

$$\mathcal{B}^{(0)}=\left\{\varepsilon^{\alpha_n}_{\mu_n}\dots\varepsilon^{\alpha_1}_{\mu_1}\mathbf{1}\;\middle|\; m^i_{j+1}\le m^i_j-\langle\alpha_i,\alpha_i\rangle \text{ for } 1\le j\le k_i-1, \text{ with } 1\le j\le n \text{ and } m^i_1\le -1-\sum_{l=1}^{i-1}k_l\langle\alpha_i,\alpha_l\rangle\right\}. \tag{4.15}$$

Now we begin proving this is a basis of W_L.

Theorem 4.8. *The set $\mathcal{B}^{(0)}$ spans W_L.*

Proof. Towards a contradiction we will assume $\mathcal{B}^{(0)}$ does not span W_L. Pick a monomial $w\notin\mathrm{span}(\mathcal{B}^{(0)})$, such that for all monomials v with $\mathrm{Ch}(v)<\mathrm{Ch}(w)$ we have $v\in\mathrm{span}(\mathcal{B}^{(0)})$, in addition, if $\mathrm{Ch}(v)=\mathrm{Ch}(w)$ and $w\prec v$ then $v\in\mathrm{span}(\mathcal{B}^{(0)})$. To summarize, monomial w has the smallest possible total charge such that $w\notin\mathrm{span}(\mathcal{B}^{(0)})$ and within total charge w is maximum in the $\prec$ ordering such that $w\notin\mathrm{span}(\mathcal{B}^{(0)})$.

As $w\notin\mathcal{B}^{(0)}$ it must either violate one or both of:

- the transition condition between colors, or
- the difference $\langle\alpha_i,\alpha_i\rangle$ condition within the i^{th} color, for some i.

This gives us two possibilities. Reading the monomial from the vacuum to the left, identify which violation occurs first.

Case 1. There is a transition condition violation first. We will write

$$w=\varepsilon^{\alpha_n}_{\mu_n}\dots\varepsilon^{\alpha_1}_{\mu_1}\mathbf{1}\in\mathcal{B}^{(0)}_0. \tag{4.16}$$

For $\beta_i,\beta_j^{'}\in\{\alpha_1,\dots,\alpha_n\}$, we can write

$$\begin{aligned} w=\varepsilon^{\alpha_n}_{\mu_n}\dots\varepsilon^{\alpha_1}_{\mu_1}\mathbf{1} =&(e^{\beta_1^{'}})_{n_1}\dots(e^{\beta_l^{'}})_{n_l}(e^{\beta_1})_{m_1}\dots(e^{\beta_k})_{m_k}\mathbf{1}\\ =&(e^{\beta_1^{'}})_{n_1}\dots(e^{\beta_l^{'}})_{n_l}w^{'}, \end{aligned}\tag{4.17}$$

where

$$w^{'}=(e^{\beta_1})_{m_1}\dots(e^{\beta_k})_{m_k}\mathbf{1} \tag{4.18}$$

and the transition condition between e^{β_1} and e^{β_2} is not satisfied. So we have

$$m_1>-1-\sum_{j=1}^{k}\langle\beta_1,\beta_j\rangle\,. \tag{4.19}$$

Now we can make use of (4.11) . We write

$$\mathcal{A}(x_1,\dots,x_k)=\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)^{-\langle\beta_i,\beta_j\rangle}Y(e^{\beta_1},x_1)\dots Y(e^{\beta_k},x_k)\mathbf{1} \tag{4.20}$$

and

$$(W_L)^{\beta_1,\dots,\beta_k}=\prod x_i^{\sum_{j=1}^{k}\langle\beta_i,\beta_j\rangle}(W_L)[[x_1,\dots,x_k]]. \tag{4.21}$$

So that (4.11) becomes

$$\mathcal{A}(x_1,\dots,x_k)\in(W_L)^{\beta_1,\dots,\beta_k}, \tag{4.22}$$

which will be easier to handle. Notice that (4.22) implies that

$$-m_1-1<\sum_{j=2}^{k}\langle\beta_1,\beta_j\rangle\,, \tag{4.23}$$

and since the elements of $(W_L)^{\beta_1,\dots,\beta_k}$ involve x_1^r, where $r\geq\sum_{j=2}^{k}\langle\beta_1,\beta_j\rangle$ we have

$$\begin{aligned} 0=&\operatorname{Coeff}_{x_1^{-m_1-1}\dots x_k^{-m_k-1}}\mathcal{A}(x_1,\dots,x_k)\\ &=(e^{\beta_1})_{m_1}\dots(e^{\beta_k})_{m_k}\mathbf{1}+\text{more terms}, \end{aligned}\tag{4.24}$$

where these extra terms come from the expansion of

$$\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)^{-\langle\beta_i,\beta_j\rangle}\in 1+x_1^{-1}\mathbb{C}[[x_1^{-1},x_2^{\pm1},\dots,x_k^{\pm1}]]. \tag{4.25}$$

So we have

$$\text{(4.26)}\quad \operatorname{Coeff}_{x_1^{-m_1-1}\ldots x_k^{-m_k-1}} \mathcal{A}(x_1,\ldots,x_k) = \sum_{\mathbf{L}} c_{\mathbf{L}} (e^{\beta_1})_{m_1+l^{(1)}} \ldots (e^{\beta_k})_{m_k+l^{(k)}} \mathbf{1},$$

where $\mathbf{L} = (l^{(1)},\ldots,l^{(k)})$ and

$$\text{(4.27)}\quad \begin{aligned} l^{(k)} &= l_{(1,k)} + \cdots + l_{(k-1,k)} \\ l^{(k-1)} &= l_{(1,k-1)} + \cdots + l_{(k-2,k-1)} - l_{(k-1,k)} \\ l^{(k-2)} &= l_{(1,k-2)} + \cdots + l_{(k-3,k-2)} - l_{(k-2,k-1)} - l_{(k-2,k)} \\ &\vdots \\ l^{(2)} &= l_{(1,2)} - l_{(2,3)} - \cdots - l_{(2,k)} \\ l^{(1)} &= -l_{(1,2)} - \cdots - l_{(1,k)} \end{aligned}$$

with $l_{(i,j)} \geq 0$ the exponent of $\frac{x_j}{x_i}$ in the expansion of $\prod_{i<j}\left(1-\frac{x_j}{x_i}\right)^{-\langle\beta_i,\beta_j\rangle}$. We will now argue that the sum in (4.26) is finite. By the truncation condition there are finitely many $l^{(k)}$, in fact we know $l^{(k)} < m_k$, such that

$$(e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1} \neq 0.$$

So we have finitely many possible values of each $l_{(i,k)}$ for $1 \leq i \leq k-1$. Fix each of these $l_{(i,k)}$ and again by the truncation condition there are finitely many $l_{(i,k-1)}$ so that

$$(e^{\beta_{k-1}})_{m_{k-1}+l^{(k-1)}}(e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1} \neq 0.$$

Continue this argument moving left away from the vacuum. In the last step we fix $l_{(i,j)}$ for $2 \leq i < j \leq k$ and there are finitely many $l_{(1,2)}$ such that

$$(e^{\beta_2})_{m_2+l^{(2)}} \ldots (e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1} \neq 0.$$

So we have shown there are finitely many $l_{(i,j)}$ such that

$$(e^{\beta_1})_{m_1+l^{(1)}} \ldots (e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1} \neq 0.$$

Thus the sum (4.26) is finite. We can now use this to write

$$(e^{\beta_1})_{m_1} \ldots (e^{\beta_k})_{m_k}\mathbf{1} = \sum_{\mathbf{L}\neq\mathbf{0}} c_{\mathbf{L}} (e^{\beta_1})_{m_1+l^{(1)}} \ldots (e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1}.$$

By the structure of the $l^{(r)}$ observed in 4.27 we see that

$$(e^{\beta_1})_{m_1} \ldots (e^{\beta_k})_{m_k}\mathbf{1} \prec (e^{\beta_1})_{m_1+l^{(1)}} \ldots (e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1}$$

for all $\mathbf{L} \neq \mathbf{0}$. So we have

$$w = -\sum_{\mathbf{L}\neq\mathbf{0}} c_{\mathbf{L}} (e^{\beta_1'})_{n_1} \ldots (e^{\beta_l'})_{n_l} (e^{\beta_1})_{m_1+l^{(1)}} \ldots (e^{\beta_k})_{m_k+l^{(k)}}\mathbf{1}.$$

Notice we have written w as a sum of terms that are higher in the $\prec$ ordering, contradicting the maximality of w.

Case 2. There is a violation of the difference condition within the i^{th} color. For $\beta_j \in \{\alpha_1, \dots, \alpha_n\}$, write

$$w = (e^{\beta_1})_{i_1} \dots (e^{\beta_k})_{i_k} (e^{\alpha_i})_l (e^{\alpha_i})_m w_0, \tag{4.28}$$

where $l > m - \langle \alpha_i, \alpha_i \rangle$. Use the results of Lemmas 3.3 and 3.4 to write

$$w = \sum_{\substack{r+t=l+m \\ r-t \geq \langle \alpha_i, \alpha_i \rangle}} c_{r,t} (e^{\beta_1})_{i_1} \dots (e^{\beta_k})_{i_k} (e^{\alpha_i})_r (e^{\alpha_i})_t w_0, \tag{4.29}$$

which is higher in the ordering than w and thus contradicts its maximality, and finishes the proof. $\square$

Now we will look at the basis for the modules $W_{L+\omega_r} = W_L \cdot e^{\omega_r}$ where ω_r is an element from the dual basis so that $\langle \alpha_i, \omega_r \rangle = \delta_{i,r}$. Similarly to when we looked at W_L we first consider a spanning set with no "transition conditions" between the particles of different color.

$$\mathcal{B}_0^{(r)} = \left\{ \varepsilon_{\mu_n}^{\alpha_n} \dots \varepsilon_{\mu_1}^{\alpha_1} e^{\omega_r} \;\middle|\; \begin{array}{l} m_{j+1}^i \leq m_j^i - \langle \alpha_i, \alpha_i \rangle \text{ for } 1 \leq j \leq k_i - 1, \\ \hfill \text{with } 1 \leq j \leq n \end{array} \right\}. \tag{4.30}$$

We will now gather the tools necessary to apply the transition condition. Instead of applying Proposition 4.6 to the vacuum as we did before we will now apply it to $e^{\omega_{i_r}}$ where i_r is such that $\beta_r = \alpha_{i_r}$. So we have:

Lemma 4.9.

$$\begin{aligned} &Y(e^{\beta_1}, x_1) \dots Y(e^{\beta_k}, x_k) e^{\omega_{i_r}} \\ &\qquad = c_r x_r \prod_{i<j} (x_i - x_j)^{\langle \beta_i, \beta_j \rangle} E^-(\beta_1, x_1) \dots E^-(\beta_k, x_k) e_{\sum_i \beta_i}, \end{aligned} \tag{4.31}$$

for some constant c_r.

This allows us to create the following relation:

$$\begin{aligned} &\prod_{i<j} \left(1 - \frac{x_j}{x_i}\right)^{-\langle \beta_i, \beta_j \rangle} Y(e^{\beta_1}, x_1) \dots Y(e^{\beta_k}, x_k) \mathbf{1} \\ &\qquad \in \prod_{i=1}^{k} x_i^{\delta_{i,r} + \sum_{j=1}^{k} \langle \beta_i, \beta_j \rangle} (W_L)[[x_1, \dots, x_k]]. \end{aligned} \tag{4.32}$$

Consider the set

$$\mathcal{B}^{(r)} = \left\{ \varepsilon_{\mu_n}^{\alpha_n} \dots \varepsilon_{\mu_1}^{\alpha_1} e^{\omega_r} \;\middle|\; m_{j+1}^i \leq m_j^i - \langle \alpha_i, \alpha_i \rangle \text{ for } 1 \leq j \leq k_i - 1, \right.$$
$$\left. \text{with } 1 \leq j \leq n \text{ and } m_1^i \leq -1 - \delta_{r,i} - \sum_{l=1}^{i-1} k_l \langle \alpha_i, \alpha_l \rangle \right\}. \tag{4.33}$$

Following the same reasoning as in the case of W_L and $\mathcal{B}^{(0)}$ we have the following.

Theorem 4.10. *The set $\mathcal{B}^{(r)}$ spans $W_{L+\omega_r}$.*

Now we show that $\mathcal{B}^{(0)}$ is a linearly independent set. Recall the $\prec$ ordering from Definition 3.1.1. The following reformulation is useful in our proofs.

Definition 4.11. Let

$$v_1 = \varepsilon^{\alpha_n}_{\mu_n^{(1)}} \varepsilon^{\alpha_{n-1}}_{\mu_{n-1}^{(1)}} \dots \varepsilon^{\alpha_2}_{\mu_1^{(1)}} \varepsilon^{\alpha_1}_{\mu_1^{(1)}} \mathbf{1}, \quad v_2 = \varepsilon^{\alpha_n}_{\mu_n^{(2)}} \varepsilon^{\alpha_{n-1}}_{\mu_{n-1}^{(2)}} \dots \varepsilon^{\alpha_2}_{\mu_1^{(2)}} \varepsilon^{\alpha_1}_{\mu_1^{(2)}} \mathbf{1}.$$

We say $v_1 \prec v_2$ if $\varepsilon^{\alpha_n}_{\mu_n^{(1)}} \prec \varepsilon^{\alpha_n}_{\mu_n^{(2)}}$ or if $\varepsilon^{\alpha_j}_{\mu_j^{(1)}} = \varepsilon^{\alpha_j}_{\mu_j^{(2)}}$ for $l+1 \leq j \leq n$ then $\varepsilon^{\alpha_l}_{\mu_l^{(1)}} \prec \varepsilon^{\alpha_l}_{\mu_l^{(2)}}$.

Let v_1 be as in the definition (a product of $\varepsilon^{\alpha_i}_{\mu_k^{(j)}}$). In the proof of Theorem 3.7 the definition of operator X_a given in (3.23). Now we generalize this concept for arbitrarily many colors. On the same token, it is not hard to show that there is a unique choice of constants $m_i^{(j)} \geq 0$, such that the operator $X_{v_1} : W_L \to W_L$ defined as

$$X_{v_1} = \left[e_{-\alpha_n} \left(e_{-\omega_n} \mathcal{Y}_c \left(e^{\omega_n}, x \right) \right)^{m_1^{(n)}} \right] \dots \left[e_{-\alpha_n} \left(e_{-\omega_n} \mathcal{Y}_c \left(e^{\omega_n}, x \right) \right)^{m_{s_n}^{(n)}} \right] \dots$$
$$\dots \left[e_{-\alpha_1} \left(e_{-\omega_1} \mathcal{Y}_c \left(e^{\omega_1}, x \right) \right)^{m_1^{(1)}} \right] \dots \left[e_{-\alpha_1} \left(e_{-\omega_1} \mathcal{Y}_c \left(e^{\omega_1}, x \right) \right)^{m_{s_1}^{(1)}} \right]$$

satisfies

$$X_{v_1}(v_1) = \lambda \mathbf{1},$$

where λ is a nonzero constant. As in Theorem 3.7 (see the claim) we can easily show

Proposition 4.12. *For $v_1, v_2, w \in \mathcal{B}^{(0)}$ with $v_1 \prec v_2$, we have $X_{v_2}(v_1) = 0$.*

Theorem 4.13. *For $0 \leq r \leq n$, the set $\mathcal{B}^{(r)}$ is linearly independent.*

Proof. Again, as in the proof of Theorem 3.7 it is sufficient to prove the assertion for $r = 0$. Suppose that the set is linearly dependent. Then we could find a relation

$$\sum_{j=1}^{m} \lambda_j v_j = 0, \tag{4.34}$$

where $\lambda_j \in \mathbb{C}$ are nonero and $v_j \in \mathcal{B}^{(0)}$ are ordered so that $v_1 \prec v_2 \prec \cdots \prec v_m$. By the previous proposition we have

$$0 = X_{v_m}\left(\sum_{j=1}^{m} \lambda_j v_j\right) = \sum_{j=1}^{m} \lambda_j X_{v_m}(v_j) = \nu\lambda_m \mathbf{1}, \tag{4.35}$$

contradicting $\nu\lambda_m \neq 0$. So we are done. □

Corollary 4.14. *The set $\mathcal{B}^{(r)}$ is a basis of $W_{L+\omega_r}$.*

5. Graded dimensions

From Part I [29], let us recall the natural multi-grading of W_L. We have an n component charge grading, by the eigenvalue of each $\omega_i(0)$, along with the standard weight grading, by the eigenvalue of $L(0)$. Recall for $v = \varepsilon^{\alpha_n}_{\mu_n} \dots \varepsilon^{\alpha_1}_{\mu_1} \cdot \mathbf{1}$, where $\mu_i = \big(m^{(i)}_{k_i}, \dots, m^{(i)}_1\big)$, we have

$$\mathrm{ch}(v) = (k_1, \dots, k_n)$$

as well as

$$\mathrm{wt}(v) = -\big(m^{(n)}_{k_n} + \cdots + m^{(1)}_1\big) + \sum_{l=1}^{n} k_l \left(\frac{\langle \alpha_l, \alpha_l\rangle}{2} - 1\right).$$

We stress that $L(0)$ is the standard Virasoro generator as in [27].

Definition 5.1. Define the $(r, k_1, \dots, k_n)$ subspace by

$$W_L^{(r,k_1,\dots,k_n)} = \big\{v \in W_L \mid \mathrm{wt}(v) = r, \quad \mathrm{ch}(v) = (k_1, \dots, k_n)\big\}. \tag{5.1}$$

Definition 5.2. Define the graded dimension of W_L by

$$\chi_{W_L}(\mathbf{x}, q) = \sum_{(k_1,\dots,k_n)\in\mathbb{N}^n, r\in\mathbb{Z}} \dim(W_L^{(r,k_1,\dots,k_n)}) q^r x_1^{k_1} \dots x_n^{k_n}. \tag{5.2}$$

Theorem 5.3. *Suppose $A = [A_{i,j}]$ is an $n \times n$ matrix with $A_{i,j} = \langle \alpha_i, \alpha_j\rangle$, and $\mathbf{k} = (k_1, k_2, \dots .k_n)$, then*

$$\chi_{W_L}(\mathbf{x}, q) = \sum_{(k_1,\dots,k_n)\in\mathbb{N}^n} \frac{q^{\frac{\mathbf{k}\cdot A\cdot \mathbf{k}^T}{2}}}{(q)_{k_1} \dots (q)_{k_n}} x_1^{k_1} \dots x_n^{k_n}. \tag{5.3}$$

Proof. Since for $v = \varepsilon^{\alpha_n}_{\mu_n} \dots \varepsilon^{\alpha_1}_{\mu_1} \cdot \mathbf{1}$, the value of $\mathrm{wt}(v)$ depends essentially on $\sum_{i=1}^{n}\big(m^{(i)}_1 + \cdots + m^{(i)}_{k_i}\big)$, we can relate the graded dimension of the subspace of charge $(k_1, \dots, k_n)$ to the generating function of colored partitions into at most $(k_1, \dots, k_n)$ (colored) parts, which is well known [3] to be

$$\frac{1}{(q)_{k_1} \dots (q)_{k_n}}.$$

Therefore, after inspecting the basis $\mathcal{B}^{(0)}$, we see that the graded dimension takes form

$$\chi_{W_L}(\mathbf{x},q) = \sum_{(k_1,\dots,k_n)\in\mathbb{N}^n} \frac{f_{(k_1,\dots,k_n)}(q)}{(q)_{k_1}\dots(q)_{k_n}} x_1^{k_1}\dots x_n^{k_n},$$

where $f_{(k_1,\dots,k_n)}(q) = q^{\mathrm{wt}(v)}$, and v is a (unique) element of this charge of the smallest possible weight. It should be also clear that this vector must be a multiple of $e^{k_1\alpha_1+\cdots+k_n\alpha_n}$. We now compute the weight of v. We have

$$m_j^{(l)} = \left(-1 - \sum_{i=1}^{l-1} k_i \langle\alpha_i,\alpha_l\rangle\right) - (j-1)\langle\alpha_l,\alpha_l\rangle, \tag{5.4}$$

for $1 \le l \le n$ and $1 \le j \le k_l$. So we have

$$\begin{aligned}\mathrm{wt}(v) &= -(m_{k_n}^{(n)} + \cdots + m_1^{(1)}) + \sum_{l=1}^{n} k_l \left(\frac{\langle\alpha_l,\alpha_l\rangle}{2} - 1\right) \\ &= \sum_{l=1}^{n}\left(k_l\left(1 + \sum_{i=1}^{l-1} k_i\langle\alpha_i,\alpha_l\rangle\right) - \frac{(k_l-1)k_l}{2}\langle\alpha_l,\alpha_l\rangle\right) \\ &\quad + \sum_{l=1}^{n} k_l\left(\frac{\langle\alpha_l,\alpha_l\rangle}{2} - 1\right).\end{aligned} \tag{5.5}$$

This simplifies to

$$\mathrm{wt}(v) = \sum_{l=1}^{n}\left(\frac{k_l^2\langle\alpha_l,\alpha_l\rangle}{2} + k_l\sum_{i=1}^{l-1} k_i\langle\alpha_i,\alpha_l\rangle\right). \tag{5.6}$$

By comparing coefficients of an arbitrary k_rk_s we see that

$$\sum_{l=1}^{n} k_l \sum_{i=1}^{n} k_i\langle\alpha_i,\alpha_l\rangle = \sum_{l=1}^{n}\left(k_l^2\langle\alpha_l,\alpha_l\rangle + 2k_l\sum_{i=1}^{l-1} k_i\langle\alpha_i,\alpha_l\rangle\right). \tag{5.7}$$

It is easy to see

$$\mathbf{k}A\mathbf{k}^T = \sum_{l=1}^{n} k_l \sum_{i=1}^{n} k_i\langle\alpha_i,\alpha_l\rangle. \tag{5.8}$$

So we have

$$\mathrm{wt}(v) = \frac{\mathbf{k}A\mathbf{k}^T}{2}. \tag{5.9}$$

Alternatively, we can reach the same conclusion by using standard vertex algebra techniques. Recall that the weight we introduced agrees with the one used in [27], given as the eigenvalue of the (standard) Virasoro operator $L(0)$. In this setup [27]

$$\mathrm{wt}(e^\beta) = \frac{1}{2}\langle\beta,\beta\rangle.$$

If we now substitute $k_1\alpha_1 + \cdots + k_n\alpha_n$ for β we immediately get again (5.9). $\square$

We now extend the notion of graded dimension to W_L-modules $W_{L+\gamma}$, $\gamma \in L^\circ$, although our definition is slightly different from the one used in the literature [27]. For $v = \varepsilon_{\mu_n}^{\alpha_n} \ldots \varepsilon_{\mu_1}^{\alpha_1} \cdot e^\gamma$, we define $\mathrm{ch}(v)$ and $\mathrm{wt}(v)$ as the above. Then we define $\chi'_{W_{L+\gamma}}(\mathbf{x}, q)$ as in [12], [29] (this is not exactly $\chi_{W_{L+\gamma}}(\mathbf{x}, q)$ because we ignore charge/weight contribution coming from e^β, which explains χ'). Then we easily get:

Corollary 5.4.

$$\chi'_{W_{L+\omega_r}}(\mathbf{x}, q) = \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \frac{q^{\frac{\mathbf{k}A\mathbf{k}^T}{2}+k_r}}{(q)_{k_1}\cdots(q)_{k_n}} x_1^{k_1}\ldots x_n^{k_n}. \tag{5.10}$$

Proof of Theorem 1.1. Let $A = (A_{i,j})$ and $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ with $\langle\alpha_i, \alpha_j\rangle = A_{i,j}$. For $B = (b_1, \ldots, b_n)$, let $e^\beta = e^{b_1\omega_1+\cdots+b_n\omega_n}$. Then we have

$$\chi'_{W_{L+\beta}}(\mathbf{x}, q)|_{\mathbf{x}=(1,\ldots,1)} = f_{A,B}(q). \qquad \square$$

Remark 5.5. All the discussion so far was in the setup of vertex algebras without a conformal vector. At first, it seems perfectly reasonable to consider the larger vertex algebra

$$W_L(\omega) := \langle\omega, e^{\alpha_1}, \ldots, e^{\alpha_n}\rangle,$$

where ω is a conformal vector (e.g., the quadratic one). But unless $n = 1$, this vertex algebra does not have very nice properties. For one, this space is extended the same way independently of the rank of L. In the next section we found a "correct" replacement for $W_L(\omega)$ at least if L is a multiple of the root lattice.

6. $\mathcal{W}$-algebras and principal subspaces

In this part we specialize $L = \sqrt{p}Q$, where $p \geq 1$ and Q is a root lattice (of ADE type) with bilinear form normalized such that $\langle\alpha, \alpha\rangle = 2$ for each root α. We should say that this restriction is not that crucial right now, and we can obtain interesting objects even if L is hyperbolic or indefinite. We equip V_L with a vertex algebra structure as earlier [12, 22, 27, 15] (by choosing an appropriate 2-cocycle). Denote by α_i, $i = 1, \ldots, n$, a set of simple roots of Q. (Important:) For the conformal vector we conveniently choose

$$\omega = \omega_{st} + \frac{p-1}{2\sqrt{p}} \sum_{\alpha\in\Delta_+} \alpha(-2)\mathbf{1}, \tag{6.1}$$

where ω_{st} is the standard (quadratic) Virasoro generator [27] and Δ_+ the set of positive roots. Then V_L is a conformal vertex algebra of central charge

$$\mathrm{rank}(L) + 12\langle\rho, \rho\rangle \left(2 - p - \frac{1}{p}\right), \tag{6.2}$$

where ρ is the half-sum of positive roots. In this section, the character of a module M is defined by using more standard formula

$$\chi_M(q) = \mathrm{tr}_M q^{L(0)-c/24}.$$

Consider the operators

$$e_0^{\sqrt{p}\alpha_i}, \quad e_0^{-\alpha_j/\sqrt{p}}, \quad 1 \leq i, j \leq \mathrm{rank}(L) \tag{6.3}$$

acting between V_L and the appropriate V_L-module. More precisely, [2], [19]:

Lemma 6.1. *For every i and j the operators $e_0^{\sqrt{p}\alpha_i}$ and $e_0^{-\alpha_j/\sqrt{p}}$ commute with each other, and they both commute with the Virasoro algebra.*

Proof. It is easy to see from the definition of ω that the vectors in question are of conformal weight one. Thus they must commute with $L(n)$, for all $n \in \mathbb{Z}$. We compute

$$\left[e_0^{\sqrt{p}\alpha_i}, e_0^{-\alpha_j/\sqrt{p}}\right] = \left(e_0^{\sqrt{p}\alpha_i} e^{-\alpha_j/\sqrt{p}}\right)_0 = \begin{cases} 0, & i \neq j, \\ \epsilon(\alpha_i(-1)e^{\alpha_i\sqrt{p}-\alpha_i/\sqrt{p}})_0, & i = j. \end{cases}$$

where ϵ is a nonzero constant. Now, we show that (for $i = j$) the vector on the right hand side is a nonzero multiple of the zero mode of

$$L(-1)e^{\left(\sqrt{p}-1/\sqrt{p}\right)\alpha_i} = \sum_{j=1}^{n} h_j(-1)h_j(0)e^{\left(\sqrt{p}-1/\sqrt{p}\right)\alpha_i}$$

where h_j, $j = 1, \ldots, n$, form an orthogonal basis of $\mathfrak{h} = L \otimes \mathbb{C}$. Indeed, the last expression is a nonzero multiple of

$$\sum_{j=1}^{n} \langle h_j, \alpha_i \rangle h_j e^{\left(\sqrt{p}-1/\sqrt{p}\right)\alpha_i} = \alpha_i(-1)e^{\left(\sqrt{p}-1/\sqrt{p}\right)\alpha_i}.$$

The assertion now follows from the relation $(L(-1)u)_0 = 0$. □

In the physics literature operators $e_0^{\sqrt{p}\alpha_i}$ and $e_0^{-\alpha_j/\sqrt{p}}$ are usually referred as the *long* and *short* screenings, respectively. It is well-known that intersection of the kernels of residues of vertex operators give rise to a vertex subalgebra (cf. [21]).

There are two main families of subalgebras of interest here:

$$\overline{M(1)}_p = \bigcap_{i=1}^{n} \mathrm{Ker}_{M(1)}\big(e_0^{-\alpha_i/\sqrt{p}}\big).$$

$$\mathcal{W}(p)_Q = \bigcap_{i=1}^{n} \mathrm{Ker}_{V_L}\big(e_0^{-\alpha_i/\sqrt{p}}\big).$$

$$\widetilde{M(1)}_p = \bigcap_{i=1}^{n} \mathrm{Ker}_{M(1)}\big(e_0^{\sqrt{p}\alpha_i}\big).$$

$$\mathcal{W}(p)_Q^{\diamond} = \bigcap_{i=1}^{n} \mathrm{Ker}_{V_L}\big(e_0^{\sqrt{p}\alpha_i}\big).$$

For $p = 1$, the vertex algebra $\overline{M(1)}$ is also known as the affine $\mathcal{W}$-algebra associated to the affine Lie algebra of type Q and is usually denoted by $\mathcal{W}_1(\mathfrak{g})$. It is known that $\overline{M(1)}_1 = \widetilde{M(1)}_1$. We also invoke a result from [21], originally due to B. Feigin and E. Frenkel, saying that $\overline{M(1)}_1$ is freely generated by the Virasoro vector and $n-1$ primary fields. Consequently, we have the well-known character formula [20]

$$\chi_{\overline{M(1)}_1}(q) = q^{-n/24} \prod_{i=1}^{\infty} \prod_{j=1}^{n} \frac{1}{1-q^{i+j}}.$$

For $p \geq 2$, there are still no rigorous results about $\overline{M(1)}_p$ and of $\widetilde{M(1)}_p$, although there are concrete conjectures about their structure [19], [1]. So let us focus first on the $p = 1$ case only and consider $\mathcal{W}(1)_Q$ and $\mathcal{W}(1)_Q^{\diamond}$. Later we shall return to $p \geq 2$ at least for $Q = A_1$, where everything can be made very explicit.

We recall first a result (discussed in [20]) regarding the structure of V_Q, viewed as $\mathfrak{g} \times \mathcal{W}_1(\mathfrak{g})$-module. It is known that V_P, here P is the weight lattice, decomposes as

$$V_P = \sum_{\lambda \in P^+} m_\lambda V(\lambda) \otimes \mathcal{W}_1(\mathfrak{g}, \lambda).$$

Here $V(\lambda)$ is the irreducible $\mathfrak{g}$-module of dominant highest weight λ, $\mathcal{W}_1(\mathfrak{g}, \lambda)$ is an irreducible $\mathcal{W}_1(\mathfrak{g})$-module, and $m_\lambda \geq 0$ is a certain multiplicity. In the case $Q = A_n$ we can actually say more (cf. Theorem 6.1 [20]).

Theorem 6.2. *The space V_P decomposes as*

$$\bigoplus_{\lambda \in P^+} V(\lambda) \otimes \mathcal{W}_1(\mathfrak{sl}_{n+1}, \lambda).$$

Moreover,

$$V_L = \bigoplus_{\lambda \in P^+ \cap Q} V(\lambda) \otimes \mathcal{W}_1(\mathfrak{sl}_{n+1}, \lambda).$$

Theorem 6.3. *Let $Q = A_n$. The vertex operator algebra $\mathcal{W}(1)_Q^\diamond$ is generated by the affine $\mathcal{W}$-algebra and finitely many primary fields.*

Proof. By using the above decomposition we get

$$V_L = \bigoplus_{\lambda \in Q \cap P^+} V(\lambda) \otimes \mathcal{W}_1(\mathfrak{sl}_{n+1}, \lambda).$$

From here and $\dim(V(\lambda)^{\mathfrak{n}_+}) = 1$, where the highest weight space is spanned by v_λ, we infer

$$V_L^{\mathfrak{n}^+} = \bigoplus_{\lambda \in Q \cap P^+} \mathbb{C} v_\lambda \otimes \mathcal{W}_1(\mathfrak{sl}_{n+1}, \lambda).$$

But v_λ can be taken to be $e^\lambda \in \mathbb{C}[L]$. To finish we have to identify all

$$\lambda \in Q \cap P^+. \tag{6.4}$$

Recall $\{\alpha_i\}_{i=1}^n$, a basis of Q. Let $\mathbf{x} = (x_1, \dots, x_n)$, $\lambda = x_1\alpha_1 + \cdots + x_n\alpha_n$. Then λ satisfies the condition (6.4) iff

$$C \cdot \mathbf{x} \geq 0,$$

where C is the Cartan matrix of Q. Thus the set of all λ forms an affine semigroup in $\mathbb{N}^n$, which is always finitely generated (as a monoid). Denote the set of minimal generators (of the monoid) by $\{\beta_1, \dots, \beta_k\}$. Because β_i all lie in the fundamental Weyl chamber we have [29]

$$[e^{\beta_i}_m, e^{\beta_j}_n] = 0.$$

Now we easily infer $e^\lambda \in \langle e^{\beta_1}, \dots, e^{\beta_k} \rangle$ for every $e^\lambda \in \mathcal{W}(1)_Q^\diamond$. □

The vertex algebra $\langle e^{\beta_1}, \dots, e^{\beta_k} \rangle$ is what we call the principal subalgebra of $\mathcal{W}(1)_Q^\diamond$.

Proposition 6.4. *The cardinality of the minimal generating set of $\mathcal{W}(1)_Q^\diamond$ is $n+$ the number of primitive nonnegative solutions of the Diophantine equation*

$$x_1 + 2x_2 + \cdots + nx_n \equiv 0 \pmod{n+1}. \tag{6.5}$$

Proof. The proof is direct from the description of fundamental weights as linear combination of simple roots. So $(x_1, \dots, x_n)$ is a nonnegative solution of (6.5) if and only if $\sum_i x_i \omega_i \in Q \cap P^+$. □

Example 6.5. If we take $Q = A_2$, the affine $\mathcal{W}$-algebra is the Zamolodchikov algebra with two generating fields $T(z)$ and $W(z)$ of conformal weights 2 and 3, which together with

$$e^{2\alpha_1+\alpha_2}(z), \quad e^{\alpha_1+2\alpha_2}(z), \quad e^{\alpha_1+\alpha_2}(z),$$

generate $\mathcal{W}(1)_{A_2}^\diamond$.

Here we consider the lattice $L = \sqrt{p}\mathbb{Z}\alpha$. We first recall results on the decomposition of $V_L \supset \mathcal{W}(p)^{\diamond}_{A_1}$ (in the continuation we omit the subscript A_1 for brevity) into indecomposable Virasoro modules of central charge $c_{p,1} = 1 - \frac{6(p-1)^2}{p}$, where the central charge is computed via (6.2). As usual we denote by $L(c,h)$ the highest weight irreducible Virasoro module of central charge c and lowest weight h. We first invoke a result of Feigin and Fuchs from (say) [1]. We have

$$\mathrm{Soc}(V_L) = \bigoplus_{n=0}^{\infty}(2n+1)L(c_{p,1}, h_{1,2n+1}), \tag{6.6}$$

$$V_L/\mathrm{Soc}(V_L) = \bigoplus_{n=1}^{\infty}(2n)L(c_{p,1}, h_{1,-2n+1}), \tag{6.7}$$

where $h_{1,m+1} = \frac{m(m-2p+2)}{4p}$.

Lemma 6.6. *As a Virasoro algebra module, the vertex algebra $\mathcal{W}(p)^{\diamond}$ is generated by the cosingular vectors $e^{n\alpha}$, $n \geq 0$. Each module*

$$V(n,p) := U(\mathrm{Vir}) \cdot e^{n\alpha}$$

is indecomposable for $p \geq 2$, and irreducible for $p = 1$. Moreover, for $p \geq 2$, we have

$$V(n,p)/L(c_{p,1}, h_{1,2n+1}) \cong L(c_{p,1}, h_{1,-2n+1})$$

and

$$\chi_{V(n,p)}(q) = \chi_{L(c_{p,1},h_{1,2n+1})}(q) + \chi_{L(c_{p,1},h_{1,-2n+1})}(q).$$

Proof. Follows immediately from Theorem 1.1 in [1]. □

Theorem 6.7. *Let $Q = A_1$ as earlier. The vertex algebra $\mathcal{W}(p)^{\diamond}$ is strongly generated by e^{α} and ω.*

Proof. We invoke Lemma 6.6, which in particular imply $\mathcal{W}(p)^{\diamond} = \langle \omega, e^{\alpha} \rangle$. Clearly, as a vector space, $\mathcal{W}(p)^{\diamond}$ is spanned by

$$L(-i_1)\dots L(-i_k)e^{\alpha}_{-2p(n-1)-1}\cdots e^{\alpha}_{-2p-1}e^{\alpha}_{-1}\mathbf{1} = L(-i_1)\dots L(-i_k)e^{n\alpha},$$

where $i_k \geq 1$ and $n \geq 1$. Thus we only have to show that $L(-1) = \omega_0$ can be eliminated from the spanning set. For that it suffices to show

$$L(-1)^s e^{n\alpha} \in \langle e^{\alpha} \rangle$$

for every $s \geq 1$ and $n \geq 0$. We show this relation by induction on n and on all s. This certainly holds for $n = 0$ for all s (a consequence of $L(-1)\mathbf{1} = 0$). Recall $[L(-1), e^{\alpha}_k] = -ke^{\alpha}_{k-1}$.

Assume $L(-1)^s e^{n\alpha} \in \langle e^{\alpha} \rangle$, which is our induction hypothesis. From

$$v = L(-1)^s e^{(n+1)\alpha} = L(-1)^s e^{\alpha}_{-2np-1}e^{n\alpha}$$

by repeatedly applying bracket relations we found out that

$$v = \sum_i \lambda_i e_i^{\alpha} L(-1)^{s_i} e^{(n-1)\alpha} \in \langle e^{\alpha} \rangle$$

by the inductive hypothesis. Thus we can reduce the spanning set of $\langle e^{\alpha}, \omega \rangle$ to be

$$L(-i_1)\dots L(-i_k) e^{\alpha}_{-n_1} \dots e^{\alpha}_{-n_s} \mathbf{1}$$

where $i_j \geq 2$ and $n_k \geq 1$. $\square$

Proposition 6.8.

$$\chi_{\mathcal{W}(p)^{\diamond}}(q) = \frac{q^{-c_{p,1}/24}}{(q)_{\infty}}.$$

Proof. As we already mentioned $\mathcal{W}(p)^{\diamond}$ is generated by $V(n,p)$, But since $V(n,p)$, $n \geq 0$, all have distinct α-charges, we obtain

$$\mathcal{W}(p)^{\diamond} = \bigoplus_{n=0}^{\infty} V(p,n).$$

For $p \geq 2$, by the above lemma

$$\begin{aligned} \chi_{\mathcal{W}(p)^{\diamond}}(q) &= \sum_{n=0}^{\infty} \chi_{L(c_{p,1},h_{1,2n+1})}(q) + \chi_{L(c_{p,1},h_{1,-2n+1})}(q) \\ &= \chi_{M(1)}(q) = \frac{q^{-c_{p,1}/24}}{(q)_{\infty}}. \end{aligned} \tag{6.8}$$

For $p = 1$,

$$\chi_{\mathcal{W}(1)^{\diamond}}(q) = \sum_{n=0}^{\infty} \chi_{L(1,n^2)}(q) = \frac{q^{-1/24}}{(q)_{\infty}}.$$ $\square$

Observe that the subalgebra $\langle e^{\alpha} \rangle \subset \mathcal{W}(p)^{\diamond}$ is also the principal subalgebra of V_L whose properties are well recorded in the literature [18], [16], [13], [29]. Therefore it is very natural to analyze the (commutative) vertex subalgebra generated by the minimal generating set $e^{\beta_1}, \dots, e^{\beta_k}$. More precisely, we would like to address the following.

Problem 6.9. As in [29], describe the vertex algebra $\langle e^{\beta_1}, \dots, e^{\beta_k} \rangle$ as a quotient of the universal commutative vertex algebra modulo relations. Find its combinatorial basis and compute the graded dimension.

As we indicated in the introduction, this is already nontrivial for A_2 where we have to analyze $\langle e^{2\alpha_1+\alpha_2}, e^{\alpha_1+\alpha_2}, e^{\alpha_1+2\alpha_2} \rangle$. The previous problem was already solved for $Q = A_1$ [18] and $Q = E_8$ [29] (in both cases β_i form a basis of L).

Furthermore, observe that vertex algebra $\mathcal{W}(p)_Q$ also contains a subalgebra analogous to $\langle e^{\beta_1}, \dots, e^{\beta_k} \rangle$. For the vertex algebra $\mathcal{W}(p)_Q$ the relevant screenings are $e^{-\alpha_j/\sqrt{p}}$ [19] [2]. They all annihilate the $e^{-\beta_i}$, where β_i are as before. Therefore

$$\langle e^{-\beta_1}, \ldots, e^{-\beta_k} \rangle \subset \mathcal{W}(p)_Q.$$

There is essentially no difference between $\langle e^{-\beta_1}, \ldots, e^{-\beta_k} \rangle$ and $\langle e^{\beta_1}, \ldots, e^{\beta_k} \rangle$, at least when it comes to generators and relation. However, due to the nonstandard conformal vector (6.1), for $p \geq 2$, we have $\mathrm{wt}(e^{\beta_i}) \neq \mathrm{wt}(e^{-\beta_i})$, thus the two vertex algebras will have different graded dimensions. This is already evident for $Q = A_1$, where

$$\chi_{\langle e^{\alpha} \rangle}(q) = q^{-c_{p,1}/24} \sum_{n=0}^{\infty} \frac{q^{pn^2-(p-1)n}}{(q)_n}$$

but

$$\chi_{\langle e^{-\alpha} \rangle}(q) = q^{-c_{p,1}/24} \sum_{n=0}^{\infty} \frac{q^{pn^2+(p-1)n}}{(q)_n}.$$

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Max Planck Institute für Mathematik, Vivatsgasse 7, Bonn, Germany

Permanent Address: Department of Mathematics and Statistics, University at Albany (SUNY), Albany, NY 12222
amilas@math.albany.edu

Department of Mathematics, University of Tennessee, Chattanooga
michael-penn@utc.edu

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